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THE RANGE OF GENERALIZED GELFAND TRANSFORMS ON C^*-ALGEBRAS

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ABSTRACT. It is shown that every Dini function on the primitive ideal space of a C^*-algebra A is the generalized Gelfand transform of an element of A. Here a Dini function on a topological space X means a non-negative lower semi-continuous function f on X with $\sup f\Big(\bigcap_\tau F_\tau\Big) = \inf_\tau \sup f(F_\tau)$ for every downward directed net $\{F_\tau\}_\tau$ of closed subsets of X.

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1. INTRODUCTION AND MAIN RESULT

We have studied in [9] the Dini functions on a T_0 space X.

DEFINITION 1.1. A map $g\colon X \to R_+ := [0,\infty)$ is a *Dini function* on a topological T_0 space X if g satisfies:

(i) g is a non-negative lower semi-continuous function on X, and

(ii) $\sup g\Big(\bigcap_{F\in\mathcal{G}} F\Big) = \inf\{\sup g(F);\, F \in \mathcal{G}\}$ for every downward directed set $\mathcal{G}$ of closed subsets of X.

Here we use the convention $\sup\varnothing = 0$, because we consider only least upper bound of subsets of $\mathbb{R}_+$. $\mathcal{G}$ is downward directed if for $F_1, F_2 \in \mathcal{G}$ there is $F_3 \in \mathcal{G}$ with $F_3 \leqslant F_1 \cap F_2$. Thus our definition implies that X is quasi-compact if X admits a Dini function f with $\inf f(X) > 0$. If the topology of X has a countable base then it suffices to consider decreasing sequences $\mathcal{G} = \{F_1 \supset F_2 \supset \cdots\}$ in (ii).

REMARK 1.2. In [9] we have shown that a *bounded* non-negative lower semi-continuous function $g\colon X \to [0,\infty)$ on a T_0 space X is a Dini function on X if and only if g fulfills the condition: *For every upward directed net $\{f_\tau\}_\tau$ of non-negative lower semi-continuous functions f_τ on X with $g(x) = \sup_\tau f_\tau(x)$ (for every $x \in X$)*

the net $\{f_\tau\}_\tau$ *converges uniformly to* g. (The latter characterizes the continuous functions on compact Hausdorff spaces by a lemma of Dini.)

If X is spectral in the sense of the below given Definitions 1.3, then every Dini function g has the property that for every closed subset $F \subset X$ there is $y \in F$ such that $g(y) = \sup g(F)$. In particular, then g is bounded.

If $g\colon X \to [0,\infty)$ is a bounded Dini function on a T_0 space X such that for every closed subset $F \subset X$ there is $y \in F$ with $g(y) = \sup(F)$, then for every $\gamma > 0$, the G_δ set $g^{-1}[\gamma,\infty) = \{y \in X : g(y) \geqslant \gamma\}$ is quasi-compact.

If $g\colon X \to [0,\infty)$ is a bounded and lower semi-continuous function on a T_0 space X and if $g^{-1}[\gamma,\infty)$ is quasi-compact for every $\gamma > 0$, then g is a Dini function on X.

But there are T_0 spaces X, bounded Dini functions $g\colon X \to [0,\infty)$ and $\gamma > 0$ such that $g^{-1}[\gamma,\infty)$ is not quasi-compact, e.g. $X := \mathbb{P} \cap [0,1]_{\mathrm{lsc}}$, $g\colon t \in X \to t \in [0,1]$ and $\gamma = 2^{-1/2}$. ($[0,1]_{\mathrm{lsc}}$ is defined below.)

The Dini functions on a locally compact Hausdorff space are just the non-negative continuous functions vanishing at infinity (cf. [9]).

DEFINITIONS 1.3. A closed subset $F \neq \emptyset$ of a T_0 space X is *prime* if F is not the union of two closed subsets F_1, F_2 of X both different from F, i.e. F is *not* "decomposable" in the sense of Hausdorff ([6], p. 231). (Here we use a terminology which is adapted to algebras: if $X = \mathrm{Prim}(A)$ then F is prime if and only if F is the hull $h(k(J))$ of the kernel $k(J)$ for a some prime ideal J of A). Since the lattice of closed subsets of X is distributive, a closed subset F of a T_0 space X is prime if $F \subset F_1 \cup F_2$ implies $F \subset F_1$ or $F \subset F_2$ for closed subsets F_1, F_2 of X (thus F is "irreducible" in the sense of Definition 4.9 in [7]).

We call a T_0 space X *spectral* or *point-complete* if every prime closed subset F of X is the closure of a point of X. (The name "spectral space" is used in Definition 4.9 of [7] for point-complete T_0 spaces. Every Hausdorff space is automatically point-complete. But $\mathbb{N}$ with the T_1-topology given by the complements of the finite sets is not point-complete, because $\mathbb{N}$ is a prime closed set.)

Recall that a topological space X is *second countable* if the topology of X has a countable base.

A subset C of a T_0 space X is *quasi-compact* if every open covering $\mathcal{V}$ of C contains a finite subset $\mathcal{V}'$ which is still a covering of C.

We use the following definition of a locally quasi-compact T_0 space:

A T_0 space X is locally quasi-compact *if for every open subset V of X and every point $x \in V$ there is a quasi-compact subset $C \subset X$ such that $C \subset V$ and x is in the interior C° of C.*

The T_0 space $\mathrm{prime}(A)$ of prime ideals of non-separable C^*-algebras A is point-complete and locally quasi-compact, but it is not second countable in general. The space $\mathrm{prime}(A)$ is the "spectral completion" of the T_0 space $\mathrm{Prim}(A)$ of primitive ideals of A, but is in general different from $\mathrm{Prim}(A)$ for non-separable

C^*-algebras, cf. [15]. In the non-separable case the adjoint of the natural map from $\mathrm{Prim}(A)$ into $\mathrm{prime}(A)$ is an isomorphism on the space of lower semi-continuous functions and maps the set of Dini functions on $\mathrm{prime}(A)$ onto the set of Dini functions on $\mathrm{Prim}(A)$, cf. [9]. $\mathbb{N}^\infty \cup \{\infty\}$ with the topology given by the open sets of $\mathbb{N}^\infty$ and the open set $\mathbb{N}^\infty \cup \{\infty\}$ is a quasi-compact, second countable and point-complete T_0 space, but is not locally quasi-compact.

