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A FREE GIRSANOV PROPERTY FOR FREE BROWNIAN MOTIONS

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ABSTRACT. A "free Girsanov" property is proved for free Brownian motions. It is reminiscent of the classical Girsanov theorem in probability theory.

In the free probability context, we prove that if $(\sigma_s)_{s\in\mathbb{R}^+}$ is a free Brownian motion in (M,τ), if x is a process free from the σ_s, if $\widetilde{\sigma_s} = \sigma_s + \int\limits_0^s x(u)\mathrm{d}u$, then there is a trace $\widetilde{\tau}$ such that$(\widetilde{\sigma_s})_{s\in\mathbb{R}^+}$ is a free Brownian motion for $\widetilde{\tau}$ and the two traces are "asymptotically equivalent". This means that τ respectively $\widetilde{\tau}$ are asymptotic limits of states Ψ_n respectively $\widetilde{\Psi}_n$ and that for each n $\widetilde{\Psi}_n$ is obtained from Ψ_n by a change of probability given by an exponential density.

KEYWORDS: *Free probability theory, free products of C^* algebras, free Brownian motion, Girsanov theorem.*

MSC (2000): 46L54, 46L09, 60J65.

1. INTRODUCTION

The context of the present work is that of free probability theory. D. Voiculescu has introduced and studied the theory of free probability, giving a meaning to free random variables, free product of states and free Brownian motions (see the book by Voiculescu, Dykema and Nica [8], for a survey).

In classical probability theory the Girsanov theorem is a very important theorem for stochastic calculus (see for exemple [3]).

In view of stochastic calculus for free Brownian motions, Biane and Speicher (see [1]) have proved an Ito formula for free stochastic integrals. The purpose of this paper is to obtain for free Brownian motions a property which is reminiscent of the classical Girsanov property.

The usual Girsanov theorem says that if one translates a Brownian motion by an adapted stochastic process ($\widetilde{W}_s = W_s + \int\limits_0^s \theta(u)\mathrm{d}u$) one can find a change of probability given by an exponential density such that $(\widetilde{W}_s)_{s\in\mathbb{R}^+}$ is a Brownian motion for this new probability.

In the context of free probability we want to prove a result which is in the same vein.

Let $(\sigma_s)_{s\in\mathbb{R}^+}$ be a free Brownian motion in (M,τ). Let x be a measurable process with values in N a commutative subalgebra of M free from the $(\sigma_s)_{s\in\mathbb{R}^+}$. Assume that $x(u) = x(u)^*$ for all u. Let $\widetilde{\sigma}_s = \sigma_s + \int_0^s x(u)\mathrm{d}u$. We want to prove the existence of a new trace $\widetilde{\tau}$ closely related to the trace τ such that $(\widetilde{\sigma}_s)_{s\in\mathbb{R}^+}$ is a free Brownian motion for the new trace $\widetilde{\tau}$ and such that the joint distribution of $(\widetilde{\sigma}_s, x(u))_{s,u\in\mathbb{R}^+}$ for $\widetilde{\tau}$ is the same as the joint distribution of $(\sigma_s, x(u))_{s,u\in\mathbb{R}^+}$ for τ.

Unfortunately as the von Neumann algebra generated by a free Brownian motion is a factor, there is only one normalized trace on it. Thus it is impossible to find a new trace on the von Neumann algebra generated by N and the σ_s satisfying the required properties.

Nevertheless notice that a free Brownian motion $(\sigma_s)_{s\in\mathbb{R}^+}$ in (M,τ) is just defined by the joint distribution of the σ_s for τ. And Voiculescu has proved that a free Brownian motion is an asymptotic limit of matrices of random processes. Using this point of view, we prove the following result:

There is a new trace $\widetilde{\tau}$ on $N * \mathbb{C}[(\sigma_s)_{s\in\mathbb{R}^+}]$ such that the joint distribution of $((\widetilde{\sigma}_s)_{s\in\mathbb{R}^+}, x(u)_{u\in\mathbb{R}^+})$ for this new trace $\widetilde{\tau}$ is the same as the joint distribution of $(\sigma_{s\in\mathbb{R}^+}, x(u)_{u\in\mathbb{R}^+})$ for the trace τ (in particular $(\widetilde{\sigma}_s)_{s\in\mathbb{R}^+}$ is a free Brownian motion for the new trace) and the two traces are asymptotically equivalent.

This has the following meaning: There is a family $(\widetilde{Z}_n(s))_{n\in\mathbb{N}^*}$ of matrices of random processes $\widetilde{Z}_n(s) \in \mathcal{M}_n(L^\infty[0,1] * L)$ and a family $(D_n(u))_{n\in\mathbb{N}^*}$ of diagonal matrices of real processes such that

$$(\mathbb{C}[\sigma_s, x(u)]_{s,u\in\mathbb{R}^+}, \widetilde{\tau}) = \lim_{n\to\infty} (\mathbb{C}[\widetilde{Z}_n(s), D_n(u)]_{s,u\in\mathbb{R}^+}, \widetilde{\Psi}_n)$$

and

$$(\mathbb{C}[\sigma_s, x(u)]_{s,u\in\mathbb{R}^+}, \tau) = \lim_{n\to\infty} (\mathbb{C}[\widetilde{Z}_n(s), D_n(u)]_{s,u\in\mathbb{R}^+}, \Psi_n)$$

where $\widetilde{\Psi}_n$ and Ψ_n are two traces on $\mathcal{M}_n(L^\infty[0,1] * L)$. $\widetilde{\Psi}_n$ is obtained from Ψ_n by a change of probability given by an exponential density h_n

$$\Psi_n = \frac{1}{n}\mathrm{Tr}_n(\phi * \phi_0) \quad \text{and} \quad \widetilde{\Psi}_n = \frac{1}{n}\mathrm{Tr}_n(\phi * \phi_0(h_n.))$$

(and for all $p \in \mathbb{N}$, $\sup_{n\in\mathbb{N}} \phi_0(h_n^p) < \infty$) i.e. the limit joint distribution of $(\widetilde{Z}_n(s), D_n(u))$ for $\widetilde{\Psi}_n$ is the joint distribution of $(\sigma_s, x(u))$ for $\widetilde{\tau}$ and the limit joint distribution of $(\widetilde{Z}_n(s), D_n(u))$ for Ψ_n is the joint distribution of $(\sigma_s, x(u))$ for τ.

In order to prove this result we make use, as already mentioned, of the asymptotic model of matrices of random processes, and we modelize the process $(x(u))$ by diagonal matrices. For each $n \in \mathbb{N}$ we can apply the classical Girsanov theorem and this gives rise to a change of probability given by an exponential density d_n. Unfortunately, these densities d_n explode as n tends to infinity and so we have to renormalize the asymptotic model of matrices of random processes in order to get densities h_n which do not explode.

The paper is organised as follows:

After a few recalls in Section 2, we construct in Section 3 a new asymptotic model of random matrices with values in a free product algebra, in order to make the renormalization. This is a technical part making use of computation of free cumulants and non crossing partitions introduced by Speicher [5].

In Section 4 making use of this new asymptotic model, we prove our main result: A free Girsanov property for free Brownian motions.

2. SOME RECALLS

FREE PROBABILITY THEORY. We recall some definitions and results in free probability theory which can be found in the references [6], [7], [8].

DEFINITION. A *$*$-free probability space* (A, ϕ) is an involutive unital algebra A over $\mathbb{C}$ with a state $\phi : A \to \mathbb{C}$ i.e. a linear functional such that $\phi(1) = 1$ and $\phi(x^*) = \overline{\phi(x)}$. Elements of A are called *random variables*.

DEFINITION. A family $(f_i)_{i\in I}$ of random variables of A is *free* if the family $(A_i)_{i\in I}$ of $*$-algebras generated by 1 and f_i is free: i.e. if $\phi(a_1a_2, \ldots, a_n) = 0$ whenever $a_j \in A_{i(j)}$ with $i(j) \neq i(j+1)$ $(1 \leqslant j \leqslant n-1)$ and $\phi(a_j) = 0$ $(1 \leqslant j \leqslant n)$.

DEFINITION. A random variable σ in (A, ϕ) is *semicircular centered of variance r^2* if the distribution of σ is

$$\phi(\sigma^\alpha) = \frac{2}{\pi r^2}\int_{-r}^{r} t^\alpha \sqrt{r^2 - t^2}\mathrm{d}t.$$

DEFINITION. A *free Brownian motion in* (A, ϕ) is a family $(\sigma_s)_{s\in\mathbb{R}^+}$ of random variables such that:

(i) $\sigma_0 = 0$;

(ii) if $0 \leqslant s' \leqslant s \leqslant t$, $\sigma_t - \sigma_s$ is semicircular centered of variance $t - s$ and is free from $\sigma_{s'}$.

