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THE NONCOMPACT BANACH-STONE THEOREM

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ABSTRACT. We give a complete description of the linear isometries between spaces of vector-valued bounded continuous functions defined on some natural families of topological spaces which may be neither compact nor locally compact. A similar study is carried out for spaces of vector-valued bounded uniformly continuous functions.

KEYWORDS: *Banach-Stone theorem, linear isometry, biseparating map, disjointness preserving, realcompact space, vector-valued bounded functions.*

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1. INTRODUCTION

The aim of this paper is the study, in some natural contexts, of linear isometries between spaces of bounded continuous functions. A classical result in this theory is the following: if X and Y are compact and Hausdorff and $T : C(X) \to C(Y)$ is a surjective linear isometry, then there exist a homeomorphism ϕ from Y onto X and $a \in C(Y)$ with $|a(y)| = 1$ for every $y \in Y$ such that $(Tf)(y) = a(y)f(\phi(y))$ for every $f \in C(X)$ and every $y \in Y$.

The above theorem has some vector-valued counterparts, and has been broadly studied. In particular, for compact X and Y, if E is a Banach space satisfying a special condition (namely, having trivial centralizer), then every linear isometry from $C(X, E)$ onto $C(Y, E)$ is a *strong Banach-Stone map* (see definition in Section 2), that is, in that case we can obtain a description of the map. This result can be extended to maps defined between spaces of continuous functions vanishing at infinity on locally compact spaces (see for instance [6]; see also [14] for recent results concerning local isometries, and [16] for related results). In general, results in this direction always include some kind of compactness of the topological spaces among the hypotheses.

Notice that if no conditions on compactness are required, then for spaces of bounded continuous functions $C_{\mathrm{b}}(X)$ and $C_{\mathrm{b}}(Y)$, a linear isometry $T : C_{\mathrm{b}}(X) \to C_{\mathrm{b}}(Y)$ always leads to a homeomorphism between βX and βY (the Stone-Čech compactifications of X and Y), as elements in $C_b(X)$ and $C_b(Y)$ can be extended to elements in $C(\beta X)$ and $C(\beta Y)$. Consequently, no direct link between X and Y can be given, and the natural context to study such isometries is that of compact spaces.

In this paper, we will see that the behaviour in the vector-valued case is essentially different, and in many important cases X and Y must be homeomorphic (see Theorem 3.2). Also, when E and F are infinite-dimensional, we show that the natural framework to carry out the study of linear isometries between spaces of bounded continuous functions $C_b(X,E)$ and $C_b(Y,F)$ is not that of compact spaces, but one containing a wider family of sets, as it is that of *realcompact* spaces (see Remark 3.3). In our approach, we will take advantage of our study of biseparating maps in previous papers [3], [2] to describe such isometries (see also [1], [12], [15] for related results in the (locally) compact case). Similar techniques can be used to study linear isometries between spaces of bounded uniformly continuous functions, providing in this case a special description of them (some papers dealing with the case of scalar-valued uniformy continuous functions have appeared recently, [4], [9], [11]).

We want to mention also that the only results related to isometries between spaces $C_b(X,E)$ and $C_b(Y,F)$ (for X and Y not locally compact) which seem to have made their way in the literature so far are contained in [5], where the author gives a representation of such isometries in the case when X and Y are *complete metric spaces* and $E = F$ is a Hilbert space. This situation will be a particular case of our Context 3 (see next section).

For a systematic account on isometries between spaces of continuous functions and related topics, the reader is referred for instance to [13] and the new book [8].

2. DEFINITIONS AND NOTATION

Throughout the paper $\mathbb{K}$ will be the field of real or complex numbers. E and F will be $\mathbb{K}$-Banach spaces.

For a completely regular space X, $C_b(X,E)$ denotes the space of E-valued *bounded* continuous functions on X. When $E = \mathbb{K}$, $C_b(X) := C_b(X,\mathbb{K})$.

On the other hand, if X is also a complete metric space, $C_b^u(X,E)$ denotes the space of uniformly continuous bounded functions defined on X, taking values in E. Also in this case $C_b^u(X) := C_b^u(X,\mathbb{K})$.