Let $[0,1]_{\mathrm{lsc}}$ denote $[0,1]$ with the T_0 topology given by the system of open sets $\{\emptyset, [0,1], (t,1]; \, t \in [0,1)\}$. The subspace $Z := \mathbb{P} \cap [0,1)$ of rational numbers $\neq 1$ in $[0,1]_{\mathrm{lsc}}$ is second countable and locally quasi-compact, but is not point-complete and has an unbounded lower semi-continuous function $g\colon Z \to [0,\infty)$ with (ii) of Definition 1.1, cf. [9].

DEFINITION 1.4. We call a T_0 space X a *Dini space* if X is point-complete, is second countable and the supports $g^{-1}(0,\infty)$ of the Dini functions $g\colon X \to [0,\infty)$ build a base of the topology of X.

It is well-known (e.g. from [3], [4] and [12], Chapter 4.3) that the T_0 space $X := \mathrm{Prim}(A)$ of primitive ideals of a *separable* C^*-algebra A with the Jacobson topology has the following properties:

(I) There is an open and continuous map from the Polish space P of pure states on A onto X.

(II) The generalized Gelfand transforms $N(a)\colon X \to [0,\infty)$ given by the norm-functions $N(a)(J) := \|a + J\|$ (J primitive ideal of A) of a are lower semi-continuous functions on X, and define the T_0 topology of X by the open sets $N(a)^{-1}(0,\infty)$.

(III) $\sup N(a)\big(\bigcap F_n\big) = \inf_n \sup N(a)(F_n)$, if $F_1 \supset F_2 \supset \cdots$ is a decreasing sequence of closed subsets F_n of X (see e.g. Lemma 3.2).

(II) and (III) show that the functions $N(a)$ are Dini functions on X in the sense of Definition 1.1. Thus (I)–(III) and Lemma 2.2 imply that $\mathrm{Prim}(A)$ is a Dini space in the sense of Definition 1.4.

The above defined space $[0,1]_{\mathrm{lsc}}$ is an example of a Dini space and has only constant continuous functions, but has many Dini functions because it is the primitive ideal space of a unital nuclear C^*-algebra, cf. [9].

The set $\mathcal{D}(X)$ of (bounded) Dini functions on a T_0 space X is closed under maximum $(f,g) \mapsto \max(f,g)$, under uniform convergence, and under compositions $f \mapsto \varphi \circ f$ with continuous increasing functions φ on $[0,\infty)$ with $\varphi(0) = 0$. In general on Dini spaces addition or multiplication of Dini functions are not possible, because each of addition, multiplication and min-operation on $\mathcal{D}(X)$ is equivalent to the property of X that the intersection of two quasi-compact G_δ subsets of X is again quasi-compact, cf. [9]. The C^*-algebra of sequences of complex 2×2-matrices which converge to a diagonal matrix is an example of an AF algebra A where this intersection property does not hold for $X = \mathrm{Prim}(A)$.

If X is a Dini space, then the set of Dini functions is closed with respect to the uniform topology, is separable with respect to the uniform topology, and there is a sequence of Dini functions $g_1, g_2, \dots$ such that their supports build a base of the topology of X and $\sup g_n(X) = 1$.

In [9] we have shown that a spectral T_0 space X is locally quasi-compact if and only if the supports of the Dini functions on X build a base of the topology of X. Thus X is a Dini space if and only if X is point-complete, locally quasi-compact, and second countable.

Our main result is the following theorem. It shows that bounded Dini functions on T_0 spaces are analogs of norm functions on primitive ideal spaces.

THEOREM 1.5. *Suppose that A is a C^*-algebra. Then every Dini function on* $\mathrm{Prim}(A)$ *is the generalized Gelfand transform (norm-function) $N(a)$ of some element $a \in A$.*

The primitive ideal space $\mathrm{Prim}(A)$ *is a Dini space if A is separable.*

The proof of Theorem 1.5 is given in Section 4. It follows that for *every C^**-algebra A the set of norm functions $N(a)$ on $\mathrm{Prim}(A)$ is closed under maximum and contains its uniform limits. Thus, the generalized Gelfand transforms $N(a)$ do not add to primitive ideal spaces $\mathrm{Prim}(A)$ any additional structure that is not automatically defined by the topology of $\mathrm{Prim}(A)$ alone.

The most interesting open question on Dini spaces is the following:

QUESTION 1.6. Is every Dini space X homeomorphic to the primitive ideal space of a separable *nuclear C^**-algebra?

If the open quasi-compact subsets of X build a base of the topology of X then there is an AF algebra A such that $\mathrm{Prim}(A) \cong X$, cf. [2].

See Section 5 for other partial answers and related questions.

2. PRELIMINARIES ON T_0 SPACES

The lemmas and remarks in this section are taken from [9]. The proofs can be found there.

DEFINITIONS 2.1. A subset Z of a T_0 space X is *pseudo-F_σ* if it can be expressed as a union $Z = \bigcup\limits_n Z_n$ of countably many intersections $Z_n = F_n \cap U_n$ of closed subsets F_n and open subsets U_n of X. A subset Z is *pseudo-G_δ* if $X \setminus Z$ is pseudo-F_σ, i.e. if Z can be expressed as an intersection $Z = \bigcap\limits_n Z_n$ of countably many unions $Z_n = F_n \cup U_n$ of closed subsets F_n and open subsets U_n of X.

