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A MODIFICATION OF READ'S TRANSITIVE OPERATOR

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ABSTRACT. In this paper we discuss an inductive process of constructing an operator on ℓ_1 without non-trivial closed invariant subspace.

KEYWORDS: *Operator, invariant subspace, transitive operator.*

MSC (2000): 47A15.

1. INTRODUCTION

The first construction of an operator without non-trivial closed invariant subspaces is due to P. Enflo. The example of that so-called, transitive operator has been produced in 1976 but got published only by 1987 [3]. In the meantime, different operators have been constructed by C. Read in [5], [6], and [7] and a simplification of the Enflo's example has been published by B. Beauzamy [2].

Although the mentioned examples provide us with the negative answer to the general invariant subspace problem, there is a vast number of related questions that remain unanswered. Existence of a topologically simple Banach algebra, of a transitive operator on a reflexive Banach space, or of a positive transitive operator are among such questions.

This paper is devoted to the modification of the example of C. Read. Our example is based on the construction from [8] and acts on l_1. The matrix of the operator consists of blocks which are defined in an inductive manner. In [8] both the length of the m-th block and the value of entries in it depend on previous $m-1$ blocks. In our paper we fix the structure of the matrix from the very beginning and only the entries of the matrix will be defined inductively.

Another modification that we make in this paper concerns the relations between entries of the matrix of Read's example. Every m-th block of Read's construction is partitioned into $4m$ smaller blocks. Among those, m contain the same entry which is essential for the proof. We eliminate this dependence from the matrix. Hence, our operator yields more degrees of freedom. Needless to say that all the other properties from [8] are preserved.

All proofs replicate the corresponding proofs of C. Read, though because we avoid working with specific coefficients, they seem simpler than the original.

2. TRANSFORMATION MATRIX

In this paper all operators under consideration will be continuous linear maps from a Banach space $X = l_1$ into itself. We consider X as the norm closure $X = (c_{00}, \|\cdot\|_1)$ of the space of finite number sequences c_{00} endowed with l_1-norm.

DEFINITION 2.1. Let $(f_i)_{i=0}^{\infty}$ denote the unit vector basis of X. For each $n \geqslant 0$, X_n will denote the linear span span$\{f_i : i = 0, \dots, n-1\}$.

Let $|p|$ denote the sum of the absolute values of the coefficients of the polynomial p. Let us agree that pairs of non-negative integers (m, n) with $m \geqslant n$ are ordered so that $(m, m) < (m, m-1) < (m, m-2) < \cdots < (m, 2) < (m, 1) < (m+1, m+1)$ for every m. Next we define an increasing sequence $u_{m,n}$ by

$$u_{1,1} = 1; \quad u_{i,j} = 16u_{m,n} + u_{n,n}$$

where the pair (i, j) is the successor of (m, n) in the linear order introduced above. This sequence partitions the set of natural numbers into the blocks $[u_{m,n}, 16u_{m,n} + u_{n,n}) \cap \mathbb{N}$ which we will refer to as block (m, n). Every such block we consider as a disjoint union of four "intervals":

$$\begin{aligned}
[1)_{m,n} &:= [u_{m,n}, 8u_{m,n}) \cap \mathbb{N}, \\
[2)_{m,n} &:= [8u_{m,n}, 8u_{m,n} + u_{n,n}) \cap \mathbb{N}, \\
[3)_{m,n} &:= [8u_{m,n} + u_{n,n}, 12u_{m,n}) \cap \mathbb{N}, \text{ and} \\
[4)_{m,n} &:= [12u_{m,n}, 16u_{m,n} + u_{n,n}) \cap \mathbb{N}.
\end{aligned}$$

Any transitive operator must be a "right shift" on a proper collection of vectors. We continue by defining that sequence of vectors $(e_i)_{i=0}^{\infty}$ using the standard basis (f_i):

1. If $i \in [1)_{m,n}$ or $i = 0$, we set
$$f_i = F_i e_i.$$
2. If $i \in [2)_{m,n}$, we set
$$f_i = G_i[H^{(2)}_{m,n} e_i - e_{i-8u_{m,n}}].$$
3. If $i \in [3)_{m,n}$, we set
$$f_i = F_i e_i.$$
4. If $i \in [4)_{m,n}$, we set
$$f_i = G_i[H^{(4)}_{m,n} e_i - e_{i-8u_{m,n}}].$$

Sometimes, to emphasize what group and interval the number i is in, we will be using additional sub/super indices. For instance, if $i \in [1)_{m,n}$ we may use the notation $F^{(1)}_{i,m,n}$ instead of just F_i.

Assuming that non-zero F-, G-, and H-coefficients are defined, we may write $f_i = \sum_{j=0}^{i} \lambda_{ij} e_j$ uniquely, for each $i \in \mathbb{Z}^+$. Since λ_{ii} is never zero, this linear relationship is invertible. So the e_i exist and are unique. In addition for each n we have

$$\operatorname{span}\{e_i : i = 0, \dots, n-1\} = \operatorname{span}\{f_i : i = 0, \dots, n-1\} = X_n.$$

Our construction is based on the order in which the coefficients are chosen and on the rate of their growth.

The rate of growth of the coefficients is governed by the functions $N_a(m,n)$ and $N_b(m,n)$. The former regulates the intervals $[1)_{m,n}$ and $[2)_{m,n}$ and the latter sets the rate of change on $[3)_{m,n}$ and $[4)_{m,n}$. The properties of N_a and N_b we will derive during the proof process and then will summarize in Theorem 6.2. To this end we assume that both functions are at least 2.

The values of $N_a(m,n)$, $N_b(m,n)$, and the coefficients for the block (m,n) are determined in the following order:

$$\begin{array}{llllll}
N_a(m,n) & \to G^{(2)}_{8u_{m,n}} & \to G^{(2)}_{8u_{m,n}+1} & \to \cdots \to G^{(2)}_{8u_{m,n}+u_{n,n}-1} & \to & \\
 & \to F^{(1)}_{u_{m,n}} & \to F^{(1)}_{u_{m,n}+1} & \to \cdots \to F^{(1)}_{8u_{m,n}-1} & \to H^{(2)}_{m,n} & \to \\
\to N_b(m,n) & \to G^{(4)}_{12u_{m,n}} & \to G^{(4)}_{12u_{m,n}+1} & \to \cdots \to G^{(4)}_{16u_{m,n}+u_{n,n}-1} & \to & \\
 & \to F^{(3)}_{8u_{m,n}+u_{n,n}} & \to F^{(3)}_{8u_{m,n}+u_{n,n}+1} & \to \cdots \to F^{(3)}_{12u_{m,n}-1} & \to H^{(4)}_{m,n} &
\end{array}$$

In setting the values of any number in this linear diagram we are free to use every number which is already defined. Using this principle we continue by defining the values of the coefficients.

