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FACTORIZATION OF A CLASS OF TOEPLITZ + HANKEL OPERATORS AND THE A_p-CONDITION

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ABSTRACT. Let $M(\phi) = T(\phi) + H(\phi)$ be the Toeplitz plus Hankel operator acting on $H^p(\mathbb{T})$ with generating function $\phi \in L^\infty(\mathbb{T})$. In a previous paper we proved that $M(\phi)$ is invertible if and only if ϕ admits a factorization $\phi(t) = \phi_-(t)\phi_0(t)$ such that ϕ_- and ϕ_0 and their inverses belong to certain function spaces and such that a further condition formulated in terms of ϕ_- and ϕ_0 is satisfied. In this paper we prove that this additional condition is equivalent to the Hunt-Muckenhoupt-Wheeden condition (or, A_p-condition) for a certain function σ defined on $[-1, 1]$, which is given in terms of ϕ_0. As an application, a necessary and sufficient criteria for the invertibility of $M(\phi)$ with piecewise continuous function ϕ is proved directly. Fredholm criteria are obtained as well.

KEYWORDS: *Toeplitz operator, Hankel operator, factorization, A_p-condition.*

MSC (2000): 47B35, 47A68.

1. INTRODUCTION

This paper is devoted to continuing the study (started in [2]) of operators of the form

$$M(\phi) = T(\phi) + H(\phi) \tag{1.1}$$

acting on the Hardy space $H^p(\mathbb{T})$ where $1 < p < \infty$. Here $\phi \in L^\infty(\mathbb{T})$ is a Lebesgue measurable and essentially bounded function on the unit circle $\mathbb{T}$. The Toeplitz and Hankel operators are defined by

$$T(\phi) : f \mapsto P(\phi f), \quad H(\phi) : f \mapsto P(\phi(Jf)), \quad f \in H^p(\mathbb{T}), \tag{1.2}$$

where is J the following flip operator,

$$J : f(t) \mapsto t^{-1} f(t^{-1}), \quad t \in \mathbb{T}, \tag{1.3}$$

acting on the Lebesgue space $L^p(\mathbb{T})$. The operator P stands for the Riesz projection,

$$(1.4) \qquad P : \sum_{n=-\infty}^{\infty} f_n t^n \mapsto \sum_{n=0}^{\infty} f_n t^n, \quad t \in \mathbb{T},$$

which is bounded on $L^p(\mathbb{T})$, $1 < p < \infty$, and whose image is $H^p(\mathbb{T})$. The complex conjugate Hardy space $\overline{H^p(\mathbb{T})}$ is the set of all functions f whose complex conjugate belongs to $H^p(\mathbb{T})$. Moreover, we denote by $L^p_{\text{even}}(\mathbb{T})$ the subspace of $L^p(\mathbb{T})$ consisting of all even functions, i.e., functions f for which $f(t) = f(t^{-1})$.

For $\phi \in L^\infty(\mathbb{T})$ we denote by $L(\phi)$ the multiplication operator acting on $L^p(\mathbb{T})$,

$$(1.5) \qquad L(\phi) : f(t) \mapsto \phi(t) f(t).$$

Obviously, $T(\phi)$ and $H(\phi)$ can be written as

$$T(\phi) = PL(\phi)P|_{H^p(\mathbb{T})}, \quad H(\phi) = PL(\phi)JP|_{H^p(\mathbb{T})}.$$

In a previous paper [2] we proved that for $\phi \in L^\infty(\mathbb{T})$ the operator $M(\phi)$ is a Fredholm operator on the space $H^p(\mathbb{T})$ if and only if the functions ϕ admits a certain kind of generalized factorization. Before recalling the underlying definitions, let us state the following simple necessary condition for the Fredholmness of $M(\phi)$ which was also established in Proposition 2.2 of [2]. Therein $GL^\infty(\mathbb{T})$ stands for the group of all invertible elements in the Banach algebra $L^\infty(\mathbb{T})$.

PROPOSITION 1.1. *Let $1 < p < \infty$ and $\phi \in L^\infty(\mathbb{T})$. If $M(\phi)$ is Fredholm on $H^p(\mathbb{T})$, then $\phi \in GL^\infty(\mathbb{T})$.*

A function $\phi \in L^\infty(\mathbb{T})$ is said to admit a *weak asymmetric factorization in* $L^p(\mathbb{T})$ if it can be written in the form

$$(1.6) \qquad \phi(t) = \phi_-(t) t^{\varkappa} \phi_0(t), \quad t \in \mathbb{T},$$

such that $\varkappa \in \mathbb{Z}$ and

(i) $(1 + t^{-1})\phi_- \in \overline{H^p(\mathbb{T})}$, $(1 - t^{-1})\phi_-^{-1} \in \overline{H^q(\mathbb{T})}$,

(ii) $|1 - t|\phi_0 \in L^q_{\text{even}}(\mathbb{T})$, $|1 + t|\phi_0^{-1} \in L^p_{\text{even}}(\mathbb{T})$.

Here $1/p + 1/q = 1$. It was proved in Proposition 3.1 of [2] that if a factorization exists, then the *index* $\varkappa$ of the weak asymmetric factorization is uniquely determined and the factors ϕ_- and ϕ_0 are uniquely determined up to a multiplicative constant. (In [2] also the notion of a *weak antisymmetric factorization in* $L^p(\mathbb{T})$ was introduced. This notion will play no role in the present paper.)

In order to introduce yet another notion, let $\mathcal{R}$ stand for the set of all trigonometric polynomials. Under the assumption that ϕ admits a weak asymmetric factorization in $L^p(\mathbb{T})$ introduce the linear spaces

$$(1.7) \qquad X_1 = \{(1 - t^{-1}) f(t) \ : \ f \in \mathcal{R}\},$$

$$(1.8) \qquad X_2 = \{(1 + t^{-1})\phi_0^{-1}(t) f(t) \ : \ f \in \mathcal{R},\ f(t) = f(t^{-1})\}.$$

It is easy to see that X_1 and X_2 are linear subspaces of $L^p(\mathbb{T})$ and that the space X_1 is dense in $L^p(\mathbb{T})$. Moreover, it was proved ([2], Lemma 4.1(a)) that

$$B := L(\phi_0^{-1})(I+J)PL(\phi_-^{-1}) \tag{1.9}$$

is a well-defined linear (not necessarily bounded) operator acting from X_1 into X_2.

We will call the above factorization (1.6) of ϕ an *asymmetric factorization in* $L^p(\mathbb{T})$ if in addition to (i) and (ii) the following condition is satisfied:

(iii) The operator $B = L(\phi_0^{-1})(I+J)PL(\phi_-^{-1})$ acting from X_1 into X_2 can be extended by continuity to a linear bounded operator acting on $L^p(\mathbb{T})$.