One has also the following very important connection between free semicircular random variables and Gaussian random matrices:

Consider a probability space $(\Sigma, \mathrm{d}\sigma)$. $L^\infty(\Sigma, \mathrm{d}\sigma)$ is a unital algebra with the state ϕ_0 defined by $\phi_0(f) = E_0(f) = \int_\Sigma f\mathrm{d}\sigma$. Let

$$L = \bigcap_{p\geqslant 1} L^p(\Sigma).$$

We denote by ϕ_n the state defined on $\mathcal{M}_n(L)$ by

$$\phi_n\Big(\sum_{1\leqslant i,j\leqslant n} b_{ij}e(i,j,n)\Big) = \frac{1}{n}\sum_{1\leqslant i\leqslant n}(\phi_0)(b_{ii}) = \frac{1}{n}\mathrm{Tr}_n((\phi_0(b_{i,j})_{1\leqslant i,j\leqslant n}))$$

(where $(e(i,j,n))_{1\leqslant i,j\leqslant n}$ is the canonical basis and $b_{i,j}\in L$).

Voiculescu has then proved the following theorem ([7], Theorem 2.2): let $Y(s,n) = \sum_{1\leqslant i,j\leqslant n} a(i,j,s,n)e(i,j,n)$ with $a(i,j,s,n)\in L$. Assume that

$$a(i,j,s,n) = \overline{a(j,i,s,n)}$$

and that $\mathrm{Re}(a(i,j,s,n)), 1\leqslant i\leqslant j\leqslant n, s\in\mathbb{N}$, $\mathrm{Im}(a(i,j,s,n)), 1\leqslant i<j\leqslant n, s\in\mathbb{N}$ are independent Gaussian random variables such that:

$$\begin{aligned} E_0(a(i,j,s,n)) &= 0,\\ E_0(\mathrm{Re}(a(i,j,s,n))^2) &= \frac{1}{2n} \quad \text{for } 1\leqslant i<j\leqslant n,\\ E_0(\mathrm{Im}(a(i,j,s,n))^2) &= \frac{1}{2n} \quad \text{for } 1\leqslant i<j\leqslant n,\\ E_0((a(i,i,s,n))^2) &= \frac{1}{n} \quad \text{for } 1\leqslant i\leqslant n. \end{aligned}$$

Consider the trace ϕ_n defined above. Let $D(j,n)$ be elements in Δ_n, the set of constant diagonal matrices, such that $\sup_{n\in\mathbb{N}}\|D(j,n)\|<\infty$, for each j; and such that for all j, $(D(j,n))$ has a limit distribution as $n\to\infty$. Then the family of subsets of random variables $\{Y(s,n) : s\in\mathbb{N}\}$ and $\{D(j,n) : j\in\mathbb{N}\}$ is asymptotically free, and the limit distributions of the $Y(s,n)$ are semicircle laws as $n\to\infty$.

It follows that a model for the free Brownian motion is the following one:

$$(\mathbb{C}[(\sigma_s)_{s\in\mathbb{R}^+}],\tau) = \lim_{n\to\infty}\mathbb{C}\Big[(B_{n,s})_{s\in\mathbb{R}^+}], \frac{1}{n}\mathrm{Tr}_n(\phi_0)\Big]$$

where $B_{n,s} = \Big(\frac{1}{\sqrt{n}}W_{n,i,j,s}\Big)_{1\leqslant i,j\leqslant n}$; the $(W_{n,i,j,s})_{1\leqslant i\leqslant j\leqslant n}$ being independent Brownian motions.

CLASSICAL GIRSANOV THEOREM. For this we refer to Karatzas and Shreve [3].

Let $(\Omega,(\mathcal{F}_s)_{0\leqslant s},P)$ be a filtered probability space. Let $(W_s)_{0\leqslant s}$ be a Brownian motion adapted to $(\mathcal{F}_s)$. Let $(\theta_u)_{0\leqslant u}$ be an adapted process such that

$$E\Big(\exp\int_0^\infty \theta_u^2 \mathrm{d}u\Big) < \infty.$$

Then $\widetilde{W}_s = W_s - \int\limits_0^s \theta_u \mathrm{d}u$ is a Brownian motion for the probability Q equivalent to the probability P defined by $Q(A) = \int\limits_A Z(s)\mathrm{d}P$ for all A in $\mathcal{F}_s$, where $Z(s) = \exp\Big(\int\limits_0^s \theta_u \mathrm{d}W_u - \frac{1}{2}\int\limits_0^s (\theta_u)^2 \mathrm{d}u\Big)$.

3. A NEW ASYMPTOTIC MODEL FOR FREE BROWNIAN MOTION

In this section we construct for the free Brownian motion an asymptotic model of random matrices with coefficients in a free product algebra. The motivation for the construction of this new model is to use a free product algebra in order to make a renormalization.

Consider a probability space $(\Sigma, \mathrm{d}\sigma)$. $L^\infty(\Sigma, \mathrm{d}\sigma)$ is a unital algebra with the state ϕ_0 defined by $\phi_0(f) = E_0(f) = \int f\mathrm{d}\sigma$. Let

$$L = \bigcap_{1\leqslant p<\infty} L^p(\Sigma).$$

Let μ be the Lebesgue measure on $[0,1]$, and the state ϕ defined on $L^\infty([0,1],\mu)$ by

$$\phi(f) = \int f\mathrm{d}\mu.$$

Now we consider the free product state $\phi * \phi_0$ on $L^\infty([0,1],\mu) * L^\infty(\Sigma, \mathrm{d}\sigma)$. We can extend $\phi * \phi_0$ to $L^\infty([0,1]) * L$. We then get a state still noted $\phi * \phi_0$ such that $L^\infty([0,1])$ is free from L for this state. We denote Ψ_n the state defined on $\mathcal{M}_n(L^\infty([0,1]) * L)$ by

$$\Psi_n\Big(\sum_{1\leqslant i,j\leqslant n} b_{ij} e(i,j,n)\Big) = \frac{1}{n}\sum_{1\leqslant i\leqslant n} (\phi * \phi_0)(b_{ii}) = \frac{1}{n}\mathrm{Tr}_n((\phi * \phi_0)(b_{i,j})_{1\leqslant i,j\leqslant n}).$$

We keep the same notations as in Section 2.

We now prove the existence of a new family of matrices of random processes with coefficients in $L^\infty([0,1]) * L$ which are asymptotically free and whose limit distributions are semi-circular laws. More precisely:

PROPOSITION 3.1. *For all $s \in \mathbb{N}$, and $n \in \mathbb{N}$, let*

$$\widetilde{Y}(s,n) = \sum_{1\leqslant i,j\leqslant n} \widetilde{a}(i,j,s,n) e(i,j,n)$$

with $\widetilde{a}(i,j,s,n) \in L^\infty([0,1]) * L$. Assume that

$$\widetilde{a}(i,j,s,n) = \sum_{k=1}^{n^2} q_{k,n} \sqrt{n} a(i,j,s,n)\, q_{k,n}$$

where the $q_{k,n}$ *are orthogonal projectors in* $L^\infty[0,1]$, $\sum_{k=1}^{n^2} q_{k,n} = 1$ *such that*

$$\phi(q_{k,n}) = \frac{1}{n^2}$$

and the $(a(i,j,s,n))_{s\in\mathbb{N},1\leqslant i\leqslant j\leqslant n,n\in\mathbb{N}}$ *are independent real normal Gaussian variables, i.e., in particular*

$$\begin{aligned} E_0(a(i,j,s,n)) &= 0, \\ E_0((a(i,j,s,n)^2) &= 1, \\ a(i,j,s,n) &= a(j,i,s,n). \end{aligned}$$

Consider the trace Ψ_n *defined above. Let* $D_n(j)$ *be elements in* Δ_n, *the set of diagonal matrices, such that* $\sup_{n\in\mathbb{N}} \|D_n(j)\| < \infty$, *for each j; and such that for all j,* $(D_n(j))$ *has a limit distribution as* $n \to \infty$.

Then the family of subsets $\{\widetilde{Y}(s,n) : s \in \mathbb{N}\}$ *and* $\{D_n(j) : j \in \mathbb{N}\}$ *are asymptotically free, and the limit distribution of the* $\widetilde{Y}(s,n)$ *are semicircular laws.*

This proposition is comparable with the Theorem 2.2 of [7] recalled in Section 2. The important property of this new asymptotic model is that it is renormalized: we have replaced the Gaussian random variables of variance $\frac{1}{n}$ of the theorem of Voiculescu by Gaussian random variables ($\sqrt{n}a(i,j,s,n)$) of variance n.

Although the proof follows the same lines of reasonning, the proof of Voiculescu must be significantly amended because the entries of these new matrices are in a free product algebra. We have to use the free calculus developped by Speicher [5].

We start with the following results.

LEMMA 3.2. *Let* $i \in \{1,\ldots,j\}$; *let* $(y_1, y_2, \ldots, y_j)$ *be random variables in L. Assume there is one* $i \in \{1,\ldots,j\}$ *such that* $y_i = az_i$, *where a is independent of all others* y_k *for* $k \neq i$ *and of* z_i, *and such that* $E_0(a) = 0$.

Then $(\phi * \phi_0)(qy_1q\cdots qy_jq) = 0$ *for each q projector in* $L^\infty([0,1])$.

