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SPECTRAL ESTIMATES FOR THE ONE-DIMENSIONAL NON-SELF-ADJOINT ANDERSON MODEL

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ABSTRACT. We obtain bounds on the spectrum of the non-self-adjoint Anderson operator in one dimension by using higher order numerical ranges and are able to determine the spectrum completely in many cases. We also develop further some previously existing methods that allow us to prove that certain curves and regions are contained in the spectrum and provide numerical examples that suggest that these curves contained in the infinite volume spectrum have a strong bearing over the finite volume cases.

KEYWORDS: *Anderson model, non-self-adjoint operator, spectrum, numerical range, ergodic operator.*

MSC (2000): 47B80, 47A12, 60H25, 65F15.

1. INTRODUCTION

The non-self-adjoint Anderson Model has been the subject of various recent papers. It originated in [15] motivated by the study of superconductivity and has also been used to model the growth of the growth of bacteria in an inhomogeneous environment, [16], [17], [23], but its purely mathematical properties have in turn been the study of many other rigorous papers. In [7] Davies considered the same operator acting on $l^2(\mathbb{Z})$ and found that in this case the spectrum is very different to that obtained in the previously cited papers. The reason for this is the high instability of spectral properties of non-self-adjoint operators; in other words, if λ is an element of the spectrum of the infinite volume non-self-adjoint Anderson operator it need not be close to the spectrum in the finite volume model.

In another series of papers, [11], [12], [13], Goldsheid and Khoruzhenko have studied in great detail the operator defined on a finite interval with periodic boundary conditions and have proved results about the spectrum in the limit as the length of the interval increases; they have also shown that the spectrum converges (almost surely) to certain curves.

A different approach is followed by Trefethen, Contedini and Embree. In [27] random bidiagonal matrices are investigated and the authors present results that fully characterize the spectra and pseudospectra, as well as the numerical range, of finite (and infinite) random bidiagonal matrices. In [3] and [4] non-self-adjoint Anderson-type models are studied in great detail and rigorous results on the connection between finite and infinite volume cases are obtained.

We will however follow the approach taken by Davies. In [7] and [8] Davies found that the spectrum of the non-self-adjoint Anderson operator is a bounded set in $\mathbb{C}$ and made considerable progress in determining the spectrum precisely. However, several questions remained open and our goal in this paper is to present methods using higher order numerical ranges that enable us to determine the spectrum of the operator more precisely. In Section 3 we study the D model of the operator (as defined by Davies in [9]) and obtain significantly tighter bounds for the spectrum and in many cases we are able to determine it completely as shown by Theorems 3.11 and 3.20. Theorems of this type cannot be found in the papers by Davies, and in fact, except for the bidiagonal case mentioned above and studied in [27] there are no other known examples in which the spectrum is fully determined. Subsection 3.3 is devoted to the study of a particular instance of the Anderson operator in which the strength of the methods developed in this paper and in [8] and [9] can be appreciated fully.

We also include two examples in which we have computed numerically the spectrum of finite operators with periodic boundary conditions. It is known that the spectra of the finite volume operators are contained in the spectra of the infinite volume operator and for completeness we include this result as Proposition 2.5 in Section 2. The results we have obtained (see Figures 6 and 9) show a high degree of correspondence between the infinite and finite volume spectra.

Let H be a bounded linear operator defined on a Hilbert space $\mathcal{H}$. We recall that the numerical range of H is the set

$$\mathrm{Num}(H) = \{\langle Hf, f\rangle : f \in \mathcal{H}, \|f\| = 1\}$$

and

$$\mathrm{Spec}(H) \subseteq \mathrm{conv}(\mathrm{Spec}(H)) \subseteq \overline{\mathrm{Num}(H)}$$

where conv denotes the closed convex hull. We also recall that if H is a normal operator then $\mathrm{conv}(\mathrm{Spec}(H)) = \overline{\mathrm{Num}(H)}$.

In [7] and [8] Davies made use of these properties to obtain bounds on the spectrum of the non-self-adjoint Anderson operator. One of the goals of this paper is to improve these bounds by generalizing the concept of numerical range to higher order numerical ranges, although it is important to note that this has no direct connections to any of the generalized ranges that Gustafson and Kao list in [14] or to the concept developed by other authors in [19] and [20].

