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SUPERCYCLIC AND HYPERCYCLIC NON-CONVOLUTION OPERATORS

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ABSTRACT. A continuous linear operator $T : X \to X$ is hypercyclic/supercyclic if there is a vector $f \in X$ such that the orbit $\mathrm{Orb}(T, f) = \{T^n f\}$/respectively the set of scalar-multiples of the orbit elements, forms a dense set. A famous theorem, due to G. Godefroy & J. Shapiro, states that every non-scalar convolution operator, on the space $\mathscr{H}$ of entire functions in d variables, is hypercyclic (and thus supercyclic). This motivates us to study cyclicity of operators on $\mathscr{H}$ outside the set of convolution operators. We establish large classes of supercyclic and hypercyclic non-convolution operators.

KEYWORDS: *Hypercyclic, backward shift, convolution operator, exponential type, PDE-preserving, Fischer pair.*

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1. INTRODUCTION AND NOTATION

Let $\mathbb{T} = (T_n)$ be a sequence of continuous linear operators on a topological vector space X. Let $\mathrm{Orb}(\mathbb{T}, f) \equiv \{T_n f : n \geqslant 0\}$ denote the orbit of $f \in X$ under $\mathbb{T}$ and by $\mathrm{Orb}_{\mathrm{l}}(\mathbb{T}, f)$ and $\mathrm{Orb}_{\mathrm{s}}(\mathbb{T}, f)$ we denote the linear hull respectively the set of scalar multiples of the elements in $\mathrm{Orb}(\mathbb{T}, f)$. Recall that $\mathbb{T}$ is said to be *cyclic/supercyclic/hypercyclic* if $\mathrm{Orb}_{\mathrm{l}}(\mathbb{T}, f)$ / $\mathrm{Orb}_{\mathrm{s}}(\mathbb{T}, f)$ / $\mathrm{Orb}(\mathbb{T}, f)$ is dense in X, respectively, for some $f \in X$. (Thus hypercyclic implies supercyclic which, in turn, implies cyclic.) The vector f is said to be of corresponding cyclic type (for $\mathbb{T}$). An operator $T : X \to X$ is cyclic (with cyclic vector f) when $\mathbb{T} \equiv (T^n)$ is cyclic (with cyclic vector f), and analogously for super- and hypercyclicity. In the case of a single operator we write, simply, $\mathrm{Orb}(T, f)$ etc. A cyclic vector manifold for T is a vector space $\mathscr{M} \subseteq X$ whose non-zero vectors are cyclic for T, supercyclic and hypercyclic vector manifolds are defined in the same way. (A full account of the significance of all these notions is given in [11] and we refer to [12] for a nice overview of the theory of hypercyclicity.)

We let d be a fixed arbitrary positive natural number and denote by $\mathscr{H}$ the Fréchet space of entire functions in d variables, equipped with the compact-open topology. Thus a generating family of semi-norms is obtained by $\|f\|_n \equiv \sup_{|z|\leqslant n} |f(z)|$, $n \in \mathbb{N} \equiv \{0, 1, \ldots\}$. ($|z| \equiv \sqrt{\sum |z_i|^2}$.) In 1929 Birkhoff proved that, in the case of one variable, every translation operator τ_a, $a \neq 0$, is hypercyclic on $\mathscr{H}$. ($\tau_a f(z) \equiv f(z+a)$.) MacLane obtained in 1952 the analogous result for the differentiation operator D (see [11] for further references to these two classical results). Note that both τ_a and D are convolution operators:

DEFINITION 1.1. The algebra of continuous linear operators on $\mathscr{H}$ is denoted by $\mathscr{L}$. A *convolution operator* on $\mathscr{H}$ is an operator $T \in \mathscr{L}$ that commutes with all translations τ_a. The set of convolution operators on $\mathscr{H}$ is denoted by $\mathscr{C}$.

In 1991 Godefroy & Shapiro generalized Birkhoff's and MacLane's results considerably. Indeed, we let Exp denote the algebra of exponential type functions, i.e., the set of all $\varphi \in \mathscr{H}$ such that $|\varphi(\xi)| \leqslant Me^{r|\xi|}$ for some $M, r > 0$, Theorem 5.1 and Proposition 5.2 in [11] contain the following (cf. Theorem 3.1 of [16]):

PROPOSITION 1.2 (Godefroy, J. Shapiro). *The mapping* $\varphi = \sum_{\alpha \in \mathbb{N}^d} \varphi_\alpha \xi^\alpha \mapsto \varphi(D) \equiv \sum_\alpha \varphi_\alpha D^\alpha$ *(standard multi-index notation, see p.* 139*) is an algebra isomorphism between* Exp *and* $\mathscr{C}$*. Every non-scalar convolution operator* $\varphi(D)$ *on* $\mathscr{H}$ *(d is arbitrary) is hypercyclic. (Non-scalar means that* φ *is not a constant mapping.)*

Their proof in [11], of the hypercyclicity, rests on the famous Hypercyclicity Criterion, which we formulate in Proposition 2.12. Godefroy-Shapiro's result motivates us, and others [1], see also [4], [6], [8], to study cyclic properties of operators outside $\mathscr{C}$. The objective in this note is to establish supercyclic and hypercyclic non-convolution operators on $\mathscr{H}$, and we shall apply results from our study on PDE-preserving operators [17]–[21]:

DEFINITION 1.3. An operator $T \in \mathscr{L}$ is PDE-preserving for a set $\mathbb{P} \subseteq$ Exp if it maps $\ker \varphi(D)$ invariantly for all $\varphi \in \mathbb{P}$. The set, and algebra, of PDE-preserving operators for $\mathbb{P}$ is denoted by $\mathscr{O}(\mathbb{P})$. (Note that $\mathscr{O}(\mathbb{P}) = \bigcap_{\varphi \in \mathbb{P}} \mathscr{O}(\varphi)$.)

Since the elements of $\mathscr{C}$ commute, $\mathscr{C}$ forms a commutative subalgebra of $\mathscr{O}(\mathbb{P})$ for *any* set $\mathbb{P}$. Consider the following result (see Theorem 2.4): An operator T is PDE-preserving for a given $\varphi \neq 0$ if and only if T "almost commutes" with $\varphi(D)$ in the sense that $\varphi(D)T = T^{(\varphi)}\varphi(D)$ for some $T^{(\varphi)} \in \mathscr{L}$. In fact, by Malgrange's Theorem [14], $\varphi(D)$ is surjective so $T^{(\varphi)}$ is unique and is called the derivative of $T \in \mathscr{O}(\varphi)$ with respect to φ. The following is now elementary:

THEOREM 1.4. *Let* $0 \neq \varphi \in \mathrm{Exp}$ *and assume that* $T \in \mathscr{O}(\varphi)$ *is cyclic and that* $f \in \mathscr{H}$ *is a corresponding cyclic vector. Then* $T^{(\varphi)}$ *is also cyclic and* $\varphi(D)f$ *forms a cyclic vector. The analogue holds for both super- and hypercyclicity.*

Thus by studying PDE-preserving properties, and corresponding derivatives, of operators of given cyclic type, it is possible to get new such operators. Unfortunately, by commutativity, any derivative of any convolution operator T is a new convolution operator (in fact, equal to T), so Theorem 1.4 does not provide us with any non-convolution operators by starting out of operators $T \in \mathscr{C}$. Thus, to apply Theorem 1.4 in this way, we must first find a, say, hypercyclic operator outside $\mathscr{C}$, and there are very few such examples in the literature. However, we shall establish a set $\mathscr{O}_{\mathrm{S}}$ of supercyclic operators on $\mathscr{H}$ and a multiplicative closed subset $\mathscr{O}_{\mathrm{H}}$ formed by hypercyclic operators, where $\mathscr{O}_{\mathrm{S}} \setminus \mathscr{C}$ and $\mathscr{O}_{\mathrm{H}} \setminus \mathscr{C}$ are large, and we can apply Theorem 1.4 in the way that we have indicated. In particular we have that $\mathscr{O}_{\mathrm{H}} \subseteq \mathscr{O}_{\mathrm{S}} \subseteq \mathscr{O}(\mathbb{H})$, where $\mathbb{H}$ denotes the set of homogeneous polynomials, so Theorem 1.4 can be applied on any $T \in \mathscr{O}_{\mathrm{S}}/\mathscr{O}_{\mathrm{H}}$ and $\varphi = P \in \mathbb{H}$, and in this way we derive some "internal" structures of the set of supercyclic/hypercyclic vectors for the operators in $\mathscr{O}_{\mathrm{S}}/\mathscr{O}_{\mathrm{H}}$.