Clearly, due to the density of X_1 in $L^p(\mathbb{T})$ an equivalent formulation for condition (iii) is the following statement:

(iii*) There exists a constant M such that $\|Bf\|_{L^p(\mathbb{T})} \leqslant M\|f\|_{L^p(\mathbb{T})}$ for all $f \in X_1$.

The main result proved in Theorem 6.4 of [1] is the following:

THEOREM 1.2. *Let $1 < p < \infty$ and $\phi \in GL^\infty(\mathbb{T})$. The operator $M(\phi)$ is a Fredholm operator on $H^p(\mathbb{T})$ if and only if the function ϕ admits an asymmetric factorization in $L^p(\mathbb{T})$. In this case, the defect numbers are given by*

$$\dim\ker M(\phi) = \max\{0, -\varkappa\}, \quad \dim\ker M^*(\phi) = \max\{0, \varkappa\}, \tag{1.10}$$

where $\varkappa$ is the index of the factorization of ϕ.

To formulate the main result of this paper we need the notion of the Hunt-Muckenhoupt-Wheeden condition (or, A_p-condition) with respect to the interval $[-1,1]$.

Let $1 < p < \infty$, $1/p + 1/q = 1$, and let $\sigma : [-1,1] \to \mathbb{R}_+$ be a Lebesgue measurable, almost everywhere nonzero function. Assume in addition that $\sigma \in L^p[-1,1]$ and $\sigma^{-1} \in L^q[-1,1]$. We say that σ satisfies the *A_p-condition on* $[-1,1]$ if

$$\sup_I \frac{1}{|I|}\Big(\int_I \sigma^p(x)\,\mathrm{d}x\Big)^{1/p}\Big(\int_I \sigma^{-q}(x)\,\mathrm{d}x\Big)^{1/q} < \infty, \tag{1.11}$$

where the supremum is taken over all subintervalls I of $[-1,1]$. The length of the interval I is denoted by $|I|$. There is an intimate connection between the A_p-condition and the boundedness of the singular integral operator, which will be stated later on.

The main result of this paper is the following:

THEOREM 1.3. *Let $1 < p < \infty$, $1/p + 1/q = 1$, and $\phi \in L^\infty(\mathbb{T})$. The operator $M(\phi)$ is a Fredholm operator on $H^p(\mathbb{T})$ if and only if the following conditions are satisfied:*

(i) $\phi \in GL^\infty(\mathbb{T})$.

(ii) *The function* ϕ *admits a weak asymmetric factorization in* $L^p(\mathbb{T})$,

$$\phi(t) = \phi_-(t)t^{\varkappa}\phi_0(t), \quad t \in \mathbb{T}. \tag{1.12}$$

(iii) *The function*

$$\sigma(\cos\theta) := |\phi_0^{-1}(e^{i\theta})| \, \frac{|1+\cos\theta|^{1/(2q)}}{|1-\cos\theta|^{1/(2p)}} \tag{1.13}$$

satisfies the A_p*-condition.*
Moreover, formulas (1.10) *hold in this case.*

We note that it is straightforward to prove that condition (ii) of the previous theorem implies $\sigma \in L^p[-1,1]$ and $\sigma^{-1} \in L^q[-1,1]$.

2. PROOF OF THEOREM 1.3

Before we are able to give the proof of Theorem 1.3 we establish some definitions and auxiliary results.

Let $C^\infty[-1,1]$ stand for the set of all infinitely differentiable functions $f : [-1,1] \to \mathbb{C}$, and denote by $C_0^\infty[-1,1]$ the subspace of all functions $f \in C^\infty[-1,1]$ such that $f(x)$ and all of its derivatives vanish at the endpoints $x = -1$ and $x = 1$,

$$C_0^\infty[-1,1] = \{f \in C^\infty[-1,1] \;:\; f^{(n)}(-1) = f^{(n)}(1) = 0 \text{ for all } n \geqslant 0\}. \tag{2.1}$$

The singular integral operator $S_{[-1,1]}$ is defined by the rule

$$(S_{[-1,1]}f)(x) = \frac{1}{\pi i}\int_{-1}^{1} \frac{f(y)}{y-x}\,dy, \quad x \in [-1,1], \tag{2.2}$$

where the integral has to be understood as the Cauchy principal value. For $f \in C_0^\infty[-1,1]$ the integral exists for each $x \in [-1,1]$. In fact,

$$(S_{[-1,1]}f)(x) = \frac{1}{\pi i}\int_{-1}^{1} \frac{f(y)-f(x)}{y-x}\,dy + \frac{f(x)}{\pi i}\ln\Big(\frac{1-x}{1+x}\Big), \quad x \in [-1,1]. \tag{2.3}$$

In particular, $S_{[-1,1]}$ is a well defined linear mapping acting from $C_0^\infty[-1,1]$ into $C^\infty[-1,1]$.

Let $\sigma : [-1,1] \to \mathbb{R}_+$ be a Lebesgue measurable, almost everywhere nonzero function. We denote by $L^p_\sigma[-1,1]$ the space consisting of all Lebesgue mesurable functions $f : [-1,1] \to \mathbb{C}$ for which

$$\|f\|_{L^p_\sigma[-1,1]} := \Big(\int_{-1}^{1} \sigma^p(x)|f(x)|^p\,dx\Big)^{1/p} < \infty. \tag{2.4}$$

For certain functions σ it is possible to extend the singular integral operator $S_{[-1,1]}$ as defined above on $C_0^\infty[-1,1]$ by continuity to a linear bounded operator acting on the Banach space $L^p_\sigma[-1,1]$. The criteria is related to the A_p-condition on $[-1,1]$.

The following theorem was established in the case of the real line $\mathbb{R}$ rather than the interval $[-1,1]$ first by Hunt, Muckenhoupt and Wheeden [6]. The theorem itself follows from the results of Coifman and Fefferman [5]. For more information about the A_p-condition and the boundedness of the singular integral operators on more general curves we refer to [3].

THEOREM 2.1. *Let $\sigma : [-1,1] \to \mathbb{R}_+$ be a Lebesgue measurable, almost everywhere nonzero function. Assume that $\sigma \in L^p[-1,1]$ and $\sigma^{-1} \in L^q[-1,1]$, where $1 < p < \infty$, $1/p + 1/q = 1$. Then $S_{[-1,1]} : C_0^\infty[-1,1] \to C^\infty[-1,1]$ can be continued by continuity to a linear bounded operator acting on $L^p_\sigma[-1,1]$ if and only if σ satisfies the A_p-condition on $[-1,1]$.*

We remark in connection with the previous theorem that the assumptions that σ is nonzero almost everywhere and that $\sigma \in L^p[-1,1]$ imply that $C_0^\infty[-1,1]$ is a dense linear subspace of $L^p_\sigma[-1,1]$.