For fixed m, the first coefficient of all $(m,*)$ blocks we set by

$$G^{(2)}_{8u_{m,m}} := m \tag{2.1}$$

to guarantee that we will "return" to e_0 with increasing precision. That is, to have

$$\|H^{(2)}_{m,m} e_{8u_{m,m}} - e_0\| = \frac{\|f_{8u_{m,m}}\|}{G_{8u_{m,m}}} = \frac{1}{m}.$$

Notice that for $n < m$ we have $(n,n) < (m,n)$ and the number of elements in $[2)_{m,n}$ is $u_{n,n}$. This allows us to set G-coefficients in $[2)_{m,n}$ equal to the corresponding coefficients from $[2)_{n,n}$:

$$G^{(2)}_{i,m,n} := G^{(2)}_{i-8u_{m,n}+8u_{n,n}}. \tag{2.2}$$

The first coefficient of $[4)_{m,n}$ we set by

$$G^{(4)}_{12u_{m,n}} := N_b(m,n). \tag{2.3}$$

If i and $i+1$ belong to the same interval of the block (m,n), the corresponding F- and G-coefficients are set by:

$$F^{(1)}_{i+1,m,n} := N_a(m,n)F^{(1)}_{i,m,n}; \tag{2.4}$$

$$G^{(2)}_{i+1,m,n} := N_a(n,n)G^{(2)}_{i,m,n}; \tag{2.5}$$

$$F^{(3)}_{i+1,m,n} := N_b(m,n)F^{(3)}_{i,m,n}; \tag{2.6}$$

$$G^{(4)}_{i+1,m,n} := N_b(m,n)G^{(4)}_{i,m,n}. \tag{2.7}$$

To finish the description of the block (m,n), we need to define H-coefficients and the values of the first F-coefficients $F_{u_{m,n}}$ and $F_{8u_{m,n}+u_{n,n}}$. To this end we set

$$F_{u_{m,n}} := N_a(m,n)G_{8u_{m,n}+u_{n,n}-1}, \tag{2.8}$$

$$F_{8u_{m,n}+u_{n,n}} := N_b(m,n)G_{16u_{m,n}+u_{n,n}-1}, \tag{2.9}$$

$$H^{(2)}_{m,n} := N_a(m,n)F_{8u_{m,n}-1}, \text{ and} \tag{2.10}$$

$$H^{(4)}_{m,n} := N_b(m,n)F_{12u_{m,n}-1}. \tag{2.11}$$

The definitions (2.4–2.11) are possible due to the order in which we define the coefficients. We finish the description of the transformation matrix by setting $F_0 = 1$.

3. SHORT OUTLINE OF THE PROOF

Using introduced vectors e_i we define the linear map $T : c_{00} \to c_{00}$ by $Te_i = e_{i+1}$. Then we show that the map T satisfies $\|T\| \leqslant 1$. Therefore, T can be extended to entire X. Next for any unit vector $x \in X$ we will find a projection $Q_{m,n}$ that maps x into a special compact subset of X. After that we will establish the existence of a polynomial q such that the numbers $\|q(T)Q_{m,n}x - e_0\|$ and $\|q(T)(I - Q_{m,n})\|$ are small. This will show that $e_0 \in \overline{\text{span}}\{T^r x : r \geqslant 0\}$.

Every inequality about the norm of operators will be obtained by estimating $\|Tf_i\|$ for every i.

4. NORM ESTIMATIONS

Let us continue by showing that T is a contraction.

LEMMA 4.1. *If for any consecutive pairs* $(m,n) < (r,s)$ *the functions* N_a *and* N_b *satisfy*

$$N_a(r,s) \geqslant 2H^{(4)}_{m,n}G^{(4)}_{16u_{m,n}+u_{n,n}-1},$$
$$N_b(m,n) \geqslant 2H^{(2)}_{m,n}G^{(2)}_{8u_{m,n}+u_{n,n}-1},$$

then $\|T\| \leqslant 1$ *holds.*

Proof. Let N_a and N_b be two functions satisfying the assumption of the lemma. Consider the operator T constructed using N_a, N_b, and the guidelines above. Let us fix (m,n) and show that $\|Tf_i\| \leqslant 1$ for every i from the block (m,n).

Since $f_0 = F_0e_0$ and $f_1 = F_1e_1$ we can estimate

$$\|Tf_0\| = \|F_0e_1\| = \frac{F_0}{F_1}\|f_1\| \leqslant \frac{1}{N_a(1,1)} \leqslant \frac{1}{2}.$$

Let us fix $i > 0$ and assume that $\|T|_{X_i}\| \leqslant 1$.

Case 1: Suppose i and $i+1$ both belong to either $[2)_{m,n}$ or $[4)_{m,n}$. Then $f_i = G_i(H_{m,n}e_i - e_{i-8u_{m,n}})$ and $f_{i+1} = G_{i+1}(H_{m,n}e_{i+1} - e_{i+1-8u_{m,n}})$. Thus,

$$\|Tf_i\| = \|G_i(H_{m,n}e_{i+1} - e_{i+1-8u_{m,n}})\| = \frac{G_i}{G_{i+1}}\|f_{i+1}\| = \frac{1}{N(m,n)} \leqslant \frac{1}{2}.$$

Here $N(m,n)$ is either $N_a(n,n)$ or $N_b(m,n)$ depending on whether i is in $[2)_{m,n}$ or in $[4)_{m,n}$ (see formulae (2.5) or (2.7) respectively). In either case $N(m,n)$ is at least 2, which implies the last inequality.

The case when both i and $i+1$ belong to either $[1)_{m,n}$ or $[3)_{m,n}$ can be checked similarly due to formulae (2.4) and (2.6) above.

Case 2: Suppose i is in $[1)_{m,n}$ or $[3)_{m,n}$ and $i+1$ is in $[2)_{m,n}$ or $[4)_{m,n}$ respectively. Then $f_i = F_ie_i$ and $f_{i+1} = G_{i+1}(H_{m,n}e_{i+1} - e_{i+1-8u_{m,n}})$. Thus,

$$\|Tf_i\| = F_i\|e_{i+1}\| = \frac{F_i}{H_{m,n}}\left\|\frac{f_{i+1}}{G_{i+1}} + e_{i+1-8u_{m,n}}\right\| \leqslant \frac{1}{N(m,n)}(1 + \|e_{i+1-8u_{m,n}}\|).$$

Here $N(m,n)$ is either $N_a(m,n)$ or $N_b(m,n)$ depending on whether i is in $[1)_{m,n}$ or in $[3)_{m,n}$ (see formulae (2.10) or (2.11) respectively). Since we have $e_{i+1-8u_{m,n}} = (T|_{X_i})^{i+1-8u_{m,n}}e_0$ by our inductive assumption we conclude $\|e_{i+1-8u_{m,n}}\| \leqslant 1$. Then the fact that $N(m,n) \geqslant 2$ and the estimate above imply $\|Tf_i\| \leqslant 1$.