Recall that a subset Z of X has the *Baire property* if for every sequence of open subsets $U_n \subset X$ with $\overline{U_n \cap Z} \supset Z$ holds $\overline{\left(\bigcap\limits_n U_n\right) \cap Z} \supset Z$, i.e. the intersection of a countable family of open and dense subsets of Z is dense in Z.

LEMMA 2.2. *Suppose that Y is a Polish space, $\psi\colon Y \to X$ is a continuous map into a T_0 space X, and Z is a pseudo-G_δ subset of X provided with the topology inherited from X.*

(i) *The set $\psi^{-1}Z$ is a G_δ subset of Y (and, hence, $\psi^{-1}Z$ is a Polish space with the topology inherited from Y).*

(ii) *If, in addition, ψ is open and $\psi(Y) = X$, then the restriction $\psi|\psi^{-1}Z$ is an open and continuous map from the Polish space $\psi^{-1}Z$ onto Z. Then Z is second countable, has the Baire property and is point-complete.*

LEMMA 2.3. *Let X and Y topological spaces and $\psi\colon Y \to X$ a map from Y onto X. Then ψ is open and continuous if and only if $\overline{\psi^{-1}(Z)} = \psi^{-1}(\overline{Z})$ for every subset $Z \subset X$.*

REMARK 2.4. Let Y and Z be T_0 spaces. We call a map Ψ from the lattice $\mathcal{O}(Y)$ of open subsets of Y into $\mathcal{O}(Z)$ a $\cup$-*preserving* map if $\Psi(U \cup V) = \Psi(U) \cup \Psi(V)$ for all open subsets of U, V of Y. Ψ is called *non-degenerate* if $\Psi(Y) = Z$ and $\Psi(\emptyset) = \emptyset$.

Let f a non-negative bounded lower semi-continuous function on Y. If one denotes by $\chi(U)$ the characteristic function of the open set U of Z or Y, then

$$V_\Psi(f)(z) := \sup\{t\chi(\Psi(f^{-1}(t,\infty)))(z);\ t > 0\}$$

defines obviously a non-negative bounded lower semi-continuous map $V_\Psi(f)$ on Z.

It is easy to check that the map $V = V_\Psi$, from the bounded non-negative lower semi-continuous functions $\mathrm{BLSC}_+(Y)$ on Y into $\mathrm{BLSC}_+(Z)$, satisfies the following conditions (i)–(v) for every $f, g \in \mathrm{BLSC}_+(X)$ and $t > 0$:

(i) $V(1) = 1$,
(ii) $V(\max(f,g)) = \max(V(f), V(g))$,
(iii) $V(f^2) = V(f)^2$,
(iv) $V(tf) = tV(f)$ and
(v) $V((f-t)_+) = (V(f)-t)_+$.

(i)–(v) imply $V(g) \leqslant V(h)$ for $g \leqslant h$, $\min(V(f), V(g)) \geqslant V(\min(f,g))$ and $V(f+g) \leqslant V(f) + V(g)$. In particular, $\|V(f) - V(g)\| \leqslant \|f - g\|$. By (i)–(v), for every continuous increasing function φ on $[0,\infty)$ with $\varphi(0) = 0$ it follows that $V(\varphi \circ f) = \varphi \circ V(f)$.

Conversely, if a map V from $\mathrm{BLSC}_+(Y)$ into $\mathrm{BLSC}_+(Z)$ with (i)–(v) is given, then V determines a non-degenerate and $\cup$-preserving map $\Psi_V\colon \mathcal{O}(Y) \to \mathcal{O}(Z)$ with $V(\chi(U)) = \chi(\Psi_V(U))$. We have $V = V_{\Psi_V}$ and $\Psi_{V_\Psi} = \Psi$.

LEMMA 2.5. *Suppose that Y and X are T_0 spaces, that $\psi\colon Y \to X$ is a continuous map, which is an open map from Y onto the subspace $\psi(Y)$. We define $V(f) := \widehat{f} :=$* $\sup f(\psi^{-1}(x))$ *for bounded lower semi-continuous non-negative functions f on Y.*

(i) *The function $\widehat{f}$ is lower semi-continuous on $\psi(Y)$, and* $\sup f(Y) = \sup \widehat{f}(\psi(Y))$.

(ii) *The map* $V\colon f \mapsto \widehat{f}$ *satisfies* $V(1) = 1$, $V(\max(f,g)) = \max(V(f), V(g))$ *and* $V(h(f)) = h(V(f))$ *for every increasing continuous function* h *on* $[0,\infty)$.

(iii) *In particular,* V *is order-preserving.*

(iv) $\widehat{g \circ \psi} = g|\psi(Y)$ *for every lower semi-continuous function* $g\colon X \to [0,\infty)$.

3. NORM FUNCTIONS $N(a)$ ON $\mathrm{Prim}(A)$

We want to identify the Dini functions on the primitive ideal space $\mathrm{Prim}(A)$ of a C^*-algebra A. Some lemmas are needed to prove that every Dini function on $\mathrm{Prim}(A)$ is a generalized Gelfand transform of an element of A.

REMARK 3.1. Recall that the space $P(A)$ of pure states of a C^*-algebra A is a Polish space if A is separable. The natural epimorphism $P(A) \to \mathrm{Prim}(A)$ from $P(A)$ onto $X := \mathrm{Prim}(A)$ is open and continuous (even if A is not separable), cf. Theorem 3.4.11 of [4] and Chapter 4.3 of [12]. Here $P(A)$ has the $\sigma(A^*, A)$-topology, and the set $\mathrm{Prim}(A)$ of kernels of irreducible representations of A carries the hull-kernel topology of Jacobson.