Both $C_b(X,E)$ and $C_b^u(X,E)$ are endowed with the sup norm.

Also, if $\mathbf{e} \in E$, then $\widehat{\mathbf{e}}$ denotes the constant function from X to E taking the value $\mathbf{e}$.

THE CONTEXTS. Our results will be valid (with the same proof) for different kinds of spaces. For this reason we first consider several situations to work in.

From now on we will assume that we are in one of the following four con texts. All definitions, results and comments given in this paper apply to these four contexts unless otherwise stated.

- CONTEXT 1. E and F are *infinite-dimensional*. X and Y are realcompact. $\mathcal{A}(X,E) = C_{\mathrm{b}}(X,E)$, $\mathcal{A}(Y,F) = C_{\mathrm{b}}(Y,F)$.
- CONTEXT 2. E and F are *infinite-dimensional*. X and Y are completely regular, and all points of X and Y are G_δ-points. $\mathcal{A}(X,E) = C_{\mathrm{b}}(X,E)$, $\mathcal{A}(Y,F) = C_{\mathrm{b}}(Y,F)$.
- CONTEXT 3. X and Y are completely regular and first countable. $\mathcal{A}(X,E) = C_{\mathrm{b}}(X,E)$, $\mathcal{A}(Y,F) = C_{\mathrm{b}}(Y,F)$.
- CONTEXT 4. X and Y are *complete* metric spaces. $\mathcal{A}(X,E) = C_{\mathrm{b}}^{\mathrm{u}}(X,E)$, $\mathcal{A}(Y,F) = C_{\mathrm{b}}^{\mathrm{u}}(Y,F)$.

This means that when we refer to spaces $X, Y, \mathcal{A}(X,E), \mathcal{A}(Y,F)$, we assume that all of them are included at the same time in one of the above four contexts. $\mathcal{A}(X,\mathbb{K})$ will have the natural meanings, that is, $\mathcal{A}(X,\mathbb{K}) = C_{\mathrm{b}}(X,\mathbb{K})$ in Contexts 1–3, and $\mathcal{A}(X,\mathbb{K}) = C_{\mathrm{b}}^{\mathrm{u}}(X,\mathbb{K})$ in Context 4. A similar comment applies to $\mathcal{A}(Y,\mathbb{K})$.

We next adapt Lemma 3.1 from [3] and give the following result.

LEMMA 2.1. *Let $\alpha, \beta \in \mathbb{R}$ satisfy $0 < \alpha < \beta$. Suppose that $f : X \to [0,+\infty)$ belongs to $\mathcal{A}(X,\mathbb{K})$, and that the sets $U := \{x \in X : f(x) \leqslant \alpha\}$ and $V := \{x \in X : f(x) \geqslant \beta\}$ are both nonempty. Then there exists $g \in \mathcal{A}(X,\mathbb{K})$ such that $0 \leqslant g \leqslant 1$, $g \equiv 0$ on U, and $g \equiv 1$ on V.*

Given $f \in \mathcal{A}(X,E)$, we define the cozero set of f as

$$c(f) := \{x \in X : f(x) \neq 0\}.$$

DEFINITION 2.2. A map $T : \mathcal{A}(X,E) \to \mathcal{A}(Y,F)$ is said to be *separating* if it is additive and $c(Tf) \cap c(Tg) = \emptyset$ whenever $f, g \in \mathcal{A}(X,E)$ satisfy $c(f) \cap c(g) = \emptyset$. Besides T is said to be *biseparating* if it is bijective and both T and T^{-1} are separating.

As for the spaces of linear functions, $L(E,F)$ and $I(E,F)$ stand for the space of continuous linear maps from E to F and the set of all linear isometries from E onto F, respectively. We consider that both $L(E,F)$ and its subset $I(E,F)$ are endowed with the strong operator topology, that is, the coarsest topology such that the mappings $S \hookrightarrow S\mathbf{e}$ are continuous for every $\mathbf{e} \in E$ (see for instance [7]).