In this paper we present the definition of this concept and some useful properties of the second order numerical range that will enable us to proceed with our goal. In [9] Davies gives a more detailed account of the subject and proves further properties of higher order numerical ranges, which are, however, more difficult to compute.

Let $p(z)$ be any polynomial defined on $\mathbb{C}$. We know that

$$p(\mathrm{Spec}(H)) = \mathrm{Spec}(p(H)) \subseteq \overline{\mathrm{Num}(p(H))}$$

and hence,

$$\mathrm{Spec}(H) \subseteq \{z : p(z) \in \overline{\mathrm{Num}(p(H))}\}. \tag{1.1}$$

Let

$$\mathrm{Num}(p, H) := \{z : p(z) \in \overline{\mathrm{Num}(p(H))}\}.$$

We define the *n-th order numerical range of H* to be

$$\mathrm{Num}_n(H) := \bigcap_{\deg(p)\leqslant n} \mathrm{Num}(p, H)$$

for any $n \in \mathbb{N}$, and set

$$\mathrm{Num}_\infty(H) := \bigcap_n \mathrm{Num}_n(H) := \bigcap_p \mathrm{Num}(p, H).$$

We note that in [24] Nevanlinna defined the n-th polynomial numerical hull of H by

$$\mathrm{Hull}_n(H) = \bigcap_{\deg(p)\leqslant n} \mathrm{Hull}(p, H)$$

where $\mathrm{Hull}(p, H) := \{z : |p(z)| \leqslant \|p(H)\|\}$ and conjectured that $\mathrm{Hull}_n(H)$ is in fact equal to the n-th order numerical range of H. This result is not obvious and was proved recently by Burke and Greenbaum in [5].

However, for our purposes in this paper we will concentrate on $\mathrm{Num}_2(H)$. In [9] Davies proves the following result whose significance will be evident in Theorem 3.5.

THEOREM 1.1. *The complement of* $\mathrm{Num}_2(H)$ *is the union of a family of hyperbolic regions and half-planes. If $z \notin \mathrm{Num}_2(H)$ then*

$$S_z = \{\theta \in [0, 2\pi) : (z + \mathrm{e}^{\mathrm{i}\theta}[0, \infty)) \cap \mathrm{Spec}(H) = \emptyset\}$$

contains an interval of length at least $\frac{\pi}{2}$ or two disjoint intervals whose combined length is at least $\frac{\pi}{2}$. On the other hand if z is in the complement of the unbounded component of $\mathbb{C} \setminus \mathrm{Spec}(H)$ then $S_z = \emptyset$.

2. THE GENERAL SETTING

In this paper we consider the Anderson operator as an operator acting on $l^2(\mathbb{Z})$ and hence limit the context of the following results to this setting. However, it is worthwhile to note that these concepts can be, and have been (see [8]), defined in a much more general context.

DEFINITION 2.1. Let M_1, M_2, M_3 be compact subsets of $\mathbb{R}$ and let H be an operator defined on $l^2(\mathbb{Z})$ such that

$$\begin{aligned}
&H(x,y) = 0 \quad \text{if } |x-y| > 1,\\
&H(x,x) \in M_1, \quad H(x,x+1) \in M_2, \quad H(x,x-1) \in M_3.
\end{aligned}$$

We say that H is $(\mathbb{Z}, M_1, M_2, M_3)$ *pseudo-ergodic* if for every $\varepsilon > 0$, every finite subset $F \subset \mathbb{Z}$ and every $W_r : F \longrightarrow M_r$, where $r = 1,2,3$, there exists $\gamma \in \mathbb{Z}$ such that

$$\begin{aligned}
|H(\gamma+x, \gamma+x) - W_1(x)| &< \varepsilon,\\
|H(\gamma+x, \gamma+x+1) - W_2(x)| &< \varepsilon,\\
|H(\gamma+x, \gamma+x-1) - W_3(x)| &< \varepsilon,
\end{aligned}$$

for all $x \in F$. If M_2 and M_3 consist solely of one point we will say that H is $(\mathbb{Z}, M_1)$-pseudo-ergodic .

