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C^*-ALGEBRAS GENERATED BY SCALING ELEMENTS

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ABSTRACT. We investigate C^*-algebras generated by scaling elements. We generalize the Wold decomposition and Coburn's theorem on isometries to scaling elements. We also completely determine when the C^*-algebra generated by a scaling element contains an infinite projection.

KEYWORDS: *Scaling elements, Wold decomposition, Coburn's theorem, topological graphs, K-groups.*

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0. INTRODUCTION

For a (complex) Hilbert space $\mathcal{H}$, we denote by $\mathfrak{B}(\mathcal{H})$ the C^*-algebra of all bounded operators on $\mathcal{H}$, and define a Hilbert space $\mathcal{H}^\infty$ by

$$\mathcal{H}^\infty = \Big\{ (\xi_n)_{n\in\mathbb{N}} \ :\ \xi_n \in \mathcal{H}, \sum_{n\in\mathbb{N}} \|\xi_n\|^2 < \infty \Big\}.$$

An isometry on $\mathcal{H}$ is a bounded operator $T \in \mathfrak{B}(\mathcal{H})$ satisfying $T^*T = I_{\mathcal{H}}$ where $I_{\mathcal{H}}$ is the identity operator in $\mathfrak{B}(\mathcal{H})$. An isometry T is said to be *proper* if $TT^* \neq I_{\mathcal{H}}$. For a Hilbert space $\mathcal{H}$, we define an operator $S_{\mathcal{H}} \in \mathfrak{B}(\mathcal{H}^\infty)$ by $S_{\mathcal{H}}(\xi_0, \xi_1, \dots) = (0, \xi_0, \xi_1, \dots)$ for $(\xi_0, \xi_1, \dots) \in \mathcal{H}^\infty$. The operator $S_{\mathcal{H}}$ is a proper isometry when $\mathcal{H} \neq 0$. Conversely, the following proposition, called the *Wold decomposition*, says that any proper isometry is essentially the direct sum of an element in this form and a unitary (see Theorem V.2.1 of [3] for a proof).

PROPOSITION 0.1. *Let T be an isometry in $\mathfrak{B}(\mathcal{H})$. Then there exist two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and an isomorphism $\mathcal{H} \cong \mathcal{H}_1^\infty \oplus \mathcal{H}_2$ such that T is unitarily equivalent to $S_{\mathcal{H}_1} \oplus U \in \mathfrak{B}(\mathcal{H}_1^\infty \oplus \mathcal{H}_2)$ for a unitary $U \in \mathfrak{B}(\mathcal{H}_2)$.*
The isometry T is proper if and only if $\mathcal{H}_1 \neq 0$.

We call $S = S_{\mathbb{C}} \in \mathfrak{B}(\mathbb{C}^\infty)$ the *unilateral shift*. The C^*-algebra generated by the unilateral shift S is called the *Toeplitz algebra* and denoted by $\mathcal{T}$. In [2], Coburn showed the following theorem using the Wold decomposition.

THEOREM 0.2 (Coburn). *For a proper isometry T, there is a $*$-isomorphism φ from $\mathcal{T}$ to the C^*-algebra $C^*(T)$ generated by T such that $\varphi(S) = T$.*

A projection P in a C^*-algebra $\mathfrak{A}$ is said to be *infinite* if there exists $U \in \mathfrak{A}$ such that $U^*U = P$ and $UU^* < P$ where $UU^* < P$ means $UU^* \leqslant P$ and $UU^* \neq P$. Otherwise we say that a projection $P \in \mathfrak{A}$ is *finite*. Existence of a proper isometry on a Hilbert space $\mathcal{H}$ shows that the unit $I_{\mathcal{H}}$ of $\mathfrak{B}(\mathcal{H})$ is infinite. It is important to determine whether a given C^*-algebra contains an infinite projection or not. To this end, Blackadar and Cuntz introduced the following notion in [1].

DEFINITION 0.3 (Blackadar, Cuntz). An element X of a C^*-algebra is called a *scaling element* if $(X^*X)X = X$ and $X^*X \neq XX^*$.

Note that the condition $(X^*X)X = X$ is equivalent to $(X^*X)(XX^*) = XX^*$. Since a partial isometry U satisfying $UU^* < U^*U$ is a scaling element, a C^*-algebra containing an infinite projection has a scaling element. The converse is true if a C^*-algebra is unital or simple (see [1] or Proposition 4.2 of [5]). However there exists a (non-unital, non-simple) C^*-algebra which has a scaling element but does not have an infinite projection (see Theorem 0.10).