Case 3: Suppose $i = 8u_{m,n} + u_{n,n} - 1$ is in $[2)_{m,n}$ and $i+1$ is in $[3)_{m,n}$. Then, as before, $Tf_i = G_{8u_{m,n}+u_{n,n}-1}(H^{(2)}_{m,n}e_{8u_{m,n}+u_{n,n}} - e_{u_{n,n}})$. Notice that $u_{n,n}$ belongs to the interval $[1)_{n,n}$ and thus satisfies $e_{u_{n,n}} = \frac{f_{u_{n,n}}}{F_{u_{n,n}}}$. We also have $G_{8u_{m,n}+u_{n,n}-1} = G_{8u_{n,n}+u_{n,n}-1}$ by formula (2.2). Since $i+1$ is in $[3)_{m,n}$ we can write $e_{8u_{m,n}+u_{n,n}} =$

$\frac{f_{8u_{m,n}+u_{n,n}}}{F_{8u_{m,n}+u_{n,n}}}$. Therefore, by (2.8), (2.9), and our assumption about $N_b(m,n)$ we obtain

$$\|Tf_i\| \leqslant \frac{\|f_{8u_{m,n}+u_{n,n}}\| G_{8u_{m,n}+u_{n,n}-1} H^{(2)}_{m,n}}{F_{8u_{m,n}+u_{n,n}}} + \frac{\|f_{u_{n,n}}\| G_{8u_{n,n}+u_{n,n}-1}}{F_{u_{n,n}}}$$
$$< \frac{G_{8u_{m,n}+u_{n,n}-1} H^{(2)}_{m,n}}{N_b(m,n)} + \frac{1}{N_a(n,n)} < 1.$$

The formulae (2.8), (2.9), and the assumption about $N_a(r,s)$ for the successor (r,s) help us to handle the last case $i = 16u_{m,n} + u_{n,n} - 1$, the end of interval $[4)_{m,n}$, in a similar manner. ∎

By the lemma above the linear map $T : c_{00} \to c_{00}$ can be extended to the completion X with $\|T\| \leqslant 1$.

Next we will discuss the relation between expansions of an element $x \in c_{00}$ with respect to (e_i) and (f_i). If $x = \sum_{i=0}^{N} \lambda_i e_i \in c_{00}$ we consider a different norm defined by $|x| = \sum_{i=0}^{N} |\lambda_i|$. Since $\|e_i\| \leqslant 1$ for every i, we conclude that $\|x\| \leqslant |x|$ for every $x \in c_{00}$. Let us introduce another auxiliary function by

$$f(m,n) := \sup\{|x| : x \in X_{8u_{m,n}+u_{n,n}}, \|x\| = 1\}.$$

From the definition of (e_i) it follows, in particular, that

$$f(m,n) > \max\{G_j : j < 8u_{m,n} + u_{n,n}\}.$$

For a fixed pair (m,n) consider projections $P_{m,n}, Q_{m,n} : X \to X_{8u_{m,n}+u_{n,n}}$, and $\tau_{m,n} : X_{8u_{m,n}+u_{n,n}} \to X_{8u_{m,n}+u_{n,n}}$ defined by

$$P_{m,n}(f_i) = \begin{cases} f_i & i \leqslant 2u_{m,n}, \\ -G_i e_{i-8u_{r,s}} & i \in [2)_{r,s} \text{ and } (s,s) \leqslant (n,n) \leqslant (m,n) \leqslant (r,s), \\ 0 & \text{otherwise (including the case } 2u_{m,n} < i \leqslant 8u_{m,n}); \end{cases}$$

$$Q_{m,n}(f_i) = \begin{cases} f_i & i < 8u_{m,n} + u_{n,n}, \\ -G_i e_{i-8u_{r,s}}, & i \in [2)_{r,s} \text{ and } (s,s) < (m,n) < (r,s), \\ 0 & \text{otherwise;} \end{cases}$$

$$\tau_{m,n}(e_i) = \begin{cases} e_i & i \leqslant 2u_{m,n}, \\ 0 & \text{otherwise.} \end{cases}$$

For every pair (m,n) let us define a compact set $K_{m,n} \subset X_{8u_{m,n}+u_{n,n}} \setminus \{0\}$ by

$$K_{m,n} = \Big\{ y \in X_{8u_{m,n}+u_{n,n}} : \|y\| \leqslant f(m,n), \|\tau_{m,n} y\| \geqslant \frac{2^{-m-n}}{16 f(m,n)} \Big\}.$$

The next lemma will help us to send any unit vector of X inside one of $K_{m,n}$.

LEMMA 4.2. *For every vector $x \in X$ with $\|x\| = 1$ and for each natural number n there exist $j \geqslant n$ and $m > j$ such that $Q_{m,j}x \in K_{m,j}$.*

Proof. First, notice that for every vector $x \in X$ with $\|x\| = 1$ and for each natural number n we have the convergence $P_{m,n}x \to x$ as $m \to \infty$. Indeed, for a fixed n and for any $y \in c_{00}$ the equation $P_{m,n}y = y$ holds for all but finitely many m. So, to prove our first claim, we need boundedness of $\|P_{m,n}\|$ where n is fixed. According to the definition of $P_{m,n}$,

$$\|P_{m,n}\| = \max\{G_i\|e_{i-8u_{r,s}}\| : i \in [2)_{r,s} \text{ and } (s,s) \leqslant (n,n) \leqslant (m,n) \leqslant (r,s)\}.$$

On the other hand, if $i \in [2)_{r,s}$ and $(s,s) \leqslant (n,n) \leqslant (m,n) \leqslant (r,s)$, then $G_i = G_{i-8u_{r,s}+8u_{s,s}}$ and $i - 8u_{r,s} + 8u_{s,s} < 9u_{s,s} \leqslant 8u_{n,n} + u_{n,n}$. This and the fact that $\|e_i\| \leqslant \|e_0\| = 1$ for every i, yield $\|P_{m,n}\| \leqslant \max\{G_j : j < 8u_{n,n} + u_{n,n}\} \leqslant f(n,n)$ for every m. Hence, we have shown that $P_{m,n}x \to x$ as $m \to \infty$.