The proof of this statement is similar to the proof of Lemma 2.2 below. Let $C^\infty(\mathbb{T})$ stand for the set of all infinitely differentiable functions on $\mathbb{T}$, and let $C_0^\infty(\mathbb{T})$ stand for the space of all $f \in C^\infty(\mathbb{T})$ such that $f(t)$ and all of its derivatives vanish at $t = 1$ and $t = -1$:

$$C_0^\infty(\mathbb{T}) = \{f \in C^\infty(\mathbb{T}) \;:\; f^{(n)}(1) = f^{(n)}(-1) = 0 \text{ for all } n \geqslant 0\}. \tag{2.5}$$

Let $\varrho : \mathbb{T} \to \mathbb{R}_+$ be a Lebesgue measurable and almost everywhere nonzero function. We denote by $L^p_\varrho(\mathbb{T})$ the space of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\|f\|_{L^p_\varrho(\mathbb{T})} := \Big(\frac{1}{2\pi}\int_0^{2\pi} \varrho^p(\mathrm{e}^{\mathrm{i}\theta})|f(\mathrm{e}^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta\Big)^{1/p} < \infty. \tag{2.6}$$

Let us remark that the dual space to $L^p_\varrho(\mathbb{T})$ can be identified with $L^q_{\varrho^{-1}}(\mathbb{T})$ by means of the sesquilinear functional

$$\langle g, f\rangle := \frac{1}{2\pi}\int_0^{2\pi} \overline{g(\mathrm{e}^{\mathrm{i}\theta})} f(\mathrm{e}^{\mathrm{i}\theta}) \,\mathrm{d}\theta \tag{2.7}$$

with $f \in L^p_\varrho(\mathbb{T})$, $g \in L^q_{\varrho^{-1}}(\mathbb{T})$, $1 < p < \infty$, $1/p + 1/q = 1$.

LEMMA 2.2. *Let $\varrho \in L^p(\mathbb{T})$, $1 < p < \infty$, and assume that ϱ is nonzero almost everywhere. Then $C_0^\infty(\mathbb{T})$ is a dense subspace of $L^p_\varrho(\mathbb{T})$.*

Proof. We introduce the set

$$X = \{\varrho f \ : \ f \in C_0^\infty(\mathbb{T})\}. \tag{2.8}$$

Obviously, $X \subseteq L^p(\mathbb{T})$, which implies that $C_0^\infty(\mathbb{T})$ is a subset of $L^p_\varrho(\mathbb{T})$. The assertion that $C_0^\infty(\mathbb{T})$ is a dense subspace in $L^p_\varrho(\mathbb{T})$ is equivalent to the statement that X is a dense subspace of $L^p(\mathbb{T})$.

We carry out the proof of this statement in several steps. First we prove that the closure of X contains all functions of the form $f = \varrho g$ with $g \in L^\infty(\mathbb{T})$. Indeed, given $g \in L^\infty(\mathbb{T})$ and $\varepsilon > 0$, there is a subset $M \subset \mathbb{T}$ of Lebesgue measure less than ε and a sequence $g_n \in C_0^\infty(\mathbb{T})$ such that $g_n \to g$ uniformly on $\mathbb{T} \setminus M$ and $\|g_n\|_{L^\infty(\mathbb{T})} \leqslant \|g\|_{L^\infty(\mathbb{T})}$. Now we can estimate

$$\|\varrho g - \varrho g_n\|_{L^p(\mathbb{T})} \leqslant 2\,\|\varrho\|_{L^p(M)}\|g\|_{L^\infty(\mathbb{T})} + \|\varrho\|_{L^p(\mathbb{T})}\|g_n - g\|_{L^\infty(\mathbb{T}\setminus M)}.$$

Since $\|\varrho\|_{L^p(M)} \to 0$ as $\varepsilon \to 0$ the assertion follows easily.

Next we prove that $L^\infty(\mathbb{T})$ is contained in the closure of X. Indeed, given $f \in L^\infty(\mathbb{T})$ we introduce the elements $g_\varepsilon = \varrho_\varepsilon f$ where

$$\varrho_\varepsilon(t) = \begin{cases} \varrho^{-1}(t) & \text{if } 0 < \varrho^{-1}(t) \leqslant \varepsilon^{-1}, \\ 0 & \text{if } \varrho^{-1}(t) > \varepsilon^{-1}. \end{cases}$$

Obviously, $\varrho_\varepsilon \in L^\infty(\mathbb{T})$ and hence $g_\varepsilon \in L^\infty(\mathbb{T})$. Now we estimate

$$\|\varrho g_\varepsilon - f\|_{L^p(\mathbb{T})} = \|(\varrho\varrho_\varepsilon - 1)f\|_{L^p(\mathbb{T})} \leqslant \|\varrho\varrho_\varepsilon - 1\|_{L^p(\mathbb{T})}\|f\|_{L^\infty(\mathbb{T})}.$$

The function $1 - \varrho\varrho_\varepsilon$ is equal to the characteristic function of

$$K_\varepsilon = \{t \in \mathbb{T} \ : \ \varrho^{-1}(t) > \varepsilon^{-1}\}.$$

Since ϱ is nonzero almost everywhere, the Lebesgue measure of K_ε tends to zero as $\varepsilon \to 0$. Hence $\|\varrho\varrho_\varepsilon - 1\|_{L^p(\mathbb{T})} \to 0$ as $\varepsilon \to 0$. It follows that ϱg_ε approximates f.

Finally, we note that $L^\infty(\mathbb{T})$ is dense in $L^p(\mathbb{T})$. ∎

Let $Q = I - P$ stand for the complementary projection to the Riesz projection P. Moreover, define the operators

$$G = \frac{1}{2}(I+J)(P-Q)(I-J), \tag{2.9}$$

$$G^* = \frac{1}{2}(I-J)(P-Q)(I+J). \tag{2.10}$$

We think of G and G^* as linear mappings acting from $C_0^\infty(\mathbb{T})$ into $C^\infty(\mathbb{T})$. Notice in this connection that P and Q map $C_0^\infty(\mathbb{T})$ into $C^\infty(\mathbb{T})$.

PROPOSITION 2.3. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Assume that $\phi \in GL^\infty(\mathbb{T})$ admits a weak asymmetric factorization $\phi(t) = \phi_-(t)t^\varkappa\phi_0(t)$ in $L^p(\mathbb{T})$. Then the following is equivalent:*

(i) *The operator $B := L(\phi_0^{-1})(I+J)PL(\phi_-^{-1}) : X_1 \to X_2$ can be continued by continuity to a linear bounded operator acting on $L^p(\mathbb{T})$.*

(ii) *The operator* $G^* : C_0^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ *can be continued by continuity to a linear bounded operator acting on* $L^q_{\omega^{-1}}(\mathbb{T})$.