Proof. The proof is done by recursion on j using the freeness.

For $j = 1$: $(\phi * \phi_0)(qy_1) = \phi(q)E_0(y_1) = \phi(q)E_0(a)E_0(z_1) = 0$.

Assume now that the result is true for j and prove it for $j+1$. From the freeness of $L^\infty([0,1])$ and L for $\phi * \phi_0$, we get that

$$(\phi * \phi_0)((q - \phi(q))(y_1 - E_0(y_1))(q - \phi(q))(y_2 - E_0(y_2))\cdots \\ (q - \phi(q))(y_{j+1} - E_0(y_{j+1}))) = 0.$$

If we develop the preceeding expression there is the term $(\phi * \phi_0)(qy_1qy_2\cdots qy_{j+1})$, and in all the other terms there is at least one $\phi(q)$ or one $E_0(y_k)$. So that all these terms can be written either $\alpha(\phi * \phi_0)(qt_1qt_2\cdots qt_k)$ with $k \leqslant j$ and $\alpha \in \mathbb{C}$; and the $(t_l)_{l\leqslant k}$ satisfy the same hypothesis as the y_l or $E_0(y_i)\alpha$ with $\alpha \in \mathbb{C}$. By recursion each of these terms is equal to 0. So $(\phi * \phi_0)(qy_1qy_2\cdots qy_{j+1}) = 0$, i.e. $(\phi * \phi_0)(qy_1qy_2\cdots qy_{j+1}q) = 0$ as $\phi * \phi_0$ is a trace and $q^2 = q$. ∎

COROLLARY 3.3. *If $s_i \neq s_1$ for any $i \neq 1$, then we have $\Psi_n(\widetilde{Y}(s_1,n)D_n(t_1)\cdots \widetilde{Y}(s_m,n)D_n(t_m)) = 0$.*

Proof. $\Psi_n(\widetilde{Y}(s_1,n)D_n(t_1)\cdots\widetilde{Y}(s_m,n)D_n(t_m))$ is a sum of terms

$$(\phi * \phi_0)(q\sqrt{n}a(i_1,j_1,s_1,n)d_1q\sqrt{n}a(i_2,j_2,s_2,n)d_2q\cdots(q\sqrt{n}a(i_m,j_m,s_m,n)d_m).$$

$a(i_1,j_1,s_1,n)$ is independent of all other $a(i_k,j_k,s_k,n)$ for $k\neq 1$. It follows that $E_0(a(i_1,j_1,s_1,n)=0$. So the result follows immediately from Lemma 3.2. ∎

We prove now the following technical lemma making use of the computation of free cumulants introduced by Speicher [5].

LEMMA 3.4. *Let $y_1,\ldots,y_j \in L$. Let $a \in L$, with $E_0(a) = 0$. Assume that a is independent of $y_1,\ldots,y_j$. Let q be a projector in $L^\infty[0,1]$. Denote $Y = y_1qy_2q\cdots qy_j$.*

(i) $k_\phi(q,q) = \phi(q) - \phi(q)^2$;
(ii) $k_{(\phi*\phi_0)}(q,Y) = (\phi*\phi_0)(qY) - \phi(q)(\phi*\phi_0)(Y)$;
(iii) $k_{(\phi*\phi_0)}(q,q,Y) = (1-2\phi(q))[(\phi*\phi_0)(qY) - \phi(q)(\phi*\phi_0)(Y)]$;
(iv) $(\phi*\phi_0)(a^2Y) = E_0(a^2)(\phi*\phi_0)(Y)$;
(v) $k_{(\phi*\phi_0)}(a,a,Y) = 0$.

Proof. (i) $\phi(q) = \phi(q^2) = \phi(q)^2 + k_\phi(q,q)$.
(ii) $(\phi*\phi_0)(qY) = k_{(\phi*\phi_0)}(q,Y) + \phi(q)(\phi*\phi_0)(Y)$.
(iii) $(\phi*\phi_0)(qY) = (\phi*\phi_0)(q^2Y) = k_\phi(q,q)(\phi*\phi_0)(Y) + 2\phi(q)k_{\phi*\phi_0}(q,Y) + k_{\phi*\phi_0}(q,q,Y) + \phi(q)^2(\phi*\phi_0)(Y)$. Using (i) and (ii) we get (iii).
(iv) From [5] as $L^\infty[0,1]$ and L are free for $\Phi*\phi_0$, we know that the cumulants mixing elements of $L^\infty[0,1]$ and L are 0; and furthermore, as $E_0(a)=0$, for a non crossing partition giving a non zero contribution, a cannot be alone. So

$$\begin{aligned}(\phi*\phi_0)(a^2Y) &= (\phi*\phi_0)(a^2y_jy_1qy_2q\cdots y_{j-1}q)\\ &= \sum_\pi k_\pi(q,q,\ldots,q)k_\pi(q,\ldots,q)k_\pi(a^2y_jy_1,y_{i_1},\ldots)k_\pi(y_{i_l},\ldots)\cdots.\end{aligned}$$

But a is independent of all the y_i so

$$k_\pi(a^2y_jy_1,y_{i_1},\ldots) = E_0(a^2)k_\pi(y_jy_1,y_{i_1},\ldots).$$

(v) $(\phi*\phi_0)(a^2Y) = k_{(\phi*\phi_0)}(a,a)(\phi*\phi_0)(Y) + k_{(\phi*\phi_0)}(a,a,Y)$ and $E_0(a^2) = (\phi*\phi_0)(a^2) = k_{(\phi*\phi_0)}(a,a)$ as $E_0(a)=0$.
It follows then from (iv) that $k_{(\phi*\phi_0)}(a,a,Y)=0$. ∎

LEMMA 3.5. *Let a,q and $Y \in L^\infty[0,1]*L$ as in Lemma 3.4. Then*

$$(\phi*\phi_0)(qaqaqY) = \phi(q)E_0(a^2)(\phi*\phi_0)(qY).$$

Proof. We have:

$$Y = y_1 q y_2 q \cdots q y_j,$$
$$(\phi * \phi_0)(qaqaqY) = \sum_{\pi \in NC(6)} k_\pi(q,a,q,a,q,Y).$$

Using the same arguments as in the proof of Lemma 3.4 and also the equality $k_{\phi*\phi_0}(a,a,Y) = 0$, we get

$$(\phi * \phi_0)(qaqaqY) = k_{\phi_0}(a,a)\phi(q)^3(\phi * \phi_0)(Y) + k_{\phi_0}(a,a)\phi(q)k_\phi(q,q)(\phi * \phi_0)(Y)$$
$$+k_{\phi_0}(a,a)\phi(q)k_{(\phi*\phi_0)}(q,q,Y) + 2k_{\phi_0}(a,a)\phi(q)^2 k_{(\phi*\phi_0)}(q,Y).$$

And now the result follows easily from the Lemma 3.4, and the equality $k_{\phi_0}(a,a) = E_0(a^2)$. ∎

LEMMA 3.6. *Let $a(i,j,s,n)_{(1\leqslant i \leqslant j \leqslant n)}$ be independent normal Gaussian variables in (L,ϕ_0). Let (B,ϕ) a $*$-free probability space.*

Let $q \in B$ be a projector such that $\phi(q) = 1/n^2$. Let $d(t,j,n)$ be elements in B commuting with q and uniformly bounded. Then

$$\phi * \phi_0(qa(i_1,i_2,s_1,n)d(t_1,i_2,n)qa(i_2,i_3,s_2,n)d(t_2,i_3,n)\cdots$$
$$qa(i_m,i_1,s_m,n)d(t_m,i_1,n)) =$$
$$\sum_\pi k_\pi[\, a(i_1,i_2,s_1,n),\ a(i_2,i_3,s_2,n),\ldots,\ a(i_m,i_1,s_m,n)]O\Big(\Big(\frac{1}{n^2}\Big)^{|\pi_B|}\Big)$$

where $|\pi_B|$ denotes the number of blocks of the restriction π_B of π to B.

If we denote by E_n the set of $(i_1,i_2,\ldots,i_m) \in \{1,\ldots,n\}^m$ such that

$$(\phi * \phi_0)(qa(i_1,i_2,s_1,n)d(t_1,i_2,n)qa(i_2,i_3,s_2,n)d(t_2,i_3,n)q\cdots$$
$$qa(i_m,i_1,s_m,n)d(t_m,i_1,n)) \neq 0,$$

then $\mathrm{Card}(E_n) = O(n^{(m/2)+1})$.

Proof. We use another time the free cumulants to compute

$$(\phi * \phi_0)(qa(i_1,i_2,s_1,n)d(t_1,i_2,n)qa(i_2,i_3,s_2,n)d(t_2,i_3,n)\cdots$$
$$qa(i_m,i_1,s_1,n)d(t_m,i_1,n)).$$

Using the hypothesis on the independence of the $a(i,j,s,n)$ and the Lemma 3.2, it follows exactly as in the proof of Theorem 2.2 of [7] that $\mathrm{Card}(E_n) = O(n^{(m/2)+1})$.