In this paper, we generalize the two results above on isometries to scaling elements, and investigate the structure of C^*-algebras generated by scaling elements. To state the main results, we need several notions.

DEFINITION 0.4. For an operator $A \in \mathfrak{B}(\mathcal{H})$, we define an operator $S_A \in \mathfrak{B}(\mathcal{H}^\infty)$ by $S_A(\xi_0, \xi_1, \ldots) = (0, A\xi_0, \xi_1, \ldots)$ for $(\xi_0, \xi_1, \ldots) \in \mathcal{H}^\infty$:

$$S_A = \begin{pmatrix} 0 & & & & \\ A & 0 & & & \\ & I_{\mathcal{H}} & 0 & & \\ & & I_{\mathcal{H}} & 0 & \\ & & & \ddots & \ddots \end{pmatrix}.$$

We have $S_{I_{\mathcal{H}}} = S_{\mathcal{H}}$. When $\mathcal{H} \neq 0$, $S_A \in \mathfrak{B}(\mathcal{H}^\infty)$ is a scaling element for any operator $A \in \mathfrak{B}(\mathcal{H})$. Our first result is the *Wold decomposition* of scaling elements.

THEOREM 0.5. *Let $X \in \mathfrak{B}(\mathcal{H})$ be an element satisfying $(X^*X)X = X$. Then there exist three Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$, and an isomorphism $\mathcal{H} \cong \mathcal{H}_1^\infty \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ such that X is unitarily equivalent to $S_A \oplus U \oplus 0$ where A is a positive operator in $\mathfrak{B}(\mathcal{H}_1)$ whose support is $I_{\mathcal{H}_1}$ and $U \in \mathfrak{B}(\mathcal{H}_2)$ is a unitary. This decomposition is unique up to unitary equivalence.*

Such X is a scaling element (i.e. $X^*X \neq XX^*$) if and only if $\mathcal{H}_1 \neq 0$.

We will prove Theorem 0.5 in Section 1. Next we state the generalization of Coburn's theorem to scaling elements. We denote by $\mathrm{sp}(A)$ the spectrum of an operator A, and by 1_C the characteristic function of a set C. It is easy to see $0, 1 \in \mathrm{sp}(|X^*|) \subset [0, \infty)$ for a scaling element X.

DEFINITION 0.6. A scaling element X is said to be *non-proper* if $\mathrm{sp}(|X^*|) \setminus \{0,1\}$ is compact, and $1_{\{1\}}(|X|) = 1_{\mathrm{sp}(|X^*|)\setminus\{0\}}(|X^*|)$. Otherwise, we say that a scaling element X is *proper*.

For a positive operator $A \in \mathfrak{B}(\mathcal{H})$, $S_A \in \mathfrak{B}(\mathcal{H}^\infty)$ is a scaling element with $\mathrm{sp}(|S_A^*|) = \mathrm{sp}(A) \cup \{0,1\}$. Using the operators S_A, we can show the following.

THEOREM 0.7 (Existence theorem). *For any compact set Ω with $0, 1 \in \Omega \subset [0,\infty)$, there exists a scaling element X with $\mathrm{sp}(|X^*|) = \Omega$. If such Ω satisfies that $\Omega \setminus \{0,1\}$ is non-empty and compact, then there exist both a proper scaling element and a non-proper one whose spectra are Ω.*

Note that a scaling element X is proper if $\mathrm{sp}(|X^*|) \setminus \{0,1\}$ is empty or non-compact. The following uniqueness theorem is a generalization of Coburn's theorem.

THEOREM 0.8 (Uniqueness theorem). *Let X, Y be scaling elements. There exists a $*$-isomorphism $\varphi\colon C^*(X) \to C^*(Y)$ with $\varphi(X) = Y$ if and only if $\mathrm{sp}(|X^*|) = \mathrm{sp}(|Y^*|)$ and X, Y are simultaneously proper or non-proper.*

Theorem 0.8 easily follows from the next proposition which concerns a hierarchy of C^*-algebras $C^*(X)$ generated by scaling elements X.

PROPOSITION 0.9. *Let X, Y be scaling elements.*

(i) *When X is proper, there exists a $*$-homomorphism $\varphi\colon C^*(X) \to C^*(Y)$ with $\varphi(X) = Y$ if and only if $\mathrm{sp}(|X^*|) \supset \mathrm{sp}(|Y^*|)$.*

(ii) *When X is non-proper, there exists a $*$-homomorphism $\varphi\colon C^*(X) \to C^*(Y)$ with $\varphi(X) = Y$ if and only if $\mathrm{sp}(|X^*|) \supset \mathrm{sp}(|Y^*|)$ and Y is non-proper.*

Clearly the $*$-homomorphism φ in the proposition above is, if it exists, unique and surjective. In Section 3 we will prove Proposition 0.9 and the following theorem, by using the theory of C^*-algebras arising from constant maps studied in Section 2.