(iii) *The operator* $G : C_0^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ *can be continued by continuity to a linear bounded operator acting on* $L^p_\omega(\mathbb{T})$.

Therein $\omega(t) = |\phi_0^{-1}(t)|$, *and* X_1 *and* X_2 *are defined by* (1.7) *and* (1.8), *respectively.*

Proof. (i)⇔(ii) Assertion (i) is equivalent to the fact that there exists a constant $M > 0$ such that

$$\|L(\phi_0^{-1})(I+J)PL(\phi_-^{-1})f\|_{L^p(\mathbb{T})} \leqslant M\,\|f\|_{L^p(\mathbb{T})}$$

for all $f \in X_1$. From the definition of $L^p_\omega(\mathbb{T})$ it follows that the last inequality can be rewritten as

$$\|(I+J)PL(\phi_-^{-1})f\|_{L^p_\omega(\mathbb{T})} \leqslant M\,\|f\|_{L^p(\mathbb{T})}.$$

Since $C_0^\infty(\mathbb{T})$ is dense in $L^q_{\omega^{-1}}(\mathbb{T})$ by Lemma 2.2, we obtain that this is in turn equivalent to the statement that

$$|\langle g, (I+J)PL(\phi_-^{-1})f\rangle| \leqslant M\,\|g\|_{L^q_{\omega^{-1}}(\mathbb{T})}\|f\|_{L^p(\mathbb{T})}$$

for all $g \in C_0^\infty(\mathbb{T})$ and all $f \in X_1$. Next notice that

$$\langle g, (I+J)PL(\phi_-^{-1})f\rangle = \langle P(I+J)g, L(\phi_-^{-1})f\rangle = \langle L((\phi_-^{-1})^*)P(I+J)g, f\rangle$$

by noting that $L(\phi_-^{-1})f \in L^q(\mathbb{T})$. Hence the above is equivalent to the statement that

$$|\langle L((\phi_-^{-1})^*)P(I+J)g, f\rangle| \leqslant M\,\|g\|_{L^q_{\omega^{-1}}(\mathbb{T})}\|f\|_{L^p(\mathbb{T})}$$

for all $g \in C_0^\infty(\mathbb{T})$ and all $f \in X_1$. Since X_1 is dense in $L^p(\mathbb{T})$ we can reformulate this by saying that

$$\|L((\phi_-^{-1})^*)P(I+J)g\|_{L^q(\mathbb{T})} \leqslant M\,\|g\|_{L^q_{\omega^{-1}}(\mathbb{T})}$$

for all $g \in C_0^\infty(\mathbb{T})$. Because $\phi_-^{-1}(t) = \phi_0(t)\phi^{-1}(t)$ the latter can be rewritten as

$$\|P(I+J)g\|_{L^q_{\omega^{-1}}(\mathbb{T})} \leqslant M\,\|g\|_{L^q_{\omega^{-1}}(\mathbb{T})}.$$

Since $\omega(t) = \omega(t^{-1})$ the operator $(I+J)$ is bounded on $L^q_{\omega^{-1}}(\mathbb{T})$. Moreover, since $P - Q = 2P - I$ we can conclude that the latter is equivalent to

$$\|(P-Q)(I+J)g\|_{L^q_{\omega^{-1}}(\mathbb{T})} \leqslant M\,\|g\|_{L^q_{\omega^{-1}}(\mathbb{T})} \tag{2.11}$$

for all $g \in C_0^\infty(\mathbb{T})$. Noting that

$$G^* = \frac{1}{2}(I-J)(P-Q)(I+J) = (P-Q)(I+J)$$

completes the proof of (i)⇔(ii).

(ii)⇔(iii) Since $C_0^\infty(\mathbb{T})$ is dense in both $L^p_\omega(\mathbb{T})$ and $L^q_{\omega^{-1}}(\mathbb{T})$ by Lemma 2.2, it is easily seen that (ii) is equivalent to the statement that

$$|\langle G^* g, f\rangle| \leqslant M\,\|g\|_{L^p_\omega(\mathbb{T})}\|f\|_{L^q_{\omega^{-1}}(\mathbb{T})}$$

for all $f, g \in C_0^\infty(\mathbb{T})$. Moreover, (iii) is equivalent to the statement that

$$|\langle g, Gf\rangle| \leqslant M\,\|g\|_{L^p_\omega(\mathbb{T})}\|f\|_{L^q_{\omega^{-1}}(\mathbb{T})}$$

for all $f, g \in C_0^\infty(\mathbb{T})$. Since $\langle G^* g, f\rangle = \langle g, Gf\rangle$ the result follows. ∎

Let $C^\infty_{\mathrm{even}}(\mathbb{T})$ stand for the space of all functions $f \in C^\infty(\mathbb{T})$ which are even, i.e., for which $f(t) = f(t^{-1})$, $t \in \mathbb{T}$. Moreover, introduce the operators

$$U : \widehat{f}(x) \in C_0^\infty[-1,1] \mapsto f(\mathrm{e}^{\mathrm{i}\theta}) := \widehat{f}(\cos\theta) \in C_0^\infty(\mathbb{T}), \tag{2.12}$$

$$V : f(t) \in C^\infty_{\mathrm{even}}(\mathbb{T}) \mapsto \widehat{f}(\cos\theta) := f(\mathrm{e}^{\mathrm{i}\theta}) \in C^\infty[-1,1], \tag{2.13}$$

and let

$$\chi(\mathrm{e}^{\mathrm{i}\theta}) = \begin{cases} 1 & \text{if } 0 < \theta < \pi, \\ -1 & \text{if } \pi < \theta < 2\pi. \end{cases} \tag{2.14}$$

Clearly, the image of U is the set of even functions defined on $\mathbb{T}$.

PROPOSITION 2.4. *For $\widehat{f} \in C_0^\infty[-1,1]$ we have*

$$S_{[-1,1]}\widehat{f} = \frac{1}{2}\,VL((1+t^{-1})^{-1})GL(\chi(1+t^{-1}))U\widehat{f}. \tag{2.15}$$

Proof. Given $\widehat{f} \in C_0^\infty[-1,1]$, we introduce the functions

$$f = L(\chi(1+t^{-1}))U\widehat{f}, \quad g = \frac{1}{2}\,Gf, \quad \widehat{g} = VL((1+t)^{-1})g.$$

