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SCHRÖDINGER OPERATORS WITH UNBOUNDED DRIFT

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ABSTRACT. Let $a_{ij} \in C^1_{\mathrm{b}}(\mathbb{R}^N)$, $i,j = 1,\dots,N$ be uniformly elliptic, and let $b \in C^1(\mathbb{R}^N)$, $V \in C(\mathbb{R}^N)$. If $\frac{\operatorname{div} b}{p} \leqslant V$, then we construct a unique minimal positive semigroup generated by a restriction of the operator A defined by the expression

$$Au = \sum_{i,j=1}^{N} D_i(a_{ij}D_j u) - \sum_{i=1}^{N} b_i D_i u - Vu$$

on $L^p(\mathbb{R}^N)$ with maximal domain. We give a criterion for $C^\infty_{\mathrm{c}}(\mathbb{R}^N)$ to be a core and we give conditions on V and b which imply that the semigroup is given by kernels allowing an upper Gaussian bound. By a specific example we show that our criteria are close to optimal.

KEYWORDS: *Schrödinger operators, positive contraction semigroups.*

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1. INTRODUCTION

Schrödinger operators of the form $\Delta - V$ with $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and their associated semigroups in $L^p(\mathbb{R}^N)$ have been studied for many properties, see e.g. the most motivating survey article by B. Simon [29]. On the other hand elliptic operators and their associated semigroups are quite well known in the case of bounded coefficients, see for example Davies' monograph [9] and also the survey [4].

In this article we consider the operator

$$A := \sum_{i,j=1}^{N} D_i(a_{ij}D_j) - \sum_{i=1}^{N} b_i D_i - V \tag{1.1}$$

under the following standing hypotheses, which we shall keep in the whole paper: $a_{ij}, b_i, V : \mathbb{R}^N \to \mathbb{R}$, with $a_{ij} \in C^1_{\mathrm{b}}(\mathbb{R}^N)$, $b_i \in C^1(\mathbb{R}^N)$, $V \in C(\mathbb{R}^N)$. Moreover, the matrix (a_{ij}) is assumed to be uniformly elliptic, i.e.,

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqslant \nu|\xi|^2,\ \nu > 0, \quad \forall\, x, \xi \in \mathbb{R}^N. \tag{1.2}$$

Notice that neither the drift $b = (b_1, \dots, b_N)$ nor the potential V are assumed to be bounded, but the unboundedness of the first order coefficients b_i can be balanced by the potential V assuming

$$\frac{\operatorname{div} b}{p} \leqslant V. \tag{1.3}$$

Let us denote by $A_{p,\max}$ the operator A endowed with its maximal domain

$$D_{p,\max} := \{u \in L^p(\mathbb{R}^N) \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N) :\ Au \in L^p(\mathbb{R}^N)\}, \quad 1 < p < \infty, \tag{1.4}$$

$$D_{1,\max} := \{u \in L^1(\mathbb{R}^N) \cap W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N) :\ Au \in L^1(\mathbb{R}^N)\}. \tag{1.5}$$

We study under which conditions there is a unique restriction A_p of the operator $A_{p,\max}$ which generates a minimal positive C_0 semigroup in $L^p(\mathbb{R}^N)$, where $1 \leqslant p < \infty$. In general, there may be other extensions generating larger positive semigroups, but we give uniqueness criteria which imply that $C_c^\infty(\mathbb{R}^N)$ is a core of the generator. We also investigate when the operator A_p has compact resolvent and give a criterion on the growth of b with respect to V which implies that the semigroup has a kernel which has an upper Gaussian bound. Such Gaussian bound has many interesting consequences, in particular, analyticity of the semigroup in $L^1(\mathbb{R}^N)$. This analyticity had been proved before under slightly more restrictive conditions by a completely different method in [24] which, however, allows to characterize the domain, see also [7] for more general results and [28] for a detailed analysis for $p = 2$. Gaussian estimates imply also that the spectrum of the generator is independent of $p \in [1, \infty[$. This property, among others, has also been studied by Liskevich, Sobol and Vogt in [19] and Sobol and Vogt in [30]. In the last section we consider the examples

$$Au(x) = u''(x) - x^3u'(x) - c|x|^\gamma u(x),$$

on $L^p(\mathbb{R})$, which show that our criteria are close to optimal. For instance, we show that for $\gamma \geqslant 6$ the semigroup has Gaussian estimates, but for $2 < \gamma < 6$ a semigroup is obtained which is not holomorphic in $L^p(\mathbb{R}^N)$ for $1 \leqslant p < \infty$.

We refrain from taking less regular coefficients a_{ij} in order to avoid technical complications which hide the basic ideas. This allows us in particular to use a very simple technique, introduced in [6], based on the Beurling-Deny criteria and Davies' trick, to prove Gaussian estimates.

Notation. For $x \in \mathbb{R}^N$, $|x|$ denotes the euclidean norm, and $B_\varrho = \{x \in \mathbb{R}^N : |x| < \varrho\}$ the open ball with radius $\varrho > 0$. For every function u we denote by u^+ and u^- the positive and negative parts, i.e., $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. The spaces $L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, are endowed with the usual norm $\|\cdot\|_{L^p(\Omega)}$ denoted also by $\|\cdot\|_p$ when $\Omega = \mathbb{R}^N$. The Sobolev space $W^{k,p}(\Omega)$ is the set of all the measurable functions in the open set $\Omega \subset \mathbb{R}^N$ which have weak derivatives p-summable in Ω up to order k, endowed with the usual norm $\|\cdot\|_{W^{k,p}(\Omega)}$, denoted by $\|\cdot\|_{k,p}$ when $\Omega = \mathbb{R}^N$. We set $u \in W^{k,p}_{\mathrm{loc}}(\Omega)$ if $\varphi u \in W^{k,p}(\Omega)$ for every $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$. We denote by $C_{\mathrm{b}}(\mathbb{R}^N)$ the space of bounded and continuous functions on $\mathbb{R}^N$, endowed with the sup norm $\|\cdot\|_\infty$. By $C^1_{\mathrm{b}}(\mathbb{R}^N)$ we denote the space of all bounded continuously differentiable functions on $\mathbb{R}^N$ with bounded derivative.

If L is a closed operator in a Banach space X, we denote by $\sigma(L)$ and $\rho(L)$ the spectrum and the resolvent set of L. The resolvent operator is denoted by $R(\lambda, L)$. We say that an operator L on L^p is *resolvent positive* if there exists $\lambda_0 \in \mathbb{R}$ such that $[\lambda_0, \infty[\subset \rho(L)$ and $R(\lambda, L)f \geqslant 0$ for $\lambda \geqslant \lambda_0$, whenever $f \in L^p$, $f \geqslant 0$.

2. PRELIMINARY RESULTS

In this section we collect some results needed for the whole paper. For simplicity, we denote by A_0 the differential operator

$$A_0 := \sum_{i,j=1}^N D_i(a_{ij}D_j).$$

In order to construct a semigroup associated with A we need the following lemmas.

LEMMA 2.1. *Let $u \in W^{2,p}(B_\varrho)$, $1 < p < \infty$ and let $\eta \in W^{1,\infty}(B_\varrho)$. Then*

$$\begin{aligned}(p-1)\int_{B_\varrho} \eta |u|^{p-2}\chi_{\{u\neq 0\}} \sum_{i,j=1}^N a_{ij}D_iuD_ju + \int_{B_\varrho} u|u|^{p-2}\sum_{i,j=1}^N a_{ij}D_iuD_j\eta \\ = \int_{\partial B_\varrho} u|u|^{p-2}\eta \sum_{i,j=1}^N a_{ij}D_iu\nu_j \mathrm{d}\sigma - \int_{B_\varrho} \eta(A_0u)u|u|^{p-2},\end{aligned}$$

where $\nu = (\nu_1, \dots, \nu_N)$ is the outward normal to ∂B_ϱ and $\mathrm{d}\sigma$ is the surface measure. In particular, if $u \in W^{1,p}_0(B_\varrho)$, taking $\eta \equiv 1$ we get

$$\int_{B_\varrho} (A_0u)u|u|^{p-2} \leqslant 0, \quad \int_{B_\varrho} (A_0u)\operatorname{sign} u \leqslant 0.$$

Proof. Even though the above equality looks obvious (formally), it is elementary only if $p \geqslant 2$, whereas a (non-trivial) argument is needed for $1 < p < 2$ to avoid the singularities of $|u|^{p-2}$ at the points where u vanishes. We refer to [25] for the details. Concerning the last inequality, note that $u \in W^{2,r}(B_\varrho) \cap W^{1,r}_0(B_\varrho)$

for all $1 < r < p$ and therefore

$$\int_{B_\varrho} (A_0 u) u |u|^{r-2} \leqslant 0.$$

Letting $r \to 1$ we obtain the claim. ∎

LEMMA 2.2. *Let $u \in W^{2,p}(B_\varrho)$, $1 \leqslant p < \infty$ and assume that $u \leqslant 0$ on ∂B_ϱ in the sense of traces. Then*

$$\int_{B_\varrho} (A_0 u)(u^+)^{p-1} \leqslant 0 \quad \text{for } 1 < p < \infty, \qquad \int_{B_\varrho} (A_0 u) \chi_{\{u>0\}} \leqslant 0 \quad \text{for } p = 1.$$

Proof. Let us first take $p > 1$ and $u \in C^2(\overline{B}_\varrho)$. Let $h_n \in C^1_{\mathrm{b}}(\mathbb{R})$ be such that $h_n(t) = 0$ for $t \leqslant 0$, $h_n' \geqslant 0$, $h_n \leqslant h_{n+1}$ and $h_n(t) \to (t^+)^{p-1}$ as $n \to \infty$ for $t \leqslant \max\limits_{\overline{B}_\varrho} u$. Then (with the same notation as in the Lemma 2.1)

$$\begin{aligned} \int_{B_\varrho} (A_0 u) h_n(u) &= - \int_{\{u>0\}} h_n'(u) \sum_{i,j=1}^N a_{ij} D_i u D_j u + \int_{\partial B_\varrho} h_n(u) \sum_{i,j=1}^N a_{ij} D_i u \nu_j \mathrm{d}\sigma \\ &\leqslant \int_{\partial B_\varrho} h_n(u) \sum_{i,j=1}^N a_{ij} D_i u \nu_j \mathrm{d}\sigma. \end{aligned}$$

Letting $n \to \infty$ we deduce

$$\int_{B_\varrho} (A_0 u)(u^+)^{p-1} \leqslant \int_{\partial B_\varrho} (u^+)^{p-1} \sum_{i,j=1}^N a_{ij} D_i u \nu_j \mathrm{d}\sigma.$$

The above equality extends by density to every $u \in W^{2,p}(B_\varrho)$, since both sides are continuous with respect to the topology of $W^{2,p}(B_\varrho)$. For $p > 1$ the claim then follows because $u^+ = 0$ on ∂B_ϱ. For $p = 1$, one proceeds similarly, approximating the characteristic function of $[0, +\infty[$ instead of $(t^+)^{p-1}$. ∎

Some regularity properties of the semigroup generated by A in $L^1(\mathbb{R}^N)$ depend on interior L^1-estimates as stated in Proposition 2.4. Since we have not been able to find a reference for them, we provide a proof inspired by Theorem 7.1.1 of [18].