An important concept, and tool, in the theory of cyclic phenomena is the notion of backward shifts [11], [13]. A general theory for cyclic properties of operators that commute with a so called generalized backward shift B, and thus with any of its powers B^n, is developed in [11] (in particular, see Theorem 3.6). Now, $B = D$ is a generalized backward shift on $\mathscr{H}$ in the case of one variable but unfortunately, in view of our purposes, an operator $T \in \mathscr{L}$ commutes with D if and only if $T \in \mathscr{C}$ ([11], Proposittion 5.2). Thus the theory in [11] is not applicable to obtain, say, hypercyclic operators outside $\mathscr{C}$. Now, our result(s), when $d = 1$, is based on the fact that, roughly, it is possible to extend their ideas on backward shifts for operators that almost commute with any power of B, i.e., with any element of $\mathscr{O}(\{1, \xi, \xi^2, \ldots\}) = \mathscr{O}(\mathbb{H})$. When $d > 1$, there is no analogue of the backward shift $B = D$. However, we can extend our one variable result(s) by showing that, for any non-constant homogeneous polynomial P, $P(D)$ may serve as some sort of a backward shift. A key to this is results from the study of so called Fischer decompositions and Fischer pairs, developed by H. Shapiro and others [10], [15], [22]:

DEFINITION 1.5 (Fischer pair). A pair (T, S) of operators $S : X \to Y$, $T : Y \to X$ is said to form a Fischer pair for X when TS maps X bijectively.

(If (T, S) is a Fischer pair, then $Y = \ker T \oplus \operatorname{Im} S$ — a Fischer decomposition. Conversely, if T is surjective and S is injective, such a decomposition of Y implies (T, S) being a Fischer pair.) We apply the fact that $(P(D), P^*)$ forms a Fischer pair for $\mathscr{H}$, where P^* denotes the homogeneous polynomial obtained by conjugating the coefficients in P and $P^* : f \mapsto P^* f$, see Proposition 2.10. (We show also that even other Fischer pairs, i.e., not necessarily built up by homogeneous polynomials, provide us with alternative "backward shifts" in the same way, see Remark 3.10 (i) and (iii) at the end.)

The paper is organized as follows: First we recall some fundamental results from our study on PDE-preserving operators. Our main results are Theorems A, B and C, which are exposed in Section 3. In Theorem A, we establish the class $\mathscr{O}_{\mathrm{S}} \subseteq \mathscr{O}(\mathbb{H})$ of supercyclic operators. Now, a characterization result for $\mathscr{O}(\mathbb{H})$ shows that $\mathscr{O}(\mathbb{H})$ is formed by the operators of the form:

$$\Phi(D)f = \sum_{n\geqslant 0} H_n(\varphi_n(D)f), \quad \Phi = (\varphi_n) \in \mathcal{S}, \tag{1.1}$$

where $\mathcal{S}$ denotes a set of sequences in Exp that satisfies a certain growth condition (see the list of notations below or Definition 2.6), and H_n denotes the projection on $\mathscr{H}$ onto the set $\mathscr{H}_n$ of n-homogeneous polynomials defined by $f = \sum_{m\geqslant 0} f_m \mapsto f_n$. (See also Proposition 2.7, and note also that $\Phi(D) \in \mathscr{C}$ if and only if $\Phi = (\varphi, \varphi, \ldots)$ for some $\varphi \in \mathrm{Exp}$ $(\Phi(D) = \varphi(D))$.) Thus, any element of $\mathscr{O}_{\mathrm{S}}$ have the explicit form (1.1), and $\mathscr{O}_{\mathrm{S}}$ is characterized by the corresponding subset $\mathcal{S}_{\mathrm{S}} \subseteq \mathcal{S}$, defined in Definition 3.2. (This explicit representation of our operators is, we think, a strength of our results.)

Next, by applying Theorem 1.4 and the fact that $\mathscr{O}_{\mathrm{S}} \subseteq \mathscr{O}(\mathbb{H})$, we prove in Theorem B that $\mathscr{O}_{\mathrm{S}}$ is stable under certain operations and establish supercyclic vector manifolds and invariant sets of supercyclic vectors, for any $T \in \mathscr{O}_{\mathrm{S}}$.

Then we consider the more delicate problem — the existence of hypercyclic non-convolution operators. Our results obtained so far motivate us to study if, in particular, there are any such operators in $\mathscr{O}_{\mathrm{S}}$. In Theorem C we establish the multiplicative closed subset $\mathscr{O}_{\mathrm{H}}$ of $\mathscr{O}_{\mathrm{S}}$ formed by hypercyclic operators, and prove an analogue of Theorem B for $\mathscr{O}_{\mathrm{H}}$, and hence, in particular, how to obtain hypercyclic vector manifolds for any $T \in \mathscr{O}_{\mathrm{H}}$. In fact, $\mathscr{O}_{\mathrm{H}} \equiv \{\Phi(D) : \Phi \in \mathcal{S}_{\mathrm{H}}\}$, where $\mathcal{S}_{\mathrm{H}} \subseteq \mathcal{S}_{\mathrm{S}} \subseteq \mathcal{S}$ and

$$\mathcal{S}_{\mathrm{H}} \equiv \bigcup_{m\geqslant 1} \{(P_n) : \{P_n\} \subseteq \mathscr{H}_m,\ c \leqslant \|P_n\|_1 \leqslant CM^n \text{ for some } c, C, M > 0\}.$$

1.1. LIST OF NOTATIONS.

$\mathscr{H}$	{Entire functions} — d variables.
Exp, $\mathscr{P}$	{Exponential type functions}, {Polynomials} — d variables.
$\mathscr{P}_n$, $\mathscr{H}_n$, $\mathbb{H}$	$\{P \in \mathscr{P}$ of degree $\leqslant n\}$, $\{n$-homogeneous $P \in \mathscr{P}\}$, $\bigcup_{n\geqslant 0} \mathscr{H}_n$.
$\mathscr{L}, \mathscr{C}$	{continuous linear operators}, {convolution operators} (on $\mathscr{H}$).
$\mathscr{O}(\mathbb{P})$	The set of PDE-preserving operators for $\mathbb{P}$, see Definition 1.3.
$\mathfrak{S}$	The set of symbols (kernels) for $\mathscr{L}$, see Definition 2.1.
$\mathcal{S}$	$\{(\varphi_n)_{n\geqslant 0} : \varphi_n \in \mathscr{H},\ \lvert\varphi_n(\xi)\rvert \leqslant CM^n e^{r\lvert\xi\rvert}, n = 0, 1, \ldots\}$.
$\mathcal{S}_{\mathrm{S}}$, $\mathcal{S}_{\mathrm{H}}$	See Definition 3.2 and 3.6 respectively.

$\mathscr{O}_{\mathrm{S}}$, $\mathscr{O}_{\mathrm{H}}$	$\{\Phi(D) : \Phi \in \mathcal{S}_{\mathrm{S}}\}$, $\{\Phi(D) : \Phi \in \mathcal{S}_{\mathrm{H}}\}$.
τ_a, e_a	$\tau_a f(z) \equiv f(a+z)$, $e_a \equiv \mathrm{e}^{\langle \cdot, a\rangle}$ where $\langle z, \xi\rangle \equiv \sum z_i \xi_i$ – $a \in \mathbb{C}^d$.
z^α, D^α	$z_1^{\alpha_1} \cdots z_d^{\alpha_d}$, $D_1^{\alpha_1} \cdots D_d^{\alpha_d}$ $(D_i \equiv \partial/\partial z_i)$ – $\alpha \in \mathbb{N}^d$.
H_n	The projector $\mathscr{H} \ni f = \sum_{m \geqslant 0} f_m \mapsto f_n \in \mathscr{H}_n$.
$\Phi^{(m)}$	$\equiv (\varphi_{n+m})_{n\geqslant 0} \in \mathcal{S}$ where $\Phi = (\varphi_n) \in \mathcal{S}$.
$\varphi(D)$	$\equiv \sum_\alpha \varphi_\alpha D^\alpha \in \mathscr{C}$ where $\varphi = \sum_\alpha \varphi_\alpha \xi^\alpha \in \mathrm{Exp}$.
$\Phi(D)$	$\equiv \sum_{n\geqslant 0} H_n \varphi_n(D) \in \mathscr{O}(\mathbb{H})$ where $\Phi = (\varphi_n) \in \mathcal{S}$.
$P(\cdot, D)$	$\equiv \sum_{\alpha,\beta} P_{\alpha,\beta} z^\alpha D^\beta \in \mathscr{L}$ where $P(z,\xi) = \sum_{\alpha,\beta} P_{\alpha,\beta} z^\alpha \xi^\beta \in \mathfrak{S}$.
$T^{(\varphi)}$	The operator $\in \mathscr{L}$ such that: $\varphi(D)T = T^{(\varphi)}\varphi(D)$, $T \in \mathscr{O}(\varphi)$.
$\\|\cdot\\|_n$, $\\|\\|\cdot\\|\\|_n$	$\sup_{\lvert z\rvert \leqslant n} \lvert\cdot\rvert$, $\sup_{\xi \in \mathbb{C}^d} \lvert\cdot\rvert \mathrm{e}^{-n\lvert\xi\rvert}$.
(P, Q), $\\|P\\|$	$\sum_\alpha P_\alpha \overline{Q}_\alpha \alpha!$, $\sqrt{(P,P)}$ where $P = \sum P_\alpha z^\alpha$, $Q = \sum Q_\alpha z^\alpha \in \mathscr{P}$.