Notice that

$$G = \frac{1}{2}(I+J)(P-Q)(I-J) = (P-Q)(I-J).$$

Hence $g(t) = t^{-1}g(1/t)$ and it follows that $L((1+t^{-1})^{-1})g$ is an even function. Moreover, it is easily seen that $Jf = -f$ whence it follows that $g = (1/2)\,Gf = (P-Q)f$. It is well known that the singular integral operator $S = P - Q$ on $\mathbb{T}$ can be written as

$$(Sf)(t) = \frac{1}{\pi\mathrm{i}}\int_{\mathbb{T}} \frac{f(s)-f(t)}{s-t}\,\mathrm{d}s + f(t)$$

for functions $f \in C^\infty(\mathbb{T})$. From this we deduce the relations

$$f(\mathrm{e}^{\mathrm{i}\theta}) = (1+\mathrm{e}^{-\mathrm{i}\theta})\chi(\mathrm{e}^{\mathrm{i}\theta})\widehat{f}(\cos\theta),$$

$$g(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(\mathrm{e}^{\mathrm{i}\varphi}) - f(\mathrm{e}^{\mathrm{i}\theta})}{1-\mathrm{e}^{\mathrm{i}(\theta-\varphi)}}\,\mathrm{d}\varphi + f(\mathrm{e}^{\mathrm{i}\theta}),$$

$$\widehat{g}(\cos\theta) = (1+\mathrm{e}^{-\mathrm{i}\theta})^{-1}g(\mathrm{e}^{\mathrm{i}\theta}).$$

We split the integral appearing in the second equation into two parts integrating on $[0,\pi]$ and $[-\pi,0]$, respectively, and make a change of variables $\varphi \mapsto -\varphi$ in the second integral. This gives

$$g(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\pi}\int_0^\pi \Big(\frac{f(\mathrm{e}^{\mathrm{i}\varphi}) - f(\mathrm{e}^{\mathrm{i}\theta})}{1-\mathrm{e}^{\mathrm{i}(\theta-\varphi)}} + \frac{f(\mathrm{e}^{-\mathrm{i}\varphi}) - f(\mathrm{e}^{\mathrm{i}\theta})}{1-\mathrm{e}^{\mathrm{i}(\theta+\varphi)}}\Big)\,\mathrm{d}\varphi + f(\mathrm{e}^{\mathrm{i}\theta}).$$

Since $f(\mathrm{e}^{-\mathrm{i}\varphi}) = -\mathrm{e}^{\mathrm{i}\varphi} f(\mathrm{e}^{\mathrm{i}\varphi})$ and since

$$\begin{aligned}\frac{1}{1-\mathrm{e}^{\mathrm{i}(\theta-\varphi)}} - \frac{\mathrm{e}^{\mathrm{i}\varphi}}{1-\mathrm{e}^{\mathrm{i}(\theta+\varphi)}} &= \frac{(1+\mathrm{e}^{-\mathrm{i}\theta})(\mathrm{e}^{\mathrm{i}\varphi}-1)}{2(\cos\varphi - \cos\theta)},\\ \frac{1}{1-\mathrm{e}^{\mathrm{i}(\theta-\varphi)}} + \frac{1}{1-\mathrm{e}^{\mathrm{i}(\theta+\varphi)}} &= \frac{\mathrm{e}^{\mathrm{i}\varphi}+\mathrm{e}^{-\mathrm{i}\varphi} - 2\mathrm{e}^{-\mathrm{i}\theta}}{2(\cos\varphi - \cos\theta)},\end{aligned}$$

it follows that

$$\begin{aligned}g(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\pi}\int_0^\pi \Big(&\frac{(1+\mathrm{e}^{-\mathrm{i}\theta})(\mathrm{e}^{\mathrm{i}\varphi}-1)f(\mathrm{e}^{\mathrm{i}\varphi})}{2(\cos\varphi-\cos\theta)} - \frac{(\mathrm{e}^{\mathrm{i}\varphi}-\mathrm{e}^{-\mathrm{i}\varphi})f(\mathrm{e}^{\mathrm{i}\theta})}{2(\cos\varphi-\cos\theta)}\Big)\,\mathrm{d}\varphi\\ &- \frac{f(\mathrm{e}^{\mathrm{i}\theta})}{\pi}\int_0^\pi \frac{\mathrm{e}^{-\mathrm{i}\varphi}-\mathrm{e}^{-\mathrm{i}\theta}}{\cos\varphi-\cos\theta}\,\mathrm{d}\varphi + f(\mathrm{e}^{\mathrm{i}\theta}).\end{aligned}$$

If we assume $0<\theta<\pi$, we obtain

$$\begin{aligned}g(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\pi}\int_0^\pi &\frac{(1+\mathrm{e}^{-\mathrm{i}\theta})(\mathrm{e}^{\mathrm{i}\varphi}-\mathrm{e}^{-\mathrm{i}\varphi})(\widehat{f}(\cos\varphi)-\widehat{f}(\cos\theta))}{2(\cos\varphi-\cos\theta)}\,\mathrm{d}\varphi\\ &- \frac{f(\mathrm{e}^{\mathrm{i}\theta})}{\pi\mathrm{i}}\int_0^\pi \frac{\sin\varphi-\sin\theta}{\cos\varphi-\cos\theta}\,\mathrm{d}\varphi.\end{aligned}$$

The first integral is equal to $(1+\mathrm{e}^{-\mathrm{i}\theta})$ times

$$\frac{\mathrm{i}}{\pi}\int_0^\pi \frac{(\widehat{f}(\cos\varphi)-\widehat{f}(\cos\theta))\sin\varphi}{\cos\varphi-\cos\theta}\,\mathrm{d}\varphi = \frac{1}{\pi\mathrm{i}}\int_{-1}^1 \frac{\widehat{f}(y)-\widehat{f}(\cos\theta)}{y-\cos\theta}\,\mathrm{d}y.$$

The second integral equals $f(\mathrm{e}^{\mathrm{i}\theta})$ times

$$\frac{1}{\pi\mathrm{i}}\int_0^\pi \cot\Big(\frac{\varphi+\theta}{2}\Big)\,\mathrm{d}\varphi = \Big[\frac{2}{\pi\mathrm{i}}\ln\sin\Big(\frac{\varphi+\theta}{2}\Big)\Big]_{\varphi=0}^{\pi} = \frac{1}{\pi\mathrm{i}}\ln\Big(\frac{1-\cos\theta}{1+\cos\theta}\Big).$$

Putting the pieces together we obtain

$$g(\mathrm{e}^{\mathrm{i}\theta}) = \frac{(1+\mathrm{e}^{-\mathrm{i}\theta})}{\pi\mathrm{i}}\int_{-1}^1 \frac{\widehat{f}(y)-\widehat{f}(\cos\theta)}{y-\cos\theta}\,\mathrm{d}y + \frac{f(\mathrm{e}^{\mathrm{i}\theta})}{\pi\mathrm{i}}\ln\Big(\frac{1-\cos\theta}{1+\cos\theta}\Big).$$

Note that the above assumption $0 < \theta < \pi$ is not an essential restriction since $\widehat{g}$ is determined by the formula

$$\widehat{g}(\cos\theta) = (1+\mathrm{e}^{-\mathrm{i}\theta})^{-1} g(\mathrm{e}^{\mathrm{i}\theta}), \quad 0<\theta<\pi.$$

It follows that

$$\widehat{g}(x) = \frac{1}{\pi\mathrm{i}} \int_{-1}^{1} \frac{\widehat{f}(y)-\widehat{f}(x)}{y-x}\,\mathrm{d}y + \frac{\widehat{f}(x)}{\pi\mathrm{i}} \ln\Big(\frac{1-x}{1+x}\Big).$$

This is equal to the singular integral operator $S_{[-1,1]}$ applied to the function $\widehat{f}$. Hence $\widehat{g} = S_{[-1,1]}\widehat{f}$, which is the assertion. ∎

Now we are able to present the proof of Theorem 1.3.