The norm-function $N(a)$ on $\mathrm{Prim}(A)$ for $a \in A$ is defined by

$$N(a)(J) := \|a + J\| := \inf_{b\in J} \|a + b\|$$

for primitive ideals $J \in \mathrm{Prim}(A)$ of A (i.e. kernels of irreducible representations). The map $a \in A_+ \to N(a) \in \mathrm{BLSC}_+(\mathrm{Prim}(A))$ generalizes the Gelfand transform on commutative C^*-algebras.

The definition of the topology on $\mathrm{Prim}(A)$ shows immediately that there is an obvious order-preserving one-to-one relation between open subsets Z of $\mathrm{Prim}(A)$ and closed ideals $I_Z := k(\mathrm{Prim}(A) \setminus Z)$ of A. If F is a closed subset of $X := \mathrm{Prim}(A)$, then $I_{X\setminus F} = k(F)$ is the intersection of the $J \in F$ and $\sup\{N(a)(J);\, J \in F\} = \|a + I_{X\setminus F}\|$.

If one considers for $a \in A_+$ the non-negative continuous function $f_a := \check{a}(\rho) := \rho(a)$ $(\rho \in P(A))$ on $P(A)$, then $N(a) = \widehat{f_a}$, where $f \in \mathrm{BLSC}_+(P(A)) \mapsto \widehat{f} \in \mathrm{BLSC}_+(X)$ is defined as in Lemma 2.5 for the open and continuous epimorphism $P(A) \to X$.

LEMMA 3.2. *Suppose that* A *is a* C^**-algebra and that* U *is an open subset of* $\mathrm{Prim}(A)$*. Then*

(i) $N(a) = N(c)$ *for* $c := (a^*a)^{1/2}$*, and*

(ii) $N(\varphi(b)) = \varphi(N(b))$ *for every increasing continuous function* φ *on* $[0,\infty)$ *with* $\varphi(0) = 0$ *if* $b \in A_+$.

(iii) *Every generalized Gelfand transformation* $N(a)\colon J \mapsto \|a + J\|$ *is a Dini function with* $\sup N(a)(\mathrm{Prim}(A)) = \|a\|$*. For every closed subset* F *of* $\mathrm{Prim}(A)$ *there is* $J \in F$ *with* $N(a)(J) = \sup N(a)(F)$*.* $N(a)^{-1}[\gamma,\infty)$ *is quasi-compact for every* $\gamma > 0$.

(iv) *There is* $a \in A_+$ *such that* U *is the support* $N(a)^{-1}(0,\infty)$ *of* $N(a)$ *if and only if* U *is the union of a countable sequence of quasi-compact subsets of* U*. (The latter is the case for all open subsets* U *of* $\mathrm{Prim}(A)$ *if* A *is separable.)*

(v) *For every bounded Dini function* $f\colon \mathrm{Prim}(A) \to [0,\infty)$ *there is* $e \in A_+$ *with same support, i.e.* $N(e)^{-1}(0,\infty) = f^{-1}(0,\infty)$*. (In particular, the support of* f *is the union of a countable sequence of quasi-compact subsets.)*

(vi) *The space* $\mathrm{Prim}(A)$ *is locally quasi-compact. It is a Dini space if* A *is in addition separable.*

Proof. (i) $N(a) = N(c)$ for $c := (a^*a)^{1/2}$ because the semi-norms $a \mapsto \|a + J\|$ have the C^*-property.

(ii) $\varphi(b) + J = \varphi(b + J)$ and $\|\varphi(b + J)\| = \varphi(\|b + J\|)$ for $b \in A_+$ if $\varphi(0) = 0$, φ is increasing and continuous.

(iii) $N(b)$ is lower semi-continuous for $b \in A_+$, because $N(b)^{-1}(t,\infty) = N((b-t)_+)^{-1}(0,\infty)$ is the open subset of $\mathrm{Prim}(A)$ which corresponds to the closed ideal J of A generated by $(b-t)_+$ for $t \in [0,\infty)$.

Thus $N(a) = N((a^*a)^{1/2})$ is lower semi-continuous for every $a \in A$.

$\sup N(a)\left(\bigcap_\tau F_\tau\right) = \inf_\tau \sup N(a)(F_\tau)$ for every decreasing net of closed subsets F_τ of X, because this is equivalent to the obvious identities $\left\|a + \overline{\bigcup_\tau J_\tau}\right\| = \left\|a + \bigcup_\tau J_\tau\right\| = \inf_\tau \|a + J_\tau\|$ for the increasing net $\{J_\tau\}$ of closed ideals J_τ of A corresponding to the complements $X \setminus F_\tau$ of the closed sets F_τ.

The rest follows from Lemma 3.3.6 and Proposition 3.3.7 of [4], but we give a proof based on Remark 1.2 as follows:

If F is a closed subset of $\mathrm{Prim}(A)$ and if $I := I_U = k(F)$ is the closed ideal of A corresponding to $\mathrm{Prim}(A) \setminus F$, then $\sup N(a)(F) = \|\pi_I(a)\|$. There is an irreducible representation $d\colon A/I \to \mathcal{L}(\mathcal{H})$ such that $\|d(a)\| = \|\pi_I(a)\|$ (e.g. the GNS construction for a pure state ρ on A which is a Hahn–Banach extension of a character χ on $C^*(\pi_I(a^*a))$ with $\rho(\pi_I(a^*a)) = \chi(\pi_I(a^*a)) = \|\pi_I(a^*a)\|$). $J := (\pi_I)^{-1}(K)$ (for the kernel K of d) is a primitive ideal of A with $J \in F$ and $N(a)(J) = \|a + J\| = \|\pi_J(a)\| = \sup N(a)(F)$.

By Remark 1.2, $N(a)^{-1}[\gamma,\infty)$ is quasi-compact for every $\gamma > 0$.