2. FUNDAMENTALS

For given $n \in \mathbb{N}$, Exp_n denotes the Banach space of functions $\varphi \in \mathscr{H}$ such that $|||\varphi|||_n \equiv \sup_{\xi\in\mathbb{C}^d} |\varphi(\xi)|\mathrm{e}^{-n|\xi|} < \infty$, equipped with the norm $|||\cdot|||_n$ thus defined. Exp is given by $\bigcup_{n\geqslant 0} \mathrm{Exp}_n$, and we provide Exp with the corresponding inductive locally convex topology. We put $e_\xi \equiv \mathrm{e}^{\langle\cdot,\xi\rangle} \in \mathscr{H}$, where $\langle z,\xi\rangle \equiv \sum z_i\xi_i$, and recall that the Fourier-Borel transform $\mathcal{F}$, defined by $\mathscr{H}' \ni \lambda \mapsto \mathcal{F}\lambda(\xi) \equiv \lambda(e_\xi)$, is a topological isomorphism between $\mathscr{H}'$ (strong topology) and Exp. (Thus, $\mathscr{H}' \simeq \mathrm{Exp} \simeq \mathscr{C}$.) Thus $\mathscr{H}$ and Exp form a dual pair by $\langle f, \varphi\rangle \equiv \mathcal{F}^{-1}\varphi(f)$ (the *Martineau-duality*), and it is convenient to note that the transpose of $\varphi(D) \in \mathscr{C}$ is the multiplication operator $\varphi : \psi \mapsto \psi\varphi$ on Exp and, as a consequence of Malgrange's (existence) Theorem, $\ker \varphi(D)^\perp = \mathrm{Im}\,\varphi = \mathrm{Exp}\cdot\varphi$.

DEFINITION 2.1. $\mathfrak{S}$ denotes the set of entire mappings $P = P(z,\xi)$, in $2d$ variables $(z,\xi) \in \mathbb{C}^d \times \mathbb{C}^d$, with the following property: For every $n \geqslant 0$ there are $m = m_n, M = M_n \geqslant 0$ such that $\|P(\cdot,\xi)\|_n \leqslant M\mathrm{e}^{m|\xi|}$ (thus $P(z,\cdot) \in \mathrm{Exp}$).

We consider Exp as a subset of $\mathfrak{S}$ by $\varphi(z,\xi) = \varphi(\xi)$, $\varphi \in \mathrm{Exp}$, and have the following Kernel-Theorem for $\mathscr{L}$:

PROPOSITION 2.2. *$T \mapsto P(z,\xi) \equiv \mathrm{e}^{-\langle z,\xi\rangle} Te_\xi(z)$ defines a bijection between $\mathscr{L}$ and $\mathfrak{S}$. P is called the* symbol *for T, we write $T = P(\cdot, D)$ and have that $Tf(z) = \langle f, P(z,\cdot)e_z\rangle = \sum_{\alpha,\beta} P_{\alpha,\beta} z^\alpha D^\beta f$ (convergence in $\mathscr{H}$) where $P(z,\xi) = \sum_{\alpha,\beta} P_{\alpha,\beta} z^\alpha \xi^\beta$. The set of convolution operators, $\mathscr{C}$, corresponds to the symbol-set* Exp $\subseteq \mathfrak{S}$, *and $\varphi(D) = \varphi(\cdot, D)$, $\varphi \in$* Exp.

Proof. Let $T \in \mathscr{L}$. We must prove that $P(z,\xi) \equiv \mathrm{e}^{-\langle z,\xi\rangle} Te_\xi(z) \in \mathfrak{S}$. Clearly, $P(\cdot,\xi) \in \mathscr{H}$ and from $Te_\xi(z) = {}^{\mathrm{t}}Te_z(\xi)$, $P(z,\cdot) \in \mathscr{H}$. By Hartogs Theorem, P is entire on $\mathbb{C}^d \times \mathbb{C}^d$ and it remains to prove that P is bounded as required. First we note that $\mathbb{C}^d \ni z \mapsto e_z \in \mathrm{Exp}$ is continuous and thus so is $\mathbb{C}^d \ni z \mapsto P(z,\cdot) \in \mathrm{Exp}$. Hence $\{P(z,\cdot) : |z| \leqslant n\}$ forms a bounded set in Exp for any $n \geqslant 0$. Now, one can prove Lemma 1 of [21] that a set in Exp is bounded if and only if it is contained and bounded in some Exp_m and hence, $P \in \mathfrak{S}$. Conversely, let $P \in \mathfrak{S}$ and define $Tf(z) \equiv \langle f, P(z,\cdot)e_z\rangle$. It is easily checked that $T \in \mathscr{L}$ and $\mathrm{e}^{-\langle z,\xi\rangle} Te_\xi(z) = P(z,\xi)$. Thus, the map $\mathscr{L} \ni T \mapsto \mathrm{e}^{-\langle z,\xi\rangle} Te_\xi(z) \in \mathfrak{S}$ is onto and since $\{e_\xi\colon \xi \in \mathbb{C}^d\}$ forms a total set in $\mathscr{H}$, it is one-to-one.

Next, formally we have that $\langle f, P(z,\cdot)e_z\rangle = \sum_{\alpha,\beta} P_{\alpha,\beta} z^\alpha D^\beta f$, and that the identity indeed follows by noting that the series converges absolutely in $\mathscr{H}$, which is easily checked by virtue of Cauchy's Estimates.

The last part follows by Proposition 1.2. ∎

Thus every operator in $\mathscr{L}$ can be written as an infinite type of differential operator with variable coefficients, and the elements of $\mathscr{C}$ are those with constant coefficients (see also Chapter 6 of [2]).

A main result in our study of PDE-preserving operators is Theorem 2.4 that follows. The technical part in our proof is the following lemma and division property for $\mathfrak{S}$ ($\mathscr{L}$):

LEMMA 2.3. *Let $0 \neq \varphi \in \mathrm{Exp}$, $P \in \mathfrak{S}$ and assume $P(z,\xi) = \varphi(\xi)Q(z,\xi)$ where $Q(z,\cdot) \in \mathrm{Exp}$ for all $z \in \mathbb{C}^d$. Then $Q \in \mathfrak{S}$.*

Proof. See Lemma 2 of [21]. ∎

THEOREM 2.4 (Characterization Theorem). *Let $\varphi \in \mathrm{Exp}$ and $T = P(\cdot, D) \in \mathscr{L}$. Then the following are equivalent:*

(i) *T is PDE-preserving for φ;*

(ii) *$\varphi(D)T = S\varphi(D)$ for some $S \in \mathscr{L}$;*

(iii) *$\varphi | \varphi(\xi + D)P(\cdot,\xi)(z)$ in $\mathfrak{S}$, i.e., $\varphi(\xi + D)P(\cdot,\xi)(z) = \varphi(\xi)Q(z,\xi)$ for some $Q \in \mathfrak{S}$.*

($\varphi(\xi + D) \equiv (\tau_\xi\varphi)(D) \in \mathscr{C}$.) If $\varphi \neq 0$ the operator S is unique and is called the derivative of $T \in \mathscr{O}(\varphi)$ with respect to φ and is denoted by $T^{(\varphi)}$.

Proof. We may assume $\varphi \neq 0$ and note that the uniqueness of S follows by the surjectivity of $\varphi(D)$. The equivalence between (ii) and (iii) follows by the observation $\varphi(D)Te_\xi(z) = \varphi(D)P(\cdot,\xi)e_\xi(z) = \mathrm{e}^{\langle z,\xi\rangle}\varphi(\xi + D)P(\cdot,\xi)(z)$. Since (ii) obviously implies (i), it remains to prove that if T is PDE-preserving for φ, then $\varphi | R$ in $\mathfrak{S}$ where $R(z,\xi) \equiv \varphi(D)Te_\xi(z)$ ($\in \mathfrak{S}$). For fixed $z \in \mathbb{C}^d$ let $\lambda_z(f) \equiv \varphi(D)Tf(z)$. Then $\lambda_z \in \mathscr{H}'$ and $\mathcal{F}\lambda_z(\xi) = R(z,\xi)$. We prove that $\mathcal{F}\lambda_z \in \mathrm{Im}\,\varphi =$

$\ker \varphi(D)^{\perp}$. But if $f \in \ker \varphi(D)$, then

$$\langle f, \mathcal{F}\lambda_z \rangle = \lambda_z(f) = \varphi(D)Tf(z) = 0$$

since $T \in \mathscr{O}(\varphi)$. Thus, for every $z \in \mathbb{C}^d$ there is a unique $Q(z, \cdot) \in \mathrm{Exp}$ such that $R(z, \xi) = \varphi(\xi)Q(z, \xi)$, $\xi \in \mathbb{C}^d$, and Lemma 2.3 completes the proof. ∎

Let $\mathscr{P}$ denote the algebra of (complex) polynomials in d variables and let $\mathscr{P}_n$ denote the vector space of polynomials in $\mathscr{P}$ of degree at most n. Recall that $\mathscr{H}_n$ denotes the set, and vector space, of n-homogeneous polynomials in $\mathscr{P}$ and $\mathbb{H}$ denotes the set $\bigcup_{n \geqslant 0} \mathscr{H}_n$ of all homogeneous polynomials. ($\mathscr{P}_0 = \mathscr{H}_0 \equiv \mathbb{C}$.)