THEOREM 2.2. *Let H be a $(\mathbb{Z}, M_1, M_2, M_3)$ pseudo-ergodic operator. If K is an operator on $l^2(\mathbb{Z})$ such that*

$$\begin{aligned}
&K(x,y) = 0 \quad \textit{if } |x-y| > 1,\\
&K(x,x) \in M_1, \quad K(x,x+1) = m_2 \in M_2, \quad K(x,x-1) = m_3 \in M_3,
\end{aligned}$$

then $\mathrm{Spec}(K) \subseteq \mathrm{Spec}(H)$.

Proof. The proof follows that of Theorem 1 in [8]. ∎

COROLLARY 2.3. *Let K_0 denote the particular instance of the operator K obtained when $K(x,x) \equiv 0$. Then* $\mathrm{Spec}(K_0)$ *is the ellipse given by*

$$\{(m_2 + m_3)\cos\theta + \mathrm{i}(m_2 - m_3)\sin\theta : 0 \leqslant \theta \leqslant 2\pi\}$$

and

$$\mathrm{Spec}(K_0) + M_1 \subseteq \mathrm{Spec}(K) \subseteq \mathrm{Spec}(H),$$

and hence, as m_2 and m_3 vary in the definition of K, it follows that

$$\bigcup_{m_2\in M_2, m_3\in M_3} \mathrm{Spec}(K_0) + M_1 \subseteq \bigcup_{m_2\in M_2, m_3\in M_3} \mathrm{Spec}(K) \subseteq \mathrm{Spec}(H).$$

Next we have the following theorem whose proof we omit as it is standard perturbation theory.

THEOREM 2.4. *Let H be a $(\mathbb{Z}, M_1, M_2, M_3)$ pseudo-ergodic operator and let $c_r \in \mathbb{R}$ $(1 \leqslant r \leqslant 3)$ be fixed. Let $d_r \in \mathbb{R}$ be non-negative and such that*

$$M_r \subseteq B(c_r, d_r) = \{x \in \mathbb{R} \;:\; |x - c_r| \leqslant d_r\}$$

for all r. Let H_{c} be the constant coefficient operator defined by

$$\begin{aligned}
&H_{\mathrm{c}}(x,y) = 0 \quad \textit{if } |x-y| > 1,\\
&H_{\mathrm{c}}(x,x) = c_1, \quad H_{\mathrm{c}}(x,x+1) = c_2, \quad H_{\mathrm{c}}(x,x-1) = c_3,
\end{aligned}$$

and let V be an operator on $l^2(\mathbb{Z})$ such that $H = H_c + V$ where

$$\|V\| \leqslant d_1 + d_2 + d_3.$$

Then,

$$\mathrm{Spec}(H) \subseteq \mathrm{Spec}(H_c) + B(0, d_1 + d_2 + d_3). \tag{2.1}$$

REMARK. It is clear that d_r is minimized by taking c_r to be the midpoint of M_r and this produces the smallest ball in the above result. On the other hand, since $\mathrm{Spec}(H_c)$ is an ellipse determined by c_1, c_2 and c_3 the best result follows from taking the intersection of the right-hand side in (2.1) over all c_r contained in the interval $\left[\min_m\{m \in M_r, r = 1,2,3\}, \max_m\{m \in M_r, r = 1,2,3\}\right]$.

Finally we include the following proposition for completeness. It is important in light of the finite examples we include in Section 3.

PROPOSITION 2.5. *Let A be an $n \times n$ matrix such that*

$$A_{i,j} = \begin{cases} \alpha & \text{if } i = j-1 \text{ or if } i = n, j = 1, \\ \beta & \text{if } i = j+1 \text{ or of } i = 1, j = n, \\ v_i & \text{if } i = j, \\ 0 & \text{otherwise ,} \end{cases}$$

where v_i is contained in a compact set $M \subset \mathbb{R}$ for all i. Let B be the infinite matrix obtained by extending A periodically to $\mathbb{Z}$ and let C denote a $(\mathbb{Z}, M, \{\alpha\}, \{\beta\})$ pseudo-ergodic operator. The following inclusions are true:

$$\mathrm{Spec}(A) \subseteq \mathrm{Spec}(B) \subseteq \mathrm{Spec}(C).$$

Proof. The first inclusion can be obtained using Bloch wave analysis and is proved by Davies in [9]. The second inclusion follows from Theorem 1 of [8]. ∎