(iv) Let I be the intersection of the primitive ideals $J \in X \setminus U$. If there is $a \in A$ such that $N(a)$ has U as its support, then U is the union of the sequence of the sets $C_n := N(a)^{-1}[1/n,\infty)$, which are quasi-compact by (iii).

If, in addition, A is separable, then there exists a strictly positive element $a \in I_+$, e.g. $a := \sum_n 2^{-n} b_n^* b_n$ for a dense sequence $b_1, b_2, \ldots$ in the unit ball of I. Then $N(a)(J) > 0$ for every primitive ideal $J \in U$, because this is equivalent to $I \not\subset J$.

If A is not separable, but if U is the union of a sequence of quasi-compact sets $C_1, C_2, \ldots \subset U$, then for every $n \in \mathbb{N}$ and every $J \in C_n$ there are contractions

$b_{n,J} \in I$ with $N(b_{n,J})(J) > 0$. Since the supports of $N(b_{n,J})$ are open and since C_n is quasi-compact, there is a sequence of contractions $b_1, b_2, \ldots \in I$ such that for every point $J \in U$ there is $n \in \mathbb{N}$ with $N(b_n)(J) > 0$. The support $N(a)^{-1}(0, \infty)$ of $N(a)$ equals U for $a := \sum_n 2^{-n} b_n^* b_n$, because $a \in I$ and $2^n N(a) \geqslant N(b_n)^2$ for every $n \in \mathbb{N}$.

(v) Let $\mathcal{G}$ denote the set of all functions $g\colon \mathrm{Prim}(A) \to [0, \infty)$ with $g \leqslant f$ and the property that there exist $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in A$ such that $g = \max(N(a_1), N(a_2), \ldots, N(a_n))$. Then $\mathcal{G}$ is an upward directed net of lower semi-continuous functions. For $J \in \mathrm{Prim}(A)$ with $f(J) =: \eta > 0$ and $\varepsilon \in (0, \eta/2)$ let $U := f^{-1}(\eta - \varepsilon, \infty)$ and $I_U := k(\mathrm{Prim}(A) \setminus U)$. Then $J \in U$, $I_U \not\subset J$, and there is $b \in (I_U)_+ \setminus J$ with $N(b)(J) = \|b + J\| = \delta > 0$. Let $\varphi(t) := \min(t, \delta)$ and $a := ((\eta - \varepsilon)/\delta)\varphi(b)$. The $g := N(a)$ satisfies $g(J) = \eta - \varepsilon$ and $g \leqslant (\eta - \varepsilon)\chi_U \leqslant f$. Thus $\mathcal{G}$ converges point-wise to f. Since f is bounded and Dini, by Remark 1.2 there are $g_n \in \mathcal{G}$ with $f - 1/n \leqslant g_n \leqslant f$ for $n = 1, 2, \ldots$. There are $b_{n,1}, \ldots, b_{n,m(n)} \in A$ with $g_n = \max(N(b_{n,1}), \ldots, N(b_{n,m(n)}))$. Thus $b_{n,j} \in I_V$ for $n \in \mathbb{N}$, $j \in \{1, \ldots, m(n)\}$, and the $b_{n,j}$ all together generate I_V as a closed ideal of A, where I_V is the closed ideal corresponding to the support $V := f^{-1}(0, \infty)$ of f.

Let $a_n := b_{n,1}^* b_{n,1} + \cdots + b_{n,m(n)}^* b_{n,m(n)}$ and $e := \sum_n (2^n \|a_n\|)^{-1} a_n$. Then we have $N(e)^{-1}(0, \infty) = \mathrm{Prim}(A) \setminus h(I)$ for the closed ideal I generated by $\{e\}$, and I is equal to the closed ideal I_V generated by $\{b_{n,j};\ n \in \mathbb{N}, 1 \leqslant j \leqslant m(n)\}$. Hence $N(e)^{-1}(0, \infty) = f^{-1}(0, \infty)$.

(vi) The supports of the Dini functions $N(a)$ on $\mathrm{Prim}(A)$ build a base of the hull-kernel topology by (iii) and (iv). This implies that $\mathrm{Prim}(A)$ is locally quasi-compact: If $N(a)(J) > \delta > 0$ then the open neighborhood $N(a)^{-1}(\delta, \infty)$ of J is contained in the quasi-compact set $C := N(a)^{-1}[\delta, \infty)$, and C is contained in the support of $N(a)$.

If A is separable, then $\mathrm{Prim}(A)$ is point-complete and second countable by Lemma 2.2, because the natural map from the Polish space $P(A)$ onto $\mathrm{Prim}(A)$ is continuous and open, cf. Chapter 4.3 of [12]. ∎

LEMMA 3.3. *Suppose that A is a C^*-algebra and that $a, b, c \in A_+$ satisfy $\|a\| \leqslant 1$, $\|b\| \leqslant 1$, $bc = c$ and $\|ab - b\| < \varepsilon$. Then $\|a\| + \|c\| - \varepsilon < \|a + c\|$.*

Proof. Suppose $\|c\| > 0$, and extend a character χ on $C^*(b, c)$ with $\chi(c) = \|c\|$ to a state ρ on the unitization of A. Then $\rho(c) = \rho(bc) = \rho(b)\rho(c)$, thus $\rho(b) = 1$, and $\rho(ab) = \rho(a)\rho(b) = \rho(a)$, which gives $|1 - \rho(a)| < \varepsilon$.

It follows $\|a\| + \|b\| - \varepsilon \leqslant 1 - \varepsilon + \|b\| < \rho(a) + \rho(b) \leqslant \|a + b\|$. ∎

In the following let $N(a)$ be the generalized Gelfand transform of a in a C^*-algebra A.

LEMMA 3.4. *Suppose that A is a C^*-algebra, f is a Dini function on* $\mathrm{Prim}(A)$ *and that $g_1, g_2, d \in A_+$ are positive contractions with $g_2 g_1 = g_1$, $N(g_1) \leqslant f \leqslant N(g_2)$, $d \in \overline{g_1 A g_1}$.*

Let J denote the closed ideal of A corresponding to the support $f^{-1}(0,\infty)$ of f.