LEMMA 2.3. *There exist constants $C, \varepsilon_0 > 0$ such that for every $u \in C_{\mathrm{c}}^\infty(\mathbb{R}^N)$ and $\varepsilon \leqslant \varepsilon_0$*

$$\|\nabla u\|_1 \leqslant \varepsilon \|A_0 u\|_1 + (C/\varepsilon)\|u\|_1.$$

Proof. Let $\phi \in C_{\mathrm{c}}^\infty$, $\|\phi\|_\infty \leqslant 1$ and for $\lambda > 0$ consider $v \in C_{\mathrm{b}}(\mathbb{R}^N) \cap W^{2,p}(\mathbb{R}^N)$ for every $p < \infty$ such that $\lambda v - A_0 v = \phi$, see Theorem 3.1.2 of [20].

Since $\lambda\|v\|_\infty \leqslant 1$, it follows from Proposition 3.1.11 of [20] that $\lambda^{1/2}\|\nabla v\|_\infty \leqslant C$, with C independent of λ. For $u \in C_c^\infty(\mathbb{R}^N)$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} D_k u\phi &= \int_{\mathbb{R}^N} D_k u(\lambda v - A_0 v) = \int_{\mathbb{R}^N} v(\lambda - A_0)D_k u \\ &= \int_{\mathbb{R}^N} vD_k(\lambda u - A_0 u) + \int_{\mathbb{R}^N} v(D_k A_0 u - A_0 D_k u) \\ &= -\int_{\mathbb{R}^N} D_k v(\lambda u - A_0 u) - \int_{\mathbb{R}^N} \sum_{i,j}(D_k a_{ij}) D_i v D_j u \\ &\leqslant C_1\lambda^{-1/2}(\|\lambda u - A_0 u\|_1 + \|\nabla u\|_1).\end{aligned}$$

It follows that

$$\|\nabla u\|_1 \leqslant C_1(\lambda^{1/2}\|u\|_1 + \lambda^{-1/2}\|A_0 u\|_1 + \lambda^{-1/2}\|\nabla u\|_1)$$

and the lemma easily follows taking $\varepsilon = C_1\lambda^{-1/2}$. ∎

PROPOSITION 2.4. *Let $\varrho > 0$ be fixed. Then there exists a constant $C > 0$ such that for every $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^N)$ the following inequality holds*

$$\|u\|_{W^{1,1}(B_\varrho)} \leqslant C(\|Au\|_{L^1(B_{2\varrho})} + \|u\|_{L^1(B_{2\varrho})}).$$

Proof. Since $a_{ij} \in C^1_b(\mathbb{R}^N)$ and the coefficients b and V are locally bounded, Lemma 2.3 provides constants $\varepsilon_0, C > 0$ such that for every $v \in W^{2,1}(\mathbb{R}^N)$ with compact support in $B_{2\varrho}$ the following inequality holds for every $0 < \varepsilon < \varepsilon_0$

$$\|v\|_{1,1} \leqslant \varepsilon\|Av\|_1 + C\varepsilon^{-1}\|v\|_1. \tag{2.1}$$

Let $\varrho_n = \varrho\sum_{j=0}^{n} 2^{-j}$ so that $\varrho_0 = \varrho$, $\lim_{n\to\infty}\varrho_n = 2\varrho$ and consider $\eta_n \in C_c^\infty(B_{\varrho_{n+1}})$ such that $\eta = 1$ on B_{ϱ_n}, $|\nabla\eta_n| \leqslant L2^n$, $|D^2\eta_n| \leqslant L4^n$ with L independent of n. Applying (2.1) to $v = \eta_n u$ we obtain for a suitable $C_1 \geqslant 1$ depending on L, ρ,

$$\begin{aligned}\|\eta_n u\|_{1,1} &\leqslant \varepsilon\|A(\eta_n u)\|_1 + C\varepsilon^{-1}\|\eta_n u\|_1 \\ &\leqslant \varepsilon(\|Au\|_{L^1(B_{2\varrho})} + C_1 4^n\|u\|_{W^{1,1}(B_{\varrho_{n+1}})}) + C\varepsilon^{-1}\|u\|_{L^1(B_{2\varrho})} \\ &\leqslant C_1\varepsilon(\|Au\|_{L^1(B_{2\varrho})} + 4^n\|\eta_{n+1}u\|_{1,1}) + C\varepsilon^{-1}\|u\|_{L^1(B_{2\varrho})}.\end{aligned}$$

Setting $\varepsilon = \gamma C_1^{-1}4^{-n}$ we get

$$\|\eta_n u\|_{1,1} \leqslant \gamma(\|Au\|_{L^1(B_{2\varrho})} + \|\eta_{n+1}u\|_{1,1}) + C_2 4^n\gamma^{-1}\|u\|_{L^1(B_{2\varrho})}.$$

Taking $\gamma = 1/8$, multiplying the inequalities above by γ^n and summing over n we obtain

$$\sum_{n=0}^{\infty}\gamma^n\|\eta_n u\|_{1,1} \leqslant C_3\|Au\|_{L^1(B_{2\varrho})} + \sum_{n=0}^{\infty}\gamma^{n+1}\|\eta_{n+1}u\|_{1,1} + C_4\|u\|_{L^1(B_{2\varrho})}$$

(the convergence of the series is easily verified). Subtracting the terms $\gamma^n \|\eta_n u\|_{1,1}$, $n \geqslant 1$, that are present in both sides, we complete the proof. ∎

We also need the following local regularity result for distributional solutions of elliptic equations. The following lemma is well-known and its proof is given here only for the sake of completeness. We refer the reader to [1], where much more general situations are treated, and also to [2] for the case $q = 2$.

LEMMA 2.5. *Let $1 < q < \infty$ and $f, w \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$ be such that*

$$\int_{\mathbb{R}^N} A\phi\, w = \int_{\mathbb{R}^N} f\phi \tag{2.2}$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$. Then $w \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$.

Proof. Let us fix $\varrho > 0$. Since $a_{ij} \in C^1_b(\mathbb{R}^N)$ and b and V are locally bounded, there exists $C > 0$ such that

$$\Big| \int_{\mathbb{R}^N} (\lambda\phi - B\phi) w \Big| \leqslant C \|\phi\|_{1,q'}$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$ with support contained in $B_{2\varrho}$. Here $B = \sum_{ij} a_{ij} D_{ij}$ and $\lambda > 0$ is fixed in such a way that $\lambda - B$ is invertible from $W^{2,q'}(\mathbb{R}^N)$ to $L^{q'}(\mathbb{R}^N)$, see Chapter 9 of [14]. Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be such that $\eta = 1$ on B_ϱ, $\eta = 0$ outside $B_{2\varrho}$. It is easily checked that $v = \eta w$ satisfies

$$\Big| \int_{\mathbb{R}^N} (\lambda\phi - B\phi) v \Big| \leqslant C_1 \|\phi\|_{1,q'}$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$. Set now $\phi = D_{-h}\psi := |h|^{-1}(\psi(\cdot - h) - \psi(\cdot))$. Using the standard properties of difference quotients and the fact that the coefficients a_{ij} have bounded derivatives we get

$$\Big| \int_{\mathbb{R}^N} (\lambda\psi - B\psi) D_h v \Big| \leqslant C_2 \|\psi\|_{2,q'}$$

for every $\psi \in C_c^\infty(\mathbb{R}^N)$ and then, by density, for every $\psi \in W^{2,q'}(\mathbb{R}^N)$. We now choose $\psi \in W^{2,q'}(\mathbb{R}^N)$ such that $\lambda\psi - B\psi = D_h v |D_h v|^{q-2}$ and $\|\psi\|_{2,q'} \leqslant C_3 \|D_{-h} v\|_q^{q-1}$ to obtain

$$\int_{\mathbb{R}^N} |D_h v|^q \leqslant C_4$$

with C_4 independent of h. The boundedness of the difference quotients $D_h v$ implies that $v \in W^{1,q}(\mathbb{R}^N)$, that is $w \in W^{1,q}_{\mathrm{loc}}(\mathbb{R}^N)$.

Next we consider A_0 instead of B and observe that $\lambda - A_0$ is invertible from $W^{2,p}(\mathbb{R}^N)$ to $L^p(\mathbb{R}^N)$ for every $\lambda > 0$ and $1 < p < \infty$. From (2.2) it easily follows that

$$\int_{\mathbb{R}^N} A_0\phi\, w = \int_{\mathbb{R}^N} f_1\phi$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$ and with $f_1 = f - \operatorname{div}(bw) + Vw \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$. Inserting $\eta\phi$ instead of ϕ in the above identity, a straightforward computation then shows that

$$\int_{\mathbb{R}^N} (\lambda\phi - A_0\phi)v = \int_{\mathbb{R}^N} g\phi \tag{2.3}$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$, where $g = \lambda v - \eta f_1 + wA_0\eta - 2\sum_{i,j} D_j(a_{ij}wD_i\eta)$ belongs to $L^q(\mathbb{R}^N)$. Let $u \in W^{2,q}(\mathbb{R}^N)$ be such that $\lambda u - A_0 u = g$. Then (2.3) is satisfied with u instead of v and we have only to prove that $u = v$. Set $z = u - v$. Then

$$\int_{\mathbb{R}^N} (\lambda\phi - A_0\phi)z = 0 \tag{2.4}$$

for every $\phi \in W^{2,q'}(\mathbb{R}^N)$, by density. Since $\lambda - A_0$ is surjective from $W^{2,q'}(\mathbb{R}^N)$ to $L^{q'}(\mathbb{R}^N)$ we infer $z = 0$. ∎

3. CONSTRUCTION OF THE SEMIGROUP

Now we construct a positive semigroup on $L^p(\mathbb{R}^N)$ whose generator is a restriction of $A_{p,\max}$.