THEOREM 0.10. *Let X be a scaling element. The C^*-algebra $C^*(X)$ has an infinite projection if and only if $[0,1] \setminus \mathrm{sp}(|X^*|) \neq \emptyset$.*

1. THE WOLD DECOMPOSITION OF SCALING ELEMENTS

In this section, we prove the Wold decomposition of scaling elements (Theorem 0.5). For an operator $X \in \mathfrak{B}(\mathcal{H})$, we denote by $l(X)$ (respectively $r(X)$) the left (respectively right) support of X. Namely $l(X)$ (respectively $r(X)$) is the smallest projection of $\mathfrak{B}(\mathcal{H})$ satisfying $l(X)X = X$ (respectively $Xr(X) = X$).

Take an element $X \in \mathfrak{B}(\mathcal{H})$ satisfying $(X^*X)X = X$. We set $P_0 = r(X)$ and $P_0' = l(X)$. Since $(X^*X)X = X$, we have $X^*XP_0' = P_0'$. Hence we get $P_0 \geqslant P_0'$. Set $P_3 = I_{\mathcal{H}} - P_0$ and $\mathcal{H}_3 = P_3\mathcal{H}$. Then we have $P_3X = XP_3 = 0$. Set $Q_0 = P_0 - P_0'$

and $X_0 = XQ_0$. Since $r(X) = P_0 \geqslant Q_0$, we have $r(X_0) = Q_0$. We define $Q_1 = l(X_0)$ and $U_1 = XQ_1$. Clearly we have $Q_1 \leqslant P_0'$. Since $X^*XP_0' = P_0'$, we have $X^*XQ_1 = Q_1$. Thus $U_1^*U_1 = Q_1$. Since $X_0^*XP_0' = Q_0X^*XP_0' = Q_0P_0' = 0$, we have $Q_1XP_0' = 0$. This implies that

$$Q_1X = Q_1XP_0 = Q_1XQ_0 + Q_1XP_0' = Q_1XQ_0 = XQ_0.$$

Recursively we define projections $Q_2, Q_3, \dots$ and partial isometries $U_2, U_3, \dots$ by $Q_n = U_{n-1}U_{n-1}^*$ and $U_n = XQ_n$. From $Q_n \leqslant P_0'$, we have $X^*XQ_n = Q_n$. Hence $U_n^*U_n = Q_n$. By the definition, we have $Q_{n+1} = XQ_nX^*$ for $n \geqslant 1$. Thus we get $Q_{n+1}X = XQ_nX^*X = XQ_n$. For $n \geqslant 1$, we have $Q_0Q_n = 0$ because $Q_n \leqslant P_0'$. For $k \geqslant 1$, we have $Q_kQ_{n+k} = (XQ_{k-1}X^*)(XQ_{n+k-1}X^*) = XQ_{k-1}Q_{n+k-1}X^*$. Thus recursively we can show that $Q_mQ_n = 0$ for $0 \leqslant m < n$. Therefore $\{Q_n\}_{n\in\mathbb{N}}$ is a family of mutually orthogonal projections. Set $P_1 = \sum_{n\in\mathbb{N}} Q_n$. We have

$$P_1X = \sum_{n=0}^{\infty} Q_nX = \sum_{n=1}^{\infty} Q_nX = \sum_{n=1}^{\infty} XQ_{n-1} = XP_1.$$

Let $X_0 = U_0|X_0|$ be a polar decomposition of X_0. Then we have $U_0^*U_0 = r(X_0) = Q_0$ and $U_0U_0^* = l(X_0) = Q_1$. Since $r(|X_0|) = l(|X_0|) = Q_0$, the restriction of $|X_0|$ on the Hilbert space $\mathcal{H}_1 = Q_0\mathcal{H}$ gives a positive operator $A \in \mathfrak{B}(\mathcal{H}_1)$ with $r(A) = l(A) = I_{\mathcal{H}_1}$. By using partial isometries $\{U_n\}_{n\in\mathbb{N}}$, we have a unitary from $P_1\mathcal{H}$ to $\mathcal{H}_1^\infty$. Via this unitary, the operator $P_1X = XP_1 \in \mathfrak{B}(P_1\mathcal{H})$ is unitarily equivalent to $S_A \in \mathfrak{B}(\mathcal{H}_1^\infty)$. We set $P_2 = I_{\mathcal{H}} - P_1 - P_3$ and $X_2 = P_2X = XP_2$. Since $P_2 \leqslant P_0' \leqslant P_0$, we have $l(X_2) = r(X_2) = P_2$. We can easily check $(X_2^*X_2)X_2 = X_2$. This shows that $X_2^*X_2 = X_2^*X_2P_2 = P_2$. Hence we get $X_2^*X_2 = X_2X_2^* = P_2$. Therefore the restriction of X on $\mathcal{H}_2 = P_2\mathcal{H}$ is a unitary in $\mathfrak{B}(\mathcal{H}_2)$. The uniqueness of this decomposition follows from the argument above and the uniqueness of polar decomposition. This shows the former part of Theorem 0.5, and the latter part is obvious.