For a subset $U \subset \mathbb{R}^+$ denote by π_U the projection such that $\pi_U(f_i) = f_i$ if $i \in U$ and zero otherwise. Our next important claim is that for every j satisfying $n < j \leqslant m$ we have $\pi_{[2)_{m,j}}\tau_{m,n}Q_{m,n} = \pi_{[2)_{m,j}}$. To show this, let us write out the formula for $\tau_{m,n}Q_{m,n}$.

Observe that if i is less than $8u_{m,n} + u_{n,n}$, then $i \leqslant 2u_{m,n}$, or $2u_{m,n} < i < 8u_{m,n}$, or $i \in [2)_{m,n}$. If $i \leqslant 2u_{m,n}$ holds, then neither $Q_{m,n}$ nor $\tau_{m,n}$ changes f_i. For any index i satisfying $2u_{m,n} < i < 8u_{m,n}$ we have $\tau_{m,n}Q_{m,n}f_i = \tau_{m,n}f_i = 0$. Finally, if $i \in [2)_{m,n}$, then $\tau_{m,n}Q_{m,n}f_i = \tau_{m,n}f_i = G_iH_{m,n}\tau_{m,n}e_i - G_i\tau_{m,n}e_{i-8u_{m,n}} = -G_ie_{i-8u_{m,n}}$. We also observe that $i \in [2)_{r,s}$ with $(s,s) < (m,n) < (r,s)$ implies $i - 8u_{r,s} < u_{s,s} < 2u_{m,n}$. It follows that for such i we have $\tau_{m,n}Q_{m,n}f_i = \tau_{m,n}(-G_ie_{i-8u_{r,s}}) = -G_ie_{i-8u_{r,s}}$.

Summarizing these comments we obtain the following.

$$\tau_{m,n}Q_{m,n}(f_i) = \begin{cases} f_i & i \leqslant 2u_{m,n}, \\ -G_ie_{i-8u_{r,s}} & i \in [2)_{r,s} \text{ and } (s,s) < (m,n) \leqslant (r,s), \\ 0 & \text{otherwise (including the case } 2u_{m,n} < i \leqslant 8u_{m,n}). \end{cases}$$

Back to our claim, for j with $n < j \leqslant m$ we have $(m,j) < (m,n)$ and, therefore, $[2)_{m,j} \subset [0, 2u_{m,n})$ holds. Since for every $i \in [0, 2u_{m,n})$ the vector f_i is unchanged by $\tau_{m,n}Q_{m,n}$ we obtain $\pi_{[2)_{m,j}} = \pi_{[2)_{m,j}}\pi_{[0,2u_{m,n})} = \pi_{[2)_{m,j}}\tau_{m,n}Q_{m,n}\pi_{[0,2u_{m,n})}$. It is left to notice that the only indexes $i \geqslant 2u_{m,n}$ for which $\tau_{m,n}Q_{m,n}f_i \neq 0$ are those satisfying $i \in [2)_{r,s}$ with $(s,s) < (m,n) \leqslant (r,s)$. But for such i we have $i - 8u_{r,s} < u_{s,s} \leqslant u_{m,m} < 8u_{m,j}$ and, hence, $\pi_{[2)_{m,j}}\tau_{m,n}Q_{m,n}\pi_{[2u_{m,n},\infty)} = 0$. We conclude that $\pi_{[2)_{m,j}}\tau_{m,n}Q_{m,n} = \pi_{[2)_{m,j}}$.

Notice that similarly to the norm-estimation of the projection $P_{m,n}$ we may obtain the following estimations

$$\|Q_{m,n}\| = \sup_i \|Q_{m,n}f_i\| \leqslant \max\{G_i : i < 8u_{m,n} + u_{n,n}\}\|e_0\| \leqslant f(m,n),$$

as well as $\|\tau_{m,n}Q_{m,n}\| \leqslant f(m,n)$.

Now, everything is ready to prove the lemma. Let x be a unit vector and let n be a fixed number. Since for every pair (m,j) we have $\|Q_{m,j}x\| \leqslant \|Q_{m,j}\| \leqslant f(m,j)$ we only need to show that there is a pair (m,j) such that $n \leqslant j < m$ and $\|\tau_{m,j}Q_{m,j}x\| \geqslant \frac{2^{-m-j}}{16f(m,j)}$. As we showed above, there is k such that $\|P_{k,n}x\| > \frac{1}{2}$. If for this k the inequality $\|\tau_{k,n}Q_{k,n}x\| > \frac{1}{4}$ holds, we are done. If not, then the inequality $\|P_{k,n}x - \tau_{k,n}Q_{k,n}x\| > \frac{1}{4}$ must hold. The image $(P_{k,n} - \tau_{k,n}Q_{k,n})f_i$ is non-zero if and only if $i \in [2)_{r,s}$ where $(n,n) < (s,s) < (k,n) < (r,s)$. Since $(k,n) < (r,s)$ with $n < s$ is possible only if $k < r$ we may say that $(P_{k,n} - \tau_{k,n}Q_{k,n})x = (P_{k,n} - \tau_{k,n}Q_{k,n})\pi_S x$ for

$$\bigcup_{(n,n)<(s,s)<(k,n)<(r,s)} [2)_{r,s} = \bigcup_{n<s\leqslant k<r} [2)_{r,s} =: S.$$

Then, since $\|(P_{k,n} - \tau_{k,n}Q_{k,n})\pi_S x\| > \frac{1}{4}$, we obtain

$$\|\pi_S x\| > \frac{1}{4\|P_{k,n} - \tau_{k,n}Q_{k,n}\|} \geqslant \frac{1}{4[f(n,n)+f(k,n)]} \geqslant \frac{1}{8f(k,n)}.$$

The fact that S is a countable union of sets $[2)_{r,s}$ suggests that there exists a pair (m,s) with $n < s \leqslant k < m$ such that $\|\pi_{[2)_{m,s}}x\| \geqslant \|\pi_S x\| 2^{-m-s}$. As discussed above, then we have $\pi_{[2)_{m,s}}\tau_{m,s-1}Q_{m,s-1}x = \pi_{[2)_{m,s}}x$ and, thus, for $n \leqslant j = s-1 < k < m$

$$\|\tau_{m,j}Q_{m,j}x\| \geqslant \|\pi_{[2)_{m,s}}\tau_{m,j}Q_{m,j}x\| = \|\pi_{[2)_{m,s}}x\| \geqslant \frac{\|\pi_S x\|}{2^{m+s}} \geqslant \frac{2^{-m-j-1}}{8f(k,n)} \geqslant \frac{2^{-m-j}}{16f(m,j)}.$$

The proof is complete. ∎

5. WHAT MAKES T TRANSITIVE?

Let us see what we need in order to show that T does not have non-trivial closed invariant subspaces. For every pair of numbers (m,n) let $K_{m,n}$ be the compact set defined above. Let $T_{m,n} : X_{8u_{m,n}+u_{n,n}} \to X_{8u_{m,n}+u_{n,n}}$ be the "truncated" version of T, namely, $T_{m,n}e_i = e_{i+1} (i < 8u_{m,n} + u_{n,n} - 1)$ or zero $(i = 8u_{m,n} + u_{n,n} - 1)$.