Now, from the Theorem 8.2 of [5] as the $a(i,j,s,n)$ are free from B for $\phi * \phi_0$, and $d(t,j,n)$ and q are in B, we can write:

$$\begin{aligned}&(\phi * \phi_0)(qa(i_1,i_2,s_1,n)d(t_1,i_2,n)qa(i_2,i_3,s_2,n)d(t_2,i_3,n)\cdots \\ &\qquad qa(i_m,i_1,s_m,n)d(t_m,i_1,n)) = \\ &\qquad \sum_{\pi} k_\pi[\,a(i_1,i_2,s_1,n),\ldots,a(i_m,i_1,s_m,n)]\phi_{\pi_B}[d(t_m,i_1,n)q,\ldots,d(t_{m-1},i_m,n)q].\end{aligned}$$

But $\phi_{\pi_B}[d(t_m,i_1,n)q,\ldots,d(t_{m-1},i_m,n)q] = O\Big(\big(\frac{1}{n^2}\big)^{|\pi_B|}\Big)$ where $|\pi_B|$ is the number of blocks of π_B (as the $d(t,j,n)$ are uniformly bounded). ∎

LEMMA 3.7. *Let k fixed. If π is a non crossing partition giving a non zero contribution in Lemma* 3.6, *the number of different blocks of π_B (i.e. $|\pi_B|$) is greater or equal to* $[\frac{m}{2}]+1$.

Proof. We do it by recursion on m.

Step 1. If $m=1$, we always obtain 0.

Step 2. If $m=2$, if the term associated to the partition π is non zero, the number of components containing the q is 2 (because $E_0(a(i,j,s,n))=0$).

Step 3. Let $m \geqslant 2$. Assume that the result is true for m and prove it for $m+1$. Let r be the minimal distance between two q which are in a same block of π. Two successive q can never be in the same component of π, (because $E_0(a(i,j,s,n)) = 0$). So $2 \leqslant r$. So there is l such that the $l^{\text{th}}q$ and the $(l+r)^{\text{th}}q$ are in the same block and all the q between are alone in one block of π. As the cumulants must be non crossing, π can be decomposed in a partition π' on

$$(a(i_l,i_{l+1},s_l,n),a(i_{l+1},i_{l+2},s_{l+1},n),\ldots,a(i_{l+r-1},i_{l+r},s_{l+r-1},n))$$

and a non crossing partition π'' on

$$\begin{aligned}&(d(t_m,i_1,n)q,a(i_1,i_2,s_1,n),d(t_1,i_2,n)q,\ldots,\\ &a(i_{l-1},i_l,s_{l-1},n),(a(i_{l+r},i_{l+r+1},s_{l+r},n),d(t_{l+r},i_{l+r+1},n)q,\ldots,a(i_m,i_1,s_m,n))\end{aligned}$$

and blocks reduced to q.

It follows that the blocks of π_B are either reduced to one element q or are blocks of the restriction of the partition π'' to B. By recursion, we know that the number of components of π'' containing the q is greater or equal to $[\frac{m-r}{2}]+1$. So the number of components of π_B is greater or equal to $[\frac{m-r}{2}]+1+r-1$; and as $2 \leqslant r$, $[\frac{m}{2}]+1 \leqslant [\frac{m-r}{2}]+1+r-1$. ∎

Before proving the Proposition 3.1, we give two other lemmas.

LEMMA 3.8. *There is a constant C_m (independent of n) such that for all π, for all (i_k,j_k,s_k),*

$$|k_\pi[a(i_1,j_1,s_1,n),\ldots,a(i_m,j_m,s_m,n)]| \leqslant C_m.$$

Proof. Since two of $a(i,j,s,n)_{i\leqslant j\leqslant n}$ are either equal or independent and taking into account that $E_0(a(i,j,s,n))=0$, and $E_0((a(i,j,s,n)^2)=1$, it follows that

$$|\phi_0(a(i'_1,j'_1,s'_1,n)\cdots a(i'_l,j'_l,s'_l,n))|\leqslant 1.$$

We now prove by recursion on l that there is a constant C_l such that for each block of length l,

$$|k_l[a(i'_1,j'_1,s'_1,n),\ldots,a(i'_l,j'_l,s'_l,n)]|\leqslant C_l.$$

Step 1. $l=1$:

$$k_1[a(i'_1,j'_1,s'_1,n)]=\phi_0(na(i'_1,j'_1,s'_1,n))=0.$$

Step 2. $l=2$:

$$\begin{aligned}&k_2[a(i'_1,j'_1,s'_1,n),a(i'_2,j'_2,s'_2,n)]\\&\quad=\phi_0(a(i'_1,j'_1,s'_1,n)a(i'_2,j'_2,s'_2,n))-\phi_0(a(i'_1,j'_1,s'_1,n))\phi_0(a(i'_2,j'_2,s'_2,n)).\end{aligned}$$

So $|k_2[a(i'_1,j'_1,s'_1,n),na(i'_2,j'_2,s'_2,n)]|\leqslant 1$.

Step 3. Assume that the result is true for l and prove it for $l+1$:

$$\begin{aligned}&k_{l+1}[a(i'_1,j'_1,s'_1,n),\ldots,a(i'_{l+1},j'_{l+1},s'_{l+1},n)]\\&\quad=\phi_0(a(i'_1,j'_1,s'_1,n)\cdots a(i'_{l+1},j'_{l+1},s'_{l+1},n))\\&\qquad-\sum_{\pi\in NC(l+1),\pi\neq 1_{l+1}}k_\pi[a(i'_1,j'_1,s'_1,n),\ldots,a(i'_{l+1},j'_{l+1},s'_{l+1},n)].\end{aligned}$$

For each π in $NC(l+1)$ such that $\pi\neq 1_{l+1}$,

$$\begin{aligned}&k_\pi[a(i'_1,j'_1,s'_1,n),\ldots,a(i'_{l+1},j'_{l+1},s'_{l+1},n)]\\&\quad=\prod_{i=1}^{r}k_{|\nu_i|}[a(i_{\alpha(1)},j_{\alpha(1)},s_{\alpha(1)},n)\cdots a(i_{\alpha(1)},j_{\alpha(1)},s_{\alpha(1)},n)].\end{aligned}$$

By recursion $|k_{|\nu_i|}[\cdots]|\leqslant C_{|\nu_i|}$ as $|\nu_i|\leqslant l$. And this proves the existence of the constant C_{l+1}. ∎

LEMMA 3.9. *Let $s_1,s_2,\ldots,s_m\in N$. Let $t_1,t_2,\ldots,t_m\in N$. Then*

$$\sup_{n\in N}|\Psi_n(\widetilde{Y}(s_1,n)D_n(t_1)\cdots\widetilde{Y}(s_m,n)D_n(t_m))|=K(m)<\infty.$$

Proof. We know that the $(q_{k,n})_{k\leqslant n^2}$ are orthogonal projectors in $(L^\infty[0,1],\phi)$ such that $\phi(q_{k,n})=\frac{1}{n^2}$. We apply Lemma 3.6:

$$\begin{aligned}&|\Psi_n(\widetilde{Y}(s_1,n)D(t_1,n)\cdots\widetilde{Y}(s_m,n)D(t_m,n))|\leqslant\\&\quad\frac{1}{n}n^2\sum_{(i_1,\ldots,i_m)\in E_n}\sum_{\pi}|k_\pi[\sqrt{n}a(i_1,i_2,s_1,n),\ldots,\sqrt{n}a(i_m,i_1,s_m,n))]|K\Big(\frac{1}{n^2}\Big)^{|\pi_B|}\Big).\end{aligned}$$

From Lemma 3.7, $|\pi_B|$ is greater or equal to $[\frac{m}{2}]+1$.

Now from Lemma 3.8, we get

$$|\Psi_n(\widetilde{Y}(s_1,n)D_n(t_1)\cdots\widetilde{Y}(s_m,n)D_n(t_m))| \leqslant n\,\mathrm{Card}(E_n)\mathrm{Card}(\{\pi\})C_m n^{\frac{m}{2}}\left(\frac{1}{n^2}\right)^{[\frac{m}{2}]+1}.$$

But from Lemma 3.6, $\mathrm{Card}(E_n) = O(n^{\frac{m}{2}+1})$. So

$$|\Psi_n(\widetilde{Y}_n(s_1)D_n(t_1)\cdots\widetilde{Y}_n(s_m)D_n(t_m))| = O(n^{m-2[\frac{m}{2}]}).$$

Case 1. If m is odd, for each π, at least one block ν of π_L is of length $|\nu|$ odd. It follows then from an obvious recursion on $|\nu|$ odd, using the fact that the $a(i,j,s,n)$ are either independent or equal, that

$$k_{|\nu|}[(\sqrt{n}a(i_1,j_1,s_1,n),\ldots,\sqrt{n}a(i_{|\nu|},j_{|\nu|},s_{|\nu|},n)] = 0$$

so

$$k_\pi[\sqrt{n}a(i_1,i_2,s_1,n),\sqrt{n}a(i_2,i_3,s_2,n),\ldots,\sqrt{n}a(i_m,i_1,s_m,n)] = 0.$$

Hence

$$\Psi_n(\widetilde{Y}(s_1,n)D_n(t_1)\cdots\widetilde{Y}(s_m,n)D_n(t_m)) = 0.$$

Case 2. If m is even, $m-2[\frac{m}{2}] = 0$. So

$$\sup_{n\in\mathbb{N}}|\Psi_n(\widetilde{Y}(s_1,n)D(t_1,n)\cdots\widetilde{Y}(s_m,n)D(t_m,n))| = K(m) < \infty. \quad ∎$$

Now we are able to prove the Proposition 3.1. We follow the proof of Theorem 2.2 of [7].