LEMMA 2.5. *If $T \in \mathscr{O}(\mathbb{H})$, then T maps every $\mathscr{P}_n$ invariantly. If $d = 1$ the converse holds true, i.e., $T \in \mathscr{O}(\mathbb{H})$ if and only if $T \in \mathscr{L}$ and maps every $\mathscr{P}_n$ invariantly.*

Proof. Let $n \geqslant 0$ and $P \in \mathscr{P}_n$. We must prove that $TP \in \mathscr{P}_n$. By Taylor's Formula, $f \in \mathscr{P}_n$ if and only if $Q(D)f = 0$ for all $Q \in \mathscr{H}_{n+1}$. So, for any such Q, $Q(D)P = 0$ and hence, $Q(D)TP = 0$ since $T \in \mathscr{O}(Q)$. The converse part, when $d = 1$, follows by the observation $\ker D^{n+1} = \mathscr{P}_n$. ∎

DEFINITION 2.6. $\mathcal{S}$ denotes the set of sequences $\Phi = (\varphi_n) = (\varphi_0, \dots)$ in Exp such that $|||\varphi_n|||_m \leqslant RM^n$, $n = 0, 1, \dots$, for some $R, M, m \geqslant 0$.

H_n denotes the projector in $\mathscr{H}$ onto $\mathscr{H}_n$ defined by $f = \sum f_m \mapsto f_n$, where $\sum f_m$ is the power series expansion of $f \in \mathscr{H}$. We have the following one-to-one correspondence between $\mathscr{O}(\mathbb{H})$ and $\mathcal{S}$:

PROPOSITION 2.7. *$\mathscr{O}(\mathbb{H})$ is formed by the operators of the form $\Phi(D)f \equiv \sum_{n \geqslant 0} H_n(\varphi_n(D)f)$, where $\Phi = (\varphi_n) \in \mathcal{S}$ and is unique. If $P \in \mathscr{H}_m$, $\Phi(D)^{(P)} = \Phi^{(m)}(D) \in \mathscr{O}(\mathbb{H})$ where $\Phi^{(m)} \equiv (\varphi_{n+m}) \in \mathcal{S}$. (Thus the derivative only depends on m, not on P.)*

Proof. A proof of the first part can be found in [17], see also [20] (in fact, the result is there extended to infinite-dimensional holomorphy). We prove the claim about the derivative. We note that, for any m-homogeneous polynomial P, $P(D)H_n = H_{n-m}P(D)$ if $n \geqslant m$ and $P(D)H_n = 0$ otherwise. Thus,

$$P(D)\Phi(D) = \sum_{n \geqslant 0} P(D)H_n\varphi_n(D) = \sum_{n \geqslant m} H_{n-m}P(D)\varphi_n(D) = \Phi^{(m)}(D)P(D)$$

since $P(D)$ and $\varphi_n(D)$ commute. ∎

EXAMPLE 2.8. With $\varphi_n = 1$ if $n \leqslant m$ and $\varphi_n = 0$ otherwise, $\Phi(D)$ is the mth *Taylor projector*, i.e. the operator obtained by mapping a function into its Taylor polynomial of order m at the origin. The *Euler operator* $\langle \cdot, D \rangle \equiv z_1D_1 + \cdots + z_dD_d$, i.e. the operator with symbol $\langle z, \xi \rangle \in \mathfrak{S}$, belongs to $\mathscr{O}(\mathbb{H})$. Indeed, for any power $m \geqslant 1$, $\langle \cdot, D \rangle^m = \Phi(D)$ where $\Phi = (\varphi_n = n^m)$.

We equip $\mathcal{S}$ with the algebra structure induced by $\mathscr{O}(\mathbb{H})$ so that $(\Phi\Psi)(D) = \Phi(D)\Psi(D)$. One can then prove Theorem 6 of [17] that if $(\xi_n) = \Phi\Psi$ in $\mathcal{S}$, then

$$\xi_n = \sum_{i=0}^{\infty} H_i(\varphi_n)\psi_{n+i}, \quad \Phi = (\varphi_n),\ \Psi = (\psi_n). \tag{2.1}$$

An element $\varphi \in \operatorname{Exp}$ is said to be non-degenerate if $\varphi(0) \neq 0$, and a sequence $\Phi = (\varphi_n)$ in Exp is non-degenerate if all the elements φ_n are. From (2.1) we deduce that the product $\Phi\Psi$ of any non-degenerate sequences Φ and Ψ in $\mathcal{S}$ is again non-degenerate ($\xi_n(0) = \varphi_n(0)\psi_n(0)$).

LEMMA 2.9. *Let $\Phi = (\varphi_n) \in \mathcal{S}$ be non-degenerate. Then $\Phi(D)$ maps every $\mathscr{P}_n$ isomorphically (cf. Lemma 2.5). Thus, the restriction of $\Phi(D)$ to $\mathscr{P}$ is an isomorphism.*

Proof. $\Phi(D)$ is surjective on $\mathscr{P}_0 = \mathbb{C}$ for $\Phi(D)1 = \varphi_0(0) \neq 0$. Next we note that if $|\alpha| \equiv \sum \alpha_i = m \geqslant 1$, then:
$(*)$ $\Phi(D)z^\alpha = \varphi_m(0)z^\alpha +$ (lower degree terms).
Assume $\Phi(D)$ is surjective on every $\mathscr{P}_m$, $m \leqslant n-1$ and let $P \in \mathscr{P}_n$. By $(*)$ we may find a $Q_n \in \mathscr{H}_n$ such that $\Phi(D)Q_n - P \in \mathscr{P}_{n-1}$ and hence, by the inductive hypothesis, $\Phi(D)Q_{n-1} = \Phi(D)Q_n - P$ for some $Q \in \mathscr{P}_{n-1}$. Thus $\Phi(D)$ maps $\mathscr{P}_n$ onto $\mathscr{P}_n$ for all n. To prove that $\Phi(D)$ is one-to-one on $\mathscr{P}_n$, it is clearly enough to prove that $\Phi(D)$ is injective on $\mathscr{P}$, which is obvious in view of $(*)$. ∎

For proofs of the following we refer to [22], the latter part is Fischer's classical Theorem from [10]:

PROPOSITION 2.10 (H. Shapiro, Fischer). *For any homogeneous polynomial $P \neq 0$, $(P(D), P^*)$ forms a Fischer pair for $\mathscr{H}$, where P^* is the homogeneous polynomial obtained by conjugating the coefficients in P and $P^* : f \mapsto P^*f$. Moreover, $P^*\mathscr{H}_n \subseteq \mathscr{H}_{n+m}$ and $P(D)\mathscr{H}_{n+m} \subseteq \mathscr{H}_n$ if $P \in \mathscr{H}_m$, and $(P(D), P^*)$ forms in this way a Fischer pair for $\mathscr{H}_n$ for any $n \geqslant 0$.*

In view of our purposes, we need estimates:

LEMMA 2.11. *For given dimension d, there is a constant $k = k(d)$ such that for any $P \in \mathscr{H}_m \setminus \{0\}$ and $Q \in \mathscr{H}_n$, $\|P^*(P(D)P^*)^{-1}Q\|_1 \leqslant k^n\|Q\|_1/m!\|P\|_1$. (Recall that $\|\cdot\|_1 \equiv \sup_{|z|\leqslant 1} |\cdot|$ and $|z| \equiv \sqrt{\sum |z_i|^2}$.)*

Proof. Consider the inner-product $(P,Q) \equiv \sum_\alpha P_\alpha \overline{Q}_\alpha \alpha!$ on $\mathscr{P}$, where $\alpha! \equiv \prod \alpha_i!$, $P = \sum_\alpha P_\alpha z^\alpha$ and the coefficients Q_α are defined analogously. By $\|\cdot\|$ we denote the corresponding (Fischer) norm. The key is to note that P^* is the Hilbert-adjoint of $P(D) : \mathscr{H}_{n+m} \to \mathscr{H}_n$, $P \in \mathscr{H}_m$, with respect to the inner-products induced by $(\cdot,\cdot)$. Indeed, let $f \in \mathscr{H}_n$ and put $g \equiv (P(D)P^*)^{-1}f \in \mathscr{H}_n$. Then, with $A \equiv P^*(P(D)P^*)^{-1}$, $P^*g = Af$ and Cauchy–Schwarz Inequality gives

$$\|f\|\|g\| = \|P(D)P^*g\|\|g\| \geqslant (P(D)P^*g, g) = \|P^*g\|^2 \geqslant \|P\|\|Af\|\|g\|,$$

since $\|P^*\| = \|P\|$ and, by the formula in the proof of Lemma 4 of [22], $\|P^*g\| \geqslant \|P^*\|\|g\|$. Thus the operator norm of $A : (\mathscr{H}_n, \|\cdot\|) \to (\mathscr{H}_{n+m}, \|\cdot\|)$ is not greater than $1/\|P\|$ and we only have to translate all this to the sup-norm $\|\cdot\|_1$. To do so we refer to p. 519 in [22], where the arguments show that

$$\|Q\|_1 \leqslant \frac{\|Q\|}{\sqrt{n!}} \leqslant (n+1)^{d/2} d^{n/2} \|Q\|_1$$

for any $Q \in \mathscr{H}_n$. (However, they are there dealing with the supremum norm over polydiscs and, in the right inequality, we have used that $\sup\limits_{\max_i |z_i| \leqslant 1} |Q| \leqslant d^{n/2}\|Q\|_1$ if $Q \in \mathscr{H}_n$.) Now, there is a constant $k = k(d)$ such that $k^n \geqslant (n+1)^{d/2} d^{n/2}$ for all $n \geqslant 0$. From $\|AQ\| \leqslant \|Q\|/\|P\|$ a straight forward computation gives the lemma. ∎

Finally we formulate the Fréchet space version of Corollary 1.4 of [11]:

PROPOSITION 2.12 (Godefroy, J. Shapiro). *Let X be a separable Fréchet space and $\mathbb{T} = (T_n)$ a sequence of continuous linear operators on X. Assume there are dense subsets $Z, Y \subseteq X$ (not necessarily subspaces) and a sequence of maps $\mathbb{S} = (S_n : Y \to X)$ (not necessarily continuous) such that:*

(i) $T_n z \to 0$ *for all* $z \in Z$;
(ii) $S_n y \to 0$ *for all* $y \in Y$;
(iii) $T_n S_n y = y$ *for all* $y \in Y$.