Given $y \in K_{m,n}$, write $y = \sum_{i=\alpha}^{8u_{m,n}+u_{n,n}-1} \lambda_i e_i$, where $\lambda_\alpha \neq 0$. Then

$$\operatorname{span}\{T_{m,n}^r y : 6u_{m,n} \leqslant r < 8u_{m,n} + u_{n,n}\} = \operatorname{span}\{e_{\alpha+6u_{m,n}}, \ldots, e_{8u_{m,n}+u_{n,n}-1}\}.$$

Since $\tau_{m,n}(y) \neq 0$ we have $\alpha \leqslant 2u_{m,n}$ and hence the vector $e_{8u_{m,n}}$ belongs to $\operatorname{span}\{T_{m,n}^r y : 6u_{m,n} \leqslant r < 8u_{m,n} + u_{n,n}\}$. Since $K_{m,n}$ is compact there is a finite number $p_1, \ldots, p_r$ of polynomials p (of form $p_j(t) = \sum_{s=6u_{m,n}}^{8u_{m,n}+u_{n,n}-1} \nu_{sj} t^s$) such that for all $y \in K_{m,n}$ there is a number j such that

$$\|p_j(T_{m,n})x - H^{(2)}_{m,n}e_{8u_{m,n}}\| < \frac{1}{n}.$$

Using the polynomials p_j that we find for a pair (m,n), we define a function $g(m,n)$ by $g(m,n) = \max_j |p_j|$. Since by the definition of $e_{8u_{m,n}}$ we have $\|H^{(2)}_{m,n}e_{8u_{m,n}} - e_0\| \leqslant \frac{1}{G_{8u_{m,n}}} = \frac{1}{G_{8u_{n,n}}} = \frac{1}{n}$ our discussion provides us with the following lemma.

LEMMA 5.1. *There is a function $g(m,n)$ with the following property. For any pair (m,n) and $y \in K_{m,n}$ there is a polynomial p such that $|p| < g(m,n)$, $p(t)$ is of the form $p(t) = \sum_{s=6u_{m,n}}^{8u_{m,n}+u_{n,n}-1} \nu_{sj}t^s$, and*

$$\|p(T_{m,n})y - e_0\| < \frac{2}{n}.$$

REMARK 5.2. The following inequalities

$$f(m,n) \leqslant [N_a(m,n)]^{h(u_{m,n})} \quad \text{and} \quad g(m,n) \leqslant [N_a(m,n)]^{h(u_{m,n})}$$

hold for a relatively simple function h. This might be interesting for explicit definition of a transitive operator. Since it is irrelevant for this paper we do not present the argument.

Our next lemma allows us to replace $T_{m,n}$ in the estimation above by T.

LEMMA 5.3. *If for any consecutive pairs $(m,n) < (r,s)$ the functions N_a and N_b satisfy*

$$N_a(r,s) \geqslant n[f(m,n)]^2 g(m,n) H^{(4)}_{m,n},$$
$$N_b(m,n) \geqslant n[f(m,n)]^2 g(m,n),$$

then the following holds:

For any $y \in K_{m,n}$, with the notation of previous lemma, the polynomial $q(t) = H^{(4)}_{m,n}t^{8u_{m,n}}p(t)$ satisfies $t^{14u_{m,n}}|q(t)$, $\deg q \leqslant 16u_{m,n} + u_{n,n}$, $|q| \leqslant H^{(4)}_{m,n}g(m,n)$, and

$$\|q(T)y - e_0\| < \frac{4}{n}.$$

Proof. Given $y \in K_{m,n}$, let p be the polynomial from the previous lemma. For convenience, let us set $M = 8u_{m,n} + u_{n,n} - 1$ for this proof. If $p(T_{m,n})y = \sum_{i=6u_{m,n}}^{M} \lambda_i e_i$, then

$$\|H^{(4)}_{m,n}T^{8u_{m,n}}p(T_{m,n})y - p(T_{m,n})y\| = \Big\| \sum_{i=6u_{m,n}}^{M} \lambda_i(H^{(4)}_{m,n}e_{8u_{m,n}+i} - e_i)\Big\|,$$

since for every $6u_{m,n} \leqslant i \leqslant M$ above we have $8u_{m,n} + i \in [4)_{m,n}$, we continue

$$= \Big\| \sum_{i=6u_{m,n}}^{M} \lambda_i \frac{f_{8u_{m,n}+i}}{G_{8u_{m,n}+i}} \Big\| \leqslant \frac{1}{G_{12u_{m,n}}} \sum_{i=6u_{m,n}}^{M} |\lambda_i| = \frac{|p(T_{m,n})y|}{G_{12u_{m,n}}} \leqslant \frac{|p||y|}{G_{12u_{m,n}}}$$
$$\leqslant \frac{g(m,n)\|y\| f(m,n)}{G_{12u_{m,n}}} \leqslant \frac{g(m,n)[f(m,n)]^2}{G_{12u_{m,n}}} = \frac{g(m,n)[f(m,n)]^2}{N_b(m,n)} \leqslant \frac{1}{n}.$$

Hence, the inequality

$$\|H^{(4)}_{m,n} T^{8u_{m,n}} p(T_{m,n})y - p(T_{m,n})y\| \leqslant \frac{1}{n}$$

holds.