THEOREM 3.1. *Let $1 < p < \infty$ and assume that*

$$p^{-1}\operatorname{div} b(x) \leqslant V(x) \quad \forall\, x \in \mathbb{R}^N. \tag{3.1}$$

Then, there exists a unique resolvent positive operator $A_p \subset A_{p,\max}$ which is minimal among the resolvent positive restrictions of $A_{p,\max}$, i.e., if $B_p \subset A_{p,\max}$ is resolvent positive, then $R(\lambda, B_p) \geqslant R(\lambda, A_p)$ for $\lambda > 0$.

Proof. Take $f \in L^p(\mathbb{R}^N)$ and consider the Dirichlet problem in $L^p(B_\varrho)$

$$\begin{cases} \lambda u - Au = f & \text{in } B_\varrho, \\ u = 0 & \text{on } \partial B_\varrho. \end{cases} \tag{3.2}$$

According to Theorem 9.15 in [14], a unique solution u_ϱ exists in $W^{2,p}(B_\varrho) \cap W_0^{1,p}(B_\varrho)$ for $\lambda > 0$. Let us multiply the above equation by $u_\varrho|u_\varrho|^{p-2}$ and integrate over B_ϱ. Since

$$\int_{B_\varrho} b\cdot\nabla u_\varrho u_\varrho|u_\varrho|^{p-2} = p^{-1}\int_{B_\varrho} b\cdot\nabla|u_\varrho|^p = -p^{-1}\int_{B_\varrho}(\operatorname{div} b)|u_\varrho|^p,$$

from Lemma 2.1 it easily follows that

$$\int_{B_\varrho}((\lambda + V - p^{-1}\operatorname{div} b)|u_\varrho|^p + (p-1)\nu|u_\varrho|^{p-2}|\nabla u_\varrho|^2\chi_{\{u_\varrho\neq 0\}}) \leqslant \int_{B_\varrho}|f||u_\varrho|^{p-1} \tag{3.3}$$

and therefore $\lambda\|u_\varrho\|_p \leqslant \|f\|_p$.

In order to show that $u_\varrho \leqslant 0$ if $f \leqslant 0$ in B_ϱ we multiply the equation

$$\lambda u_\varrho - Au_\varrho = f$$

by $(u_\varrho^+)^{p-1}$ and integrate over B_ϱ. Since

$$\int_{B_\varrho} b\cdot\nabla u_\varrho(u_\varrho^+)^{p-1} = p^{-1}\int_{B_\varrho} b\cdot\nabla(u_\varrho^+)^p = -p^{-1}\int_{B_\varrho}(\operatorname{div} b)(u_\varrho^+)^p,$$

from Lemma 2.2 it follows that

$$\int_{B_\varrho}(\lambda + V - p^{-1}\operatorname{div} b)(u_\varrho^+)^p \leqslant \int_{B_\varrho} f(u_\varrho^+)^{p-1} \leqslant 0$$

and hence $u_\varrho \leqslant 0$.

To show the convergence of u_ϱ as $\varrho \to \infty$, we may assume that $f \geqslant 0$. In this case, $0 \leqslant u_\varrho \leqslant u_r$ in B_ϱ for every $r > \varrho$. In fact, the function $v = u_\varrho - u_r$ belongs to $W^{2,p}(B_\varrho)$, is negative on ∂B_ϱ in the sense of traces, and satisfies $\lambda v - Av = 0$ on B_ϱ. Multiplying this equation by $(v^+)^{p-1}$, integrating on B_ϱ and using Lemma 2.2 it follows that $v \leqslant 0$.

This shows that the functions u_ϱ increase pointwise with ϱ, hence we may define a function $u = \lim_{\varrho\to\infty} u_\varrho$. The Beppo Levi theorem implies that $\lambda\|u\|_p \leqslant \|f\|_p$ and hence u_ϱ converges to u in $L^p(\mathbb{R}^N)$. Let us fix two radii $\varrho_1 \leqslant \varrho_2$ and use the interior L^p-estimate ([14], Theorem 9.11)

$$\|u\|_{W^{2,p}(B_{\varrho_1})} \leqslant C[\|\lambda u - Au\|_{L^p(B_{\varrho_2})} + \|u\|_{L^p(B_{\varrho_2})}].$$

Applying it to the differences $u_\varrho - u_r$ with $\varrho, r > \varrho_2$, we deduce that the family (u_ϱ) converges to u in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ and therefore $u \in D_{p,\max}(A)$ and $\lambda u - Au = f$.

We can now define an operator $A_p = (A, D_p)$ with $D_p \subset D_{p,\max}(A)$ such that, for every $\lambda > 0$, $\lambda - A$ is bijective from D_p onto $L^p(\mathbb{R}^N)$. Setting $A_\varrho := (A, W^{2,p}(B_\varrho) \cap W^{1,p}_0(B_\varrho))$ the functions u_ϱ are given by $u_\varrho = R(\lambda, A_\varrho)f$. Let us define a family of bounded operators $(R(\lambda))_{\lambda>0}$ on $L^p(\mathbb{R}^N)$ by the formula $R(\lambda)f = \lim_{\varrho\to\infty} R(\lambda, A_\varrho)f$. Clearly $\|\lambda R(\lambda)\| \leqslant 1$ and $R(\lambda)f \geqslant 0$ if $f \geqslant 0$. Moreover, $R(\lambda)f \in D_{p,\max}(A)$ and $(\lambda - A)R(\lambda)f = f$. Let us verify that the family $(R(\lambda))_{\lambda>0}$ satisfies the resolvent identity $R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$. Since this is true for the families $(R(\lambda, A_\varrho))_{\lambda>0}$, it is sufficient to show that, for every

$f \in L^p(\mathbb{R}^N)$, $R(\lambda)R(\mu)f = \lim_{\varrho\to\infty} R(\lambda, A_\varrho)R(\mu, A_\varrho)f$. We may assume that $f \geqslant 0$. Then $R(\lambda)R(\mu)f \geqslant \limsup_{\varrho\to\infty} R(\lambda, A_\varrho)R(\mu, A_\varrho)f$. Conversely, for every fixed ϱ_1 we have

$$\liminf_{\varrho\to\infty} R(\lambda, A_\varrho)R(\mu, A_\varrho)f \geqslant \liminf_{\varrho\to\infty} R(\lambda, A_{\varrho_1})R(\mu, A_\varrho)f = R(\lambda, A_{\varrho_1})R(\mu)f$$

and hence, letting $\varrho_1 \to \infty$, $\liminf_{\varrho\to\infty} R(\lambda, A_\varrho)R(\mu, A_\varrho)f \geqslant R(\lambda)R(\mu)f$.

Since $(\lambda - A)R(\lambda)f = f$, the operators $R(\lambda)$ are injective, and therefore there exists an operator $A_p = (A, D_p)$, $D_p \subset D_{p,\max}(A)$, such that $R(\lambda)$ is the resolvent of A_p (see Chapter III, Proposition 4.6 of [13]).

Finally, let us show the minimality of $u = R(\lambda)f$, $f \geqslant 0$, among the positive solutions of the equation $\lambda w - Aw = f$ in $D_{p,\max}(A)$. Let w be such a solution and consider the difference $v = u_\varrho - w$ in B_ϱ. With the same argument used to prove the monotonicity of the net (u_ϱ) it follows that $v \leqslant 0$, that is $u_\varrho \leqslant w$ in B_ϱ. Letting $\varrho \to \infty$ we obtain $u \leqslant w$. ∎

In the sequel we write A_p for (A, D_p).

Let us point out a simple consequence of our construction for $1 < p \leqslant 2$ which is probably false, in general, for $p > 2$.

COROLLARY 3.2. *Assume that the hypotheses of Theorem* 3.1 *hold for* $1 < p \leqslant 2$. *Then* $D_p \subset W^{1,p}(\mathbb{R}^N)$.

Proof. From (3.3) it follows that

$$\int_{B_\varrho} |u_\varrho|^{p-2}|\nabla u_\varrho|^2 \chi_{\{u_\varrho \neq 0\}} \leqslant C\|f\|_p^p.$$

Letting $\varrho \to \infty$ we deduce from Fatou's lemma

$$\int_{\mathbb{R}^N} |u|^{p-2}|\nabla u|^2 \chi_{\{u \neq 0\}} \leqslant C\|f\|_p^p$$

and then, using Hölder's inequality, we obtain

$$\int_{\mathbb{R}^N} |\nabla u|^p \leqslant \Big(\int_{\mathbb{R}^N} |u|^{p-2}|\nabla u|^2 \chi_{\{u \neq 0\}}\Big)^{p/2} \Big(\int_{\mathbb{R}^N} |u|^p\Big)^{1-p/2} \leqslant C_1\|f\|_p^p. \quad ∎$$

We can now prove generation in $L^p(\mathbb{R}^N)$ for $1 < p < \infty$.

THEOREM 3.3. *Under the hypotheses of Theorem* 3.1, *the operator* A_p *generates a positive and contractive semigroup* T_p. *Moreover, if condition* (3.1) *holds with* $q \neq p$ *in place of* p, *so that there exists also the semigroup* T_q, *then* $T_p f = T_q f$ *for every* $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$.

Proof. Let $u \in C_c^\infty(\mathbb{R}^N)$, $f = \lambda u - Au$. If B_ϱ contains the support of u the function u_ϱ constructed in the proof of Theorem 3.1 coincides with u. Letting

$\varrho \to \infty$ it follows that $u = R(\lambda, A_p)f$ and hence $C_c^\infty(\mathbb{R}^N) \subset D_p$ and D_p is dense in $L^p(\mathbb{R}^N)$.

The first statement is now an immediate consequence of the Lumer-Phillips Theorem and of the positivity of the resolvent of A_p.

Concerning the second statement, we simply notice that for $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, the functions u_ϱ are independent of p, q, hence $R(\lambda, A_p)f = R(\lambda, A_q)f$ and the claim follows. ∎

REMARK 3.4. Notice that if the potential V is nonnegative and condition (3.1) holds, the analogous one, with $q > p$, holds as well. As a consequence, the semigroups T_q exist for every $q \geqslant p$ and are consistent. Moreover, they are also contractive with respect to the sup-norm.

Let us now deal with generation in $L^1(\mathbb{R}^N)$. Consider the operator $A_{1,\max}$ on $L^1(\mathbb{R}^N)$ defined by $A_{1,\max}u = Au$ with domain $D(A_{1,\max})$ given by (1.5).