Then $\mathbb{T}$ is hypercyclic.

3. THE MAIN RESULTS

We are now ready to prove our first main result — Theorem A. We shall first prove the statement in the case of one variable, where the proof is based on the theory of backward shifts. (An alternative proof can be obtained by applying Corollary 3.3 of [3].)

THEOREM A (one variable). *Let $\Phi = (\varphi_n) \in \mathcal{S}$ be a sequence such that $\varphi_n = \xi^m \psi_n$, i.e. $\Phi(D) = \Psi(D)D^m$, for some $m \geqslant 1$ and non-degenerate $\Psi = (\psi_n) \in \mathcal{S}$. Then $\Phi(D)$ is supercyclic.*

Proof. Let $e_n \equiv z^n/n!$ denote the monomial basis vectors in $\mathscr{P}$ and define the "forward shift" $A : \mathscr{P} \to \mathscr{P}$ by: $Ae_n \equiv e_{n+1}$ and then extended linearly. Then BA is the identity on $\mathscr{P}$ where $B \equiv D$ (backward shift). We can find a non-degenerate sequence $\Phi_0 = (\phi_n) \in \mathcal{S}$ such that $\Phi(D) = B^m \Phi_0(D) = \Phi_0^{(m)}(D) B^m$ (i.e. $\Phi_0^{(m)} = \Psi$). Indeed, let ϕ_n, $n = 0, \ldots, m-1$, be arbitrary non-degenerate elements in Exp and put $\phi_n \equiv \psi_{n-m}$ for $n \geqslant m$. Then $\Phi_0 \equiv (\phi_n)$ is non-degenerate and $\Phi_0^{(m)} = \Psi$. From this point we apply the technique of Godefroy and Shapiro in the proof of Theorem 3.6.b of [11] (however, they are dealing with Banach spaces and we must complement with some arguments). Let $\Phi_0^{-1}(D)$ denote the inverse of $\Phi_0(D)$ as a mapping $\mathscr{P} \to \mathscr{P}$ (Lemma 2.9) and put $C \equiv \Phi_0^{-1}(D)A^m$ so that (∗) $\Phi(D)C = B^m A^m = \mathrm{Id}_{\mathscr{P}}$. C maps $\mathscr{P}_n$ into $\mathscr{P}_{n+m}$ and, with notation as in [11], we let $\sigma(n)$ denote the operator norm of this restriction of C. Here we assume that every finite-dimensional space $\mathscr{P}_n$ is equipped with the norm $|P|_n \equiv \sum_0^n \|H_i(P)\|_1$. Now, C^n maps $\mathscr{P}_k$ into $\mathscr{P}_{k+nm}$ with norm $\leqslant \sigma(k+(n-1)m)^n \equiv \sigma_{k,n}$ ([11], p. 246). Let $r_n \equiv n!\sigma_{n,n}$ and put $T_n \equiv r_n \Phi(D)^n$. Then $T_n \in \mathscr{L}$ and it suffices to prove that $\mathbb{T} \equiv (T_n)$ is hypercyclic. We shall apply Proposition 2.12. Define $S_n \equiv r_n^{-1}C^n : \mathscr{P} \to \mathscr{P}$. Then with $Z = Y = \mathscr{P}$, $T_n \to 0$ pointwise on Z, since $m \geqslant 1$, and $T_n S_n = \mathrm{Id}_Y$ in view of (∗). (In fact, $T_n P = 0$ for all n sufficiently large if $P \in \mathscr{P}$.) Thus, by virtue of Proposition 2.12, we only have to prove that $S_n \to 0$ pointwise on $\mathscr{P}$. But if $0 \neq P \in \mathscr{P}_k$ and $n \geqslant k$,

$$\frac{|S^n P|_{k+nm}}{|P|_k} \leqslant r_n^{-1}\sigma_{k,n} \leqslant r_n^{-1}\sigma_{n,n} = \frac{1}{n!},$$

since σ is increasing. Hence, for any given semi-norm $\|\cdot\|_\nu$ $(\nu \geqslant 1)$,

$$\|S_n P\|_\nu \leqslant \sum_{i=0}^{k+nm} \nu^i \|H_i(S_n P)\|_1 \leqslant \nu^{k+nm}|S_n P|_{k+nm} \leqslant \frac{\nu^{k+nm}|P|_k}{n!} \to 0,$$

as $n \to \infty$. ∎

The following example shows that some of the operators in Theorem A are in fact hypercyclic, and we shall pursue this later (Theorem C).

EXAMPLE 3.1. Aron and Markose proved recently [1] that, in the case of one variable, T_λ, $T_\lambda f(z) \equiv f'(\lambda z)$, is a hypercyclic operator for any $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$. They also discuss hypercyclicity of $T_{\lambda:a} \equiv T_\lambda \tau_a : f \mapsto f'(\lambda z + a)$ (see below). Note that $T_{1:a} = D\tau_a \in \mathscr{C}$ and, in fact, $T_{\lambda:a} \in \mathscr{C}$ if and only if $\lambda = 1$. We note now that, for arbitrary λ and a, $T_{\lambda:a} = \Phi(D) \in \mathscr{O}(\mathbb{H})$ where $\Phi = (\varphi_n(\xi) = \xi \mathrm{e}^{\langle a,\xi\rangle}\lambda^n)$. Thus, for any $\lambda \neq 0$, $T_{\lambda:a}$ belongs to the class of operators in Theorem A and is thus supercyclic. However, assume $|\lambda| \geqslant 1$ so that $T_\lambda = T_{\lambda:0}$ is hypercyclic. By Theorem 1.4, $T_\lambda^{(P)}$ also forms a hypercyclic operator for any $P = \xi^m \in \mathbb{H}$. We deduce that $T_\lambda^{(P)} = \lambda^m T_\lambda$ so $\lambda^m T_\lambda$ is a hypercyclic operator, and moreover, $P(D)f = f^{(m)}$ is a hypercyclic vector for any such vector f for T_λ. A simple argument [1] shows that $T_{\lambda:a}$ is hypercyclic for any root of unity λ, $\lambda^m = 1$. We note that, for such λ, $T_{\lambda:a} \in \mathscr{O}(P)$ for any $P = \sum_i a_i z^i \in \mathscr{P}$ such that $m|i$ whenever $i, a_i \neq 0$. However, $T_{\lambda:a}^{(P)} = T_{\lambda:a}$ so this does not provide us with any new hypercyclic operator. On the other hand, Theorem 1.4 gives that $P(D)$ maps the set of hypercyclic vectors for $T_{\lambda:a}$ invariantly.

Finally, an interesting phenomenon is that T_λ (and presumably $T_{\lambda:a}$) is *not* hypercyclic if $|\lambda| < 1$ ([1], Proposition 14), but we know that T_λ is supercyclic for any $\lambda \neq 0$.

Let us note that Theorem A covers some facts we already know. We know that $\varphi(D)$ is hypercyclic, and thus supercyclic, for any non-constant $\varphi \in \mathrm{Exp}$. In particular, if $\varphi(0) = 0$, $\varphi(\xi) = \xi^m \psi(\xi)$ for some unique $m > 0$ and non-degenerate $\psi \in \mathrm{Exp}$ and now, $\varphi(D) = \psi(D)D^m = \Psi(D)D^m$ where $\Psi \equiv (\psi, \psi, \ldots)$. Thus the class of operators in Theorem A above contains all $T = \varphi(D) \in \mathscr{C}$ such that $\varphi(0) = 0$, i.e., $T1 = 0$.

Next we shall extend Theorem A to an arbitrary number of variables d.

DEFINITION 3.2. $\mathcal{S}_\mathrm{S}$ denotes the set of sequences $\Phi \in \mathcal{S}$ of the form $\Phi = (\psi_n P_n)$ where $\Psi = (\psi_n)$ is a non-degenerate sequence in Exp and $\{P_n\} \subseteq \mathscr{H}_m \setminus \{0\}$ for some $m \geqslant 1$. $\mathscr{O}_\mathrm{S}$ denotes the corresponding class of operators $\Phi(D)$, $\Phi \in \mathcal{S}_\mathrm{S}$.