Then for every $\delta > 0$ there is a positive contraction $e = e_\delta \in J \cap \overline{g_2 A g_2}$ with $(1-\delta) g_1 \leqslant e$, $(1-\delta) d \leqslant e$, $\|ed - d\| < \delta$ and $(f-\delta)_+ \leqslant N(e)$.

Proof. Let $X := \mathrm{Prim}(A)$ and $m \in \mathbb{N}$ with $m \geqslant 1/\delta^2$, and $U := f^{-1}(0,\infty) \in \mathcal{O}(X)$. Since $f \leqslant N(g_2)$, the intersection D of the ideal J (corresponding to the support U of f) with the hereditary C^*-subalgebra $\overline{g_2 A g_2}$ is full in J, i.e. $\overline{\mathrm{span}(ADA)} = J$. The element g_1 is in D, because $g_2 g_1 = g_1$ and $N(g_1) \leqslant f$, i.e. because $N(g_1)^{-1}(0,\infty) \subset U$. Thus $g_1, d \in \overline{g_1 A g_1} \subset D$.

Now we use the natural isomorphism $\mathrm{Prim}(D) \cong U$ given by $J \in U \mapsto J \cap D \in \mathrm{Prim}(D)$. By Lemma 3.2(v), there is a positive element $k \in D_+$ with $\|k\| = 1/2$ such that $N(k)^{-1}(0,\infty) = U$.

By the proof of Theorem 1.4.2 in [12], there is a contraction $h \in D_+$ with $(1-\delta) g_1 \leqslant h$, $(m/(m+1)) d^{1/m} \leqslant h$ and $k \leqslant h$.

Then $N(h)^{-1}(0,\infty) = U$, $(1-\delta) g_1 \leqslant h^{1/n}$ and $\|h^{1/n} d - d\| < \delta$ for all $n \in \mathbb{N}$, because $h \leqslant h^{1/n} \leqslant 1$ and

$$\|h^{1/n} d - d\|^2 \leqslant \|d^{1/2}(1-h) d^{1/2}\| \leqslant \left\| d - \frac{m}{m+1} d^{1/m} d \right\| \leqslant \frac{1}{m+1} < \delta^2 .$$

Furthermore, $\min(N(h^{1/n}), f)$ is an increasing sequence of lower semi-continuous functions on X which converges point-wise to f, because $f \leqslant 1$, $N(h^{1/n}) = N(h)^{1/n}$ and $N(h)^{-1}(0,\infty) = f^{-1}(0,\infty)$. By Remark 1.2 on Dini functions, there is $n \in \mathbb{N}$ such that $f - \delta \leqslant N(h^{1/n})$. The element $e := h^{1/n}$ is as desired. ∎

LEMMA 3.5. *Suppose that $f_1, f_2, \ldots, f_n$ are Dini functions on* $\mathrm{Prim}(A)$ *with norm $\leqslant 1$ such that $f_{k+1} f_k = f_k$ for $k = 1, \ldots, n-1$, and that $a_1, \ldots, a_n$ are positive contractions in A_+ with $a_{k+1} a_k = a_k$ and $N(a_k) \leqslant f_k$ for $k = 1, \ldots, n$, and that there is $m < n$ such that $f_j \leqslant N(a_{j+m})$ for $j = 1, \ldots, n-m$. Let $\delta > 0$ fixed.*

There are positive contractions b_k and d_k in A_+ with the following properties:

(i) *$b_k \in J_k \cap \overline{a_{k+m} A a_{k+m}}$ for $k = 1, \ldots, n-m$, $b_k \in J_k$ for $k = n-m+1, \ldots, n-1$, where J_k is the closed ideal of A corresponding to the support $f_k^{-1}(0,\infty)$ of f_k.*

(ii) *$\|b_k d_{k-1} - d_{k-1}\| < \delta$ for $k > 1$.*

(iii) *$(f_k - \delta)_+ \leqslant N(b_k)$.*

(iv) *$(1-\delta) a_k \leqslant b_k$.*

(v) *$d_k (b_k + b_{k-1} + \cdots + b_1 - \delta)_+ = (b_k + b_{k-1} + \cdots + b_1 - \delta)_+$, and*

(vi) *$d_k \in J_k \cap \overline{a_{k+m} A a_{k+m}}$ for $k = 1, \ldots, n-m$, and $d_k \in J_k$ for $k = n-m+1, \ldots, n$.*

The elements $b := b_{n-1} + \cdots + b_1$, $a := a_n + \cdots + a_1$ and the function $f := f_n + \cdots + f_1$ satisfy

$$(1-\delta)(a-1)_+ \leqslant b \leqslant a + m \quad \textit{and} \quad (f-1)_+ - 3n\delta \leqslant N(b) \leqslant f .$$

Proof. For $k = 1$ let $d_0 := 0$. By Lemma 3.4 there is $b_1 \in J_1 \cap \overline{a_{1+m}Aa_{1+m}}$ with $(f_1 - \delta)_+ \leqslant N(b_1)$ and $(1-\delta)a_1 \leqslant b_1$: consider $f_1, a_1, a_{1+m}, 0$ in place of f, g_1, g_2, d in Lemma 3.4. Let $d_1 := \delta^{-1}(b_1 - (b_1 - \delta)_+)$, then (i)–(vi) are satisfied for b_1 and d_1.

Suppose $b_1, \dots, b_k$ and $d_1, \dots, d_k$ have been found with (i)–(vi).