Proof of Theorem 1.3. It is obvious from Proposition 1.1 and Theorem 1.2 that the Fredholmness of $M(\phi)$ implies assertions (i) and (ii).

Hence it is sufficient to prove the following. If conditions (i) and (ii) are fulfilled, then $M(\phi)$ is Fredholm if and only if σ satisfies the A_p-condition.

Under these assumptions we deduce from Theorem 1.2 that the Fredholmness of $M(\phi)$ on $H^p(\mathbb{T})$ is equivalent to the existence of a bounded continuation of the operator $B = L(\phi_0^{-1})(I+J)PL(\phi_-^{-1}) : X_1 \to X_2$ on $L^p(\mathbb{T})$. We apply Proposition 2.3 and see that this existence is equivalent to the condition that

$$\|Gg\|_{L^p_\omega(\mathbb{T})} \leqslant M\,\|g\|_{L^p_\omega(\mathbb{T})}$$

for all $g \in C_0^\infty(\mathbb{T})$ where $\omega(t) := |\phi_0^{-1}(t)|$.

Next we claim that this, in turn, is equivalent to the condition that

$$\|Gg\|_{L^p_\omega(\mathbb{T})} \leqslant M\,\|g\|_{L^p_\omega(\mathbb{T})}$$

for all $g \in C_0^\infty(\mathbb{T})$ for which $Jg = -g$. In order to prove the non-trivial part of this equivalence, we decompose an arbitrarily given $g \in C_0^\infty(\mathbb{T})$ into $g = g_1 + g_2$ where $g_1 = (1/2)(I+J)g$ and $g_2 = (1/2)(I-J)g$. The function g_1 lies in the kernel of G while $g_2(t) = (g(t) - t^{-1}g(t^{-1}))/2$ belongs to $C_0^\infty(\mathbb{T})$ and satisfies the relation $Jg_2 = -g_2$. Moreover, since ω is an even function, the operator $(I-J)/2$ is bounded on $L^p_\omega(\mathbb{T})$. We obtain the estimate

$$\|Gg\|_{L^p_\omega(\mathbb{T})} = \|Gg_2\|_{L^p_\omega(\mathbb{T})} \leqslant M\,\|g_2\|_{L^p_\omega(\mathbb{T})} \leqslant M\,\|g\|_{L^p_\omega(\mathbb{T})},$$

which proves this claim.

Next we remark that the operator $L(\chi(1+t^{-1}))U$ maps the space $C_0^\infty[-1,1]$ onto the subspace of functions $g \in C_0^\infty(\mathbb{T})$ satisfying $Jg = -g$. This allows us to make the substitution $g = L(\chi(1+t^{-1}))Uf$ with $f \in C_0^\infty[-1,1]$. We obtain the equivalent estimate

$$\|GL(\chi(1+t^{-1}))Uf\|_{L^p_\omega(\mathbb{T})} \leqslant M\,\|L(\chi(1+t^{-1}))Uf\|_{L^p_\omega(\mathbb{T})}$$

for all $f \in C_0^\infty[-1,1]$. Clearly, the last estimate can be written in the form

$$\|VL((1+t^{-1})^{-1})GL(\chi(1+t^{-1}))Uf\|_{L^p_\sigma[-1,1]} \leqslant M\,\|f\|_{L^p_\sigma[-1,1]}$$

for all $f \in C_0^\infty[-1,1]$, where

$$\sigma(\cos\theta) = \frac{\omega(e^{i\theta})|1+e^{-i\theta}|}{\sqrt{2}\,|\sin\theta|^{1/p}} = |\phi_0^{-1}(e^{i\theta})|\,\frac{|1+\cos\theta|^{1/(2q)}}{|1-\cos\theta|^{1/(2p)}}.$$

Along with Proposition 2.4 and Theorem 2.1 this completes the proof. ∎

3. APPLICATIONS TO PIECEWISE CONTINUOUS FUNCTIONS

We now apply the previous results in order to obtain necessary and sufficient conditions for the operator $M(\phi)$ to be invertible or Fredholm on $H^p(\mathbb{T})$ for piecewise continuous functions ϕ with finitely many jumps. These results have already been established in [2] (and in [1] for the case $p = 2$). The proofs given in [1] and [2] rely on the results establish in [7] by help of Banach algebra methods. The proof which we will give here is more direct and relies entirely on the factorization methods developed here and in [2] in connection with the A_p-condition.

We restrict to piecewise continuous functions with a *finite* number of discontinuities because these functions can be written in a convenient manner which is useful in many instances. It is well known that any piecewise continuous and nonvanishing function with a finite number of discontinuities at the points $\theta_1, \dots, \theta_R$ can be written as a product

$$\phi(e^{i\theta}) = b(e^{i\theta}) \prod_{r=1}^{R} t_{\beta_r}(e^{i(\theta-\theta_r)}) \tag{3.1}$$

where b is a nonvanishing continuous function and

$$t_\beta(e^{i\theta}) = \exp(i\beta(\theta-\pi)), \quad 0<\theta<2\pi. \tag{3.2}$$

Notice that the parameters $\beta_1, \dots, \beta_R$ in this formula are uniquely determined up to an additive integer. In fact,

$$\frac{\phi(e^{i\theta_r}-0)}{\phi(e^{i\theta_r}+0)} = \frac{t_{\beta_r}(1-0)}{t_{\beta_r}(1+0)} = \exp(2\pi i\beta_r). \tag{3.3}$$

Moreover, the formula

$$t_{\beta+n}(t) = (-t)^n t_\beta(t), \quad t \in \mathbb{T}, \tag{3.4}$$

holds for $n \in \mathbb{Z}$.

The parameters in the representation (3.1) are useful to decide Fredholmness and invertibility. For example, the Toeplitz operators $T(\phi)$ with a piecewise continuous function ϕ with finitely many jump discontinuities is invertible on $H^p(\mathbb{T})$ if and only if the function ϕ can be represented in the form (3.1) with

$-1/q < \operatorname{Re}\beta_r < 1/p$ and the winding number of b equal to zero. If the operator is Fredholm with Fredholm index $\varkappa$, then the zero is replaced by $-\varkappa$. ([4], Chapter 6).