Similarly as [2], we can deduce Proposition 0.9 from Theorem 0.5. However the computations we need here are much harder than the ones in [2]. In this paper, we will prove Proposition 0.9 by using the general theory of C^*-algebras arising from constant maps.

2. C^*-ALGEBRAS ARISING FROM CONSTANT MAPS

In this section, we study the structures of C^*-algebras arising from constant maps. In the next section, we apply the results of this section to the C^*-algebras generated by scaling elements.

Take a locally compact space Ω and a point v of Ω. We fix them throughout this section. An *(Ω, v)-pair* is a pair (π, t) consisting of a $*$-homomorphism π and a linear map t from $C_0(\Omega)$ to some C^*-algebra $\mathfrak{A}$ satisfying

$$t(f)^*t(g) = \pi(\overline{f}g), \quad t(f)\pi(g) = t(fg), \quad \pi(f)t(g) = f(v)t(g)$$

for $f, g \in C_0(\Omega)$. Note that the second condition automatically follows from the first one (see [6]).

DEFINITION 2.1. Let us denote by $(\widehat{\pi}, \widehat{t})$ the universal (Ω, v)-pair, and by $\mathcal{T}(\Omega, v)$ the C^*-algebra generated by the images of $\widehat{\pi}$ and $\widehat{t}$.

The universality means that for any (Ω, v)-pair (π, t) on a C^*-algebra $\mathfrak{A}$, there exists a $*$-homomorphism $\rho\colon \mathcal{T}(\Omega, v) \to \mathfrak{A}$ with $\pi = \rho \circ \widehat{\pi}$ and $t = \rho \circ \widehat{t}$. The existence of the universal (Ω, v)-pair follows from a standard argument, and the uniqueness is easy to see.

LEMMA 2.2. *Both $\widehat{\pi}$ and $\widehat{t}$ are injective.*

Proof. It suffices to find one (Ω, v)-pair (π, t) such that π and t are injective. Let us take a faithful representation $\varphi\colon C_0(\Omega) \to \mathfrak{B}(\mathcal{H})$. We define an injective $*$-homomorphism $\pi\colon C_0(\Omega) \to \mathfrak{B}(\mathcal{H}^\infty)$ by

$$\pi(f)(\xi_0, \xi_1, \xi_2, \ldots) = (\varphi(f)\xi_0, f(v)\xi_1, f(v)\xi_2, \ldots)$$

for $f \in C_0(\Omega)$ and $(\xi_0, \xi_1, \ldots) \in \mathcal{H}^\infty$. We define an injective linear map $t\colon C_0(\Omega) \to \mathfrak{B}(\mathcal{H}^\infty)$ by $t(f) = S_{\mathcal{H}}\pi(f)$ for $f \in C_0(\Omega)$. Then the pair (π, t) is an (Ω, v)-pair. We are done. ∎

For a while, we suppose that $\Omega \setminus \{v\}$ is compact. We denote by V the isometry $\widehat{t}(1_\Omega) \in \mathcal{T}(\Omega, v)$. The C^*-algebra $\mathcal{T}(\Omega, v)$ is generated by the isometry V and $\widehat{\pi}(C(\Omega))$. By the proof of Lemma 2.2, the projection $P = \widehat{\pi}(1_{\{v\}}) - VV^*$ is non-zero. Let us define $I \subset \mathcal{T}(\Omega, v)$ to be the closure of the linear span of

$$\mathcal{E} = \{V^n P (V^*)^m : n, m \in \mathbb{N}\}.$$

LEMMA 2.3. *The subset I is the ideal of $\mathcal{T}(\Omega, v)$ generated by P.*