THEOREM 3.5. *Assume that* $\operatorname{div} b(x) \leqslant V(x)$ *for every* $x \in \mathbb{R}^N$. *Then there is a unique minimal resolvent positive operator* $A_1 \subset A_{1,\max}$ *in the same sense as Theorem* 3.1. *Moreover,* A_1 *generates a positive contraction semigroup* T_1 *in* $L^1(\mathbb{R}^N)$. *Its domain* D_1 *satisfies* $C_c^\infty(\mathbb{R}^N) \subset D_1 \subset D_{1,\max} \cap D(V - \operatorname{div} b)$, *where*

$$D(V - \operatorname{div} b) = \{u \in L^1(\mathbb{R}^N) : (V - \operatorname{div} b)u \in L^1(\mathbb{R}^N)\}.$$

Proof. We proceed as in the proof of Theorem 3.1. Let $f \in C_c^\infty(\mathbb{R}^N)$ and consider the Dirichlet problem in $L^2(B_\varrho)$

$$\begin{cases} \lambda u - Au = f & \text{in } B_\varrho, \\ u = 0 & \text{on } \partial B_\varrho. \end{cases}$$

According to Theorem 9.15 of [14], a unique solution u_ϱ exists in $W^{2,2}(B_\varrho) \cap W_0^{1,2}(B_\varrho)$ for $\lambda > 0$. Let us multiply the above equation by $\operatorname{sign} u_\varrho$ and integrate over B_ϱ. Since

$$\int_{B_\varrho} b \cdot \nabla u_\varrho \operatorname{sign} u_\varrho = \int_{B_\varrho} b \cdot \nabla |u_\varrho| = -\int_{B_\varrho} (\operatorname{div} b)|u_\varrho|,$$

from Lemma 2.1 it follows that

$$\int_{B_\varrho} (\lambda + V - \operatorname{div} b)|u_\varrho| \leqslant \int_{B_\varrho} |f|. \tag{3.4}$$

In particular, for $\lambda > 0$, $\lambda\|u_\varrho\|_1 \leqslant \|f\|_1$.

Setting $A_\varrho := (A, W^{2,2}(B_\varrho) \cap W_0^{1,2}(B_\varrho))$, the functions u_ϱ are given by $u_\varrho = R(\lambda, A_\varrho)f$ and (3.4) shows that the operators $R(\lambda, A_\varrho)$ can be continuously extended to bounded operators $R_\varrho(\lambda)$ on $L^1(B_\varrho)$ satisfying $\|\lambda R_\varrho(\lambda)f\|_1 \leqslant \|f\|_1$. As in Theorem 3.1, one shows that $R(\lambda)f := \lim_{\varrho\to\infty} R_\varrho(\lambda)f$ exists in $L^1(\mathbb{R}^N)$, $\|\lambda R(\lambda)f\|_1 \leqslant \|f\|_1$ and the family $(R(\lambda))_{\lambda>0}$ satisfies the resolvent identity.

Moreover, $R(\lambda)f \in D(V - \operatorname{div} b)$, letting $\varrho \to \infty$ in (3.4), and $R(\lambda)f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$, by Proposition 2.4.

In order to complete the proof, we have to show that $R(\lambda)$ has dense range and is injective. As in Theorem 3.3 one shows that the range of $R(\lambda)$ contains $C_c^\infty(\mathbb{R}^N)$ and that $R(\lambda)f = u$ if $u \in C_c^\infty(\mathbb{R}^N)$ and $\lambda u - Au = f$. This proves the required density. For f in the range of $R(\lambda)$ one has $\lim\limits_{\lambda\to+\infty} \lambda R(\lambda)f = f$ in $L^1(\mathbb{R}^N)$ by the resolvent identity. It follows from the density of the range that $\lim\limits_{\lambda\to+\infty} \lambda R(\lambda)f = f$ for all $f \in L^1(\mathbb{R}^N)$. Since the kernel of $R(\lambda)$ is independent of λ we conclude that $R(\lambda)$ is injective for all $\lambda > 0$. Consequently, there exists an operator A_1 such that $(0,\infty) \subset \rho(A_1)$ and $R(\lambda, A_1) = R(\lambda)$ for all $\lambda > 0$. It follows from the Hille-Yosida theorem that A_1 generates a C_0-semigroup T_1 which is positive since $R(\lambda, A_1) \geqslant 0$.

It remains to prove the minimality. Let $B \subset A_{1,\max}$ be resolvent positive and let $\lambda_0 > 0$ be such that $[\lambda_0, \infty) \subset \rho(B)$. For $\lambda \geqslant \lambda_0$ and $f \in C_c^\infty(\mathbb{R}^N)$ we consider $u = R(\lambda, A_1)f$, $v = R(\lambda, B)f$. Then $v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ (hence in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for some $q > 1$), $v \geqslant 0$ and $\lambda v - Av = f$ in $\mathcal{D}'(\mathbb{R}^N)$. It follows applying iteratively Lemma 2.5 that $v \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for all $1 < p < \infty$. In particular, v is continuous. Since $u = \lim\limits_{\varrho\to\infty} u_\varrho$, it suffices to prove that $u_\varrho \leqslant v$, where $u_\varrho \in W^{2,2}(B_\varrho) \cap W_0^{1,2}(B_\varrho)$, $\lambda u_\varrho - Au_\varrho = f$. Let $w = u_\varrho - v \in W^{2,2}(B_\varrho) \cap C(\overline{B}_\varrho)$. Then $w \leqslant 0$ on ∂B_ϱ and $\lambda w - Aw = 0$ in B_ϱ. Hence

$$\lambda \int\limits_{B_\varrho} w\phi + \int\limits_{B_\varrho} \sum_{i,j=1}^N a_{ij} D_i w D_j \phi + \int\limits_{B_\varrho} \sum_{j=1}^N b_j D_j w \phi + \int\limits_{B_\varrho} V w \phi = 0$$

for all $\phi \in W_0^{1,2}(B_\varrho)$. Observe that $w^+ \in W_0^{1,2}(B_\varrho)$, hence taking $\phi = w^+$ we conclude that

$$\lambda \int\limits_{B_\varrho} (w^+)^2 + \int\limits_{B_\varrho} \sum_{i,j=1}^N a_{ij} D_i w^+ D_j w^+ + \int\limits_{B_\varrho} \sum_{j=1}^N b_j D_j w^+ w^+ + \int\limits_{B_\varrho} V (w^+)^2 = 0.$$

Since

$$\int\limits_{B_\varrho} \sum_{j=1}^N b_j D_j w^+ w^+ = \frac{1}{2} \int\limits_{B_\varrho} \sum_{j=1}^N b_j D_j (w^+)^2 = -\frac{1}{2} \int\limits_{B_\varrho} (w^+)^2 \sum_{j=1}^N D_j b_j \leqslant \int\limits_{B_\varrho} V (w^+)^2,$$

we obtain from (1.2) that

$$\lambda \int\limits_{B_\varrho} (w^+)^2 + \nu \int\limits_{B_\varrho} |\nabla w^+|^2 \leqslant 0.$$

This implies that $w^+ = 0$. ∎

In Theorems 3.1 and 3.5 we have constructed the resolvent of A_p as the limit of $R(\lambda, A_\varrho)$ for $\varrho \to \infty$. In a suitable sense, the same convergence also holds for the semigroups generated by A_ϱ, as we explain in the next result, where we use the notation introduced in the proof of Theorem 3.1.

PROPOSITION 3.6. *Let* $1 \leqslant p < \infty$. *Assume that* $\frac{\operatorname{div} b}{p} \leqslant V$. *For every* $\varrho > 0$, *let* $(T_{p,\varrho}(t))_{t\geqslant 0}$ *be the semigroup generated by* A_ϱ *in* $L^p(B_\varrho)$. *For every* $f \in L^p(\mathbb{R}^N)$ *let us define* $\widetilde{T}_{p,\varrho}(\cdot)f : [0,+\infty[\to L^p(\mathbb{R}^N)$ *setting*

$$\widetilde{T}_{p,\varrho}(t)f = \begin{cases} T_{p,\varrho}(t)f & \text{in } B_\varrho, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\widetilde{T}_{p,\varrho}(t)f \to T_p(t)f$ *as* $\varrho \to \infty$, *uniformly on compact sets of* $[0,+\infty[$.

Proof. By density, and since the semigroups $(T_{p,\varrho}(t))_{t\geqslant 0}$, $(T_p(t))_{t\geqslant 0}$ are contractive, we may assume that $f \in C_c^\infty(\mathbb{R}^N)$. Let us take a sequence ϱ_n going to $+\infty$; the statement will follow from the arbitrariness of the sequence. Since the Laplace transform of $\widetilde{T}_{p,\varrho_n}(\cdot)f$ is given by

$$g_{\varrho_n}(\lambda) = \begin{cases} R(\lambda, A_{\varrho_n})f & \text{in } B_\varrho, \\ 0 & \text{otherwise,} \end{cases}$$

and the sequence (g_{ϱ_n}) is pointwise convergent for $\lambda > 0$ to $R(\lambda, A_p)f$, by Theorems 3.3, 3.5, the claim follows from Theorem 1.7.5 in [5] if we verify that for every $t_0 \geqslant 0$, the sequence $(\widetilde{T}_{p,\varrho_n}(t_0)f)$ is equicontinuous. Take n so that the support of f is contained in B_{ϱ_n}, and notice that for $0 \leqslant t_0 < t < \infty$ we have

$$\widetilde{T}_{p,\varrho_n}(t)f - \widetilde{T}_{p,\varrho_n}(t_0)f = \int_{t_0}^{t} \widetilde{T}_{p,\varrho_n}(s)Af\,\mathrm{d}s,$$

whence

$$\|\widetilde{T}_{p,\varrho_n}(t) - \widetilde{T}_{p,\varrho_n}(t_0)\|_p \leqslant |t - t_0|\|Af\|_p$$

and the equicontinuity follows. ∎

We can describe the semigroup T_p by a minimality property.

COROLLARY 3.7. *Let* $1 \leqslant p < \infty$. *Let* $B \subset A_{p,\max}$ *be the generator of a positive* C_0*-semigroup* S *on* $L^p(\mathbb{R}^N)$. *Then* $T_p(t) \leqslant S(t)$ *for all* $t \geqslant 0$.

Proof. Since B generates a positive C_0-semigroup, B is resolvent positive. It follows from Theorem 3.1 that for large λ, $R(\lambda, A_p) \leqslant R(\lambda, B)$. Consequently,

$$T_p(t)f = \lim_{n\to\infty} \left(I - \frac{t}{n}A_p\right)^{-n} f \leqslant \lim_{n\to\infty} \left(I - \frac{t}{n}B\right)^{-n} f = S(t)$$

for all $0 \leqslant f \in L^p(\mathbb{R}^N)$, $t > 0$. ∎

In the following result we investigate the compactness of the resolvent of A_p.

THEOREM 3.8. *Let* $1 \leqslant p < \infty$ *and assume that*

$$\lim_{|x|\to\infty} (V(x) - p^{-1}\operatorname{div} b(x)) = +\infty; \tag{3.5}$$

then the resolvent operator $R(\lambda, A_p)$ is compact in $L^p(\mathbb{R}^N)$.