Lemma 3.4 applies to $f_{k+1}, a_{k+1}, a_{k+m+1}, d_k$, if $k < n-m$, and to $f_{k+1}, a_{k+1}, 1_{\mathcal{M}(A)}, d_k$, if $k \geqslant n-m$. It gives b_{k+1} with (i)–(iv). (Note here that $\mathrm{Prim}(A)$ is an open subspace of $\mathrm{Prim}(\mathcal{M}(A))$ and that f_{k+1} is also a Dini function on $\mathrm{Prim}(\mathcal{M}(A))$.)

Then $c := b_{k+1} + \cdots + b_1$ is in $J_{k+1} \cap \overline{a_{k+1+m}Aa_{k+1+m}}$ for $k < n-m$, and is in J_{k+1} for $k \geqslant n-m$. Thus $d_{k+1} := \delta^{-1}(c - (c-\delta)_+)$ satisfies (v) and (vi).

The inequality $(1-\delta)(a-1)_+ \leqslant b := b_{n-1} + \cdots + b_1$ follows from (iv), because $(a-1)_+ = a_{n-1} + \cdots + a_1$.

Since $a_{k+1}a_k = a_k$ and b_k is a contraction in $\overline{a_{k+m}Aa_{k+m}}$ by (i), we have $b_k \leqslant a_{k+1+m}$ for $k < n-m$. Thus $b \leqslant m + b_{n-m-1} + \cdots + b_1 \leqslant a + m$.

By (ii) and Lemma 3.3 we have

$$N(b_{k+1} + ((b_k + \cdots + b_1) - \delta)_+) \geqslant N(b_{k+1}) + (N(b_k + \cdots + b_1) - \delta)_+ - \delta\,.$$

Since $N\colon A_+ \to \mathrm{BLSC}_+(\mathrm{Prim}(A))$ is order-monotone, it follows

$$N(b_{n-1}) + \cdots + N(b_1) - 2(n-2)\delta \leqslant N(b) \leqslant N(b_{n-1}) + \cdots + N(b_1)$$

and $(f-1)_+ - 3n\delta \leqslant N(b)$ by (iii), because $(f-1)_+ = f_{n-1} + \cdots + f_1$.

It holds $N(b_k) \leqslant f_{k+1}$, because $f_{k+1}f_k = f_k$ and the support of $N(b_k)$ is contained in the support of f_k by (i). Thus also $N(b) \leqslant f$. ∎

4. PROOF OF THEOREM 1.5

First let g a be *bounded* Dini function on $X := \mathrm{Prim}(A)$. We can suppose that $\sup g(X) = 1$.

We show that for $c \in A_+$ and $t \in (0,1]$ with $N(c) \leqslant g \leqslant N(c) + t$ there is $e \in A_+$ with $N(e) \leqslant g \leqslant N(e) + t/2$ and $c - t/2 \leqslant e \leqslant c + 3t/2$. Then one gets by induction a convergent sequence $a_0 = 0, a_1, a_2, \ldots \in A_+$ with $N(a_n) \leqslant g \leqslant N(a_n) + 2^{-n}$ and $a_n - 2^{-n-1} \leqslant a_{n+1} \leqslant a_n + 2^{-n-1}3$.

The existence of e reduces to Lemma 3.5 as follows:

Take $n \in \mathbb{N}$ and $\delta > 0$ such that $n > 4/t$ and $\delta < t/12$. Let $m \leqslant n$ denote the smallest integer $\geqslant nt$.

We use the continuous increasing functions from $[0,1]$ into $[0,1]$ given by $\varphi_1(t) := (nt - (n-1))_+$ and $\varphi_k(t) := (nt - (n-k))_+ - (nt - (n-k+1))_+$ (It is indexed from top to bottom.)

We have $N(\varphi_k(c)) = \varphi_k(N(c))$. Thus n, m, $a_k := \varphi_k(c)$, $f_k := \varphi_k(g)$ for $k = 1, \ldots, n$ satisfy the assumptions of Lemma 3.5. It holds $a = nc$ and $f = ng$.

Let $e := n^{-1}b$. Then $(1-\delta)(c-1/n)_+ \leqslant e \leqslant c+m/n$ and $(g-1/n)_+ - 3\delta \leqslant N(e) \leqslant g$.

Thus e is as desired, by the choice of n and δ.

Now we show that *every* Dini function $g\colon X := \mathrm{Prim}(A) \to [0,\infty)$ is *bounded*.

Let $\psi(t) := t/(1+t)$ for $t \in [0,\infty)$ and $\psi(\infty) := 1$. The function ψ is continuous on $[0,\infty]$, strictly increasing, $\psi(0) = 0$, $\psi(\sup Z) = \sup \psi(Z)$, and $\psi(\inf Z) = \inf \psi(Z)$ for every subset $Z \subset [0,\infty]$. It follows that $\psi \circ g$ is a bounded Dini function on X and $\sup \psi \circ g(X) \leqslant 1$. Thus there is $a \in A$ with $\psi \circ g = N(a)$ and $\|a\| \leqslant 1$. Since g has values in $[0,\infty)$, there is no $J \in X$ with $\|a+J\| = 1$. Thus $\|a\| < 1$ and $\sup g(X) \leqslant \psi^{-1}(\|a\|) = \|a\|/(1-\|a\|) < \infty$. Which ends the proof of Theorem 1.5. ∎

5. REMARKS AND QUESTIONS ABOUT PRIMITIVE IDEAL SPACES

QUESTION 5.1. Is every Dini space (at least) the primitive ideal space of a separable C^*-algebra?

Recent joint works with H. Harnisch and M. Rørdam give a partial answer. They show that the following properties (I)–(IV) of a point-complete second countable T_0 space X are equivalent. (Note for the following that $\mathcal{F}(Y)$ means the lattice of closed subsets of a topological space Y. The greatest lower bound (g.l.b., inf) of a family in the lattice $\mathcal{F}(Y)$ is simply the intersection of the closed sets in the family, and the least upper bound (l.u.b., sup) is the closure of the union of the sets in the family.)