Before stating the analogue of this result for Toeplitz plus Hankel operators $M(\phi)$ we have to establish the following auxiliary result.

LEMMA 3.1. *Let $1 < p < \infty$, $1/p + 1/q = 1$, $-1 = x_0 < x_1 < \cdots < x_R < x_{R+1} = 1$ and*

$$\sigma(x) = \prod_{r=0}^{R+1} |x - x_r|^{\alpha_r}. \tag{3.5}$$

If $-1/p < \alpha_r < 1/q$ for each $0 \leqslant r \leqslant R+1$, then $\sigma \in L^p[-1,1]$, $\sigma^{-1} \in L^q[-1,1]$ and σ satisfies the A_p-condition.

Proof. This can be verified straightforwardly. ∎

The promised result is the following:

THEOREM 3.2. *Let $1 < p < \infty$, $1/p + 1/q = 1$. Suppose that ϕ has finitely many jump discontinuities. Then $M(\phi)$ is Fredholm on $H^p(\mathbb{T})$ if and only if ϕ can be written in the form*

$$\phi(\mathrm{e}^{\mathrm{i}\theta}) = b(\mathrm{e}^{\mathrm{i}\theta}) t_{\beta^+}(\mathrm{e}^{\mathrm{i}\theta}) t_{\beta^-}(\mathrm{e}^{\mathrm{i}(\theta-\pi)}) \prod_{r=1}^{R} t_{\beta_r^+}(\mathrm{e}^{\mathrm{i}(\theta-\theta_r)}) t_{\beta_r^-}(\mathrm{e}^{\mathrm{i}(\theta+\theta_r)}) \tag{3.6}$$

where b is a continuous nonvanishing function on $\mathbb{T}$, the numbers $\theta_1, \ldots, \theta_R \in (0, \pi)$ are distinct, and

(i) $-1/q < \operatorname{Re}(\beta_r^+ + \beta_r^-) < 1/p$ *for each* $1 \leqslant r \leqslant R$,
(ii) $-1/2 - 1/(2q) < \operatorname{Re}\beta^+ < 1/(2p)$ *and* $-1/(2q) < \operatorname{Re}\beta^- < 1/2 + 1/(2p)$.

Moreover, in this case,

$$\begin{aligned} \dim\ker M(\phi) &= \max\{0, -\operatorname{wind}(b)\}, \\ \dim\ker M^*(\phi) &= \max\{0, \operatorname{wind}(b)\}. \end{aligned} \tag{3.7}$$

Proof. In the first step we prove that $M(\psi)$ is a Fredholm operator on $H^p(\mathbb{T})$ with Fredholm index zero if ψ is of the form

$$\psi(\mathrm{e}^{\mathrm{i}\theta}) = t_{\beta^+}(\mathrm{e}^{\mathrm{i}\theta}) t_{\beta^-}(\mathrm{e}^{\mathrm{i}(\theta-\pi)}) \prod_{r=1}^{R} t_{\beta_r^+}(\mathrm{e}^{\mathrm{i}(\theta-\theta_r)}) t_{\beta_r^-}(\mathrm{e}^{\mathrm{i}(\theta+\theta_r)}) \tag{3.8}$$

and the parameters satisfy the conditions (i) and (ii). In regard to Theorem 1.3 it suffices to construct a weak asymmetric factorization of ψ and to prove that the corresponding weight σ satisfies the A_p-condition. For this purpose we introduce the functions

$$\eta_\beta(t) = (1-t)^\beta, \quad \xi_\beta(t) = (1-t^{-1})^\beta.$$

Notice that $t_\beta(t) = \eta_\beta(t)\xi_{-\beta}(t)$. Then we can factor $\psi(t) = \psi_-(t)\psi_0(t)$ with

$$\begin{aligned}\psi_-(e^{i\theta}) &= \xi_{-2\beta^+}(e^{i\theta})\xi_{-2\beta^-}(e^{i(\theta-\pi)})\\ &\qquad \times \prod_{r=1}^{R} \xi_{-\beta_r^+-\beta_r^-}(e^{i(\theta-\theta_r)})\xi_{-\beta_r^+-\beta_r^-}(e^{i(\theta+\theta_r)}),\\ \psi_0(e^{i\theta}) &= \eta_{\beta^+}(e^{i\theta})\xi_{\beta^+}(e^{i\theta})\eta_{\beta^-}(e^{i(\theta-\pi)})\xi_{\beta^-}(e^{i(\theta-\pi)})\\ &\qquad \times \prod_{r=1}^{R} \eta_{\beta_r^+}(e^{i(\theta-\theta_r)})\xi_{\beta_r^+}(e^{i(\theta+\theta_r)})\eta_{\beta_r^-}(e^{i(\theta+\theta_r)})\xi_{\beta_r^-}(e^{i(\theta-\theta_r)}).\end{aligned}$$

Because of conditions (i) and (ii), it can be checked straightforwardly that the function

$$\begin{aligned}(1+e^{-i\theta})\psi_-(e^{i\theta}) &= \xi_{-2\beta^+}(e^{i\theta})\xi_{-2\beta^-+1}(e^{i(\theta-\pi)})\\ &\qquad \times \prod_{r=1}^{R} \xi_{-\beta_r^+-\beta_r^-}(e^{i(\theta-\theta_r)})\xi_{-\beta_r^+-\beta_r^-}(e^{i(\theta+\theta_r)})\end{aligned}$$

belongs to $\overline{H^p(\mathbb{T})}$ and the function

$$\begin{aligned}(1-e^{-i\theta})\psi_-^{-1}(e^{i\theta}) &= \xi_{2\beta^++1}(e^{i\theta})\xi_{2\beta^-}(e^{i(\theta-\pi)})\\ &\qquad \times \prod_{r=1}^{R} \xi_{\beta_r^++\beta_r^-}(e^{i(\theta-\theta_r)})\xi_{\beta_r^++\beta_r^-}(e^{i(\theta+\theta_r)})\end{aligned}$$

belongs to $\overline{H^q(\mathbb{T})}$. From the fact that ψ_0 is even and that $\psi_0(t) = \psi_-(t)^{-1}\psi(t)$, it follows that the function ψ_0 fulfills all the necessary conditions in regard to a weak asymmetric factorization. Hence $\psi(t) = \psi_-(t)\psi_0(t)$ is indeed a weak asymmetric factorization with index zero.