Proof. It suffices to see that the linear span of $\mathcal{E}$ is closed under the multiplication by V, V^* and $\widehat{\pi}(C(\Omega))$ from left. For $f \in C(\Omega)$, we have $\widehat{\pi}(f)V = f(v)V$ and $\widehat{\pi}(f)P = \widehat{\pi}(f1_{\{v\}}) - \widehat{\pi}(f)VV^* = f(v)P$. Hence the linear span of $\mathcal{E}$ is closed under the multiplication by $\widehat{\pi}(C(\Omega))$ from left. Clearly it is closed under the multiplication by V from left. We have $V^*P = V^*\widehat{\pi}(1_{\{v\}}) - V^*VV^* = V^* - V^* = 0$. This shows that the linear span of $\mathcal{E}$ is also closed under the multiplication by V^* from left. We are done. ∎

DEFINITION 2.4. When $\Omega \setminus \{v\}$ is compact, we define a C^*-algebra $\mathcal{O}(\Omega, v)$ by $\mathcal{O}(\Omega, v) = \mathcal{T}(\Omega, v)/I$.

We denote by $\mathcal{K}$ the C^*-algebra of compact operators on the Hilbert space $\mathbb{C}^\infty$. The matrix unit of $\mathcal{K}$ is denoted by $\{U_{n,m} : n, m \in \mathbb{N}\}$.

LEMMA 2.5. *The map $V^n P (V^*)^m \mapsto U_{n,m}$ induces an isomorphism $I \cong \mathcal{K}$.*

Proof. This follows from routine computation. ∎

Let (π, t) be an (Ω, v)-pair, and $\mathfrak{A}$ be the C^*-algebra generated by the images of π and t. By the universality, there exists a surjection $\rho\colon \mathcal{T}(\Omega, v) \to \mathfrak{A}$ with $\pi = \rho \circ \widehat{\pi}$ and $t = \rho \circ \widehat{t}$. When $\Omega \setminus \{v\}$ is compact and $\pi(1_{\{v\}}) = t(1_\Omega)t(1_\Omega)^*$, the surjection $\rho\colon \mathcal{T}(\Omega, v) \to \mathfrak{A}$ factors through a surjection $\overline{\rho}\colon \mathcal{O}(\Omega, v) \to \mathfrak{A}$. By using results in [6], we get the following.

PROPOSITION 2.6. (i) *When $\Omega \setminus \{v\}$ is not compact, the surjection ρ is an isomorphism if and only if π is injective.*

(ii) *When $\Omega \setminus \{v\}$ is compact, the surjection ρ is an isomorphism if and only if π is injective and $\pi(1_{\{v\}}) \neq t(1_\Omega)t(1_\Omega)^*$.*

(iii) *When $\Omega \setminus \{v\}$ is a non-empty compact set and $\pi(1_{\{v\}}) = t(1_\Omega)t(1_\Omega)^*$, the surjection $\overline{\rho}\colon \mathcal{O}(\Omega, v) \to \mathfrak{A}$ is an isomorphism if and only if π is injective.*

Proof. We define a continuous map $r\colon \Omega \to \Omega$ by $r(x) = v$ for all $x \in \Omega$, and set a topological graph $E = (\Omega, \Omega, \mathrm{id}, r)$ (see [6]). If $\Omega \setminus \{v\}$ is not compact, then we have $E^0_{\mathrm{rg}} = \emptyset$. Thus $\mathcal{T}(\Omega, v) = \mathcal{T}(E) = \mathcal{O}(E)$. When $\Omega \setminus \{v\}$ is compact, we have $E^0_{\mathrm{rg}} = \{v\}$. Hence we get $\mathcal{T}(\Omega, v) = \mathcal{T}(E)$ and $\mathcal{O}(\Omega, v) = \mathcal{O}(E)$. It is easy to verify that the topological graph E is topologically free when $\Omega \neq \{v\}$. Therefore (i) and (iii) follows from Theorem 5.12 of [6], and (ii) follows from Proposition 3.16 of [8]. ∎

When $\Omega = \{v\}$, we have $\mathcal{T}(\Omega, v) \cong \mathcal{T}$ and $\mathcal{O}(\Omega, v) \cong C(\mathbb{T})$ where $\mathbb{T}$ is the one-dimensional torus. Thus in this case, (ii) in the proposition above is Coburn's Theorem introduced in the introduction, and the corresponding statement of (iii) does not hold.

Let us take a closed subset Ω' of Ω with $v \in \Omega'$. The universal (Ω', v)-pair is denoted by $(\widehat{\pi}', \widehat{t}')$. By the universality, there exists a surjection $\rho\colon \mathcal{T}(\Omega, v) \to \mathcal{T}(\Omega', v)$ with $\widehat{\pi}' \circ \sigma = \rho \circ \widehat{\pi}$ and $\widehat{t}' \circ \sigma = \rho \circ \widehat{t}$ where $\sigma\colon C_0(\Omega) \to C_0(\Omega')$ is the natural surjection. The kernel of σ is $C_0(\Omega \setminus \Omega')$.