DEFINITION 2.3. A surjective linear isometry $T : \mathcal{A}(X,E) \to \mathcal{A}(Y,F)$ is said to be a *strong Banach-Stone map* if there exist a continuous map $J : Y \to I(E,F)$ and a surjective homeomorphism $\phi : Y \to X$ such that for every $f \in \mathcal{A}(X,E)$ and $y \in Y$, $(Tf)(y) = (Jy)(f(\phi(y)))$. In Context 4, a strong Banach-Stone map $T : C_{\mathrm{b}}^{\mathrm{u}}(X,E) \to C_{\mathrm{b}}^{\mathrm{u}}(Y,F)$ is said to be *uniform* if both ϕ and ϕ^{-1} are uniformly continuous.

MULTIPLIERS AND CENTRALIZER. For a Banach space B, we denote by Ext_B the set of extreme points of the closed unit ball of its dual B'.

DEFINITION 2.4. Given a Banach space B, a continuous linear operator $T : B \to B$ is said to be a *multiplier* if every $p \in \mathrm{Ext}_B$ is an eigenvector for the transposed operator T', i.e. if there is a function $a_T : \mathrm{Ext}_B \to \mathbb{K}$ such that $p \circ T = a_T(p)p$ for every $p \in \mathrm{Ext}_B$.

DEFINITION 2.5. Let B be a Banach space. Given two multipliers $T, S : B \to B$, we say that S is an *adjoint* for T if $a_S = \overline{a}_T$, that is, if a_S coincides with a_T when $\mathbb{K} = \mathbb{R}$, and with the complex conjugate of a_T when $\mathbb{K} = \mathbb{C}$.

The centralizer of B is the set of those multipliers $T : B \to B$ for which an adjoint exists.

When it exists, the adjoint operator for T, which must be unique, will be denoted by T^*. On the other hand, the centralizer of B will be denoted by $Z(B)$ (notice that when $\mathbb{K} = \mathbb{R}$, the centralizer of B consists of the set of all multipliers of B).

Given $h \in \mathcal{A}(X, \mathbb{K})$, we define the operator $M_h : \mathcal{A}(X, E) \to \mathcal{A}(X, E)$ as $M_h(f) := hf$ for each $f \in \mathcal{A}(X, E)$.

3. MAIN RESULTS

We first state a proposition which will be crucial to prove the main result of the paper. It is not the first time that the relation "linear isometry–biseparating map" is used to prove Banach-Stone theorems between spaces of vector-valued continuous functions defined on (locally) compact spaces (see for instance [12], where the case of E and F strictly convex or with strictly convex dual is covered; see also [1] for related results).

PROPOSITION 3.1. *Suppose that $Z(E)$ and $Z(F)$ are one-dimensional. If $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ is a surjective linear isometry, then T is biseparating.*

Finally we state our main result.

THEOREM 3.2. *Suppose that $Z(E)$ and $Z(F)$ are one-dimensional. If $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ is a surjective linear isometry, then it is a strong Banach-Stone map. If we are in Context 4, it is also uniform.*

REMARK 3.3. Suppose that X and Y are any completely regular spaces (not necessarily included in Contexts 1–4), and E and F are infinite-dimensional Banach spaces. It is easy to see that the proof of Proposition 3.1 (see next section) is also valid to show that if $T : C_{\mathrm{b}}(X, E) \to C_{\mathrm{b}}(Y, F)$ is a surjective linear isometry, then it is a biseparating map. According to Theorem 3.4 in [2], we have that the realcompactifications of X and Y are homeomorphic. We conclude that the natural setting to study such isometries for infinite-dimensional E and F is that where X and Y are realcompact.

Consequently Theorem 3.2 cannot be stated in general if we assume that our spaces are not included in Contexts 1–4. Example 3.4 below shows that even if X implies to be included in one of our contexts and Y is included in a different one, then the theorem is no longer valid. As for Example 3.5, we see that in Context 2, the fact that E and F are infinite-dimensional is essential to get the description given in Theorem 3.2. Finally, in Example 3.6, we see that in Context 4, the requirement of completeness of X and Y is necessary to get the homeomorphism ϕ given in Theorem 3.2.