3. MAIN RESULTS

In this section we consider the one-dimensional non-self-adjoint Anderson operator given by

$$Hf_n = \pm\beta f_{n-1} + v_n f_n + \alpha f_{n+1} \tag{3.1}$$

acting on $l^2(\mathbb{Z})$ where $0 < \beta < \alpha$. The potential V given by $Vf_n = v_n f_n$ satisfies the conditions given in Definition 2.1 so that H is a $(\mathbb{Z}, M)$ pseudo-ergodic operator where M is a real compact set. That is,

$$H(x,y) = 0 \quad \text{if } |x-y| > 1 \quad \text{and} \quad H(x,x) \in M$$

and for every $\varepsilon > 0$, every finite subset $F \subset \mathbb{Z}$ and $W : F \longrightarrow M$, there exists $\gamma \in \mathbb{Z}$ such that

$$|H(\gamma + x, \gamma + x) - W(x)| < \varepsilon$$

for all $x \in F$.

The Anderson Model usually considered in the literature (see for example [6]) involves potentials in which the values v_n at each point are independent and identically distributed according to a probability law with compact support. A random operator is pseudo-ergodic with probability 1 and this enables us to work within the pseudo-ergodicity framework with no mention of probability. Thus, the results we obtain are true in an absolute sense for the operator H which is pseudo-ergodic by definition and hence almost surely random.

We will divide the study of this operator into two different cases, namely

$$Hf_n = \beta f_{n-1} + v_n f_n + \alpha f_{n+1} \tag{3.2}$$

and

$$Hf_n = -\beta f_{n-1} + v_n f_n + \alpha f_{n+1}. \tag{3.3}$$

Goldsheid and Khoruzhenko in [11] and [12] have shown that in the finite case great differences arise when studying the spectrum of the operator given by (3.2) and the operator given by (3.3). We will show that in the infinite case the same methods can be used to study these two cases but the results obtained when $\alpha\beta > 0$ or when $\alpha\beta < 0$ are genuinely distinct. We will follow Davies and first look at the case given by (3.2).

3.1. THE CASE $Hf_n = \beta f_{n-1} + v_n f_n + \alpha f_{n+1}$. Without loss of generality we can assume $\beta < \alpha$ and after multiplying H by a suitable constant we can also assume that $\alpha - \beta = 1$. We will thus suppose this to be true throughout this section.

We rewrite H in the following forms

$$H = A + \mathrm{i}B = (C + V) + \mathrm{i}B = H_0 + V \tag{3.4}$$

where

$$Af_n = af_{n-1} + v_n f_n + af_{n+1}, \qquad Bf_n = bf_{n-1} - bf_{n+1},$$
$$Cf_n = af_{n-1} + af_{n+1}, \qquad Vf_n = v_n f_n,$$

with $a = \frac{1}{2}(\alpha + \beta)$ and $b = \frac{\mathrm{i}}{2}$. Clearly A, B, C and V are self-adjoint, $\|B\| = 1$, V is diagonal and H_0 is normal with spectrum

$$\mathrm{Spec}(H_0) = \{\alpha e^{\mathrm{i}\theta} + \beta e^{-\mathrm{i}\theta} : \theta \in [0, 2\pi]\},$$

in other words, $\mathrm{Spec}(H_0)$ coincides with the ellipse given parametrically by

$$((\alpha + \beta)\cos\theta, \sin\theta) \quad \text{for } 0 \leqslant \theta \leqslant 2\pi.$$

In [8] Davies has shown that

$$\mathrm{Spec}(H_0) + M \subseteq \mathrm{Spec}(H) \subseteq \mathrm{conv}(\mathrm{Spec}(H_0)) + \mathrm{conv}(M) \tag{3.5}$$

and when more information on the nature of the set M is available we have the following results also due to Davies which we include here for completeness.