Proof. We keep the notation introduced in the proof of Theorem 3.1 and write $u = R(\lambda, A_p)f = \lim\limits_{\varrho\to\infty} u_\varrho$. Letting $\varrho \to \infty$ in (3.3) and using the inequality $\lambda\|u\|_p \leqslant \|f\|_p$, we deduce

$$\int\limits_{\mathbb{R}^N} (V - p^{-1}\mathrm{div}\, b)|u|^p \leqslant \frac{\|f\|_p}{\lambda^{p-1}}.$$

By the assumption, given $\varepsilon > 0$, we can choose $\varrho > 0$ such that, for every $f \in L^p(\mathbb{R}^N)$ with $\|f\|_p \leqslant 1$,

$$\int\limits_{\mathbb{R}^N\setminus B_\varrho} |R(\lambda, A_p)f|^p \leqslant \varepsilon^p.$$

The interior estimate

$$\|R(\lambda, A_p)f\|_{W^{1,p}(B_\varrho)} \leqslant C(\|f\|_p + \|R(\lambda, A_p)f\|_p) \leqslant C(1+\lambda^{-1})\|f\|_p$$

(which follows as in the proof of Theorem 3.1 for $p > 1$ and from Lemma 2.5 for $p = 1$) and the compactness of the embedding of $W^{1,p}(B_\varrho)$ into $L^p(B_\varrho)$ imply that the family $\{R(\lambda, A_p)f, \quad \|f\|_p \leqslant 1\}$ is relatively compact in $L^p(B_\varrho)$. Let $\{g_1, \dots, g_k\}$ be an ε-net for this family in $L^p(B_\varrho)$. Then it is immediate to check that the same functions, extended to 0 outside B_ϱ, are a 2ε-net in $L^p(\mathbb{R}^N)$. ∎

4. UNIQUENESS

In this section we investigate uniqueness in $L^p(\mathbb{R}^N)$ for $1 \leqslant p < \infty$. The results are based upon the existence of suitable control functions for the drift term (see (4.1) and Theorem 4.3 below). There is a wide literature on uniqueness of diffusion operators, i.e., when $V = 0$. We refer the reader to [11], see also [10], for a discussion of several notions of uniqueness in L^p, $1 < p < \infty$, and related results which are valid also in the case of singular coefficients. We refer also the reader to [31] for uniqueness in L^1. The question of uniqueness is well understood in the case of Schrödinger operators, see e.g. [12], [16], [17]. However, we are not aware of results dealing with the general second order operator, even in the case of smooth (unbounded) coefficients.

Our first result is in the same line as in Chapter 2.c of [10].

THEOREM 4.1. *Let $1 < p < \infty$ and suppose that condition* (3.1) *holds. Assume that there is a positive function $z \in C^2(\mathbb{R}^N)$ such that $z(x) \to +\infty$ as $|x| \to \infty$, $|\nabla z| \leqslant c(1+z)$ and*

$$b \cdot \nabla z \leqslant c(1+z)(1 + (V - p^{-1}\mathrm{div}\, b)^\alpha), \tag{4.1}$$

for suitable constants $c > 0$, $0 \leqslant \alpha < 1$. Then $C_c^\infty(\mathbb{R}^N)$ is a core for A_p.

Proof. Since $C_c^\infty(\mathbb{R}^N)$ is contained in D_p and A_p has non-empty resolvent set, it suffices to show that $(\lambda - A_p)C_c^\infty(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$ for λ sufficiently large.

Let q be such that $1/p + 1/q = 1$ and $w \in L^q(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (\lambda\phi - A\phi)w = 0$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$. By Lemma 2.5, $w \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$ and hence $\lambda w - A^* w = 0$, where

$$A^* := \sum_{i,j=1}^N D_i(a_{ij}D_j) + \sum_{i=1}^N b_i D_i - (V - \operatorname{div} b) \tag{4.2}$$

is the formal adjoint of A.

Let $g \in C^\infty(0,+\infty)$ be such that $0 \leqslant g \leqslant 1$, $g(r) = 1$ for $r \leqslant 1$, $g(r) = 0$ for $r \geqslant 2$, $g' \leqslant 0$ and define $\eta_n \in C_c^\infty(\mathbb{R}^N)$ by $\eta_n(x) = g(z(x)/n)$. Multiplying the identity $\lambda w - A^* w = 0$ by $\eta_n^s w|w|^{q-2}$, with $s \geqslant 2$, and integrating by parts, we obtain from Lemma 2.1

$$\int_{\mathbb{R}^N} ((\lambda + V - p^{-1}\operatorname{div} b)|w|^q \eta_n^s + \nu(q-1)\eta_n^s |w|^{q-1}|\nabla w|^2) = I_1 + I_2 \tag{4.3}$$

where

$$|I_1| = \Big| \int_{\mathbb{R}^N} s\eta_n^{s-1}|w|^{q-2}w \sum_{i,j=1}^N a_{ij}D_i w D_j \eta_n \Big| \leqslant sK \int_{\mathbb{R}^N} \eta_n^{s-1}|w|^{q-1}|\nabla w||\nabla \eta_n|\,,$$

$$I_2 = -\frac{s}{q}\int_{\mathbb{R}^N} \eta_n^{s-1}|w|^q b\cdot\nabla\eta_n\,,$$

and $K = N^2 \max_{i,j} \|a_{ij}\|_\infty$. Observe that

$$\nabla\eta_n(x) = n^{-1}g'(z(x)/n))\nabla z(x)\chi_{\{n\leqslant z\leqslant 2n\}}$$

hence $|\nabla\eta_n| \leqslant C$ for a suitable $C > 0$, independent of n, since $|\nabla z| \leqslant c(1+z)$. By Hölder's inequality and since $s \geqslant 2$ we get

$$\begin{aligned}
|I_1| &\leqslant CKs\Big(\int_{\{n\leqslant z\leqslant 2n\}} |w|^q\Big)^{1/2}\Big(\int_{\mathbb{R}^N} \eta_n^{2s-2}|w|^{q-2}|\nabla w|^2\chi_{\{w\neq 0\}}\Big)^{1/2} \\
&\leqslant \varepsilon CKs \int_{\mathbb{R}^N} \eta_n^s |w|^{q-2}|\nabla w|^2 \chi_{\{w\neq 0\}} + \frac{CKs}{4\varepsilon}\int_{\{n\leqslant z\leqslant 2n\}} |w|^q.
\end{aligned}$$

As regards I_2 we have

$$\begin{aligned} I_2 &= -\frac{s}{q} \int\limits_{\{n\leqslant z\leqslant 2n\}} \eta_n^{s-1}|w|^q n^{-1} g'(z(x)/n) b\cdot\nabla z \\ &\leqslant C_1 s \int\limits_{\{n\leqslant z\leqslant 2n\}} \eta_n^{s-1} n^{-1}(1+z)|w|^q (V-p^{-1}\operatorname{div} b)^\alpha \\ &\leqslant C_2 s \int\limits_{\{n\leqslant z\leqslant 2n\}} \eta_n^{s-1}|w|^q (V-p^{-1}\operatorname{div} b)^\alpha \end{aligned}$$

and Hölder's inequality with $r=\alpha^{-1}$ and $r'=r/(r-1)$ yields

$$\begin{aligned} I_2 &\leqslant C_2 s \int\limits_{\{n\leqslant z\leqslant 2n\}} \eta_n^{s-1}|w|^{q/r'}|w|^{q/r}(V-p^{-1}\operatorname{div} b)^\alpha \\ &\leqslant C_2 s \Big(\int\limits_{\{n\leqslant z\leqslant 2n\}} |w|^q \Big)^{1/r'} \Big(\int\limits_{\mathbb{R}^N} \eta_n^{r(s-1)}|w|^q (V-p^{-1}\operatorname{div} b)\Big)^{1/r}. \end{aligned}$$

Fixing $s\geqslant 2$ such that $r(s-1)\geqslant s$ we obtain from (4.3) and for every $\varepsilon>0$

$$\int\limits_{\mathbb{R}^N} ((\lambda+V-p^{-1}\operatorname{div} b)\eta_n^s|w|^q + (\nu(q-1)-\varepsilon CKs)\eta_n^s|w|^{q-2}|\nabla w|^2\chi_{\{w\neq 0\}})$$

$$\leqslant C_\varepsilon \int\limits_{\{n\leqslant z\leqslant 2n\}} |w|^q + \varepsilon \int\limits_{\mathbb{R}^N} \eta_n^s |w|^q (V-p^{-1}\operatorname{div} b).$$

Taking ε small enough we deduce

$$\lambda \int\limits_{\mathbb{R}^N} |w|^q \eta_n^s \leqslant C \int\limits_{\{n\leqslant z\leqslant 2n\}} |w|^q$$

and, letting $n\to\infty$, $w=0$. ∎

To deal with uniqueness in $L^1(\mathbb{R}^N)$ we need the following maximum principle.

PROPOSITION 4.2. *Assume that the potential V for the operator A defined in* (1.1) *is nonnegative and that there exists a positive function $z\in C^2(\mathbb{R}^N\setminus B_\varrho)$ for some $\varrho>0$ such that $z(x)\to+\infty$ as $|x|\to\infty$ and $Az\leqslant\lambda z$ for some $\lambda>0$. If $w\in C_b(\mathbb{R}^N)\cap W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$* for all *$q<\infty$ and $\lambda w = Aw$, then $w=0$.*

Proof. Let us show that $w\leqslant 0$. A similar argument shows that $w\geqslant 0$, hence $w=0$.

For $|x|\geqslant\varrho$ we consider the function $w_\varepsilon = w-\varepsilon z$. Since $\lambda w_\varepsilon - Aw_\varepsilon\leqslant 0$ and $V\geqslant 0$, using e.g. Lemma 3.2 of [21] it is readily seen that the function w_ε cannot have a positive maximum for $|x|>\varrho$, and hence $w_\varepsilon(x)\leqslant \max\limits_{|x|=\varrho} w_\varepsilon^+ \leqslant \max\limits_{|x|=\varrho} w^+$ for

$|x| \geqslant \varrho$. Letting $\varepsilon \to 0$ we obtain $w(x) \leqslant \max\limits_{|x|=\varrho} w^+$ for $|x| \geqslant \varrho$. The same argument applies directly to w for $|x| \leqslant \varrho$ and yields $w(x) \leqslant \max\limits_{|x|=\varrho} w^+$ for $|x| \leqslant \varrho$, hence for every $x \in \mathbb{R}^N$. Since w cannot have a positive maximum, $w^+(x) = 0$ for $|x| = \varrho$, hence $w \leqslant 0$. ∎

The following is our uniqueness result in $L^1(\mathbb{R}^N)$.