Next we replace $T_{m,n}$ by T inside the polynomial p. To this end notice that

$$\begin{aligned} T^{8u_{m,n}}(p(T) - p(T_{m,n}))y &\in T^{8u_{m,n}} \mathrm{span}\{e_j : M < j \leqslant 2M\} \\ &\subset \mathrm{span}\{e_j : 16u_{m,n} + u_{n,n} \leqslant j < 8(16u_{m,n} + u_{n,n})\} \\ &= \mathrm{span}\{e_j : j \in [1)_{r,s}\} \end{aligned}$$

where the block (r,s) is the successor of (m,n). Since $F_j e_j = f_j$ for every j in $[1)_{r,s}$ we obtain

$$\|e_j\| \leqslant \frac{1}{F_{u_{r,s}}}$$

for every $j \in [1)_{r,s}$. So, with p as above

$$\begin{aligned} \|H^{(4)}_{m,n}[T^{8u_{m,n}} p(T_{m,n})y - T^{8u_{m,n}} p(T)y]\| &\leqslant \frac{H^{(4)}_{m,n}|p||y|}{F_{u_{r,s}}} \leqslant \frac{H^{(4)}_{m,n} f(m,n) g(m,n)\|y\|}{F_{u_{r,s}}} \\ &< \frac{H^{(4)}_{m,n}[f(m,n)]^2 g(m,n)}{N_a(r,s)} < \frac{1}{n}. \end{aligned}$$

Summarizing we obtain for $q(t) = H^{(4)}_{m,n} t^{8u_{m,n}} p(t)$ the inequality $\|q(T)y - e_0\| < \frac{4}{n}$ provided the conditions on N_a and N_b are satisfied. ∎

By this point we have proved the following. For every $x \in X$ such that $\|x\| = 1$ and for any number n_0 we can find numbers $m > n \geqslant n_0$ such that $y = Q_{m,n}x$ belongs to the compact set $K_{m,n}$. Then for that $y \in K_{m,n}$ we find a polynomial q with $t^{14u_{m,n}}|q$ and $|q| < H^{(4)}_{m,n} g(m,n)$ such that $\|q(T)y - e_0\| \leqslant \frac{4}{n}$. Thus, we obtain the estimation

$$\begin{aligned} \|q(T)x - e_0\| &\leqslant \|q(T)x - q(T)Q_{m,n}x\| + \|q(T)Q_{m,n}x - e_0\| \\ &\leqslant \|\widetilde{q}(T)T^{14u_{m,n}}(I - Q_{m,n})x\| + \frac{4}{n} \end{aligned}$$

where $\widetilde{q}$ is such that $q(t) = t^{14u_{m,n}}\widetilde{q}(t)$. So, in order to show that e_0 belongs to the closure of $\mathrm{span}\{T^r x : r \geqslant 0\}$ we need to show that the first term can be made arbitrarily small. To this end observe that

$$\begin{aligned}\|\widetilde{q}(T)T^{14u_{m,n}}(I-Q_{m,n})x\| &\leqslant |q|\|T^{14u_{m,n}}(I-Q_{m,n})\| \\ &\leqslant g(m,n)H^{(4)}_{m,n}\|T^{14u_{m,n}}(I-Q_{m,n})\|.\end{aligned}$$

After that we will prove the following estimate of $g(m,n)H^{(4)}_{m,n}\|T^{14u_{m,n}}(I-Q_{m,n})\|$.

LEMMA 5.4. *If for any, not necessary consecutive, pairs $(m,n) < (r,s)$ the functions N_a and N_b satisfy*

$$\begin{aligned}N_a(r,s) &\geqslant 2ng(m,n)[H^{(4)}_{m,n}]^2,\\ N_b(r,s) &\geqslant 2ng(m,n)H^{(4)}_{m,n}H^{(2)}_{r,s},\end{aligned}$$

then for every pair (m,n) the inequality

$$g(m,n)H^{(4)}_{m,n}\|T^{14u_{m,n}}(I-Q_{m,n})\| < \frac{1}{n}$$

holds.

Proof. Let us fix a pair (m,n) and consider the corresponding projection $Q_{m,n}$. We will prove the claim of the lemma by proving the inequality

$$g(m,n)H^{(4)}_{m,n}\|T^{14u_{m,n}}(I-Q_{m,n})f_i\| < \frac{1}{n} \tag{5.1}$$

for every i.

By the definition of the projection $Q_{m,n}$ we have $(I-Q_{m,n})f_i = 0$ for every $i < 8u_{m,n} + u_{n,n}$ and, hence, (5.1) holds trivially. Thus, for our estimations we consider $i \geqslant 8u_{m,n} + u_{n,n}$. Observe that in this case

$$(I-Q_{m,n})f_i = \begin{cases} G_iH^{(2)}_{r,s}e_i & \text{if } i \in [2)_{r,s} \text{ and } (s,s) < (m,n) < (r,s),\\ f_i & \text{otherwise.}\end{cases}$$

Case 1: If $i \in [2)_{r,s}$ where $(s,s) < (m,n) < (r,s)$, then $(I-Q_{m,n})f_i = G_iH^{(2)}_{r,s}e_i$. Notice that $(s,s) < (m,n) < (r,s)$ implies $u_{s,s} < u_{m,n} < 14u_{m,n} < u_{r,s}$, hence, every i such that $8u_{r,s} \leqslant i < 8u_{r,s} + u_{s,s}$ satisfies $8u_{r,s} + u_{s,s} \leqslant i + 14u_{m,n} < 12u_{r,s}$. It follows that $i + 14u_{m,n} \in [3)_{r,s}$. Therefore, using $e_{i+14u_{m,n}} = \frac{f_{i+14u_{m,n}}}{F^{(3)}_{i+14u_{m,n}}}$, $F^{(3)}_{i+14u_{m,n}} > [N_b(r,s)]^2$, and $G^{(2)}_i < N_a(r,s)H^{(2)}_{r,s}$ we obtain

$$\begin{aligned}\|T^{14u_{m,n}}(I-Q_{m,n})f_i\| &= \|T^{14u_{m,n}}G^{(2)}_iH^{(2)}_{r,s}e_i\| = G^{(2)}_iH^{(2)}_{r,s}\|e_{i+14u_{m,n}}\|\\ &< \frac{[H^{(2)}_{r,s}]^2}{F^{(3)}_{i+14u_{m,n}}N_a(r,s)} < \frac{[N_b(r,s)]^2}{N_a(r,s)[N_b(r,s)]^2} \leqslant \frac{1}{N_a(r,s)}\\ &< \frac{1}{ng(m,n)H^{(4)}_{m,n}}\end{aligned}$$

and that will make the inequality (5.1) hold.

For the rest of the proof we assume that $i \geqslant 8u_{m,n} + u_{n,n}$ and $(I - Q_{m,n})f_i = f_i$ since the other cases have been verified.