THEOREM 3.1. *If $M = [-\gamma, \gamma]$ and $\gamma \geqslant \alpha + \beta$, then*

$$\mathrm{Spec}(H) = \mathrm{Spec}(H_0) + [-\gamma, \gamma].$$

Furthermore, $0 \in \mathrm{Spec}(H) \Longleftrightarrow \gamma \geqslant \alpha - \beta = 1$.

THEOREM 3.2. *If $M = \{\pm\gamma\}$ and $\gamma > \alpha + \beta$ then*

$$\mathrm{conv}(\mathrm{Spec}(H_0) + \gamma) \cup \mathrm{conv}(\mathrm{Spec}(H_0) - \gamma) \subseteq \mathrm{Spec}(H) \subseteq \mathrm{conv}(\mathrm{Spec}(H_0)) + [-\gamma, \gamma]$$

and

$$\mathrm{Spec}(H) \subseteq B(\gamma, \alpha + \beta) \cup B(-\gamma, \alpha + \beta).$$

COROLLARY 3.3. *If $M = \{\pm\gamma\}$ then $0 \in \mathrm{Spec}(H)$ if and only if*

$$1 \leqslant \gamma \leqslant \alpha + \beta.$$

Our first objective is to determine whether similar conditions can be obtained for other points on the imaginary axis to be contained in $\mathrm{Spec}(H)$. We will begin by considering the case when M consists solely of two points, that is, $v_n = \pm\gamma$ for all n. It thus follows that $\|V\| = \gamma$.

From the definition of H and (3.5) above, we know that

$$\mathrm{Spec}(H) \subset \{z = x + \mathrm{i}y \in \mathbb{C} : |y| \leqslant 1\}.$$

LEMMA 3.4. *The following identity holds:*

$$B^2 + \frac{1}{4a^2}C^2 = I.$$

Proof. Let L denote the left-shift operator and R the right-shift operator, then $B = \frac{\mathrm{i}}{2}(L - R)$ and $C = a(L + R)$ and the proof follows. ∎

In order to proceed with the spectral analysis of H we will look at H^2, and in particular at the numerical range of this operator for we know that the set $\{z : z^2 \in \overline{\mathrm{Num}(H^2)}\}$ contains $\mathrm{Spec}(H)$ by (1.1).

THEOREM 3.5. *If $\eta(\gamma) = -1 + \frac{1}{4a^2+1}\gamma^2$, then $\mathrm{Re}(H^2) \geqslant \eta(\gamma)I$ and hence,* $\mathrm{Num}(H^2) \subseteq \{z : \mathrm{Re}(z) \geqslant \eta(\gamma)\}$.

Proof. From Lemma 3.4 it follows that $\mathrm{Re}(H^2)$ equals

$$\begin{aligned} A^2 - B^2 &= (V + C)^2 - \Big(I - \frac{1}{4a^2}C^2\Big) \\ &= -I + \Big(\frac{2a}{\sqrt{4a^2+1}}V + \frac{\sqrt{4a^2+1}}{2a}C\Big)^2 + \Big(1 - \frac{4a^2}{4a^2+1}\Big)V^2 \\ &\geqslant -I + \Big(1 - \frac{4a^2}{4a^2+1}\Big)V^2 = -I + \frac{1}{4a^2+1}V^2. \end{aligned}$$

Thus, as $|v_n| = \gamma$ for all n,

$$A^2 - B^2 \geqslant \Big(-1 + \frac{1}{4a^2+1}\gamma^2 \Big) I = \eta(\gamma) I. \quad \blacksquare$$

It now follows that if $x^2 - y^2 < \eta(\gamma)$ then $z \notin \operatorname{Spec}(H)$, and we have the following result in terms of the spectrum of H.