THEOREM 4.3. *Let A^* be as in* (4.2) *and assume that* $\operatorname{div} b \leqslant V$ *and that there exists a positive function $z \in C^2(\mathbb{R}^N \setminus B_\varrho)$ for some $\varrho > 0$ such that $z(x) \to +\infty$ as $|x| \to \infty$ and $A^* z \leqslant \lambda z$ for some $\lambda > 0$. Then $C_c^\infty(\mathbb{R}^N)$ is a core for A_1.*

Proof. We proceed as in the proof of Theorem 4.1 and show that for λ sufficiently large $(\lambda - A_1) C_c^\infty(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N)$.

Let $w \in L^\infty(\mathbb{R}^N)$ be such that

$$\int_{\mathbb{R}^N} (\lambda\phi - A\phi) w = 0$$

for every $\phi \in C_c^\infty(\mathbb{R}^N)$. By Lemma 2.5, $w \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $q < \infty$ and then w is a bounded solution of the equation $\lambda w - A^* w = 0$, where A^* is the formal adjoint of A defined in (4.2).

From Proposition 4.2, applied to A^*, we infer that $w = 0$ and the proof is complete. ∎

REMARK 4.4. Let us point out some explicit examples of the hypotheses in Theorems 4.1 and 4.3. For instance, if $z(x) = 1 + |x|^2$, then condition (4.1) reads

$$b(x) \cdot x \leqslant c(1 + |x|^2)(1 + (V - p^{-1} \operatorname{div} b)^\alpha).$$

Analogously, plugging $z(x) = \log|x|$ in A^* and imposing that $A^* z(x) \leqslant \lambda z(x)$ for large $|x|$ and λ, we obtain the condition

$$b(x) \cdot x \leqslant c(1 + |x|^2)(1 + (V - \operatorname{div} b)). \tag{4.4}$$

A slightly better condition can be found plugging $z(x) = \log|x|$ ($|x| \geqslant 1$) in Theorems 4.1, 4.3. In fact one obtains

$$b(x) \cdot x \leqslant c(1 + |x|^2 \log|x|)(1 + (V - p^{-1} \operatorname{div} b)^\alpha)$$

for some $\alpha < 1$ if $1 < p < \infty$ and

$$b(x) \cdot x \leqslant c(1 + |x|^2 \log|x|)(1 + (V - \operatorname{div} b))$$

if $p = 1$.

Further results and comments on uniqueness in $C_b(\mathbb{R}^N)$ can be found in [21].

5. GAUSSIAN ESTIMATES

In this section we show that under suitable conditions on the coefficients the semigroup generated by A admits a Gaussian estimate.

DEFINITION 5.1. A positive semigroup T on $L^p(\mathbb{R}^N)$ admits a *Gaussian estimate* if there exists a measurable kernel $k_t : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ satisfying

$$k_t(x,y) \leqslant ct^{-N/2}\mathrm{e}^{\omega t}\mathrm{e}^{-b|x-y|^2/t}$$

$(t > 0)$ for some $c \geqslant 0,\ b > 0,\ \omega \in \mathbb{R}$ such that

$$(T(t)f)(x) = \int\limits_{\mathbb{R}^N} k_t(x,y)f(y)\,\mathrm{d}y$$

a.e. for all $f \in L^p(\mathbb{R}^N),\ t > 0$.

Keeping the general assumptions of the Introduction we consider the following additional hypotheses on the potential V and the drift b. We assume that $V \geqslant 0$ and

$$\text{(H1)} \qquad |b| \leqslant \gamma V^{1/2}$$

for some $\gamma \geqslant 0$ and

$$\text{(H2)} \qquad \operatorname{div} b \leqslant \beta V$$

for some constant $0 < \beta < 1$. Note that the apparently more general assumptions $|b| \leqslant \gamma V^{1/2} + C$ and $\operatorname{div} b \leqslant \beta V + C$ easily reduce to (H1), (H2) considering $V + \lambda$ for a suitable $\lambda > 0$.

Condition (H2) implies in particular that condition (3.1) is satisfied for all $1 \leqslant p < \infty$. Thus, by the results of Section 3, we obtain consistent C_0-semigroups T_p on $L^p(\mathbb{R}^N),\ 1 \leqslant p < \infty$ whose generators we denote by A_p.

THEOREM 5.2. *Under assumptions* (H1), (H2) *the semigroup T_p admits a Gaussian estimate.*

We note some consequences. Assume (H1), (H2). Then the spectrum of the generators A_p is independent of p,

$$\sigma(A_p) = \sigma(A_1) \quad (1 \leqslant p < \infty)\,.$$

The semigroups T_p are all holomorphic $(1 \leqslant p < \infty)$. For $1 < p < \infty$ each operator $-A_p$ admits a bounded H^∞-calculus; in particular $(-A_p)^{\mathrm{i}s}$ is a bounded operator on $L^p(\mathbb{R}^N)$ for all $s \in \mathbb{R},\ 1 < p < \infty$. Moreover the operator A_p has the *maximal regularity property* for $1 < p < \infty$. We refer to Chapter 7 of [4] for this and other consequences of Gaussian estimates.

REMARK 5.3. In the preceeding sections as well as in Theorem 5.2 we considered the semigroup T_p and its generator A_p on the real space $L^p(\mathbb{R}^N)$. But of course, saying that T_p is holomorphic means that T_p is a holomorphic semigroup on the complex space L^p. Also the resolvent set $\varrho(A_p)$ and the notion of functional calculus are defined with respect to the complex space.

For the proof of Theorem 5.2 we use a strategy introduced in [6], based on the Beurling-Deny criterion. At first we show that A_2 is associated with a closed form. Let us set $D(V) = \{u \in L^2(\mathbb{R}^N) : Vu \in L^2(\mathbb{R}^N)\}$ and consider the Hilbert space

$$W := W^{1,2}(\mathbb{R}^N) \cap D(V)$$

endowed with the inner product

$$(u \mid v)_W = (u \mid v)_{W^{1,2}(\mathbb{R}^N)} + \int_{\mathbb{R}^N} Vuv.$$

Using mollifiers in a standard way one sees that $C_c^\infty(\mathbb{R}^N)$ is dense in W. Moreover, W is continuously embedded into $L^2(\mathbb{R}^N)$ and dense in $L^2(\mathbb{R}^N)$. For the proof of Theorem 5.2 more general operators will be needed. Let $c_j : \mathbb{R}^N \to \mathbb{R}$ be differentiable, satisfying

$$\text{(H1)}' \qquad |c| \leqslant \gamma V^{1/2}$$

where $c = (c_1, \dots, c_N)$ and $\gamma \geqslant 0$ is the same constant as in (H1), as well as

$$\text{(H2)}' \qquad \operatorname{div} c \leqslant \beta V$$

where $0 < \beta < 1$ is the same constant as in (H2). Then

$$\text{(5.1)} \qquad a(u,v) := \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij} D_i u D_j v + \int_{\mathbb{R}^N} (v\, b \cdot \nabla u + u\, c \cdot \nabla v) + \int_{\mathbb{R}^N} Vuv$$

defines a continuous bilinear form on W. This follows from the Cauchy-Schwarz inequality

$$\Big| \int_{\mathbb{R}^N} v\, b \cdot \nabla u \Big| \leqslant \gamma \int_{\mathbb{R}^N} V^{1/2} |v| |\nabla u| \leqslant \gamma \Big(\int_{\mathbb{R}^N} |v|^2 V \Big)^{1/2} \Big(\int_{\mathbb{R}^N} |\nabla u|^2 \Big)^{1/2} \leqslant \gamma \|u\|_W \|v\|_W$$

and similarly for the other terms. Moreover, in virtue of (H2), (H2') we have for $u \in C_c^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} (u\, b \cdot \nabla u + u\, c \cdot \nabla u) = \int_{\mathbb{R}^N} \sum_{j=1}^N \frac{1}{2} (b_j + c_j) D_j u^2 = -\frac{1}{2} \int_{\mathbb{R}^N} (\operatorname{div} b + \operatorname{div} c) u^2$$
$$\geqslant - \int_{\mathbb{R}^N} \beta V u^2.$$

Thus

$$a(u,u) \geqslant \alpha \int_{\mathbb{R}^N} |\nabla u|^2 + (1-\beta) \int_{\mathbb{R}^N} Vu^2$$

for all $u \in C_c^\infty(\mathbb{R}^N)$ and hence for all $u \in W$ by density. This implies that the form a with domain W is closed (see [15]). Denote by $-A$ the operator associated with a, i.e., for $u, f \in L^2(\mathbb{R}^N)$ one has $u \in D(A), -Au = f$ if and only if $u \in W$ and

$$a(u,v) = \int_{\mathbb{R}^N} fv \quad \text{for all } v \in W.$$

If $c = 0$, then $A = A_2$. To see this we note that, because of (H1) we may take $z(x) = \sqrt{1+|x|^2}$, $\alpha = 1/2$ in Theorem 4.1 to obtain that $C_c^\infty(\mathbb{R}^N)$ is core of A_2. But A is closed and coincides with A_2 on $C_c^\infty(\mathbb{R}^N)$ so that both operators coincide.

The introduction of the auxiliary term containing c allows to deal with A and A^* simultaneously. In fact, if $u \in C_c^\infty(\mathbb{R}^N)$, then

$$Au = \sum_{i,j=1}^N D_i(a_{ij}D_j u) + (-b \cdot \nabla u + \nabla(cu)) - Vu$$

and

$$A^* u = \sum_{i,j=1}^N D_i(a_{ij}D_j u) + (\nabla(bu) - c \cdot \nabla u) - Vu$$

so that the role of b and c interchanges, passing from A to its adjoint.