Case 2: If both i and $i + 14u_{m,n}$ belong to the same interval. Observe that these two indices cannot lie in the same interval of the block (m, n). Hence, let them both belong to some block (r, s) with $(r, s) > (m, n)$. Then, similar to the proof of $\|T\| \leqslant 1$ we express $T^{14u_{m,n}} f_i$ as follows. If i is in $[1)_{r,s}$ or in $[3)_{r,s}$, then $T^{14u_{m,n}} f_i = \frac{F_i}{F_{i+14u_{m,n}}} f_{i+14u_{m,n}}$, otherwise, that is, if i is in $[2)_{r,s}$ or in $[4)_{r,s}$, we have $T^{14u_{m,n}} f_i = \frac{G_i}{G_{i+14u_{m,n}}} f_{i+14u_{m,n}}$. In either case we obtain

$$\|T^{14u_{m,n}}(I - Q_{m,n})f_i\| = \|T^{14u_{m,n}} f_i\| < \frac{1}{[N(r,s)]^{14u_{m,n}}} < \frac{1}{N(r,s)} \leqslant \frac{1}{ng(m,n)H^{(4)}_{m,n}}$$

where the function N is $N_a(r,s), N_b(r,s)$, or $N_a(s,s)$ with $(s,s) > (m,n)$ depending on the situation. This guarantees the inequality (5.1).

Case 3: If i belongs to the block (m, n). Let (r, s) be the successor of (m, n). Then since $i \geqslant 8u_{m,n} + u_{n,n}$ we have $i + 14u_{m,n} > 16u_{m,n} + u_{n,n} + 1 = u_{r,s} + 1$. We also have $i + 14u_{m,n} < 8(16u_{m,n} + u_{n,n})$ and, thus, $i + 14u_{m,n} \in [1)_{r,s}$. Moreover, $F^{(1)}_{i+14u_{m,n},r,s} > [N_a(r,s)]^2$. Therefore, if $i \in [3)_{m,n}$, then we get the estimate

$$\|T^{14u_{m,n}}(I - Q_{m,n})f_i\| = \|T^{14u_{m,n}} f_i\| = \frac{F^{(3)}_{i,m,n}}{F^{(1)}_{i+14u_{m,n},r,s}} < \frac{H^{(4)}_{m,n}}{[N_a(r,s)]^2} < \frac{1}{N_a(r,s)}$$

which leads to the inequality (5.1) as before.

If $i \in [4)_{m,n}$, then $i \geqslant 12u_{m,n}$ which implies

$$i - 8u_{m,n} + 14u_{m,n} \geqslant 18u_{m,n} > 16u_{m,n} + u_{n,n} + 1 = u_{r,s} + 1$$

and, thus, $i - 8u_{m,n} + 14u_{m,n}$ lies in $[1)_{r,s}$. Now inequality (5.1) follows from

$$\begin{aligned}\|T^{14u_{m,n}}(I - Q_{m,n})f_i\| &= \|G^{(4)}_i H^{(4)}_{m,n} e_{i+14u_{m,n}} - G^{(4)}_i e_{i-8u_{m,n}+14u_{m,n}}\| \\ &\leqslant \frac{G^{(4)}_i H^{(4)}_{m,n}}{F^{(1)}_{i+14u_{m,n}}} + \frac{G^{(4)}_i}{F^{(1)}_{i-8u_{m,n}+14u_{m,n}}} \\ &< \frac{[H^{(4)}_{m,n}]^2}{[N_a(r,s)]^2} + \frac{H^{(4)}_{m,n}}{[N_a(r,s)]^2} < \frac{2}{N_a(r,s)}.\end{aligned}$$

For the rest of the proof we assume that i and $i + 14u_{m,n}$ belong to different intervals and that i is in the block (r, s) with $(r, s) > (m, n)$. In particular, this means $u_{r,s} > 16u_{m,n}$.

Case 4: If i belongs to $[1)_{r,s}$, then $i + 14u_{m,n} < 9u_{r,s}$ and, therefore, $i + 14u_{m,n}$ lies in $[2)_{r,s}$ or in $[3)_{r,s}$. For the latter, the inequality (5.1) easily follows from

$$\|T^{14u_{m,n}}(I - Q_{m,n})f_i\| = \frac{F^{(1)}_i}{F^{(3)}_{i+14u_{m,n}}} < \frac{H^{(2)}_{r,s}}{[N_b(r,s)]^2} < \frac{1}{N_b(r,s)}.$$

For the former, the inequality (5.1) follows from (2.10) and

$$\|T^{14u_{m,n}}(I-Q_{m,n})f_i\| = \Big\|\frac{F_i^{(1)}f_{i+14u_{m,n}}}{G^a_{i+14u_{m,n}}H^{(2)}_{r,s}} + \frac{F_i^{(1)}e_{i+14u_{m,n}-8u_{r,s}}}{H^{(2)}_{r,s}}\Big\| < \frac{2F_i^{(1)}}{H^{(2)}_{r,s}} < \frac{2}{N_a(r,s)}.$$

Case 5: If i belongs to $[2)_{r,s}$, then $i+14u_{m,n}$ lies in $[3)_{r,s}$ because of the inequality $i+14u_{m,n} < 12u_{r,s}$. Due to Case 1 above we consider only the case $(s,s) > (m,n)$, that is, $u_{s,s} > 16u_{m,n}$. Let us determine in what interval the number $i-8u_{r,s}+14u_{m,n}$ lies. Since $i+14u_{m,n} \geqslant 8u_{r,s}+u_{s,s}$ we conclude that $i-8u_{r,s}+14u_{m,n} \geqslant u_{s,s}$. We also have $i < 8u_{r,s}+u_{s,s}$ which implies $i-8u_{r,s}+14u_{m,n} < u_{s,s}+14u_{m,n} < 8u_{s,s}$. Altogether this gives us that $i-8u_{r,s}+14u_{m,n}$ belongs to $[1)_{s,s}$. Using (2.2) we estimate

$$\begin{aligned}\|T^{14u_{m,n}}(I-Q_{m,n})f_i\| &\leqslant \frac{G_i^{(2)}H^{(2)}_{r,s}}{F^{(3)}_{i+14u_{m,n}}} + \frac{G_i^{(2)}}{F^{(1)}_{i-8u_{r,s}+14u_{m,n}}} < \frac{[H^{(2)}_{r,s}]^2}{N_a(r,s)[N_b(r,s)]^2} + \frac{1}{N_a(s,s)}\\ &< \frac{1}{N_a(r,s)} + \frac{1}{N_a(s,s)}\end{aligned}$$

which together with the fact that $(r,s) > (s,s) > (m,n)$ yield the inequality (5.1).

Case 6: If i belongs to $[3)_{r,s}$, then $i+14u_{m,n}$ lies in $[4)_{r,s}$ and we get as usually

$$\|T^{14u_{m,n}}(I-Q_{m,n})f_i\| = \Big\|\frac{F_i^{(3)}f_{i+14u_{m,n}}}{G^{(4)}_{i+14u_{m,n}}H^{(4)}_{r,s}} + \frac{F_i^{(3)}e_{i+14u_{m,n}-8u_{r,s}}}{H^{(4)}_{r,s}}\Big\| < \frac{2F_i^{(3)}}{H^{(4)}_{r,s}} < \frac{2}{N_b(r,s)}.$$

So, the inequality (5.1) holds.