PROPOSITION 2.7. *The kernel of ρ is isomorphic to $C_0(\Omega \setminus \Omega') \otimes \mathcal{K}$.*

Proof. Let J be the closure of the linear span of elements in the form

$$\widehat{t}(f_1)\widehat{t}(f_2)\cdots\widehat{t}(f_n)\widehat{\pi}(h)\widehat{t}(g_m)^*\cdots\widehat{t}(g_2)^*\widehat{t}(g_1)^*$$

where $f_1, \ldots, f_n, g_1, \ldots, g_m \in C_0(\Omega)$ and $h \in C_0(\Omega \setminus \Omega')$. It is routine to check that J is the ideal generated by $\widehat{\pi}(C_0(\Omega \setminus \Omega'))$. It is also routine to see that the map

$$\begin{aligned} J \ni\ & \widehat{t}(f_1)\cdots\widehat{t}(f_n)\widehat{\pi}(h)\widehat{t}(g_m)^*\cdots\widehat{t}(g_1)^* \\ & \mapsto f_1(v)\cdots f_n(v)\overline{g_m(v)}\cdots\overline{g_1(v)}\big(h \otimes U_{n,m}\big) \in C_0(\Omega \setminus \Omega') \otimes \mathcal{K} \end{aligned}$$

induces an isomorphism. We will show that $J = \ker\rho$. Since $\widehat{\pi}(C_0(\Omega \setminus \Omega')) \subset \ker\rho$, we have $J \subset \ker\rho$. Hence the surjection $\rho\colon \mathcal{T}(\Omega, v) \to \mathcal{T}(\Omega', v)$ factors through $\varphi\colon \mathcal{T}(\Omega,v)/J \to \mathcal{T}(\Omega',v)$. Since $\widehat{\pi}(C_0(\Omega\setminus\Omega')), \widehat{t}(C_0(\Omega\setminus\Omega')) \subset J$, we have a $*$-homomorphism $\pi'\colon C_0(\Omega') \to \mathcal{T}(\Omega,v)/J$ and a linear map $t'\colon C_0(\Omega') \to \mathcal{T}(\Omega,v)/J$ such that $\widehat{\pi}' = \varphi\circ\pi'$ and $\widehat{t}' = \varphi\circ t'$. The pair (π',t') is an (Ω',v)-pair. Hence we get a $*$-homomorphism $\mathcal{T}(\Omega',v) \to \mathcal{T}(\Omega,v)/J$ which is clearly the inverse of the surjection φ. Thus $J = \ker(\rho)$. ∎

When $\Omega\setminus\{v\}$ is compact, $\Omega'\setminus\{v\}$ is also compact. In this case, the surjection $\rho\colon \mathcal{T}(\Omega,v)\to\mathcal{T}(\Omega',v)$ induces the $*$-homomorphism $\overline{\rho}\colon \mathcal{O}(\Omega,v)\to\mathcal{O}(\Omega',v)$. By Lemma 2.3, the kernel of the surjection $\overline{\rho}$ is also isomorphic to $C_0(\Omega\setminus\Omega')\otimes\mathcal{K}$.

By taking $\Omega' = \{v\}$, we get the following commutative diagram with two exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(\Omega\setminus\{v\}) & \longrightarrow & C_0(\Omega) & \xrightarrow{\mathrm{ev}_v} & \mathbb{C} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle\widehat{\pi}} & & \big\downarrow & & \\
0 & \longrightarrow & C_0(\Omega\setminus\{v\})\otimes\mathbb{K} & \longrightarrow & \mathcal{T}(\Omega,v) & \xrightarrow{\rho} & \mathcal{T} & \longrightarrow & 0
\end{array}.$$

By this diagram, we see that $\mathcal{T}(\Omega,v)$ is a type I C^*-algebra. By Lemma 2.5, $\mathcal{O}(\Omega,v)$ is also type I when $\Omega\setminus\{v\}$ is compact.