(I) X is isomorphic to the primitive ideal space of a separable *nuclear* C^*-algebra.

(II) $\mathcal{F}(X)$ is lattice-isomorphic to a sub-lattice $\mathcal{G}$ of $\mathcal{F}(Y)$ which is closed under forming of l.u.b. and g.l.b. for some *locally compact* Polish space Y. Equivalently, this means that there is a map Ψ from the open subsets $\mathcal{O}(X)$ of X into the open subsets $\mathcal{O}(Y)$ of Y with following properties (i)–(iv):

(i) $\Psi(\bigcup_\tau U_\tau) = \bigcup_\tau \Psi(U_\tau)$.

(ii) $\Psi\Big(\big(\bigcap_\tau U_\tau\big)^\circ\Big) = \Big(\bigcap_\tau \Psi\big(U_\tau\big)\Big)^\circ$. ($Z^\circ$ denotes the interior of Z.)

(iii) $\Psi(X) = Y$, $\Psi(\emptyset) = \emptyset$.

(iv) $\Psi(U) = \Psi(V)$ implies $U = V$.

(III) $\mathcal{F}((0,1]_{\mathrm{lsc}} \times X)$ is (in a lattice sense) the projective limit of $\mathcal{F}(P_n \setminus \{q_n\})$ for pointed finite one-dimensional polyhedra (P_n, q_n). With $Y_n = P_n \setminus \{q_n\}$ the connecting maps $\Phi_n\colon \mathcal{F}(P_{n+1} \setminus \{q_{n+1}\}) \to \mathcal{F}(P_n \setminus \{q_n\})$ satisfy:

(i) $\Phi_n\Big(\overline{\bigcup_\tau F_\tau}\Big) = \overline{\bigcup_\tau \Phi_n(F_\tau)}$ for every family $\{F_\tau\}_\tau$ of closed subsets in $\mathcal{F}(Y_{n+1})$, (ii*) $\Phi_n\Big(\bigcap_k F_k\Big) = \bigcap_k \Phi_n(F_k)$ for every *decreasing* sequence $F_1 \supset F_2 \supset \cdots$ in $\mathcal{F}(Y_{n+1})$, and

(iii) $\Phi_n(Y_{n+1}) = Y_n\,,\ \Phi_n(\emptyset) = \emptyset$.

(IV) There are a locally compact Polish space Y and a continuous map $\varphi\colon Y \to X$ such that, for closed subset $F \subset G$ of X with $F \neq G$, the set $G \setminus F$ contains a point of $\varphi(Y)$, and that

$$\overline{\bigcup_n \varphi^{-1}(F_n)} = \varphi^{-1}\Big(\overline{\bigcup_n F_n}\Big)$$

for every increasing sequence of closed subsets of X.

One can show (with the methods of [9]) that every Dini space X is the image of an open and continuous map φ from a Polish space Y onto X. Then $F \in \mathcal{F}(X) \to \varphi^{-1}F \in \mathcal{F}(Y)$ defines an complete order isomorphism onto an sup- and inf-closed sublattice of $\mathcal{F}(Y)$. Unfortunately, our construction gives in general not a locally compact space Y. (But we know that $[0,1]_{\mathrm{lsc}}$ is a continuous and open image of the Hilbert cube $[0,1]^\infty$. The map can be defined by a suitable increasing family $\{C_t\,;\, t \in [0,1]\}$ of compact convex subsets of the Hilbert space.)

REMARK 5.2. (i) If Ω is a closed subset of $[0,1]$ with $0,1 \in \Omega$, then Ω and $\Omega \setminus \{0\}$ considered as subspaces of $[0,1]_{\mathrm{lsc}}$ are primitive ideal spaces of separable nuclear C^*-algebras A in the UCT class, as follows from [11], or even of a C^*-algebra A, that is an inductive limit of $C_0((0,1], M_{2^n})$, $n = 1,2,\ldots$, cf. [13]. One could construct also a suitable Cuntz–Pimsner algebra, by the above mentioned general result.
(ii) Another explicit construction of an A with $\mathrm{Prim}(A) \cong \Omega_{\mathrm{lsc}}$ goes as follows: $C(\Omega)$ is a subalgebra of the Cantor algebra $C(\{0,1\}^\infty) \subset M_{2^\infty} \subset \mathcal{O}_2$. Let $h\colon C(\Omega) \to C(\Omega \times \Omega) \subset C(\Omega) \otimes B$ for $B = M_{2^\infty}$ or $B = \mathcal{O}_2$ and $h(f)(s,t) := f(\min(s,t))$. There is a unital isomorphism $\iota\colon B \otimes B \hookrightarrow B$. $\varphi := (\mathrm{id}_{C(\Omega)} \otimes \iota) \circ (h \otimes \mathrm{id}_B)$ is a unital endomorphism of $D := C(\Omega) \otimes B$. The inductive limit A of $\varphi^n\colon D \to D$ has primitive ideal space $\Omega \subset [0,1]_{\mathrm{lsc}}$, as one can easily see.

QUESTION 5.3. Does there exist (up to homeomorphisms) a Dini space X_∞ which contains (up to homeomorphisms) every other Dini space as a *closed* subspace of X_∞?

Every primitive ideal space of a separable C^*-algebra is a closed subspace of the primitive ideal space $\mathrm{Prim}(J)$ of the kernel J of the trivial character on the full group C^*-algebra $C^*(F_2)$ of the free group F_2 on two generators.

QUESTION 5.4. Suppose that X is a second countable T_0 space, and that *every* pseudo-G_δ subset of X satisfies the Baire property. Is there an open and continuous map from a Polish space onto X? (The converse is trivial, see Lemma 2.2(ii). There are non-Polish second countable metrizable Hausdorff spaces X with Baire property, as follows from capacity theory.)

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