EXAMPLE 3.4. Take $X = W(\omega_1) := \{\sigma : \sigma < \omega_1\}$, where ω_1 denotes the first uncountable ordinal ([10], 5.12), and let Y be its Stone-Čech compactification, which coincides with its realcompactification. It is clear that X is first countable and completely regular (but not realcompact). Suppose $E = l^2$, which satisfies that $Z(E)$ is one-dimensional. Since E is realcompact ([10], 8.2), every $f \in C_{\mathrm{b}}(X, E)$ can be extended to a map $f' \in C_{\mathrm{b}}(Y, E)$.

EXAMPLE 3.5. Take $X = \mathbb{N}$ and $Y = \mathbb{N} \cup \{\sigma\}$, where $\sigma \in \beta\mathbb{N} \setminus \mathbb{N}$. Clearly each $f \in C_{\mathrm{b}}(X)$ admits a continuous extension $f' : Y \to \mathbb{K}$. We have that X and Y are not homeomorphic (see 4M in [10]).

EXAMPLE 3.6. Take $X = (0, 1)$, $Y = [0, 1]$, and E any Banach space. It is easy to see that each $f \in C^{\mathrm{u}}_{\mathrm{b}}(X, E)$ can be extended to a map $f' \in C^{\mathrm{u}}_{\mathrm{b}}(Y, E)$.

In the three examples above, the operator sending each f into its extension f' turns out to be a surjective linear isometry which is not a strong Banach-Stone map.

4. PROOFS

With a proof similar to that of Proposition 4.7 (i) in [6], we have the following result.

LEMMA 4.1. *For each $h \in \mathcal{A}(X, \mathbb{K})$, the operator M_h belongs to $Z(\mathcal{A}(X, E))$.*

Notice that when $\mathbb{K} = \mathbb{C}$, given $p \in \mathrm{Ext}_{\mathcal{A}(X,E)}$, its real part $\mathrm{Re}\, p$ belongs to $\mathrm{Ext}_{\mathcal{A}(X,E)_{\mathbb{R}}}$ (where $\mathcal{A}(X, E)_{\mathbb{R}}$ is $\mathcal{A}(X, E)$ viewed as a *real* space). Next, considering the real and imaginary parts of h, $\mathrm{Re}\, h$ and $\mathrm{Im}\, h$, we have that $(\mathrm{Re}\, p) \circ M_{\mathrm{Re}\, h} = a_{M_{\mathrm{Re}\, h}}(\mathrm{Re}\, p)\mathrm{Re}\, p$ and $(\mathrm{Re}\, p) \circ M_{\mathrm{Im}\, h} = a_{M_{\mathrm{Im}\, h}}(\mathrm{Re}\, p)\mathrm{Re}\, p$. Taking into account that $p(f) = \mathrm{Re}\, p(f) - \mathrm{i}\mathrm{Re}\, p(\mathrm{i}f)$ for every $f \in \mathcal{A}(X, E)$, it is straightforward to see that $a_{M_h}(p) = a_{M_{\mathrm{Re}\, h}}(\mathrm{Re}\, p) + \mathrm{i}a_{M_{\mathrm{Im}\, h}}(\mathrm{Re}\, p)$. We deduce that the adjoint for M_h is $M_{\overline{h}}$.

Next we are going to see that the converse of Lemma 4.1 is also true when $Z(E)$ is trivial (i.e., one-dimensional). First we will state the following lemma, whose proof is easy.

LEMMA 4.2. *Let $r, s \in (0,1]$ satisfy $s > 1 - r/100$. Let $\alpha, \beta \in \mathbb{K}$ be such that $\alpha/\beta \in (-\infty, 0)$. If $|\alpha|, |\beta| > r/3$, then either $|s - \alpha| > 1$ or $|s - \beta| > 1$.*

LEMMA 4.3. *Suppose that $Z(E)$ is one-dimensional. Given an operator $T \in Z(\mathcal{A}(X,E))$, there exists $h \in \mathcal{A}(X,\mathbb{K})$ such that $T = M_h$.*

Proof. Suppose that $T \in Z(\mathcal{A}(X,E))$. Then there exists a map

$$a_T : \mathrm{Ext}_{\mathcal{A}(X,E)} \to \mathbb{K}$$

such that $q \circ T = a_T(q)q$ for every $q \in \mathrm{Ext}_{\mathcal{A}(X,E)}$.