PROPOSITION 2.8. *The $*$-homomorphism $\widehat{\pi}\colon C_0(\Omega)\to\mathcal{T}(\Omega,v)$ induces an isomorphism between K-groups.*

Proof. It is well-known that the left and right vertical maps in the diagram above induce isomorphisms on K-groups. Hence the Five Lemma shows that $\widehat{\pi}\colon C_0(\Omega)\to\mathcal{T}(\Omega,v)$ also induces an isomorphism between K-groups. ∎

REMARK 2.9. By using the isomorphism $\mathcal{T}(\Omega,v) = \mathcal{T}(E)$ explained in the proof of Proposition 2.6, Proposition 2.8 follows from Lemma 6.5 of [6] (see also Theorem 4.4 of [9] or Proposition 8.2 of [7]). By Corollary 6.10 of [6] with some computation, we see that the K-groups of the C^*-algebra $\mathcal{O}(\Omega,v) = \mathcal{O}(E)$ are isomorphic to the ones of $C(\Omega\setminus\{v\})$ when $\Omega\setminus\{v\}$ is compact and non-empty.

The next two lemmas are standard, hence we omit the proofs.

LEMMA 2.10. *Let Ω be a compact space, and v,w be points in Ω. The two evaluation maps $\mathrm{ev}_v,\mathrm{ev}_w\colon C(\Omega)\to\mathbb{C}$ induce same maps on K-groups if and only if v and w are in a same connected component.*

LEMMA 2.11. *A projection P of $\mathcal{T}$ defines* 0 *in $K_0(\mathcal{T})$ if and only if P is finite.*

PROPOSITION 2.12. *The C^*-algebra $\mathcal{T}(\Omega,v)$ has an infinite projection if and only if there exists a compact open subset C of Ω containing v.*

Proof. Suppose that there exists a compact open subset C of Ω containing v. Set $U = \widehat{t}(1_C) \in \mathcal{T}(\Omega,v)$. We will show that this partial isometry U satisfies $UU^* < U^*U$. When $\Omega = \{v\}$, the C^*-algebra $\mathcal{T}(\Omega,v)$ is isomorphic to the Toeplitz algebra and U is a proper isometry. Thus $UU^* < U^*U$. Otherwise, we can find $w_1 \in C \setminus \{v\}$ or $w_2 \in \Omega \setminus C$. When there exists $w_1 \in C \setminus \{v\}$, we can find $f \in C_0(\Omega)$ satisfying $0 \leqslant f \leqslant 1$, $f(v) = 1$, $f(w_1) = 0$ and $f(w) = 0$ for $w \in \Omega \setminus C$. Then we have

$$UU^* = \widehat{\pi}(f)UU^*\widehat{\pi}(f) \leqslant \widehat{\pi}(f^2) < \widehat{\pi}(1_C) = U^*U.$$

Hence $UU^* < U^*U$. When there exists $w_2 \in \Omega \setminus C$, we can find $g \in C_0(\Omega)$ satisfying $0 \leqslant g \leqslant 1$, $g(w_2) = 1$, and $g(w) = 1$ for $w \in C$. Then we have $UU^* = \widehat{t}(g)\widehat{\pi}(1_C)\widehat{t}(g)^* \leqslant \widehat{t}(g)\widehat{t}(g)^*$. Take $h \in C_0(\Omega)$ with $h(w_2) = 1$ and $h(w) = 0$ for $w \in C$. Since $UU^*\widehat{t}(h) = 0$ and $\widehat{t}(g)\widehat{t}(g)^*\widehat{t}(h) = \widehat{t}(g\overline{g}h) \neq 0$, we have $UU^* \neq \widehat{t}(g)\widehat{t}(g)^*$. Hence $UU^* < \widehat{t}(g)\widehat{t}(g)^* \leqslant U^*U$. Therefore, if there exists a compact open subset C of Ω containing v then the C^*-algebra $\mathcal{T}(\Omega, v)$ has an infinite projection.

Conversely suppose that there exist no compact open subsets of Ω containing v. To derive a contradiction, we assume that the C^*-algebra $\mathcal{T}(\Omega, v)$ has an infinite projection P. Take $U \in \mathcal{T}(\Omega, v)$ with $U^*U = P$ and $UU^* < P$. Set $Q = UU^*$ and $P_0 = P - Q$. By Lemma 2.10, the evaluation map $\mathrm{ev}_v \colon C_0(\Omega) \to \mathbb{C}$ at $v \in \Omega$ induces 0 map between K-groups. Hence the natural surjection $\rho \colon \mathcal{T}(\Omega, v) \to \mathcal{T}$ also induces 0 map between K-groups, because the three vertical maps in the following commutative diagram induce isomorphisms on K-groups:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(\Omega \setminus \{v\}) & \longrightarrow & C_0(\Omega) & \xrightarrow{\mathrm{ev}_v} & \mathbb{C} & \longrightarrow & 0 \\
 & & \big\downarrow \widehat{\pi} & & \big\downarrow \widehat{\pi} & & \big\downarrow & & \\
0 & \longrightarrow & \ker \rho & \longrightarrow & \mathcal{T}(\Omega, v) & \xrightarrow{\rho} & \mathcal{T} & \longrightarrow & 0
\end{array}.$$