Proof of Proposition 3.1. *Step 1.* It is to prove that

$$\sup_{n\in\mathbb{N}}|\Psi_n(\widetilde{Y}(s_1,n)D(t_1,n)\cdots\widetilde{Y}(s_m,n)D(t_m,n))| < \infty.$$

This is exactly Lemma 3.9.

Step 2. (i) First we prove that

$$\Psi_n(\widetilde{Y}(s_{\alpha(1)},n)D(t_1,n)\widetilde{Y}(s_{\alpha(2)},n)D(t_2,n)\cdots\widetilde{Y}(s_{\alpha(m)},n)D(t_m,n)) = 0$$

if $\alpha(1)\neq\alpha(j)$ for all $j\neq 1$.

This is Corollary 3.3.

(ii) Let $\alpha(1)=\alpha(2)$ and $\mathrm{Card}(\alpha^{-1}(p))\leqslant 2$ for all p in $\mathbb{N}$. We want to prove that

$$\Psi_n(\widetilde{Y}(s_{\alpha(1)},n)D(t_1,n)\widetilde{Y}(s_{\alpha(2)},n)D(t_2,n)\cdots\widetilde{Y}(s_{\alpha(m)},n)D(t_m,n)) =$$
$$\Psi_n(D(t_1,n)\Psi_n(D(t_2,n)\widetilde{Y}(s_{\alpha(3)},n)D(t_3,n)\cdots\widetilde{Y}(s_{\alpha(m)},n)D(t_m,n)).$$

Indeed

$$\Psi_n(\widetilde{Y}(s_{\alpha(1)},n)D(t_1,n)\widetilde{Y}(s_{\alpha(2)},n)D(t_2,n)\cdots\widetilde{Y}(s_{\alpha(m)},n)D(t_m,n)) =$$
$$\frac{1}{n}\sum_{k=1}^{n^2}\sum_{i_1,i_2,\ldots,i_m}(\phi*\phi_0)(q_{k,n}a(i_1,i_2,s_{\alpha(1)},n)d(t_1,i_2)q_{k,n}a(i_2,i_3,s_{\alpha(1)},n)d(t_2,i_3)\cdots$$
$$q_{k,n}a(i_m,i_1,s_{\alpha(m)},n)d(t_m,i_1))n^{\frac{m}{2}}.$$

From Lemma 3.2 and Lemma 3.5, this is equal to

$$\frac{1}{n}\sum_{k=1}^{n^2}\phi(q_{k,n})\sum_{i_1,i_2,i_4,\dots,i_m} d(t_1,i_2)(E_0(a(i_1,i_2,s_{\alpha(1)},n)^2)(\phi*\phi_0)(d(t_2,i_1)q_{k,n}a(i_1,i_4,s_{\alpha(3)},n)$$
$$d(t_3,i_4))\cdots q_{k,n}a(i_m,i_1,s_{\alpha(m)},n)d(t_m,i_1))n^{\frac{m}{2}} =$$
$$\frac{1}{n^3}\sum_{k=1}^{n^2}\sum_{i_2}d(t_1,i_2)\sum_{i_1,i_4,\dots,i_m}(\phi*\phi_0)(d(t_2,i_1)q_{k,n}a(i_1,i_4,s_{\alpha(3),n}d(t_3,i_4))$$
$$\cdots q_{k,n}a(i_m,i_1,s_{\alpha(m)},n)d(t_m,i_1))n^{\frac{m}{2}} =$$
$$\frac{1}{n}\sum_{i_2}d(t_1,i_2)\frac{1}{n}\sum_{k=1}^{n^2}\sum_{i_1,i_4,\dots,i_m}(\phi*\phi_0)(d(t_2,i_1)q_{k,n}a(i_1,i_4,s_{\alpha(3)},n)d(t_3,i_4))$$
$$\cdots q_{k,n}a(i_m,i_1,s_{\alpha(m)},n)d(t_m,i_1))n^{\frac{m}{2}-1} =$$
$$\Psi_n(D(t_1,n))\Psi_n(D(t_2,n)\widetilde{Y}(s_{\alpha(3)},n)D(t_3,n)\cdots\widetilde{Y}(s_{\alpha(m)},n)D(t_m,n)). \quad \blacksquare$$

(iii) One proves then that

$$\lim_{n\to\infty}\Psi_n(\widetilde{Y}(s_1,n)D(t_1,n)\cdots\widetilde{Y}(s_m,n)D(t_m,n)) = 0$$

if $s_k \neq s_{k+1}$ $(1 \leqslant k \leqslant m-1)$ and $s_m \neq s_1$.

As in the proof of Theorem 2.2 of [7] if

$$\Psi_n(\widetilde{Y}(s_1,n)D(t_1,n)\cdots\widetilde{Y}(s_m,n)D(t_m,n)) \neq 0$$

there is an automorphism γ of order 2 of $1,\dots,m$ without fixed point such that for $p \neq q$, $s_p = s_q$ if and only if $p = \gamma(q)$. And then as in Lemma 3.6

$$\Psi_n(\widetilde{Y}(s_1,n)D(t_1,n)\cdots\widetilde{Y}(s_m,n)D(t_m,n)) =$$
$$\frac{1}{n}\sum_{k=1}^{n^2}\sum_{(i_1,i_2,\dots,i_m)\in E_n(\gamma)}\sum_{\pi}k_\pi[\, a(i_1,i_2,s_1,n),\; a(i_2,i_3,s_2,n),$$
$$\cdots\; a(i_m,i_1,s_m,n)]O\Big(\Big(\frac{1}{n^2}\Big)^{|\pi_B|}\Big)n^{\frac{m}{2}}.$$

As in [7] $E_n(\gamma)$ denotes the set of $(i_1,i_2,\dots,i_m) \in (1,2,\dots,n)^m$ such that $i_k = i_{\gamma(k)+1}$, $i_{k+1} = i_{\gamma(k)}$ where $\gamma(k)$ and $\gamma(k)+1$ are considered modulo m. From [7] $\mathrm{Card}(E_n(\gamma)) \leqslant n^{\frac{m}{2}}$. So from Lemma 3.6, Lemma 3.7 and Lemma 3.8, we get

$$\Psi_n(\widetilde{Y}(s_1,n)D(t_1,n)\cdots\widetilde{Y}(s_m,n)D(t_m,n)) = O(n^{(m-2[\frac{m}{2}]-1)}).$$

Case 1. So if m is even,

$$\Psi_n(\widetilde{Y}(s_1,n)D(t_1,n)\cdots\widetilde{Y}(s_m,n)D(t_m,n)) = O\Big(\frac{1}{n}\Big)$$

and we get the result.

Case 2. On the other hand, if m is odd, as in the proof of Lemma 3.9,

$$\Psi_n(\widetilde{Y}(s_1,n)D(t_1,n)\cdots\widetilde{Y}(s_m,n)D(t_m,n)) = 0.$$

This ends Step 2.

Step 3. Exactly as in Step 3 of the proof of Theorem 2.2 of [7], we apply the Theorem 2.1 of [7], to prove that the family of random variables $\widetilde{Y}(s,n)$ and $D(j,n)$ are asymptotically free, with the $\widetilde{Y}(s,n)$ having limit distributions given by semicircular laws of variance 1.

This ends the proof of Proposition 3.1.

This gives now the following renormalized model for the free Brownian motion:

THEOREM 3.10. *For all $s \in \mathbb{R}^+$, and $n \in \mathbb{N}^*$, let*

$$\widetilde{Z}_n(s) = \sum_{1\leqslant i,j\leqslant n} \widetilde{W}_{(i,j,n)}(s)e(i,j,n)$$

with $\widetilde{W}_{(i,j,n)}(s) \in L^\infty([0,1]) * L$. Assume that

$$\widetilde{W}_{(i,j,n)}(s) = \sum_{k=1}^{n^2} q_{k,n}\sqrt{n}W_{(i,j,n)}(s)\ q_{k,n}$$

where the $q_{k,n}$ are orthogonal projectors in $L^\infty[0,1]$, $\sum\limits_{k=1}^{n^2} q_{k,n} = 1$ such that

$$\phi(q_{k,n}) = \frac{1}{n^2}$$

and the $(W_{(i,j,n)}(s)_{s\in R^+})_{1\leqslant i\leqslant j\leqslant n,n\in\mathbb{N}^*}$ are independent Brownian motions, in particular

$$\text{if } s_0 = 0 < s_1 < s_2 \cdots < s_k, \quad (W_{(i,j,n)}(s_{l+1}) - W_{(i,j,n)}(s_l))_{0\leqslant l\leqslant k-1}$$

are independent Gaussian random variables centered of variance $s_{l+1} - s_l$ and

$$W(i,j,n)(s) = W(j,i,n)(s).$$

Consider the trace Ψ_n defined at the begining of the section. Let $D_n(j)$ be elements in Δ_n, the set of diagonal matrices, such that $\sup\limits_{n\in N} \|D_n(j)\| < \infty$, for each j; and such that for all j, $(D_n(j))$ has a limit distribution as $n \to \infty$.