COROLLARY 3.6. *The spectrum of H satisfies*

$$\operatorname{Spec}(H) \subseteq \{z \in \mathbb{C} : |y| \leqslant 1\} \setminus \{z \in \mathbb{C} : x^2 - y^2 < \eta(\gamma)\}.$$

The set of complex numbers $z \in \mathbb{C}$ such that $x^2 - y^2 = \eta(\gamma)$ lie on a hyperbola which opens about the real axis or the imaginary axis depending on the sign of $\eta(\gamma)$. If $\eta(\gamma)$ is negative then the hyperbola opens about the imaginary axis and has vertices at $\pm\mathrm{i}\sqrt{1 - \frac{1}{4a^2+1}\gamma^2}$. If $\eta(\gamma)$ is positive the hyperbola opens to the left and to the right and has vertices at $\pm\sqrt{-1 + \frac{1}{4a^2+1}\gamma^2}$.

The latter case (together with Corollary 3.6) implies that $\operatorname{Spec}(H)$ lies in the band $\{z \in \mathbb{C} : |y| \leqslant 1\}$ but away from the origin, and the former implies that while the spectrum is still contained in $\{z \in \mathbb{C} : |y| \leqslant 1\}$ there exist neighbourhoods of $\pm\mathrm{i}$ that do not lie in the spectrum as the vertices of the hyperbola lie symmetrically about the origin on the imaginary axis between $-\mathrm{i}$ and i.

Now, for any real number s such that $-\gamma < s < \gamma$ define a new potential $\widetilde{V}$ by setting $\widetilde{v}_n = v_n - s$; that is, consider the operator $\widetilde{H} = H - sI$ which we write as $\widetilde{H} = \widetilde{A} + \mathrm{i}\widetilde{B} = \widetilde{C} + \widetilde{V} + \mathrm{i}\widetilde{B}$ where $\widetilde{A} = A - sI, \widetilde{B} = B$ and $\widetilde{C} = C$.

It follows trivially from the definition of spectrum that

$$-\mathrm{i} \notin \operatorname{Spec}(\widetilde{H}) \Leftrightarrow -\mathrm{i} + s \notin \operatorname{Spec}(H) \tag{3.6}$$

and hence we obtain an analogous result to that of Corollary 3.6 for each operator $H - sI$. Namely, a hyperbola centered at the point s which determines a region in the complex plane in which the spectrum of the operator is contained.

If we consider the case when $\eta(\gamma)$ is negative, we can extend this result as follows:

THEOREM 3.7. *Let $|v_n| = \gamma > 0$ for all n as above. If in addition $\gamma < \sqrt{4a^2+1}$, then* $\operatorname{Spec}(H)$ *does not intersect the set*

$$\left\{ z \in \mathbb{C} : \frac{1}{4a^2+1}\gamma \leqslant x \leqslant \gamma \text{ and } \sqrt{1 - \frac{(x-\gamma)^2}{(2a)^2}} \leqslant |y| \leqslant 1 \right\} \cup \left\{ z \in \mathbb{C} : -\gamma \leqslant x \leqslant -\frac{1}{4a^2+1}\gamma \text{ and } \sqrt{1 - \frac{(x+\gamma)^2}{(2a)^2}} \leqslant |y| \leqslant 1 \right\}.$$

Proof. We note that if $|v_n| \geqslant \gamma > 0$ then $|\widetilde{v}_n| \geqslant \gamma - |s| > 0$ and hence, reproducing the proof of Theorem 3.5 for the operator $\widetilde{H}$ and in light of (3.6) it follows that

$$\operatorname{Spec}(H) \subseteq \{z \in \mathbb{C} : |y| \leqslant 1\} \setminus \{z \in \mathbb{C} : (x-s)^2 - y^2 < \eta(\gamma - |s|)\}$$

where $\eta(\gamma - |s|) = -1 + \frac{1}{4a^2+1}(\gamma - |s|)^2$.

The family of inequalities $(x-s)^2 - y^2 < \eta(\gamma - |s|)$ describes regions in the complex plane delimited by the hyperbolae $(x-s)^2 - y^2 = \eta(\gamma - |s|)$. We will assume that $0 < s < \gamma$ as the case when $-\gamma < s < 0$ is symmetric and thus consider only $(x-s)^2 - y^2 = \eta(\gamma - s)$.