It is convenient to clarify the following. Let $\Psi = (\psi_n)$ be a non-degenerate sequence in Exp and let $0 \neq P_n, P \in \mathscr{H}_m$ where $m \geqslant 1$, then:

(i) If $\Psi \in \mathcal{S}$ and $\|P_n\|_1 \leqslant MR^n \ \forall n$, then $\Phi \equiv (\psi_n P_n) \in \mathcal{S}_\mathrm{S}$;
(ii) $\Phi \equiv (\psi_n P) \in \mathcal{S}_\mathrm{S}$ if and only if $\Psi \in \mathcal{S}$;
(iii) $(P_n) \in \mathcal{S}$ if and only if $\|P_n\|_1 \leqslant MR^n \ \forall n$.

(Implication (i) is elementary and, by Cauchy's Estimates, a sequence (φ_n) in Exp belongs to $\mathcal{S}$ if and only if $\|H_i(\varphi_n)\|_1 \leqslant MR^n r^i / i!$ for some $r, R, M \geqslant 0$, hence equivalence (iii), and the one in (ii), is an easy consequence of the following: If $P \in \mathscr{H}_n$ and $Q \in \mathscr{H}_m$, then $\|P\|_1 \|Q\|_1 \leqslant (2\mathrm{e})^{n+m} \|PQ\|_1$ ([9], p. 72).) In particular, equivalence (ii) implies that when $d = 1$ then $\mathscr{O}_\mathrm{S}$ is precisely the class of operators in Theorem A above, that we thus extend by:

THEOREM A. *Every operator $\Phi(D) \in \mathscr{O}_\mathrm{S}$ is supercyclic. Thus, in particular, any operator $\Phi(D) = \Psi(D)P(D)$, where $\Psi \in \mathcal{S}$ is non-degenerate and $0 \neq P \in \mathscr{H}_m$, $m \geqslant 1$, is supercyclic.*

Proof. Let us first prove the special case, i.e., assume all the homogeneous P_n in Φ are equal to some $P \in \mathscr{H}_m$ so that $\Phi(D) = \Psi(D)P(D)$, $\Psi \in \mathcal{S}$. First of all we note that $\Phi(D)^n P = 0$ for all n sufficiently large if $P \in \mathscr{P}$, since $m \geqslant 1$. Next, as in the one variable proof, we can find a non-degenerate $\Phi_0 \in \mathcal{S}$ such that $\Phi_0^{(m)} = \Psi$ and thus $\Phi(D) = P(D)\Phi_0(D) = \Phi_0^{(m)}(D)P(D)$. Next, $P(D)P^*$ is a bijection on $\mathscr{H}$ and $P(D)P^*$ maps $\mathscr{P}_n$ into $\mathscr{P}_n$ isomorphically (Proposition 2.10). Thus, if $(P(D)P^*)^{-1}$ denotes the inverse of the restriction of $P(D)P^*$ to $\mathscr{P}$, $A \equiv P^*(P(D)P^*)^{-1} : \mathscr{P} \to \mathscr{P}$ maps $\mathscr{P}_n$ into $\mathscr{P}_{n+m}$. Now, $\Phi_0^{-1}(D) : \mathscr{P} \to \mathscr{P}$ exists by Lemma 2.9 and with $C \equiv \Phi_0^{-1}(D)A : \mathscr{P} \to \mathscr{P}$, $\Phi(D)C = \mathrm{Id}_{\mathscr{P}}$ and $C\mathscr{P}_n \subseteq \mathscr{P}_{n+m}$. From this point the arguments in the proof above for $d = 1$ prove the theorem for this particular case.

Next consider the general case, i.e. $\Phi = (\psi_n P_n)$ where $P_n \in \mathscr{H}_m$. Again, as a starting point we conclude that $\Phi(D)^n P = 0$ for large n if $P \in \mathscr{P}$. We define $\mathcal{B} : \mathscr{P} \to \mathscr{P}$ by $\mathcal{B}Q = \sum_{n \geqslant m} P_{n-m}(D)Q_n$ where $Q = \sum Q_n$, $Q_n \in \mathscr{H}_n$.

Let $\Phi_0 = (\phi_n)$ be a non-degenerate sequence in Exp with $\Phi_0^{(m)} = \Psi$. Since Ψ may not be in $\mathcal{S}$, it is possible that $\Phi_0 \notin \mathcal{S}$, however, $\Phi_0(D) = \sum_{n \geqslant 0} H_n \phi_n(D)$ is a well-defined map on $\mathscr{P}$, and we claim that $\Phi(D) = \mathcal{B}\Phi_0(D)$ on $\mathscr{P}$. Indeed,

$$\mathcal{B}\Phi_0(D) = \sum_{n \geqslant m} P_{n-m}(D) H_n \phi_n(D) = \sum_{n \geqslant m} H_{n-m} P_{n-m}(D) \phi_n(D) = \Phi(D)$$

since $\phi_n = \psi_{n-m}$ for $n \geqslant m$. Moreover, from the proof of Lemma 2.9, it is clear that $\Phi_0(D)^{-1} : \mathscr{P} \to \mathscr{P}$ exists. By Proposition 2.10, we can define a map $\mathcal{A} : \mathscr{P} \to \mathscr{P}$ by $\mathcal{A}Q = \sum P_n^*(P_n(D)P_n^*)^{-1} Q_n$ where $Q_n \equiv H_n Q$. We deduce that $\mathcal{B}\mathcal{A} = \mathrm{Id}_{\mathscr{P}}$ so with $\mathcal{C} \equiv \Phi_0^{-1}(D)\mathcal{A} : \mathscr{P} \to \mathscr{P}$, $\Phi(D)\mathcal{C} = \mathrm{Id}_{\mathscr{P}}$ and, again, from this point the arguments in the one variable proof complete the proof. ∎

Let $\mathscr{O}_{\mathrm{S}}^*$ denote the set of operators $\Phi(D) = \Psi(D)P(D)$ in Theorem A, i.e. where $\Psi \in \mathcal{S}$ is non-degenerate and $0 \neq P \in \mathscr{H}_m$, $m > 0$. By $\mathcal{S}_{\mathrm{S}}^*$ we denote the corresponding set of sequences $\Phi = (P\psi_n) = P\Psi$ in $\mathcal{S}$. Note that in the case of one variable, $\mathscr{O}_{\mathrm{S}}^* = \mathscr{O}_{\mathrm{S}}$.

THEOREM B. *Assume $\Phi(D) \in \mathscr{O}_{\mathrm{S}}$ ($\Phi = (\varphi_n) \in \mathcal{S}_{\mathrm{S}}$). Then:*

(i) *$\Phi^{(m)}(D) \in \mathscr{O}_{\mathrm{S}}$ for any $m \geqslant 0$, and conversely.*
(ii) *For any $m \geqslant 0$ there is a $\Psi(D) \in \mathscr{O}_{\mathrm{S}}$ such that $\Psi^{(m)}(D) = \Phi(D)$.*
(iii) *For any $\Psi \in \mathcal{S}_{\mathrm{S}}^*$ or non-degenerate $\Psi \in \mathcal{S}$, $\Phi(D)\Psi(D) \in \mathscr{O}_{\mathrm{S}}$.*

$\mathscr{O}_{\mathrm{S}}^*$ forms a multiplicative closed subset of $\mathscr{O}_{\mathrm{S}}$ and is stable in the sense of (i)–(iii). *For any set $A \subseteq \mathcal{S} \times \mathbb{H}$ such that $P \neq 0$, $\Psi^{(m)} = \Phi$ if $P \in \mathscr{H}_m$, for all $(\Psi, P) \in A$:*

$$\mathscr{I}_A \equiv \bigcup_A \{P(D)f : f \text{ supercyclic for } \Psi(D)\} \tag{3.1}$$

forms an invariant set (under $\Phi(D)$) of supercyclic vectors for $\Phi(D)$. In particular, for any $m \geqslant 1$ there is a vector $f \in \mathscr{H}$ such that

$$\mathscr{M}_m \equiv \{P(D)f : P \in \mathscr{H}_m\}$$

forms an $\binom{m+d-1}{d-1}$-dimensional (i.e. $\simeq \mathscr{H}_m$) supercyclic vector manifold for $\Phi(D)$.

Proof. Property (i) is elementary and so is property (ii). Indeed, let $m \geqslant 0$, then $\Psi^{(m)} = \Phi$ and $\Psi = (\psi_n) \in \mathcal{S}_{\mathrm{S}}$ where $\psi_{n+m} \equiv \varphi_n P_n$ if $n \geqslant 0$ and $\psi_n \equiv P_n \phi_n$, $n < m$, where $\phi_n \in$ Exp are arbitrary non-degenerate elements. Property (iii) follows by formula (2.1). For assume $\Psi = (\psi_n Q) \in \mathcal{S}_{\mathrm{S}}^*$ and let $(\xi_n) \equiv \Phi\Psi$. Then (2.1) gives that $\xi_n = P_n Q \sum_{i \geqslant m} H_{i-m}(\varphi_n)\psi_{i+n} = P_n Q \phi_n$ if $P_n \in \mathscr{H}_m$. $R_n \equiv P_n Q$ are all homogeneous of the same degree > 0 and every ϕ_n is non-degenerate. Thus $\mathscr{O}_{\mathrm{S}}\mathscr{O}_{\mathrm{S}}^* \subseteq \mathscr{O}_{\mathrm{S}}$ and the other property in (iii) follows in the same way.