In order to prove Gaussian estimates, we consider the set

$$G := \{\psi \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) : \|D_i\psi\|_\infty \leqslant 1,\ \|D_iD_j\psi\|_\infty \leqslant 1,\ i,j = 1,\dots,N\}.$$

Denote by T the semigroup generated by A. For $\psi \in G, \varrho \in \mathbb{R}$ we consider the C_0-semigroup T^ϱ given by

$$T^\varrho(t)f = \mathrm{e}^{-\varrho\psi}T(t)(\mathrm{e}^{\varrho\psi}f).$$

By Davies' trick (see Proposition 3.3 of [6]) we have to show that there exist $c > 0$, $\omega \in \mathbb{R}$ such that

$$\|T^\varrho(t)\|_{\mathcal{L}(L^1,L^\infty)} \leqslant c\mathrm{e}^{\omega(1+\varrho^2)t}t^{-N/2} \quad (t > 0) \tag{5.2}$$

for all $\varrho \in \mathbb{R}, \psi \in G$. Since the generator of T^ϱ is $\mathrm{e}^{-\varrho\psi}A\mathrm{e}^{\varrho\psi}$, it follows that T^ϱ is associated with the bilinear form $a^\varrho : W \times W \to \mathbb{R}$ defined by

$$a^\varrho(u,v) = \int_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}D_iuD_jv + \int_{\mathbb{R}^N} \sum_{i=1}^N (b_i^\varrho vD_iu + c_i^\varrho uD_iv) + \int_{\mathbb{R}^N} V^\varrho uv$$

where

$$b_i^\varrho = b_i - \varrho \sum_{j=1}^N a_{ij}\psi_j$$

$$c_i^\varrho = c_i + \varrho \sum_{k=1}^N a_{ki}\psi_k, \quad i = 1, \dots, N$$

$$V^\varrho = V - \varrho^2 \sum_{i,j=1}^N a_{ij}\psi_i\psi_j + \varrho \sum_{i=1}^N b_i\psi_i - \varrho \sum_{i=1}^N c_i\psi_i$$

and $\psi_j = D_j\psi$, see e.g. Lemma 3.6 of [6]. Note that the form a^ϱ is closed by the arguments given above.

Let $b : W \times W \to \mathbb{R}$ be a closed bilinear form on $L^2(\mathbb{R}^N)$ and denote by S the associated semigroup on $L^2(\mathbb{R}^N)$. In order to show (5.2) we use two criteria. The first is due to Beurling and Deny and is stated below.

PROPOSITION 5.4. *The semigroup S is submarkovian if and only if for any $u \in W$*

(a) $b(u^+, u^-) \leqslant 0$,
(b) $b(u \wedge 1, (u-1)^+) \geqslant 0$.

Here we call S *submarkovian* if $0 \leqslant f \leqslant 1 \Rightarrow 0 \leqslant S(t)f \leqslant 1$ for all $f \in L^2(\mathbb{R}^N), t > 0$. This is equivalent to saying that S is positive and L^∞-contractive. Note that $u \in W$ implies $u^+, u^-,\ u \wedge 1,\ (u-1)^+ \in W$. We refer to [27] for the proof of the proposition.

The next criterion allows to deduce ultracontractivity, that is the boundedness of $S(t)$, $t > 0$, from $L^1(\mathbb{R}^N)$ to $L^\infty(\mathbb{R}^N)$, from Nash's inequality

$$\|u\|_2^{2+4/N} \leqslant c_N \|u\|_{1,2}^2 \|u\|_1^{4/N},$$

$u \in W^{1,2}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, see p. 78–79 in [9]. Since the form domain W is continuously injected into $W^{1,2}(\mathbb{R}^N)$, there is a constant c such that

$$\|u\|_2^{2+4/N} \leqslant c \|u\|_W^2 \|u\|_1^{4/N},$$

for every $u \in W \cap L^1(\mathbb{R}^N)$.

PROPOSITION 5.5. *Consider a continuous bilinear form $b : W \times W \to \mathbb{R}$ which is coercive, i.e.,*

$$b(u,u) \geqslant \alpha \|u\|_W^2$$

for every $u \in W$ and some $\alpha > 0$. Assume that b and b^* (defined by $b^*(u,v) = b(v,u)$) satisfy the Beurling-Deny criterion of Proposition 5.4. Then the associated semigroup S on $L^2(\mathbb{R}^N)$ satisfies

$$\|S(t)\|_{\mathcal{L}(L^1,L^\infty)} \leqslant c_N \alpha^{-N/2} t^{-N/2} \quad (t > 0) \tag{5.3}$$

where the constant c_N does not depend on b.

This follows from Proposition 3.8 in [6] where a proof and further references are given.

Now observe that the semigroup $(\mathrm{e}^{-\omega(1+\varrho^2)t}T(t))_{t\geqslant 0}$ is associated with the bilinear form b^ϱ on W given by

$$b^\varrho(u,v) = a^\varrho(u,v) + \omega(1+\varrho^2)(u \mid v)_{L^2(\mathbb{R}^N)} \,.$$

Thus, the following lemma together with Proposition 5.5 shows that (5.2) holds, which proves Theorem 5.2 to hold.

LEMMA 5.6. *There exist $\mu > 0, \omega \in \mathbb{R}$ such that for every $u \in W$*

$$a^\varrho(u,u) + \omega(1+\varrho^2)\|u\|_2^2 \geqslant \mu\|u\|_W^2\,, \tag{5.4}$$

$$a^\varrho(u^+,u^-) + \omega(1+\varrho^2)(u^+ \mid u^-)_{L^2} = 0, \tag{5.5}$$

$$a^\varrho(u\wedge 1,(u-1)^+) + \omega(1+\varrho^2)(u\wedge 1 \mid (u-1)^+)_{L^2} \geqslant 0, \tag{5.6}$$

and

$$a^\varrho((u-1)^+,u\wedge 1) + \omega(1+\varrho^2)((u-1)^+ \mid u\wedge 1)_{L^2} \geqslant 0 \tag{5.7}$$

for all $0 \leqslant u \in W$.

Proof. Let $\psi \in G$, $\varrho \in \mathbb{R}$. Let $u \in W$. Then

$$\int\limits_{\mathbb{R}^N} \sum_{i,j=1}^N a_{ij}D_iuD_ju \geqslant \nu \int\limits_{\mathbb{R}^n} |\nabla u|^2$$

and

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} \sum_{i=1}^N (b_i^\varrho uD_iu + c_i^\varrho uD_iu)\\
&\qquad = -\frac{1}{2}\int\limits_{\mathbb{R}^N} \sum_{i=1}^N D_i(b_i^\varrho + c_i^\varrho)u^2\\
&\qquad = -\frac{1}{2}\int\limits_{\mathbb{R}^N} \sum_{i=1}^N D_i(b_i + c_i)u^2 + \frac{\varrho}{2}\int\limits_{\mathbb{R}^N} \sum_{i=1}^N D_i\Big(\sum_{j=1}^N a_{ij}\psi_j - \sum_{k=1}^N a_{ki}\psi_k\Big)u^2\\
&\qquad \geqslant -\beta\int\limits_{\mathbb{R}^N} Vu^2 - w_1(1+\varrho^2)\|u\|_2^2\,,
\end{aligned}$$

where w_1 is independent of u, ϱ and ψ, by (H2) and $\|D_i\psi_k\|_\infty \leqslant 1$. Moreover,

$$\int\limits_{\mathbb{R}^N} V^\varrho|u|^2 = \int\limits_{\mathbb{R}^N} V|u|^2 - \varrho^2\int\limits_{\mathbb{R}^N}\Big(\sum_{i,j=1}^N a_{ij}\psi_i\psi_j\Big)|u|^2 + \int\limits_{\mathbb{R}^N}\sum_{i=1}^N \varrho\psi_i(b_i - c_i)|u|^2.$$

Using (H1), (H1)′ as well as Young's inequality $2ab \leqslant \varepsilon a^2 + \frac{b^2}{\varepsilon}$ we estimate the last term as follows

$$\int_{\mathbb{R}^N} \sum_{i=1}^N \varrho\psi_i(b_i - c_i)|u|^2 \geqslant -\int_{\mathbb{R}^N} \varrho N^{1/2} 2\gamma V^{1/2}|u|^2 \geqslant -\varepsilon \int_{\mathbb{R}^N} V|u|^2 - \frac{1}{\varepsilon}\varrho^2 N\gamma^2 \int_{\mathbb{R}^N} |u|^2.$$

We choose $\varepsilon = \frac{1-\beta}{2}$. Then

$$\begin{aligned} a^\varrho(u,u) \geqslant{}& \nu \int_{\mathbb{R}^N} |\nabla u|^2 - \beta \int_{\mathbb{R}^N} V|u|^2 - w_1(1+\varrho^2)\|u\|_2^2 \\ &+ \int_{\mathbb{R}^N} V|u|^2 - \varrho^2 \int_{\mathbb{R}^N} \Big(\sum_{i,j=1}^N a_{ij}\psi_i\psi_j\Big)|u|^2 \\ &- \varepsilon \int_{\mathbb{R}^N} V|u|^2 - \frac{1}{\varepsilon}\varrho^2 N\gamma^2 \int |u|^2 \\ \geqslant{}& \nu \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1-\beta}{2} \int_{\mathbb{R}^N} V|u|^2 - \omega_2(1+\varrho^2)\|u\|_2^2 \\ \geqslant{}& \mu\|u\|_W^2 - \omega_2(1+\varrho^2)\|u\|_2^2 \end{aligned}$$

for all $u \in W$, $\varrho \in \mathbb{R}$, $\psi \in G$, where $\mu = \min\{\nu, \frac{1-\beta}{2}\} > 0$ and $\omega_2 \in \mathbb{R}$ suitable (recall that $\|\psi_i\|_\infty \leqslant 1$ for all $\psi \in G$). Thus (5.4) is satisfied for $\omega = \omega_2$. (5.5) holds for all $\omega, \varrho \in \mathbb{R}$ since

$$D_j(u^+) = \chi_{\{u>0\}} D_j u, \quad D_j u^- = -\chi_{\{u>0\}} D_j u\,.$$

Next we show (5.6) replacing ω_2 by a larger constant ω. Let $0 \leqslant u \in W$. Observe that $D_j(u \wedge 1) = \chi_{\{u>1\}} D_j u$, $D_j(u-1)^+ = \chi_{\{u>1\}} D_j u$. Thus $D_i(u \wedge 1) D_j(u-1)^+ = 0$ and $D_i(u \wedge 1)(u-1)^+ = 0$ a.e. Hence

$$\begin{aligned} a^\varrho(u \wedge 1,\ (u-1)^+) &= \int_{\mathbb{R}^N} \sum_{i=1}^N c_i^\varrho (u\wedge 1) D_i(u-1)^+ + \int_{\mathbb{R}^N} V^\varrho (u \wedge 1)(u-1)^+ \\ &= -\int_{\mathbb{R}^N} \sum_{i=1}^N (D_i c_i^\varrho)(u\wedge 1)(u-1)^+ + \int_{\mathbb{R}^N} V^\varrho (u\wedge 1)(u-1)^+\,. \end{aligned}$$

Thus we have to show that there exists $\omega \in \mathbb{R}$ such that

$$-\sum_{i=1}^N D_i c_i^\varrho + V^\varrho \geqslant -(1+\varrho^2)\omega$$

for all $\psi \in G$, $\varrho \in \mathbb{R}$. By (H2)′ we have

$$\begin{aligned} V^{\varrho} - \sum_{i=1}^{N} D_i c_i^{\varrho} &= V - \varrho^2 \sum_{i,j=1}^{N} a_{ij}\psi_i\psi_j + \varrho \sum_{i=1}^{N}(b_i - c_i)\psi_i - \operatorname{div} c - \varrho \sum_{i=1}^{N} D_i\Big(\sum_{k=1}^{N} a_{ki}\psi_k\Big) \\ &\geqslant V - \varrho^2\omega_3 + \varrho\sum_{i=1}^{N}(b_i - c_i)\psi_i - \beta V - (1+\varrho^2)\omega_4 \\ &\geqslant (1-\beta)V - (1+\varrho^2)(\omega_3+\omega_4) - \varrho 2\gamma V^{1/2}N^{1/2} \\ &\geqslant (1-\beta)V - (1+\varrho^2)(\omega_3+\omega_4) - \varepsilon V - \frac{1}{\varepsilon}\varrho^2\gamma^2 N \\ &\geqslant \varepsilon V - (1+\varrho^2)\omega_5 \end{aligned}$$

for all $\varrho \in \mathbb{R}$, $\psi \in \mathbb{R}$, where ω_5 is a suitable constant. This proves (5.6). Inequality (5.7) is proved as (5.6) since the conditions on b and on c are the same. This finishes the proof. ∎

REMARK 5.7. We observe that our proof shows that the semigroup T associated with the closed form a given (5.1) on $L^2(\mathbb{R}^N)$ admits a Gaussian bound whenever $0 \leqslant V \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ and $c, b \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$ satisfy (H1), (H2), (H1)′, (H2)′.