For $x \in X$, define $T_x : E \to E$ as $T_x\mathbf{e} := (T\widehat{\mathbf{e}})(x)$ for each $\mathbf{e} \in E$, that is, $T_x = e_x \circ T \circ \mathbf{i}$, where $e_x : \mathcal{A}(X,E) \to E$ is the evaluation map at x, and $\mathbf{i} : E \to \mathcal{A}(X,E)$ is the natural embedding.

CLAIM 1. *If $p \in \mathrm{Ext}_E$ and $x_0 \in X$, then $p \circ e_{x_0} \in \mathrm{Ext}_{\mathcal{A}(X,E)}$.*

Suppose that $p \circ e_{x_0} = \alpha p_1 + (1-\alpha)p_2$, where p_1, p_2 are points in the closed unit ball of the dual space $\mathcal{A}(X,E)'$, and $0 < \alpha < 1$. We have to prove that $p_1 = p_2 = p \circ e_{x_0}$.

Notice that $\mathcal{A}(X,E)$ can be expressed as the direct sum of the closed subspaces $E_1 := \{\widehat{\mathbf{e}} : \mathbf{e} \in E\}$ and $E_2 := \{f \in \mathcal{A}(X,E) : f(x_0) = 0\}$. It is easy to see that if we define, for $i = 1,2$, $q_i : E \to \mathbb{K}$ as $q_i(\mathbf{e}) := p_i(\widehat{\mathbf{e}})$ for every $\mathbf{e} \in E$, then q_1 and q_2 belong to the closed unit ball of E'. Clearly we have that, for $\mathbf{e} \in E$,

$$p(\mathbf{e}) = (p \circ e_{x_0})(\widehat{\mathbf{e}}) = \alpha p_1(\widehat{\mathbf{e}}) + (1-\alpha)p_2(\widehat{\mathbf{e}}) = \alpha q_1(\mathbf{e}) + (1-\alpha)q_2(\mathbf{e}),$$

that is, $p = \alpha q_1 + (1-\alpha)q_2$. Next, since $p \in \mathrm{Ext}_E$, we deduce that $q_1 = q_2 = p$. This implies that $p_1 = p_2 = p \circ e_{x_0}$ in the subspace E_1. Our next step will be to prove that $p_1 = p_2 = 0$ in the subspace E_2.

We clearly have that, in E_2,

$$p_2 = \alpha_0 p_1,$$

where $\alpha_0 := \alpha/(\alpha - 1) < 0$. Assume without less of generality that $|\alpha_0| \geqslant 1$, that is, $1 - \alpha \leqslant \alpha$ (otherwise we would work with p_2 instead of p_1). Suppose that there exists $f_0 \in E_2$ with $\|f_0\| = 1$ and $r := p_1(f_0) > 0$. Next take $\mathbf{e} \in E$ with $\|\mathbf{e}\| = 1$ and such that $p(\mathbf{e}) > 1 - r/100$. By Lemma 2.1, we can find $g \in \mathcal{A}(X,\mathbb{K})$, $0 \leqslant g \leqslant 1$, such that $g(\{x_0\}) = 1$ and

$$g(\{x \in X : \|f_0(x)\| \geqslant r/100\}) \equiv 0.$$

Suppose now that $t := p_1((1-g)\mathbf{e}) \in \mathbb{K}$ satisfies $|t| > r/3$. Then we have that also $|p_2((1-g)\mathbf{e})| = |\alpha_0 t| > r/3$, and by Lemma 4.2, either

$$|p_1(g\mathbf{e})| = |p(\mathbf{e}) - t| > 1 \quad \text{or} \quad |p_2(g\mathbf{e})| = |p(\mathbf{e}) - \alpha_0 t| > 1,$$

which is impossible because $\|g\mathbf{e}\| \leqslant 1$ and $\|p_i\| \leqslant 1$ ($i = 1,2$).