That (3.1) is formed by supercyclic vectors follows by Theorem 1.4. We must prove that $\mathscr{I}_A$ is invariant. So let $P(D)f \in \mathscr{I}_A$, $(\Psi, P) \in A$. Then $\Phi(D)P(D)f = P(D)\Psi(D)f$. Since f is supercyclic for $\Psi(D)$, it is elementary that $\Psi(D)f$ also forms a supercyclic vector for $\Psi(D)$, hence $\Phi(D)P(D)f \in \mathscr{I}_A$.

In particular, given m, then, in view of property (ii), there is a $\Psi \in \mathcal{S}_{\mathrm{S}}$ with $\Psi^{(m)} = \Phi$ and by Theorem A we can find a supercyclic vector f for $\Psi(D)$. So from (3.1), $\{P(D)f : 0 \neq P \in \mathscr{H}_m\}$ is formed by supercyclic vectors for $\Phi(D)$ and we deduce that $\mathscr{H}_m \ni P \mapsto P(D)f \in \mathscr{M}_m$ defines a linear isomorphism ℓ. Indeed, $P(D)f \neq 0$ for all $P \neq 0$, for otherwise 0 would be a supercyclic vector, so ℓ is one-to-one and hence a bijection. ∎

EXAMPLE 3.3. Fix m and $\Psi \in \mathcal{S}_{\mathrm{S}}$ such that $\Psi^{(m)} = \Phi \in \mathcal{S}_{\mathrm{S}}$. Then, with $A \equiv \{(\Psi, P) : 0 \neq P \in \mathscr{H}_m\}$, we obtain the invariant set $\mathscr{I}_A = \bigcup_{P \in \mathscr{H}_m \setminus \{0\}} P(D)\mathrm{SC}(\Psi)$ of supercyclic vectors for $\Phi(D)$. Here $\mathrm{SC}(\Psi)$ denotes the set of supercyclic vectors for $\Psi(D)$.

REMARK 3.4. Note that the arguments in the proof concerning the invariant set $\mathscr{I}_A$ hold more generally: Let $S \in \mathscr{L}$ and let $A = \{(T, \varphi)\}$ be any family of pairs $(T, \varphi) \in \mathscr{L} \times \mathrm{Exp}$ such that $\varphi \neq 0$, $T \in \mathscr{O}(\varphi)$ and $T^{(\varphi)} = S$. Then $\bigcup_A \varphi(D)\{f$: f supercyclic for $T\}$ (possibly empty) forms an invariant set of supercyclic vectors for S. The analogue holds for hypercyclicity (but *not* for cyclicity, cf. p. 235 in [11]).

EXAMPLE 3.5. The example of Aron and Markose (Example 3.1) is easily extended to d variables. Indeed, let $\lambda \in \mathbb{C}^d$ and consider the affine map $\Lambda : z \mapsto \lambda \cdot z \equiv (\lambda_i z_i)$. (We assume $a = 0$.) Define $M_\Lambda f \equiv f(\lambda \cdot z)$. Then we claim that if $|\lambda_i| \geqslant 1$ for all i, $T \equiv M_\Lambda D^\alpha$ (i.e. $Tf(z) = f^{(\alpha)}(\lambda \cdot z)$) is hypercyclic for any $\alpha \neq 0$. (The proof runs parallel to that of Theorem 13 in [1].) Now, if all λ_i are equal, $\lambda_i = \lambda$, but where λ is arbitrary, we have that $T = \Phi(D) \in \mathscr{O}(\mathbb{H})$ where $\Phi = (\varphi_n = \xi^\alpha \lambda^n)$. Thus, if $|\lambda| \geqslant 1$, T is a hypercyclic operator in $\mathscr{O}^*_{\mathrm{S}} \subseteq \mathscr{O}_{\mathrm{S}}$ and is outside $\mathscr{C}$ if $\lambda \neq 1$ (since then Φ is not a constant sequence). If λ_i not all are equal, it follows that $T \notin \mathscr{O}(\mathbb{H})$ and consequently $T \notin \mathscr{O}_{\mathrm{S}}$, so if also $|\lambda_i| \geqslant 1$ for all i, T is a hypercyclic non-convolution operator outside $\mathscr{O}_{\mathrm{S}}$.

Example 3.5 shows that there are examples of hypercyclic non-convolution operators in $\mathscr{O}_{\mathrm{S}}$ and $\mathscr{O}^*_{\mathrm{S}}$ also when $d > 1$, we now extend this fact.

DEFINITION 3.6. $\mathcal{S}_{\mathrm{H}}$ denotes the set of sequences $(P_n) \in \mathcal{S}$ such that $\{P_n\} \subseteq \mathscr{H}_m \setminus \{0\}$ for some $m \geqslant 1$ and $\|P_n\|_1 \geqslant c$ for some $c > 0$. (Recall that $(P_n) \in \mathcal{S}$ is equivalent to $\|P_n\|_1 \leqslant CM^n$).) $\mathscr{O}_{\mathrm{H}}$ denotes the corresponding class of operators $\Phi(D)$, $\Phi \in \mathcal{S}_{\mathrm{H}}$.

It is clear that $\mathcal{S}_{\mathrm{H}} \subseteq \mathcal{S}_{\mathrm{S}}$, and accordingly $\mathscr{O}_{\mathrm{H}} \subseteq \mathscr{O}_{\mathrm{S}}$, and now:

THEOREM C. *The following hold for* $\mathscr{O}_{\mathrm{H}}$*:*

(i) *The elements of* $\mathscr{O}_{\mathrm{H}}$ *are hypercyclic.*

(ii) $\mathscr{O}_{\mathrm{H}}$ *is multiplicative closed and satisfies the analogue of* (i)–(ii) *in Theorem* B.

(iii) *For any* $T \in \mathscr{O}_{\mathrm{H}}$*, invariant sets of hypercyclic vectors for* T *are obtained analogous to* (3.1) *and, in particular, for every* $m \geqslant 1$ *there is an* $f \in \mathscr{H}$ *such that* $\mathscr{M}_m = \{P(D)f : P \in \mathscr{H}_m\}$ *forms an* $\binom{m+d-1}{d-1}$*-dimensional hypercyclic vector manifold for* T.

Proof. We prove that any $T = \Phi(D) \in \mathscr{O}_{\mathrm{H}}$ is hypercyclic and intend to apply Proposition 2.12 with $Z = Y = \mathscr{P}$. We define, as in the proof of Theorem A, $\mathcal{A} : \mathscr{P} \to \mathscr{P}$ by $\mathcal{A} \equiv \sum_{n\geqslant 0} A_n H_n$, where $A_n \equiv P_n^*(P_n(D)P_n^*)^{-1}$ and $\Phi = (P_n)$. Then $\Phi(D)\mathcal{A} = \mathrm{Id}_{\mathscr{P}}$ and, if $\{P_n\} \subseteq \mathscr{H}_m$,

$$\mathcal{A}^n Q = \sum_{i\geqslant 0} A_{i+m(n-1)} \cdots A_{i+m} A_i Q_i, \quad Q_i \equiv H_i Q \in \mathscr{H}_i,$$

and thus, by Lemma 2.11,

$$\begin{aligned}
\|\mathcal{A}^n Q\|_r &\leqslant \sum_{i\geqslant 0} r^{i+nm} \|A_{i+m(n-1)} \cdots A_i Q_i\|_1 \\
&\leqslant \sum_{i\geqslant 0} r^{i+nm} \frac{k(d)^n}{\|P_i\|_1 \cdots \|P_{i+m(n-1)}\|_1} \frac{\|Q_i\|_1}{m!^n} \\
&\leqslant \frac{r^{nm} c^{-n} k(d)^n}{m!^n} \sum_{i\geqslant 0} r^i \|Q_i\|_1 \to 0
\end{aligned}$$

as $n \to \infty$. So $S_n \equiv \mathcal{A}^n \to 0$ pointwise on Y, $T^n S_n = \mathrm{Id}_Y$ and, since $m > 0$, $T^n \to 0$ pointwise on $Z = \mathscr{P}$. Thus T is hypercyclic by Proposition 2.12.

That the analogues of (i) and (ii) in Theorem B hold true for $\mathscr{O}_{\mathrm{H}}$ is elementary and we prove that $\mathscr{O}_{\mathrm{H}}$ is multiplicative closed. But, by formula (2.1), $(P_n)(Q_n) = (P_n Q_{n+m})$ if $P_n \in \mathscr{H}_m$ for all n, hence $\mathscr{O}_{\mathrm{H}}$ is stable under multiplication.

The arguments in the last part of the proof of Theorem B prove (iii). ∎

REMARK 3.7. It is well known, see [5], [7], that every hypercyclic operator on a real or complex locally convex space, has a dense invariant hypercyclic vector manifold. Accordingly, any $T \in \mathscr{O}_{\mathrm{H}}$ admits such a hypercyclic vector manifold. (The existence of dense invariant supercyclic vector manifolds, for supercyclic operators (like any $T \in \mathscr{O}_{\mathrm{S}}$), is more delicate.)