Then the family of subsets $\{\widetilde{Z}_n(s)\}$ and $\{D_n(j) : j \in N\}$ are asymptotically free, and the limit distribution of the $\widetilde{Z}_n(s)$ is the distribution of the free Brownian motion.

Proof. For $i \in \{0,\ldots,k-1\}$ and $n \in \mathbb{N}$, let $0 = s_0 < s_1 < s_2 < \cdots < s_k$. Let $\widetilde{Y}(i,n) = \frac{1}{\sqrt{s_{i+1}-s_i}}(\widetilde{Z}_n(s_{i+1}) - \widetilde{Z}_n(s_i))$.

We apply the Proposition 3.1 to $\widetilde{Y}(i,n)$ and we get the result. ∎

4. A FREE GIRSANOV PROPERTY

Hypothesis: Let $(\sigma_s)_{s\in\mathbb{R}^+}$ be a free Brownian motion in (M,τ). Let N be a commutative C^*-subalgebra of M free from the $(\sigma_s)_{s\in\mathbb{R}^+}$. Let x be a measurable process with values in N. Assume that $x(u) = x(u)^*$ for all u and that $\int\limits_0^\infty \|x(u)\|^2 \mathrm{d}u < \infty$. Let $\widetilde{\sigma}_s = \sigma_s + \int\limits_0^s x(u)\mathrm{d}u$.

We want first to associate to the system $(\sigma_s, x(u))_{s,u\in R^+}$ an asymptotic system $(\widetilde{Z}_n(s), D_n(u))$ in the set of random matrices with coefficients in a free product algebra $\mathcal{M}_n(L^\infty[0,1] * L)$, and then to define on $\mathcal{M}_n(L^\infty[0,1] * L)$ two traces Ψ_n and $\widetilde{\Psi}_n$ such that their asymptotic limits are respectively the traces τ and $\widetilde{\tau}$, where τ is the given trace and $\widetilde{\tau}$ is a new trace such that $(\widetilde{\sigma}_s)_{s\in\mathbb{R}^+}$ is a free Brownian motion for $\widetilde{\tau}$.

AN ASYMPTOTIC SYSTEM IN $\mathcal{M}_n(L^\infty[0,1] * L)$. $((\sigma_s)_{s\in\mathbb{R}^+})$ is a free Brownian motion and the $x(u) = x(u)^*$ belong to a commutative subalgebra of M free from the σ_s. In view of Theorem 3.10, we will associate to the process σ_s the process of random matrices $(\widetilde{Z}_n(s)_{s\in\mathbb{R}^+})$ and we want to associate to the process $x(u)_{u\in\mathbb{R}^+}$ a process of diagonal matrices with real coefficients. We construct now this process.

LEMMA 4.1. *Let N be a commutative C^*-algebra with a finite trace τ. There is a family of homomorphisms H_n from N to Δ_n (the set of diagonal matrices with complex coefficients) such that for all $x \in N$,*

$$\tau(x) = \lim_{n\to\infty} \frac{1}{n} \mathrm{Tr}_n(H_n(x)).$$

Proof. From the Chapter 2 of [2] the set of states on N is the weak* closed convex hull of the set of pure states on N. Furthermore as N is commutative, the pure states are the characters. Denote $\mathcal{X}$ the set of the characters of N. Denote $\overline{S}$ the weak* closure of

$$S = \Big\{ \frac{1}{n} \sum_{1\leqslant i\leqslant n} \chi_i : n \in \mathbb{N}^*, \chi_i \in \mathcal{X} \Big\}.$$

Using the density of $\{\frac{k}{n} : n \in \mathbb{N}^*, 1 \leqslant k \leqslant n\}$ in $[0,1]$, it is easy to verify that $\overline{S}$ is a convex set; so $\overline{S}$ is equal to the set of all the states on N. It follows that there is a sequence S_n of elements of S such that the limit of S_n for the weak* topology of N is equal to the trace τ, i.e. for all x in N, $S_n(x) \to \tau(x)$ as $n \to \infty$, with

$$S_n = \frac{1}{n} \sum_{1\leqslant i\leqslant n} \chi_{i,n}\,.$$

Define now the homomorphism H_n from N to Δ_n by:

$$H_n(x) = \begin{pmatrix} \chi_{1,n}(x) & 0 & \cdots & 0 \\ 0 & \chi_{2,n}(x) & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \chi_{n,n}(x) \end{pmatrix}.$$

Then

$$\|H_n(x)\| \leqslant \sup(|\chi_{i,n}(x)|) \leqslant \|x\|$$

and for every $x \in N$,

$$\frac{1}{n}\mathrm{Tr}_n(H_n(x)) = \frac{1}{n}\sum_{1\leqslant i\leqslant n} \chi_{i,n}(x) = S_n(x).$$

So

$$\lim_{n\to\infty} \frac{1}{n}\mathrm{Tr}_n(H_n(x)) = \tau(x). \quad \blacksquare$$

For all $u \in \mathbb{R}^+$ denote $D_n(u)$ the diagonal matrix $D_n(u) = H_n(x(u))$. It is a real matrix because $x(u)$ is selfadjoint.

DEFINITION 4.2. The *asymptotic system* associated to $(\sigma_s, x(u))_{s,u\in\mathbb{R}^+}$ is the system of random matrices $(\widetilde{Z}_n(s), D_n(u))_{s,u\in\mathbb{R}^+}$ in $\mathcal{M}_n(L^\infty[0,1] * L)$ where $(\widetilde{Z}_n(s))_{s\in\mathbb{R}^+}$ is the process of random matrices defined in Theorem 3.10 and $(D_n(u))_{u\in\mathbb{R}^+}$ is the process of diagonal matrices defined above.

TWO TRACES ON $\mathcal{M}_n(L^\infty[0,1] * L)$. In the preceding section we have associated to $(\sigma_s, x(u))_{s,u\in\mathbb{R}^+}$ an asymptotic system in $\mathcal{M}_n(L^\infty[0,1] * L)$. Define now two traces Ψ_n and $\widetilde{\Psi}_n$ on $\mathcal{M}_n(L^\infty[0,1] * L)$ such that their asymptotic limits will give the two traces τ and $\widetilde{\tau}$.

Denote

$$h_n = \exp -\Big[\int_0^\infty \sum_{1\leqslant i\leqslant n} \frac{1}{\sqrt{n}} D_n(u)_{i,i} \mathrm{d}W_{i,i,n}(u) + \int_0^\infty \sum_{1\leqslant i\leqslant n} \frac{D_n(u)^2_{i,i}}{2n}\mathrm{d}u\Big].$$

LEMMA 4.3. *For all $n \in \mathbb{N}^*$ $\phi_0(h_n) = 1$. For all $p \geqslant 2$ $\sup_{n\in\mathbb{N}^*} \phi_0(h_n^p) < \infty$, $\lim_{n\to\infty} \phi_0(h_n^p) = \exp\Big[\frac{p^2-p}{2}\int_0^\infty \tau(x(u)^2)\mathrm{d}u\Big]$ and the family*

$$\Big(\Big[W_{(i,j,n)}(s) + \int_0^s \frac{1}{\sqrt{n}}(D_n(u))_{i,i}\delta_{i,j}\mathrm{d}u\Big]_{s\in\mathbb{R}^+}\Big)_{1\leqslant i\leqslant j\leqslant n}$$

is a family of independent Brownian motions for $\phi_0(h_n.)$

Proof. We have:

$$\begin{aligned}
&\phi_0(h_n^p)\\
&= E\Big(\exp -p\Big[\int_0^\infty \frac{1}{\sqrt{n}}\sum_{1\leqslant i\leqslant n}(D_n(u))_{i,i}\mathrm{d}W_{i,i,n}(u)+\frac{1}{2n}\int_0^\infty\sum_{1\leqslant i\leqslant n}(D_n(u))_{i,i}^2\Big]\mathrm{d}u\Big)\\
&= \exp\Big[\frac{1}{2}\int_0^\infty\sum_{1\leqslant i\leqslant n}\frac{(D_n(u))_{i,i}^2p^2}{n}\mathrm{d}u-\frac{p}{2n}\int_0^\infty\sum_{1\leqslant i\leqslant n}(D_n(u))_{i,i}^2\mathrm{d}u\Big]\\
&= \exp\Big[\frac{p^2-p}{2}\int_0^\infty\frac{1}{n}\mathrm{Tr}_n(D_n(u)^2)\mathrm{d}u\Big].
\end{aligned}$$

Since $D_n(u) = H_n(x(u))$, we have, for all n, $\frac{1}{n}\mathrm{Tr}_n(D_n(u)^2) \leqslant \|x(u)\|^2$. From Lemma 4.1, $\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}_n(D_n(u)^2) = \tau(x(u)^2)$, so we get the result for $\phi_0(h_n^p)$ applying the dominated convergence theorem of Lebesgue.