Let $F(x, y, s) = (x-s)^2 - y^2 + 1 - \frac{1}{4a^2+1}(\gamma - s)^2$. Solving

$$F(x, y, s) = 0, \tag{3.7}$$

$$\frac{\partial F}{\partial s} = 0, \tag{3.8}$$

gives us the envelope of the family of hyperbolae just described and this will in turn be the curve required in the statement of the theorem. Then

$$\frac{\partial F}{\partial s} = -2(x-s) + \frac{2}{4a^2+1}(\gamma - s).$$

Setting this equal to 0 yields $s = \frac{1}{4a^2}((4a^2+1)x - \gamma)$ and substituting this into (3.7) gives

$$\left[\frac{1}{4a^2}\gamma - \frac{1}{4a^2}x\right]^2 - y^2 + 1 - \frac{1}{4a^2+1}\left[\frac{4a^2+1}{4a^2}\gamma - \frac{4a^2+1}{4a^2}x\right]^2 = 0$$

or equivalently,

$$\frac{(x-\gamma)^2}{(2a)^2} + y^2 = 1 \tag{3.9}$$

and the case $-\gamma < s < 0$ yields the curve

$$\frac{(x+\gamma)^2}{(2a)^2} + y^2 = 1.$$

However the restrictions on s, namely, $0 < s < \gamma$ in the first case and $-\gamma < s < 0$ in the second, imply that only a certain part of each of the ellipses is actually obtained as the envelope. In the case of (3.9) the restrictions on s imply that x varies between $\frac{1}{4a^2+1}\gamma$ and γ and these values, together with those obtained symmetrically in the second case ($-\gamma \leqslant x \leqslant -\frac{1}{4a^2+1}\gamma$), render the sets in the statement of the theorem. ∎

REMARK. The ellipse defined by (3.9) coincides with that defined by $\frac{(x-\gamma)^2}{(\alpha+\beta)^2}$ $+y^2 = 1$, that is with the set $\operatorname{Spec}(H_0) + \gamma$ and this simple observation allows us to note that the previous theorem improves upon the results of Davies we have cited from [8]; and in fact, we have the following corollaries.

COROLLARY 3.8. *If in addition to the hypotheses of Theorem* 3.7, *we have* $\gamma < 2a$, *then* $\operatorname{Spec}(H)$ *is contained in the set*

$$\left\{ z \in \mathbb{C} : 0 \leqslant |x| \leqslant \frac{1}{4a^2+1}\gamma \text{ and } |y| \leqslant \sqrt{x^2 + 1 - \frac{1}{4a^2+1}\gamma^2} \right\} \\ \cup \left\{ z \in \mathbb{C} : \frac{1}{4a^2+1}\gamma < |x| \leqslant \gamma + 2a \text{ and } |y| \leqslant \sqrt{1 - \frac{(x-\gamma)^2}{(2a)^2}} \right\}.$$

Proof. The inclusion follows immediately by combining Theorem 3.7 and Corollary 3.6. ∎

Now, if $2a \leqslant \gamma < \sqrt{4a^2+1}$ we can improve the result of Theorem 3.2 as follows:

COROLLARY 3.9. *If in addition to the hypotheses of Theorem* 3.7, *we have* $2a \leqslant \gamma < \sqrt{4a^2+1}$, *then* $\operatorname{Spec}(H)$ *is contained in the set*

$$\left\{ z \in \mathbb{C} : \gamma - 2a \leqslant |x| \leqslant \frac{1}{4a^2+1}\gamma \text{ and } |y| \leqslant \sqrt{x^2 + 1 - \frac{1}{4a^2+1}\gamma^2} \right\} \\ \cup \left\{ z \in \mathbb{C} : \frac{1}{4a^2+1}\gamma < |x| \leqslant \gamma + 2a \text{ and } |y| \leqslant \sqrt{1 - \frac{(x-\gamma)^2}{(2a)^2}} \right\}.$$

Proof. The result follows from Theorems 3.2, 3.7 and Corollary 3.6. ∎

Figures 1, 2 and 3 show the results of Corollaries 3.8 and 3.9 graphically. In each case the spectrum of H is contained in the union of the two ellipses and the small area between the ellipses and the hyperbola. There are three qualitatively different cases for $\gamma < \sqrt{4a^2+1}$, namely $\gamma < 2a$, $\gamma = 2a$ and $\gamma > 2a$ and they are shown in order.