Case 7: If i belongs to $[4)_{r,s}$, then $i+14u_{m,n}$ lies in $[1)_{l,j}$, where (l,j) is the successor of (r,s). Then by our assumptions we obtain $8u_{r,s}+u_{s,s} \leqslant i-8u_{r,s}+14u_{m,n} < 12u_{r,s}$ meaning that $i-8u_{r,s}+14u_{m,n}$ lies in $[3)_{r,s}$. So, the inequalities (2.6), (2.8), and (2.9) give us

$$\begin{aligned}\|T^{14u_{m,n}}(I-Q_{m,n})f_i\| &\leqslant \frac{G_i^{(4)}H^{(4)}_{r,s}}{F^{(1)}_{i+14u_{m,n}}} + \frac{G_i^{(4)}}{F^{(3)}_{i-8u_{r,s}+14u_{m,n}}} < \frac{[H^{(4)}_{r,s}]^2}{N_b(r,s)N_a(l,j)} + \frac{1}{N_b(r,s)}\\ &< \frac{N_a(l,j)}{N_b(r,s)N_a(l,j)} + \frac{1}{N_b(r,s)} < \frac{2}{N_b(r,s)}.\end{aligned}$$

Since we have considered all possibilities, the lemma has been proved. ∎

6. CONCLUSION

Let us finish with the demonstration of the quasinilpotence of our construction. After that we will summarize our demands for the functions N_a and N_b.

Similarly to [8] we obtain:

THEOREM 6.1. *If the functions N_a and N_b satisfy the conditions of Lemma* 5.4 *and, in addition, for any, not necessary consecutive, pairs $(m,n) < (r,s)$ the functions N_a and N_b satisfy*

$$N_a(r,s), N_b(r,s), \quad \textit{and} \quad N_b(m,n) \geqslant n^{14u_{m,n}}[f(m,n)]^2,$$

then the resulting operator T is quasinilpotent.

Proof. Consider the functions N_a and N_b satisfying the condition of the theorem. Let T be constructed using N_a and N_b. Fix a pair of natural numbers (m,n) with $m > n$. Then similarly to Lemma 5.4 we obtain

$$\|T^{14u_{m,n}}\| \leqslant \|T^{14u_{m,n}}(I - Q_{m,n})\| + \|T^{14u_{m,n}}Q_{m,n}\| < \frac{2}{N(r,s)} + \|T^{14u_{m,n}}Q_{m,n}\|$$

where (r,s) is some pair satisfying $(r,s) > (m,n)$ and N is either N_a or N_b. Recall that $Q_{m,n}(X) = X_{8u_{m,n}+u_{n,n}}$ and $\|Q_{m,n}\| \leqslant f(m,n)$ which provides us with $\|T^{14u_{m,n}}Q_{m,n}\| \leqslant f(m,n)\|T^{14u_{m,n}}|_{X_{8u_{m,n}+u_{n,n}}}\|$. Using the facts that $\|T\| \leqslant 1$ and that for every $x \in X_{8u_{m,n}+u_{n,n}}$ we have $|x| \leqslant f(m,n)\|x\|$ we estimate

$$\begin{aligned}\|T^{14u_{m,n}}|_{X_{8u_{m,n}+u_{n,n}}}\| &\leqslant f(m,n)\|T^{14u_{m,n}} : (X_{8u_{m,n}+u_{n,n}}, |\cdot|) \to X\| \\ &\leqslant f(m,n)\max\{\|e_{i+14u_{m,n}}\| : i < 8u_{m,n} + u_{n,n}\} \\ &\leqslant f(m,n)\|e_{8u_{m,n}+u_{n,n}}\| = \frac{f(m,n)}{F_{8u_{m,n}+u_{n,n}}} < \frac{f(m,n)}{N_b(m,n)}.\end{aligned}$$

We combine everything above into

$$\|T^{14u_{m,n}}\| \leqslant \frac{2}{N(r,s)} + \frac{[f(m,n)]^2}{N_b(m,n)} < \frac{3}{n^{14u_{m,n}}}.$$

Since the last inequality holds for every pair (m,n), the operator T is quasinilpotent. ∎

We may summarize the conditions of all lemmas above, for instance, as follows.

THEOREM 6.2. *An operator $T : l_1 \to l_1$ described in Section* 2 *above is a quasinilpotent operator without non-trivial closed invariant subspaces provided the functions N_a and N_b satisfy the following inductive conditions:*

$$N_a(r,s) > \max\{n^{14u_{m,n}} g(m,n) H^{(2)}_{m,n} [f(m,n) H^{(4)}_{m,n}]^2 : (m,n) < (r,s)\} \textit{ and}$$

$$N_b(r,s) > N_a(r,s) s^{14u_{r,s}} g(r,s) H^{(2)}_{r,s} [f(r,s)]^2.$$

It should be noticed that our modification is less rigid than the original construction. We eliminated the connection between transitivity and the form of the operator. As a result we may consider more involved matrices preserving transitivity. Another degree of freedom can be obtained from the order of the blocks (m,n). For our paper we have fixed it in convenient way at the beginning. Despite this the only place where the order of the blocks has been used was

Lemma 4.2. Hopefully these new degrees of freedom might help to produce new interesting examples.

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REFERENCES

[1] Y.A. ABRAMOVICH, C.D. ALIPRANTIS, *Invitation to Operator Theory*, Grad. Stud. Math., vol. 50, Amer. Math. Soc., Providence, RI 2002.

[2] B. BEAUZAMY, An operator with no invariant subspace: simplification of the example of P. Enflo [French], *Integral Equations Operator Theory* **8**(1985), 314–384.

[3] P. ENFLO, On the invariant subspace problem for Banach spaces, *Acta Math.* **158**(1987), 213–313.

[4] H. RADJAVI, P. ROSENTHAL, *Invariant Subspaces*, Springer-Verlag, New York-Berlin 1973.

[5] C.J. READ, A solution to the invariant subspace problem, *Bull. London Math. Soc.* **16**(1984), 337–401.

[6] C.J. READ, A solution to the invariant subspace problem on the space l_1, *Bull. London Math. Soc.* **17**(1985), 305–317.

[7] C.J. READ, A short proof concerning the invariant subspace problem, *J. London Math. Soc.* **34**(1986), 335–348.

[8] C.J. READ, Quasinilpotent operators and the invariant subspace problem, *J. London Math. Soc.* **56**(1997), 595–606.

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