REMARK 5.8. (Arbitrary domains). In some applications the operator A is defined on exterior domains. The results we obtained in Sections 3 and 5 remain valid if $\mathbb{R}^N$ is replaced by an arbitrary open set Ω and the generated semigroup satisfies homogenuous Dirichlet boundary conditions on $\partial\Omega$. However, in the proofs the balls B_ϱ should be replaced by a sequence of bounded open sets Ω_n with C^∞ boundary such that $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. The maximal operator $A_{p,\max}$ may be defined as in (1.4) for $1 < p < \infty$ and as in (1.5) for $p = 1$, with Ω in place of $\mathbb{R}^N$. Notice also that in this more general situation we have to use an approximation argument for forms to show that the operator A_2 and the one defined by the form a coincide, since the uniqueness results of Section 4 clearly hold only in $\mathbb{R}^N$. For this approximation argument we refer to [3], [8].

6. AN EXAMPLE

In order to test our results in a concrete situation, we discuss in detail the one-dimensional operator $A = D^2 - x^3 D - c|x|^{\gamma}$, with $c > 0$, $\gamma \geqslant 0$. The generalization to exponents different from 3 (but bigger than 1) in the power appearing in the drift term is straightforward. Moreover, some of the negative results proved below can be generalized to the higher dimensional case.

We start by showing that a condition like (3.1) is needed, in general, to generate a semigroup in L^p.

PROPOSITION 6.1. *A restriction of the operator $A_{p,\max}$ generates a semigroup in $L^p(\mathbb{R})$, $1 \leqslant p < \infty$, if $\gamma > 2$ or $\gamma = 2$ and $cp \geqslant 3$. On the other hand, if $\gamma < 2$ or $\gamma = 2$ and $cp \leqslant 1$, then no restriction of $A_{p,\max}$ is a generator in $L^p(\mathbb{R})$.*

Proof. If $\gamma > 2$ or $\gamma = 2$ and $cp \geqslant 3$, then Theorem 3.3 applies and yields a restriction A_p of $A_{p,\max}$ which generates a semigroup in $L^p(\mathbb{R})$.

Fix now $1 \leqslant p < \infty$ and assume that $A_{p,\max}$ has a restriction generating a semigroup in $L^p(\mathbb{R})$. In particular, $\lambda - A_{p,\max}$ is surjective for large λ. Given $\phi \in C_c^\infty(\mathbb{R})$, $\phi \geqslant 0$, $\phi \neq 0$, let $u \in D(A_{p,\max})$ be such that $\lambda u - Au = \phi$. In particular, $\lambda u - Au = 0$ for $|x| \geqslant b$, where $[-b, b]$ contains the support of ϕ. However, if $\gamma < 2$ or $\gamma = 2$ and $cp \leqslant 1$, no non-zero solution of the equation $\lambda u - Au = 0$ belongs to $L^p([b, +\infty[)$ for every λ sufficiently large (see Lemma 6.2 below) and hence $u = 0$ in $[b, +\infty[$ and, by the same argument, in $]-\infty, -b]$. Therefore u has compact support and, since it belongs to $C^2(\mathbb{R})$, the maximum principle yields $u \geqslant 0$ everywhere. Finally note that u attains its minimum and therefore, by the strong minimum principle, $u = 0$ everywhere, in contrast with $\phi \neq 0$. This shows that $\lambda - A_{p,\max}$ is not surjective and concludes the proof. ∎

Let us prove the lemma used above.

LEMMA 6.2. *Assume that $\gamma < 2$ or $\gamma = 2$ and $cp \leqslant 1$. If $\lambda > 0$ is sufficiently large, no solution of the differential equation $\lambda u - Au = 0$ belongs to $L^p([b, +\infty[)$ for every $b \in \mathbb{R}$.*

Proof. We give all the details for $\gamma = 2$ and $cp \leqslant 1$, the other case being similar. With the substitution $u(x) = v(x)\exp\{x^4/8\}$ the equation $\lambda u - Au = 0$ becomes

$$v'' = \left(\lambda + \frac{1}{4}x^6 + \left(c - \frac{3}{2}\right)x^2\right)v = fv. \tag{6.1}$$

Fix now any $\lambda > 0$ such that $f \geqslant 0$ and observe that the function $f^{-1/4}D^2(f^{-1/4})$ belongs to $L^1(\mathbb{R})$. Using Theorem 2.1 in [26] we see that (6.1) has two linearly independent solutions v_1, v_2 in $[b, +\infty[$ such that

$$v_1(x) \approx x^{c-3}e^{x^4/8}, \quad v_2(x) \approx x^{-c}e^{-x^4/8}$$

as $x \to +\infty$. This yields

$$u_1(x) \approx x^{c-3}e^{x^4/4}, \quad u_2(x) \approx x^{-c}$$

as $x \to +\infty$ and the statement follows. ∎

Next we show that our uniqueness results are quite precise for $p = 1$.

PROPOSITION 6.3. *If $\gamma > 2$ or $\gamma = 2$ and $cp > 3$, then $C_c^\infty(\mathbb{R})$ is a core for A_p. However, if $p = 1$ and $\gamma = 2, c = 3$, then $C_c^\infty(\mathbb{R})$ is not a core for A_1.*

Proof. The first statement follows immediately from Remark 4.4. Assume now that $p = 1$ and consider $A = D^2 - x^3D - 3x^2$ so that $A^* = D^2 + x^3D$. For every $\lambda > 0$ there exists a bounded function $0 \neq u \in C^2(\mathbb{R})$ such that $\lambda u - A^* u = 0$. This follows from Feller's theory on one dimensional diffusions, since $\pm\infty$ are exit boundaries for the operator A^*, see Chapter VI.4.c of [13] for an introduction to Feller's theory and a proof of the above result. Integrating by parts we get for every $\phi \in C_c^\infty(\mathbb{R})$

$$\int_{\mathbb{R}} (\lambda\phi - A_1\phi)u = \int_{\mathbb{R}} \phi(\lambda u - A^* u) = 0$$

and therefore $C_c^\infty(\mathbb{R})$ is not a core for A_1. ∎

Finally we show that, in general, Gaussian estimates fail when condition (H1) is violated.

PROPOSITION 6.4. *If $\gamma \geqslant 6$ the generated semigroup T_p admits a Gaussian estimate. On the other hand, if $\gamma < 6$, then T_p is not analytic in $L^p(\mathbb{R})$.*

Proof. If $\gamma \geqslant 6$ the first statement is a immediate consequence of Theorem 5.2. To prove the second one we proceed as in [23] and fix $1 \leqslant p < \infty$. Given β such that $\max\{3,\gamma\} < \beta < 6$, let $I_n : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ be defined by $(I_n u)(x) = u\left((x-n)/\lambda_n\right)$ where $\lambda_n = n^{3-\beta}$. Clearly, $\left((I_n)^{-1}v\right)(x) = v(n+\lambda_n x)$ and $\|I_n u\|_p = (\lambda_n)^{1/p}\|u\|_p$. We consider the differential operators $A_n = r_n (I_n)^{-1} A I_n$, with $r_n = n^{-\beta}$ and observe that for every $u \in C_c^\infty(\mathbb{R})$

$$A_n u(x) = \frac{1}{n^{6-\beta}} u''(x) - \frac{1}{n^3}(n + n^{3-\beta}x)^3 u'(x) - \frac{c}{n^\beta}|n + n^{3-\beta}x|^\gamma u(x)$$

hence $A_n u \to -u'$ in $L^p(\mathbb{R})$ for every $u \in C_c^\infty(\mathbb{R})$. Next observe that the operator A_n with domain $I_n^{-1}D(A_p)$ is the generator of the semigroup $T_n(t) = I_n^{-1}T_p(r_n t)I_n$ and that, since $r_n \to 0$, we have $\|T_n(t)\| \leqslant M e^{\omega t}$ for suitable M, ω independent of t. Moreover, $C_c^\infty(\mathbb{R})$ is a core for the operator $Bu = -u'$ and hence from the Trotter-Kato theorem, see Theorem 4.8, Chapter II of [13], we deduce that $R(\lambda, A_n)f \to R(\lambda, B)f$ for every $f \in L^p(\mathbb{R})$ and $\operatorname{Re}\lambda > \omega$. Assume now, by contradiction, that T_p is analytic. Then $\|R(\lambda, A_p)\| \leqslant C|\lambda|^{-1}$ if $\operatorname{Re}\lambda$ is sufficiently large. Since

$$R(\lambda, A_n) = I_n^{-1}(\lambda - r_n A_p)^{-1} I_n = \frac{1}{r_n} I_n^{-1} R\Big(\frac{\lambda}{r_n}, A_p\Big) I_n$$

it follows that $\|R(\lambda, A_n)\| \leqslant C|\lambda|^{-1}$ and hence

$$\|R(\lambda, B)f\|_p \leqslant \liminf_{n\to\infty} \|R(\lambda, A_n)f\|_p \leqslant C|\lambda|^{-1}\|f\|_p$$

for large $\operatorname{Re}\lambda$. Since the semigroup generated by B is not analytic, this is a contradiction and the proof is complete. ∎

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