Thus $\rho(P)$ defines 0 in $K_0(\mathcal{T})$. By Lemma 2.11, $\rho(P)$ is finite. Hence we have $\rho(P) = \rho(Q)$. Thus $P_0 \in \ker \rho$. Since the map $\widehat{\pi} \colon C_0(\Omega \setminus \{v\}) \to \ker \rho$ is an isomorphism onto a full corner, there exists a non-zero projection P_0' in $M_n(C_0(\Omega \setminus \{v\}))$ for some $n \in \mathbb{N}$ whose image by $\widehat{\pi}$ defines the same element as P_0 in $K_0(\ker \rho)$. Since every non-zero projection in $M_n(C_0(\Omega))$ defines non-zero elements in $K_0(C_0(\Omega))$, the image of P_0' by the natural embedding $C_0(\Omega \setminus \{v\}) \to C_0(\Omega)$ defines a non-zero element in $K_0(C_0(\Omega))$. This shows that $P_0 \in \mathcal{T}(\Omega, v)$ defines a non-zero element in $K_0(\mathcal{T}(\Omega, v))$. This is a contradiction because $P_0 = U^*U - UU^*$. Therefore if there exist no compact open subsets of Ω containing v, then $\mathcal{T}(\Omega, v)$ has no infinite projections. ∎

REMARK 2.13. The proof of the previous proposition shows that the C^*-algebra $\mathcal{T}(\Omega, v)$ is stably finite when there exist no compact open subsets of Ω containing v. However its unitization has an infinite projection because it has a scaling element. Hence the C^*-algebra $\mathcal{T}(\Omega, v)$ is not quasi-diagonal.

PROPOSITION 2.14. *When $\Omega \setminus \{v\}$ is a non-empty compact set, the C^*-algebra $\mathcal{O}(\Omega, v)$ has an infinite projection.*

Proof. Let U be the image of $\widehat{t}(1_{\{v\}}) \in \mathcal{T}(\Omega, v)$ via the natural surjection $\mathcal{T}(\Omega, v) \to \mathcal{O}(\Omega, v)$. Then similarly as in the proof of Proposition 2.12, we can show that U is a partial isometry with $UU^* < U^*U$. ∎

When $\Omega = \{v\}$, the C^*-algebra $\mathcal{O}(\Omega, v) \cong C(\mathbb{T})$ has no infinite projections.

3. C^*-ALGEBRAS GENERATED BY SCALING ELEMENTS

In the last section, we will show Proposition 0.9 and Theorem 0.10 by using results in the previous section.

Take a scaling element X in some C^*-algebra $\mathfrak{A}$. Set $\Omega = \mathrm{sp}(|X^*|) \setminus \{0\}$ which is a locally compact space. We define an injective $*$-isomorphism $\pi\colon C_0(\Omega) \to \mathfrak{A}$ by $\pi(f) = f(|X|)$ for $f \in C_0(\Omega)$. Take $g \in C_0(\Omega)$ and define $f \in C_0(\Omega)$ by $f(x) = xg(x)$. Then the formula $t(f) = Xg(|X|)$ extends a well-defined linear map $t\colon C_0(\Omega) \to \mathfrak{A}$ satisfying $t(f)^*t(g) = \pi(\overline{f}g)$ and $t(f)\pi(g) = t(fg)$ for $f, g \in C_0(\Omega)$. Since X satisfies $(X^*X)X = X$, we have $\pi(f)t(g) = f(1)t(g)$ for $f, g \in C_0(\Omega)$. Hence the pair (π, t) is an $(\Omega, 1)$-pair. When Ω is compact, we have $t(1_\Omega)t(1_\Omega)^* = 1_\Omega(|X^*|)$. Therefore Proposition 2.6 shows that the C^*-algebra $C^*(X)$ generated by the scaling element X is isomorphic to $\mathcal{T}(\Omega, 1)$ if X is proper, and to $\mathcal{O}(\Omega, 1)$ if X is non-proper. This completes the proof of Proposition 0.9. Theorem 0.10 follows from Proposition 2.12 and Proposition 2.14. We also see that the C^*-algebra $C^*(X)$ is type I and that $\pi\colon C_0(\Omega) \to C^*(X)$ induces an isomorphism between K-groups if X is proper. If a scaling element X is non-proper, the K-groups of the C^*-algebra $C^*(X)$ are isomorphic to the ones of $C(\Omega \setminus \{1\})$.

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