As the Brownian motions $(W_{i,j,n}(s))_{1\leqslant i\leqslant j\leqslant n}$ are independent and as for all $i\leqslant n$, $\int_0^\infty H_n(x(u))_{i,i}^2\mathrm{d}u < \infty$, it results from the usual Girsanov theorem that

$$\Big(\Big[W_{(i,j,n)}(s)+\int_0^s\frac{1}{\sqrt{n}}(D_n(u))_{i,i}\delta_{i,j}\mathrm{d}u\Big]_{s\in\mathbb{R}^+}\Big)_{1\leqslant i\leqslant j\leqslant n}$$

are independent Brownian motions for $\phi_0(h_n.)$. ∎

DEFINITION 4.4. Define now the traces Ψ_n and $\widetilde{\Psi}_n$ on $\mathcal{M}_n(L^\infty([0,1]) * L)$ by

$$\Psi_n\Big(\sum_{1\leqslant i,j\leqslant n}x_{ij}e(i,j,n)\Big)=\frac{1}{n}\sum_{1\leqslant i\leqslant n}(\phi*\phi_0)(x_{ii})$$

and

$$\widetilde{\Psi}_n\Big(\sum_{1\leqslant i,j\leqslant n}x_{ij}e(i,j,n)\Big)=\frac{1}{n}\sum_{1\leqslant i\leqslant n}(\phi*\phi_0(h_n.))(x_{ii}).$$

For simplicity we will denote

$$\Psi_n=\frac{1}{n}\mathrm{Tr}_n(\phi*\phi_0)\quad\text{and}\quad\widetilde{\Psi}_n=\frac{1}{n}\mathrm{Tr}_n(\phi*\phi_0(h_n.)).$$

Ψ_n is the same state as in Section 3.

PROPOSITION 4.5. *The joint distribution of* $((\widetilde{Z}_n(s)_{s\in\mathbb{R}^+}, H_n(x)_{x\in N})$ *for* Ψ_n *is the same as the joint distribution of* $\Big(\Big(\widetilde{Z}_n(s)+\int_0^s D_n(u)\mathrm{d}u\Big)_{s\in\mathbb{R}^+}, H_n(x)_{x\in N}\Big)$ *for* $\widetilde{\Psi}_n$.

Proof. We have:

$$\Big(\widetilde{Z}_n(s)+\int_0^s D_n(u)\mathrm{d}u\Big)_{i,j} = \sum_{k=1}^{n^2} q_{k,n}\sqrt{n}W_{(i,j,n)}(s)\, q_{k,n} + \int_0^s (D_n(u))_{i,i}\delta_{i,j}\mathrm{d}u$$
$$= \sum_{k=1}^{n^2} q_{k,n}\Big[\sqrt{n}W_{(i,j,n)}(s) + \int_0^s (D_n(u))_{i,i}\delta_{i,j}\mathrm{d}u\Big]\, q_{k,n}.$$

To compute the joint distribution of $\Big(\big(\widetilde{Z}_n(s)+\int_0^s D_n(u)\mathrm{d}u\big)_{s\in\mathbb{R}^+}, H_n(x)_{x\in N}\Big)$ for $\widetilde{\Psi}_n$, it is enough, as N is a unital C^*-algebra, to compute for all $p\in\mathbb{N}$, $s_i\geqslant 0$ and $x_i\in N$

$$\widetilde{\Psi}_n\Big(H_n(x_1)\Big(\widetilde{Z}_n(s_1)+\int_0^{s_1} D_n(u)\mathrm{d}u\Big)H_n(x_2)\cdots H_n(x_p)\Big(\widetilde{Z}_n(s_p)+\int_0^{s_p} D_n(u)\mathrm{d}u\Big)\Big)$$
$$= \frac{1}{n}\sum_{1\leqslant k\leqslant n^2}\sum_{1\leqslant i_1\leqslant n}\sum_{1\leqslant j_1\leqslant n}\cdots\sum_{1\leqslant j_{p-1}\leqslant n}(\phi*\phi_0(h_n.))\Big[(H_n(x_1))_{i_1,i_1}\cdot$$
$$\Big(q_{k,n}\Big[\sqrt{n}W_{(i_1,j_1,n)}(s_1)+\int_0^{s_1}(D_n(u))_{i_1,i_1}\delta_{i_1,j_1}\mathrm{d}u\Big]q_{k,n}\Big)\cdot$$
$$(H_n(x_2))_{j_1,j_1}\Big(q_{k,n}\Big[\sqrt{n}W_{(j_1,j_2,n)}(s_2)+\int_0^{s_2}(D_n(u))_{j_1,j_1}\delta_{j_1,j_2}\mathrm{d}u\Big]q_{k,n}\Big)\cdots$$
$$(H_n(x_p))_{j_{p-1},j_{p-1}}\Big(q_{k,n}\Big[\sqrt{n}W_{(j_{p-1},i_1,n)}(s_p)+\int_0^{s_p}(D_n(u))_{j_{p-1},j_{p-1}}\delta_{j_{p-1},i_1}\mathrm{d}u\Big]q_{k,n}\Big)\Big].$$

Remark now that the $q_{k,n}$ are free from L for $\phi*\phi_0(h_n.)$ and also for $\phi*\phi_0$. Furthermore, Lemma 4.3 implies that the joint distribution of $\Big(\Big[\sqrt{n}W_{(i,j,n)}(s) + \int_0^s (D_n(u))_{i,i}\delta_{i,j}\mathrm{d}u\Big]_{s\in\mathbb{R}^+}\Big)_{1\leqslant i\leqslant j\leqslant n}$ for $\phi_0(h_n.)$ is equal to the joint distribution of $([\sqrt{n}W_{(i,j,n)}(s)]_{s\in\mathbb{R}^+})_{1\leqslant i\leqslant j\leqslant n}$ for ϕ_0. We then get that the preeceding sum is equal to

$$= \frac{1}{n}\sum_{1\leqslant k\leqslant n^2}\sum_{1\leqslant i_1\leqslant n}\sum_{1\leqslant j_1\leqslant n}\cdots\sum_{1\leqslant j_{p-1}\leqslant n}(\phi*\phi_0)((H_n(x_1))_{i_1,i_1}$$
$$(q_{k,n}[\sqrt{n}W_{(i_1,j_1,n)}(s_1)]q_{k,n})(H_n(x_2))_{j_1,j_1}(q_{k,n}[\sqrt{n}W_{(j_1,j_2,n)}(s_2)]q_{k,n})\cdots$$
$$(H_n(x_p))_{j_{p-1},j_{p-1}}(q_{k,n}[\sqrt{n}W_{(j_{p-1},i_1,n)}(s_p)]q_{k,n}))$$
$$= \Psi_n(H_n(x_1)\widetilde{Z}_n(s_1)H_n(x_2)\widetilde{Z}_n(s_2)\cdots H_n(x_p)\widetilde{Z}_n(s_p)).$$

This ends the proof of the Proposition 4.5. ∎

MAIN RESULT. We can now prove our main result: a free Girsanov property for the free Brownian motion.

THEOREM 4.6. *Let $(\sigma_s)_{s\in\mathbb{R}^+}$, be a free Brownian motion in (M,τ). Let N be a commutative C^*-subalgebra of M such that N is free from $(\sigma_s)_{s\in\mathbb{R}^+}$. Let $x:\mathbb{R}^+\to N$ measurable such that $x(u)=x(u)^*$ for all u and $\int_0^\infty \|(x(u)\|^2\mathrm{d}u<\infty$. Let $\widetilde{\sigma}_s = \sigma_s+\int_0^s x(u)\mathrm{d}u$.*

Then there is a trace $\widetilde{\tau}$ on the free product algebra $N*\mathbb{C}[(\sigma_s)_{s\in\mathbb{R}^+}]$ such that the joint distribution of $((\widetilde{\sigma}_s)_{s\in\mathbb{R}^+}, x(u)_{u\in\mathbb{R}^+})$ for $\widetilde{\tau}$ is the same as the joint distribution of $((\sigma_s)_{s\in\mathbb{R}^+}, x(u)_{u\in\mathbb{R}^+})$ for τ (in particular, $(\widetilde{\sigma}_s)_{s\in\mathbb{R}^+}$ is a free Brownian motion for the new trace $\widetilde{\tau}$). Furthermore the two traces are asymptotically equivalent in the following sense: There is a family $\widetilde{Z}_n(s)$ of random matrices in $\mathcal{M}_n(L^\infty[0,1]*L)$ and a family $D_n(u)$ in Δ_n such that:

(i) $(\mathbb{C}[\sigma_s,x(t)]_{s,t\in\mathbb{R}^+},\tau)=\lim_{n\to\infty}(\mathbb{C}[\widetilde{Z}_n(s),D_n(t)]_{s,t\in\mathbb{R}^+},\Psi_n)$;

(ii) $(\mathbb{C}[\sigma_s,x(t)]_{s,t\in\mathbb{R}^+},\widetilde{\tau})=\lim_{n\to\infty}(\mathbb{C}[\widetilde{Z}_n(s),D_n(t)]_{s,t\in\mathbb{R}^+},\widetilde{\Psi}_n)$;

where $\widetilde{\Psi}_n$ is obtained from Ψ_n by a change of probability with exponential density h_n

$$\Psi_n=\frac{1}{n}\mathrm{Tr}_n(\phi*\phi_0),\quad \widetilde{\Psi}_n=\frac{1}{n}\mathrm{Tr}_n(\phi_*\phi_0(h_n.)).$$

Furthermore for all p, $\sup_{n\in\mathbb{N}}\phi_0(h_n^p)<\infty$.