Consequently $|t| \leqslant r/3$ and $|p_1(g\mathbf{e})| \geqslant 1 - r/100 - r/3$. Next, multiplying by a number of modulus one if necessary, we may assume that $p_1(g\mathbf{e}) > 0$. Thus, $\|f_0 + g\mathbf{e}\| \leqslant 1 + r/100$ and

$$p_1(f_0 + g\mathbf{e}) \geqslant r + 1 - \frac{r}{100} - \frac{r}{3} > 1 + \frac{r}{100},$$

which contradicts the fact that $\|p_1\| \leqslant 1$.

As a consequence $p_1 = p_2 = p \circ e_{x_0}$, and $p \circ e_{x_0} \in \mathrm{Ext}_{\mathcal{A}(X,E)}$. The claim is proved.

CLAIM 2. *For $x_0 \in X$, T_{x_0} belongs to $Z(E)$.*

By Claim 1, if $p \in \mathrm{Ext}_E$, then $(p \circ e_{x_0}) \circ T = a_T(p \circ e_{x_0}) p \circ e_{x_0}$, which gives us that, for every $f \in \mathcal{A}(X,E)$,

$$p((Tf)(x_0)) = a_T(p \circ e_{x_0}) p(f(x_0)).$$

Consequently we have that, whenever $p \in \mathrm{Ext}_E$ and $\mathbf{e} \in E$,

$$p(T_{x_0}(\mathbf{e})) = a_T(p \circ e_{x_0}) p(\mathbf{e}).$$

In this way, if we define $a_{T_{x_0}} : \mathrm{Ext}_E \to \mathbb{K}$ as $a_{T_{x_0}}(p) := a_T(p \circ e_{x_0})$, then we will have that, for every $p \in \mathrm{Ext}_E$,

$$p \circ T_{x_0} = a_{T_{x_0}}(p) p.$$

As a consequence, T_{x_0} is a multiplier.

But notice that working as above we can prove that the operator $e_{x_0} \circ T^* \circ \mathbf{i} : E \to E$ is also a multiplier. On the other hand it is straightforward to see that (in the complex setting) it is the adjoint for T_{x_0}. Consequently T_{x_0} belongs to $Z(E)$, and the claim is proved.

Now, as $Z(E) = \mathbb{K}\,\mathrm{Id}_E$ (where $\mathrm{Id}_E : E \to E$ stands for the identity map on E), we have that, for each $x \in X$, there exists $a_x \in \mathbb{K}$ such that $T_x = a_x \mathrm{Id}_E$, and this implies clearly that, for every $p \in \mathrm{Ext}_E$, $a_{T_x}(p) = a_x$, that is, a_{T_x} is a constant function for each $x \in X$.

Thus, given $f \in \mathcal{A}(X,E)$, we saw above that, for every $p \in \mathrm{Ext}_E$, $p((Tf)(x)) = a_T(p \circ e_x) p(f(x))$, that is,

$$p((Tf)(x)) = a_x p(f(x)) = p(a_x f(x)).$$

This implies that

$$(Tf)(x) = a_x f(x),$$

because Ext_E separates the points of E. Since this is true for every $x \in X$, we conclude that, if we define $h : X \to \mathbb{K}$ as $h(x) := a_x$ for each $x \in X$, then $Tf = hf$ for every $f \in \mathcal{A}(X,E)$. Finally, since for $\mathbf{e} \in E \setminus \{0\}$, $T\widehat{\mathbf{e}} = h\widehat{\mathbf{e}}$ belongs to $\mathcal{A}(X,E)$, we deduce that $h \in \mathcal{A}(X,\mathbb{K})$. Consequently we can say that $T = M_h$. ∎

The proof of the following lemma is an adaptation of the one given for Lemma 4.13 (i) in [6].