REMARK 3.8. $\mathcal{S}_{\mathrm{H}}$ is a special class of sequences $\Phi = (\psi_n P_n) \in \mathcal{S}_{\mathrm{S}}$ where we can choose $\psi_n = 1$ for all n. Thus, it is a natural question to ask: For what sequences $\Psi = (\psi_n)$ and (P_n), where Ψ is non-degenerate and $\{P_n\} \subseteq \mathscr{H}_m \setminus \{0\}$ for some $m \geqslant 1$, do we have that $\Phi = (\psi_n P_n) \in \mathcal{S}_{\mathrm{S}}$ and $\Phi(D)$ is hypercyclic?

In particular, if $\Phi = (\varphi_n)$ is a sequence of scalars such that: (b) $0 < c \leqslant |\varphi_n| \leqslant CR^n$, then $\Phi \in \mathcal{S}$ and $\Phi(D)P(D)$ ($\in \mathscr{O}_{\mathrm{H}}$) is hypercyclic for any non-constant $P \in \mathbb{H}$. ($P_n = \varphi_n P$ for all n in Theorem C.) In the case of one variable, every $\mathscr{H}_n$ is one dimensional and every element of $\mathscr{O}_{\mathrm{H}}$ can be factorized to this form $\Phi(D)P(D)$ (cf. $\mathscr{O}_{\mathrm{S}}^* = \mathscr{O}_{\mathrm{S}}$).

EXAMPLE 3.9. Consider the Euler operator $\langle \cdot, D \rangle \in \mathscr{O}(\mathbb{H})$. We recall from Example 2.8 that $\langle \cdot, D\rangle^m = \Phi(D)$ where $\Phi = (\varphi_n = n^m)$. Thus Φ satisfies the bound condition (b) above except when $n = 0$ ($\varphi_0 = 0$). But if we add a sequence $(c, 0, \ldots)$, $c \neq 0$, to Φ we obtain a sequence satisfying (b) and conclude:

For any $m \geqslant 1$, $c \neq 0$ and non-constant $P \in \mathbb{H}$, $(\langle \cdot, D \rangle^m + c\delta_0)P(D)$ is hypercyclic. ($\delta_0(f) \equiv f(0) = H_0 f$.)

Further, any derivative $\Phi^{(n)}(D)$ of $\langle \cdot, D \rangle^m$ corresponds in $\mathcal{S}$ to a sequence $\Phi^{(n)}$ of constants satisfying (b) and hence: $P(D)\langle \cdot, D \rangle^m$ is hypercyclic for any $m \geqslant 1$ and non-constant $P \in \mathbb{H}$. Thus, for example, if $d = 1$ then $f \mapsto D(zDf) = zf''(z) + f'(z)$ forms a hypercyclic operator.

Note that with $|\lambda| \geqslant 1$ and $\Phi = (\varphi_n \equiv \lambda^n)$, $T \equiv \Phi(D)D^\alpha \in \mathscr{O}_{\mathrm{H}}$ if $\alpha \neq 0$. In fact, T is precisely the hypercyclic operator in Example 3.5 with $\lambda_i = \lambda$ for all i. In particular, if $d = 1$ and $\alpha = 1$, our result that T is hypercyclic is that of Aron and Markose, saying that T_λ is hypercyclic provided $|\lambda| \geqslant 1$ (Example 3.1).

REMARK 3.10. (i) We note that the example due to Aron and Markose, and all our examples so far, of cyclic type operators T outside $\mathscr{C}$ degenerates in the sense that $T1 = 0$. Thus, the question is if there is any, say, hypercyclic $T \in \mathscr{L} \setminus \mathscr{C}$ with $T1 \neq 0$. The answer is affirmative, and we illustrate this by, once again, showing how Fischer pairs provide us with alternative "backward shifts". (For simplicity we let $d = 1$, and since the arguments run parallel to those in the proof of Theorem A, we shall be quite brief.)

Proposition 2.10 admits the following generalization: $(P(D) - c, P^*)$ forms a Fischer pair for $\mathscr{H}$, for any constant c and homogeneous $P \in \mathbb{H} \setminus \{0, c\}$ ([22], Theorem 3).

Let $P \equiv \xi$, then $P(D) - c = D - c$ and $P^* = P$. Put $\mathscr{E}_n \equiv \ker(D-c)^{n+1}$, i.e., $\mathscr{E}_n = \mathscr{P}_n e_c = \mathscr{P}_n \mathrm{e}^{cz}$ (finite-dimensional). Then $\mathscr{E} \equiv \bigcup_{n \geqslant 0} \mathscr{E}_n$ is dense in $\mathscr{H}$ and an operator T is PDE-preserving for $\mathbb{E} \equiv \{1, \xi - c, (\xi - c)^2, \ldots\}$ if and only if it maps every $\mathscr{E}_n$ invariantly (cf. Lemma 2.5). It is now easy to prove that $P(\cdot, D) \in \mathscr{O}(\mathbb{E})$ if and only if $P_c(\cdot, D) \in \mathscr{O}(\mathbb{H})$ where $P_c \equiv P(z, \xi + c) \in \mathfrak{S}$. From this and Proposition 2.7 we deduce the following. Let E_n denote the map $E_n \equiv e_c H_n e_{-c}$, i.e. $E_n f \equiv (D-c)^n f(0) z^n e_c / n! \in \mathscr{E}_n$. Then $\Phi \mapsto \Phi[D] \equiv \sum_{n \geqslant 0} E_n \varphi_n(D)$ defines a one-to-one correspondence between $\mathcal{S}$ and $\mathscr{O}(\mathbb{E})$ and $\Phi[D]^{(P)} = \Phi^{(n)}[D]$ if $P = (\xi - c)^n$. If Φ is non-degenerate at c, i.e., $\varphi_n(c) \neq 0$ for all n, then $\Phi[D]$ maps every $\mathscr{E}_n$ isomorphically. We obtain: *Every operator T of the form $\Phi[D](D-c)^m$ is supercyclic if Φ is non-degenerate at c and $m \geqslant 1$.* Indeed, $(D-c)\mathscr{E}_{n+1} \subseteq \mathscr{E}_n$ so $T^n E = 0$ for large n if $E \in \mathscr{E}$. Further, there is a factorization $T = (D - c)^m \Phi_0^{(m)}[D]$ where $\Phi_0 \in \mathcal{S}$ is non-degenerate at c. $(P(D) - c)P^*$ maps every $\mathscr{E}_n$ isomorphically and we put $A \equiv P^*[(P(D) - c)P^*]^{-1} : \mathscr{E} \to \mathscr{E}$. Thus with $C \equiv \Phi_0^{-1}[D]A^m : \mathscr{E} \to \mathscr{E}$, $TC = \mathrm{Id}_{\mathscr{E}}$ and we deduce that there is a sequence $r = (r_n)$ such that $\mathbb{T} = (T_n \equiv r_n T^n)$ is hypercyclic, hence T is supercyclic. In particular we note that $T1 \neq 0$ for a suitable Φ (see below for a concrete example), on the other hand, $Te_c = 0$. In the same way, with smaller modifications of the proof of Theorem C we obtain: $\Phi[D](D-c)^m$ *is hypercyclic for any scalar sequence* $\Phi = (\varphi_n)$ *that satisfies the bound condition* (b) (thus Φ is non-degenerate at c).

With $\Phi = (\varphi_n = n+1)$ and $m = 1$ we obtain the hypercyclic operator: $T = zD^2 - 2czD + c^2z + D - c$, which with $c = 0$ reduces to the operator in the latter part of Example 3.9 and $T1 \neq 0$ if $c \neq 0$.

(ii) We suggest a study on to what degree the converse of Theorem 1.4 holds: Is every, say, hypercyclic $S \in \mathscr{L}$ the derivative, $T^{(\varphi)}$, of some hypercyclic $T \in \mathscr{O}(\varphi)$? (Note that this is true for any S in $\mathscr{C} \setminus \mathbb{C}$ and in $\mathscr{O}_{\mathrm{H}}$.) Or even stronger, are there φ and $T \in \mathscr{O}(\varphi)$ such that $T^{(\varphi)} = S$ and such that every hypercyclic vector g for S is of the form $\varphi(D)f$ for some hypercyclic vector f for T? (This is, as far as we know, an open problem even for $S \in \mathscr{C} \setminus \mathbb{C}$.)

(iii) Our technique, based on Fischer pairs, should work for other spaces and, in particular, for other power series spaces. Indeed, Fischer decompositions have also been studied for: Exp, germs of analytic functions, the entire ring of formal power series etc. [15], [22]. This is interesting in view of the fact that these spaces do not in general admit backward shifts. In particular, Fischer splittings have been studied for entire function spaces in an infinite number of variables [18] and we believe that some of the results in this note are extendible to infinite-dimensional holomorphy in this way. (Cf. [16] where an infinite-dimensional analogue of Godefroy-Shapiro's Theorem (Proposition 1.2) is obtained.)

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