Proof. $\widetilde{\sigma}_s=\sigma_s+\int_0^s x(u)\mathrm{d}u$. Denote $y(s)=\int_0^s x(u)\mathrm{d}u$; then $y(s)$ is an element of the C^*-algebra N and $H_n(y(s))=\int_0^s D_n(u)\mathrm{d}u$.

From Proposition 4.5, the joint distribution of $((\widetilde{Z}_n(s)+H_n(y(s))_{s\in\mathbb{R}^+}$, $H_n(x)_{x\in N})$ for $\widetilde{\Psi}_n$ is the same as the joint distribution of $(\widetilde{Z}_n(s)_{s\in\mathbb{R}^+}, H_n(x)_{x\in N})$ for Ψ_n. Hence for every non commutative polynomial P, $\widetilde{\Psi}_n(P(\widetilde{Z}_n(s_i)+H_n(y(s_i)),H_n(x_j)) = \Psi_n(P(\widetilde{Z}_n(s_i),H_n(x_j))$. From Theorem 3.10, and Lemma 4.1 this last quantity has a limit as n tends to ∞ and this limit is equal to $\tau(P(\sigma_{s_i},x_j))$. So this gives (i).

It follows also that there is a trace $\widetilde{\tau}$ well defined on $\mathbb{C}[\widetilde{\sigma}_s]*N$ by

$$\widetilde{\tau}(P(\widetilde{\sigma}_{s_i},x_j))=\lim_{n\to\infty}\widetilde{\Psi}_n(P(\widetilde{Z}_n(s_i)+H_n(y(s_i)),H_n(x_j))$$

and that the joint distribution of $((\widetilde{\sigma}_s)_{s\in\mathbb{R}^+},x_{x\in N})$ for $\widetilde{\tau}$ is the same as the joint distribution $((\sigma_s)_{s\in\mathbb{R}^+},x_{x\in N})$ for τ. This gives also the equality (ii). ∎

Now we finish by the following remark: if we replace in the preceding theorem the random process $\widetilde{Z}_n(s)$ by the random process $B_{n,s}=\left(\frac{1}{\sqrt{n}}W_{n,i,j,s}\right)_{1\leqslant i,j\leqslant n}$ considered by Voiculescu (cf. Section 2), the asymptotic limits for Ψ_n and $\widetilde{\Psi}_n$ give both the trace τ. This is why we were obliged to construct a matrix random process with values in a free product algebra. More precisely we have the following result.

PROPOSITION 4.7. *Let $B_{n,s}$ be the matrix random process $B_{n,s}=\left(\frac{1}{\sqrt{n}}W_{n,i,j,s}\right)_{1\leqslant i,j\leqslant n}$ where $(W_{n,i,j,s})_{1\leqslant i\leqslant j\leqslant n}$ are independent Brownian motions. Let Ψ_n and $\widetilde{\Psi}_n$ be the traces of Theorem 4.6.Then:*

(i′) $(\mathbb{C}[\sigma_s, x(t)]_{s,t\in\mathbb{R}^+}, \tau) = \lim_{n\to\infty}(\mathbb{C}[B_{n,s}, D_n(t)]_{s,t\in\mathbb{R}^+}, \Psi_n)$;

(ii′) $(\mathbb{C}[\sigma_s, x(t)]_{s,t\in\mathbb{R}^+}, \tau) = \lim_{n\to\infty}(\mathbb{C}[B_{n,s}, D_n(t)]_{s,t\in\mathbb{R}^+}, \widetilde{\Psi}_n)$.

Proof. Step 1. The equality (i′) results from the Theorem 2.2 of [7] as it is recalled in Section 2 and from the Lemma 4.1.

Notice that $B_{n,s}$ and $D_n(t) = H_n(x(t))$ are matrices with coefficients in L so here Ψ_n respectively $\widetilde{\Psi}_n$ are simply equal to $\frac{1}{n}\mathrm{Tr}_n(\phi_0)$ respectively $\frac{1}{n}\mathrm{Tr}_n(\phi_0(h_n.))$; Ψ_n restricted to $\mathcal{M}_n(L)$ is equal to the trace ϕ_n of Section 2. As in the proof of Theorem 4.6 denote $y(s) = \int_0^s x(u)\mathrm{d}u$.

Step 2. From Lemma 4.3, the joint distribution of $\left(B_{n,s}+\frac{1}{n}H_n(y(s)), H_n(x(t))\right)$ for $\widetilde{\Psi}_n$ is the same as the joint distribution of $(B_{n,s}, H_n(x(t)))$ for Ψ_n.

Step 3. Let P be a non commutative polynomial. Compute now:

$$\begin{aligned}\Psi_n\Big(&\Big[B_{n,s_1} + \frac{1}{n}H_n(y(s_1))\Big]^{\alpha_1} H_n(x_1)\Big[B_{n,s_2} + \frac{1}{n}H_n(y(s_2)))\Big]^{\alpha_2} H_n(x_2)\cdots\\ &\quad\Big[B_{n,s_m} + \frac{1}{n}H_n(y(s_m))\Big]^{\alpha_m} H_n(x_m)\Big)\\ &= \Psi_n((B_{n,s_1})^{\alpha_1}H_n(x_1)(B_{n,s_2})^{\alpha_2}H_n(x_2)\cdots(B_{n,s_m})^{\alpha_1}H_n(x_m))\\ &\quad + \sum_{i=1}^{\alpha_1+\cdots+\alpha_m}\left(\frac{1}{n}\right)^i(\Psi_n(Q_i(B_{n,s_1},\ldots,B_{n,s_m},H_n(x_1),\ldots,H_n(x_m)),\end{aligned}$$

where Q_i is a non commutative polynomial.

From the theorem of Voiculescu recalled in Section 2, $\lim_{n\to\infty}\Psi_n(Q_i(B_{n,s_1},\ldots, B_{n,s_m}, H_n(x_1),\ldots,H_n(x_m)) = \tau(Q_i(\sigma_{s_1},\ldots,\sigma_{s_m},x_1,\ldots,x_m))$, for all i, and i.e. for every non commutative polynomial P we have $\Psi_n\left(P\left(B_{n,s_i}+\frac{1}{n}H_n(y(s_i)), H_n(x_j)\right)\right)$ $-\Psi_n(P(B_{n,s_i}, H_n(x_j)))$ tends to zero as n tends to ∞, and

$$\begin{aligned}\Big|\widetilde{\Psi}_n\Big(\Big(P\Big(&B_{n,s_i} + \frac{1}{n}H_n(y(s_i)), H_n(x_j)\Big)\Big) - \widetilde{\Psi}_n(P(B_{n,s_i}, H_n(x_j)))\Big| \leqslant\\ &\phi_0(h_n^2)^{\frac{1}{2}}\Psi_n\Big(\Big[P\Big(B_{n,s_i} + \frac{1}{n}H_n(y(s_i)), H_n(x_j)\Big) - P(B_{n,s_i}, H_n(x_j))\Big]^*\cdot\\ &\Big[P\Big(B_{n,s_i} + \frac{1}{n}H_n(y(s_i)), H_n(x_j)\Big) - P(B_{n,s_i}, H_n(x_j))\Big]\Big)^{1/2},\end{aligned}$$

and we know from Lemma 4.3 that $\lim_{n\to\infty}\phi_0(h_n^2) = \exp(\tau(a^2))$.

It follows that the limit joint distribution of $(B_{n,s}, H_n(x(t)))$ for $\widetilde{\Psi}_n$ is the same as the limit joint distribution of $\left(B_{n,s} + \frac{1}{n}H_n(y(s)), H_n(x(t))\right)$ for $\widetilde{\Psi}_n$. Applying now Step 2 and 1 it follows that this limit is equal to the joint distribution of $(\sigma_s, x(t))$ for τ. So we get (ii′). ∎

The generalization of this Girsanov property (Theorem 4.6) to the case where the process x is adapted to the free Brownian motion (σ_s) is a work in progress.

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