LEMMA 4.4. *If $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ is a surjective linear isometry, then for each $h \in \mathcal{A}(Y, \mathbb{K})$, the map $\widehat{T}M_h$, defined as $(\widehat{T}M_h)(f) := T^{-1}(hTf)$ for each $f \in \mathcal{A}(X, E)$, belongs to $Z(\mathcal{A}(X, E))$.*

Proof. First notice that the transposed operator $(T^{-1})' : \mathcal{A}(X, E)' \to \mathcal{A}(Y, F)'$ is a linear surjective isometry, and consequently it maps $\mathrm{Ext}_{\mathcal{A}(X,E)}$ onto $\mathrm{Ext}_{\mathcal{A}(Y,F)}$. So, for $p \in \mathrm{Ext}_{\mathcal{A}(X,E)}$, $(T^{-1})'(p)$ belongs to $\mathrm{Ext}_{\mathcal{A}(Y,F)}$, and then, by Lemma 4.1, it is clear that $((T^{-1})'(p)) \circ M_h = a_{M_h}((T^{-1})'(p)) \cdot (T^{-1})'(p)$. But this means that, for every $p \in \mathrm{Ext}_{\mathcal{A}(X,E)}$, $p \circ (T^{-1} \circ M_h \circ T) = a_{M_h}(p \circ T^{-1}) \cdot p$. As a consequence, if we define $a_{T^{-1} \circ M_h \circ T}(p) := a_{M_h}(p \circ T^{-1})$, then we have that $T^{-1} \circ M_h \circ T$ is a multiplier.

So the lemma is proved if $\mathbb{K} = \mathbb{R}$. Now, if $\mathbb{K} = \mathbb{C}$, we just have to find an adjoint for $T^{-1} \circ M_h \circ T$. But notice that if $h \in \mathcal{A}(Y, \mathbb{K})$, then $\overline{h}$ also belongs to $\mathcal{A}(Y, \mathbb{K})$. We deduce that, in the same way as above, $T^{-1} \circ M_{\overline{h}} \circ T$ is also a multiplier and, for every $p \in \mathrm{Ext}_{\mathcal{A}(X,E)}$, $a_{T^{-1} \circ M_{\overline{h}} \circ T}(p) = a_{M_{\overline{h}}}(p \circ T^{-1})$. Finally, since the adjoint for M_h is $M_{\overline{h}}$, as we remarked after Lemma 4.1, we conclude that $T^{-1} \circ M_{\overline{h}} \circ T$ is the adjoint for $T^{-1} \circ M_h \circ T$, and we are done. ∎

Proof of Proposition 3.1. Clearly, it is enough to prove that T^{-1} is separating, because a similar argument allows us to conclude that T is also separating.

Suppose that g_1 and g_2 in $\mathcal{A}(Y, F) \setminus \{0\}$ satisfy $c(g_1) \cap c(g_2) = \emptyset$, and take $x_0 \in X$ with

$$r := \|(T^{-1}g_1)(x_0)\| > 0.$$

We will see that $(T^{-1}g_2)(x_0) = 0$. First, by Lemma 2.1, we can take $k \in \mathcal{A}(Y, \mathbb{K})$ such that $0 \leqslant k \leqslant 1$, $k \equiv 1$ on $\{y \in Y : \|g_1(y)\| \geqslant r/2\}$ and $k \equiv 0$ on $\{y \in Y : \|g_1(y)\| \leqslant r/3\}$. Now define $g_1' := kg_1$. It is clear that $\|g_1' - g_1\| \leqslant r/2$. Consequently, as T^{-1} is an isometry, we see that $(T^{-1}g_1')(x_0) \neq 0$. Next, using again Lemma 2.1, take $h \in \mathcal{A}(Y, \mathbb{K})$, $0 \leqslant h \leqslant 1$, such that

$$h \equiv 0 \text{ on } \{y \in Y : \|g_1(y)\| \leqslant r/4\} \quad \text{and} \quad h \equiv 1 \text{ on } \{y \in Y : \|g_1(y)\| \geqslant r/3\}.$$

It is clear that $hg_1' = g_1'$ and $hg_2 = 0$.