In order to calculate the corresponding weight function (1.13) consider

$$\begin{aligned}\psi_0(e^{i\theta}) &= |1-e^{i\theta}|^{2\beta^+}|1+e^{i\theta}|^{2\beta^-}\\ &\times \prod_{r=1}^{R} |1-e^{i(\theta-\theta_r)}|^{\beta_r^++\beta_r^-}|1-e^{i(\theta+\theta_r)}|^{\beta_r^++\beta_r^-} t_{\frac{\beta_r^+-\beta_r^-}{2}}(e^{i(\theta-\theta_r)})t_{\frac{\beta_r^--\beta_r^+}{2}}(e^{i(\theta+\theta_r)}),\end{aligned}$$

and observe that $|1-e^{i\theta}| = (2-2\cos\theta)^{1/2} = 2|\sin(\frac{\theta}{2})|$ and $2\sin\frac{\theta-\theta_r}{2}\sin\frac{\theta+\theta_r}{2} = \cos\theta_r - \cos\theta$. Hence

$$\psi_0^{-1}(e^{i\theta}) = \sigma_0(\cos\theta)(1-\cos\theta)^{-\beta^+}(1+\cos\theta)^{-\beta^-}\prod_{r=1}^{R}|\cos\theta-\cos\theta_r|^{-\beta_r^+-\beta_r^-},$$

where $\sigma_0(x) \in GL^\infty(\mathbb{T})$ is a function which comes from collecting the terms $t_{\frac{\beta_r^+-\beta_r^-}{2}}(e^{i(\theta-\theta_r)})$, $t_{\frac{\beta_r^--\beta_r^+}{2}}(e^{i(\theta+\theta_r)})$ and certain constants. It follows that σ evaluates to

$$\sigma(x) = |\sigma_0(x)|(1-x)^{-\operatorname{Re}\beta^+-1/(2p)}(1+x)^{-\operatorname{Re}\beta^-+1/(2q)} \times \prod_{r=1}^{R} |x-\cos\theta_r|^{-\operatorname{Re}\beta_r^+-\operatorname{Re}\beta_r^-}. \tag{3.9}$$

It suffices to apply Lemma 3.1 in order to see that σ satisfies the A_p-condition.

In the second step we prove that $M(\phi)$ is a Fredholm operator if ϕ is given by (3.6) with conditions (i) and (ii) being fulfilled and if the function b is continuous and nonvanishing. We can write $\phi(t) = b(t)\psi(t)$ where ψ is as above. From well-known identities for Toeplitz and Hankel operators,

$$T(\phi) = T(b)T(\psi) + H(b)H(\widetilde{\psi}),$$
$$H(\phi) = T(b)H(\psi) + H(b)T(\widetilde{\psi}),$$

where $\widetilde{\psi}(t) = \psi(t^{-1})$, it follows that

$$M(\phi) = T(b)M(\psi) + H(b)M(\widetilde{\psi}).$$

Under the assumption on b the operator $H(b)$ is compact and the operator $T(b)$ is Fredholm with Fredholm index equal to $-\text{wind}\,(b)$. Since we have just proved that $M(\psi)$ is Fredholm with Fredholm index zero, it follows that $M(\phi)$ is Fredholm with Fredholm index equal to $-\text{wind}\,(b)$.

Hence we have proved the "if" part of the theorem and also computed the Fredholm index of $M(\phi)$. Now we apply Theorem 1.2 with formula (1.10). This formula implies that the Fredholm index is equal to $-\varkappa$, where $\varkappa$ is the index of the asymmetric factorization of ϕ. Hence $\varkappa = \text{wind}\,(b)$ and formula (3.7) follows.

In the last step we are going to prove the "only if" part of the theorem. It is settled by a well-known perturbation argument. Suppose that $M(\phi)$ is a Fredholm operator with index $\varkappa$, say. We conclude from Proposition 1.1 that $\phi \in GL^\infty(\mathbb{T})$. Since ϕ has only a finite number of jump discontinuities, this implies that ϕ can be written in the form (3.1) or in the form (3.6) if we put some of the β-parameters equal to zero if necessary. Therein the function b is continuous and nonvanishing on $\mathbb{T}$. Moreover, due to formula (3.3) and (3.4) we can choose the β-parameters to satisfy the conditions

(i*) $-1/q < \operatorname{Re}(\beta_r^+ + \beta_r^-) \leqslant 1/p$ for each $1 \leqslant r \leqslant R$,
(ii*) $-1/2 - 1/(2q) < \operatorname{Re}\beta^+ \leqslant 1/(2p)$ and $-1/(2q) < \operatorname{Re}\beta^- \leqslant 1/2 + 1/(2p)$.

Assume contrary to what we want to prove, namely, that in at least one instance we have equality in the above conditions (i*) and (ii*). We are going to perturbate the β-parameters (and thus the function ϕ) in two different ways in order to arrive at a contradiction. Remark that since $M(\phi)$ is assumed to be Fredholm, the Fredholm index is constant with respect to any small perturbation.

We first perturbate by replacing all β-parameters by $\beta - \varepsilon$ where $\varepsilon > 0$ is sufficiently small. This turns the conditions (i*) and (ii*) into (i) and (ii). Applying the "if" part of the theorem with the formula for the Fredholm index, it follows that $\varkappa = -\mathrm{wind}\,(b)$. In the second perturbation we do the same substitution except for one of the instances of equality in (i*) and (ii*) where we replace the corresponding $\beta^{\pm}$ by $\beta^{\pm} - 1 + \varepsilon$, or, the corresponding $\beta_r^+ + \beta_r^-$ by $\beta_r^+ + \beta_r^- - 1 + \varepsilon$, respectively. Moreover, we have to replace $b(t)$ by $t\,b(t)$ times a certain constant due to formula (3.4). This is again a small perturbation of ϕ, which leaves the Fredholm index unchanged. The corresponding parameters fulfill (i) and (ii), but we obtain $\varkappa = -\mathrm{wind}\,(t\,b(t)) = -1 - \mathrm{wind}\,(b)$ contradicting the above formula. This completes the proof of the "only if" part. ∎

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REFERENCES

[1] E.L. BASOR, T. EHRHARDT, On a class of Toeplitz + Hankel operators, *New York J. Math.* **5**(1999), 1–16.

[2] E.L. BASOR, T. EHRHARDT, Factorization theory for a class of Toeplitz + Hankel operators, *J. Operator Theory* **51**(2004), 411–433.

[3] A. BÖTTCHER, Y. KARLOVICH, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*, Birkhäuser, Basel 1997.

[4] A. BÖTTCHER, B. SILBERMANN, *Analysis of Toeplitz Operators*, Springer, New York 1990.

[5] R.R. COIFMAN, C. FEFFERMAN, Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* **51**(1974), 241–250.

[6] R. HUNT, B. MUCKENHOUPT, R. WHEEDEN, Weighted norm inequalities for the conjugate function and Hilbert transform, *Trans. Amer. Math. Soc.* **176**(1973), 227–251.

[7] S. ROCH, B. SILBERMANN, *Algebras of Convolution Operators and their Image in the Calkin Algebra*, Rep. MATH, vol. 90-05, Akademie der Wissenschaften der DDR, Karl-Weierstrass-Institut für Mathematik, Berlin 1990.

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