On the one hand, by Lemma 4.4 we know that $\widehat{T}M_h \in Z(\mathcal{A}(X, E))$, and then by Lemma 4.3 $\widehat{T}M_h = M_f$ for some $f \in \mathcal{A}(X, \mathbb{K})$, that is,

$$(\widehat{T}M_h)(T^{-1}g_1') = fT^{-1}g_1' \quad \text{and} \quad (\widehat{T}M_h)(T^{-1}g_2) = fT^{-1}g_2.$$

On the other hand, by definition,

$$(\widehat{T}M_h)(T^{-1}g_1') = T^{-1}(hg_1') = T^{-1}g_1' \quad \text{and} \quad (\widehat{T}M_h)(T^{-1}g_2) = T^{-1}(hg_2) = 0.$$

This implies that $fT^{-1}g_1' = T^{-1}g_1'$ and $fT^{-1}g_2 = 0$, which gives us that $f \equiv 1$ on $c(T^{-1}g_1')$ and $f \equiv 0$ on $c(T^{-1}g_2)$. We deduce that $x_0 \notin c(T^{-1}g_2)$. It is easy to see now that T^{-1} is separating, as we wanted to see. ∎

Now we are in a position to prove the main theorem. But first, let us recall a necessary result from [3].

THEOREM 4.5 ([3], Theorem 3.5 and Corollary 4.3). *Suppose that $T : \mathcal{A}(X, E) \to \mathcal{A}(Y, F)$ is a linear and continuous biseparating map. Then there exist a surjective homeomorphism $\phi : Y \to X$ and a continuous map $J : Y \to L(E, F)$ such that $(Tf)(y) = (Jy)(f(\phi(y))$ for every $f \in \mathcal{A}(X, E)$ and $y \in Y$; also Jy is bijective for every $y \in Y$. On the other hand, if we are in Context* 4*, then ϕ is also a uniform homeomorphism.*

Proof of Theorem 3.2. Since T is biseparating, we can use the description given in Theorem 4.5, and we have just to prove that, for each $y \in Y$, $Jy \in I(E, F)$. Take any $y \in Y$ and $\mathbf{e} \in E$. Then $\|(Jy)(\mathbf{e})\| = \|(T\widehat{\mathbf{e}})(y)\|$. We are going to see that $\|(T\widehat{\mathbf{e}})(y)\| = \|\mathbf{e}\|$. Of course, if this is not the case for some $y_0 \in Y$, then $\|(T\widehat{\mathbf{e}})(y_0)\| < \|\mathbf{e}\|$. Let

$$r \in (\|(T\widehat{\mathbf{e}})(y_0)\|, \|\mathbf{e}\|).$$

Next, using Lemma 2.1, we can take $g \in \mathcal{A}(Y, \mathbb{K})$ such that $0 \leqslant g \leqslant 1$, $g \equiv 0$ on $\{y \in Y : \|(T\widehat{\mathbf{e}})(y)\| \geqslant r\}$, and $g(y_0) = 1$. It is clear that there exists $\alpha > 0$ such that $\|(1 + \alpha g(y))(T\widehat{\mathbf{e}})(y)\| \leqslant \|\mathbf{e}\|$ for every $y \in Y$, that is,

$$\|T\widehat{\mathbf{e}} + \alpha g T\widehat{\mathbf{e}}\| \leqslant \|\mathbf{e}\|.$$

Now, if $f_0 \in \mathcal{A}(X, E)$ satisfies $Tf_0 = \alpha g T\widehat{\mathbf{e}}$, then since $g(y_0) = 1$,

$$(Tf_0)(y_0) = \alpha g(y_0)(T\widehat{\mathbf{e}})(y_0) = (T\alpha\widehat{\mathbf{e}})(y_0).$$

We deduce that $(T(f_0 - \alpha\widehat{\mathbf{e}}))(y_0) = 0$, which implies by Theorem 4.5 that $(Jy_0)(f_0(\phi(y_0)) - \alpha\mathbf{e}) = 0$. As a consequence, since Jy_0 is bijective, $f_0(\phi(y_0)) = \alpha\mathbf{e}$ and

$$\|\widehat{\mathbf{e}} + f_0\| \geqslant \|(\widehat{\mathbf{e}} + f_0)(\phi(y_0))\| = \|(1 + \alpha)\mathbf{e}\| > \|\mathbf{e}\|,$$

which contradicts the fact that T is an isometry. We conclude $\|(T\widehat{\mathbf{e}})(y)\| = \|\mathbf{e}\|$ for every $y \in Y$, and we are done. ∎

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