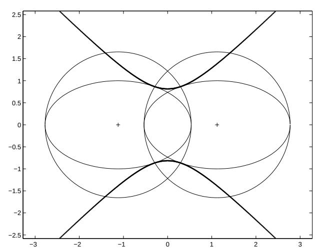

FIGURE 1. $\gamma = 1.15 < 1.65 = 2a$.

We now turn to the case when $\eta(\gamma) \geqslant 0$. Following the same approach as when $\eta(\gamma)$ is negative, we can improve upon the result of Davies in [8] which we have cited here as Theorem 3.2 as the restriction $\gamma > 2a$ is not stronger than the restriction $\eta(\gamma) \geqslant 0$.

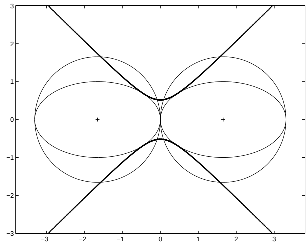

FIGURE 2. $\gamma = 1.65 = 2a$.

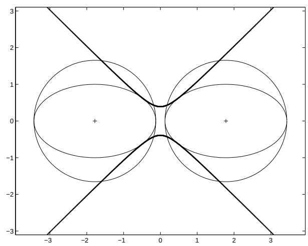

FIGURE 3. $\gamma = 1.85 > 1.65 = 2a$.

The case $\eta(\gamma) = 0$, that is, $\gamma = \sqrt{4a^2+1}$, represents a critical case for the hyperbola described by the equation $x^2 - y^2 = \eta(\gamma)$: it becomes two straight lines through the origin, $y = \pm x$. However, this does not create a significantly different case in our treatment here. Nevertheless, there exists another critical value for γ, namely $\gamma = \frac{4a^2+1}{2a}$, that does play an important role as shown in our next results.

THEOREM 3.10. *Let $|v_n| = \gamma > 0$ for all n and $\sqrt{4a^2+1} \leqslant \gamma \leqslant \frac{4a^2+1}{2a}$ then* $\mathrm{Spec}(H)$ *is contained in*

$$\left\{z \in \mathbb{C} : \gamma - 2a \leqslant |x| \leqslant \frac{1}{4a^2+1}\gamma \text{ and } |y| \leqslant \sqrt{x^2 + 1 - \frac{1}{4a^2+1}\gamma^2}\right\}$$
$$\cup \left\{z \in \mathbb{C} : \frac{1}{4a^2+1}\gamma < |x| \leqslant \gamma + 2a \text{ and } |y| \leqslant \sqrt{1 - \frac{(x-\gamma)^2}{(2a)^2}}\right\}.$$

Proof. For $-\gamma < s < \gamma$ we know that

$$\mathrm{Spec}(H) \subseteq \{z \in \mathbb{C} : |y| \leqslant 1\} \setminus \{z \in \mathbb{C} : (x-s)^2 - y^2 < \eta(\gamma - |s|)\} \tag{3.10}$$

where $\eta(\gamma - |s|) = -1 + \frac{1}{4a^2+1}(\gamma - |s|)^2$. We will again assume that s lies between 0 and γ and consider only $(x-s)^2 - y^2 = \eta(\gamma - s)$.

Let $F(x,y,s) = (x-s)^2 - y^2 + 1 - \frac{1}{4a^2+1}(\gamma - s)^2$. Solving

$$F(x,y,s) = 0 = \frac{\partial F}{\partial s}$$

gives the envelope of the family of hyperbolae which can be written parametrically as

$$x(s) = \frac{1}{4a^2+1}(\gamma + 4a^2 s), \quad y(s) = \pm\frac{1}{4a^2+1}\sqrt{(4a^2+1)^2 - 4a^2(\gamma - s)^2}$$

where the condition $\sqrt{4a^2+1} \leqslant \gamma \leqslant \frac{4a^2+1}{2a}$ ensures that $y(s)$ is well defined. Eliminating s from these equations gives

$$\frac{(x-\gamma)}{(2a)^2}^2 + y^2 = 1$$

and the case $-\gamma < s < 0$ yields the curve

$$\frac{(x+\gamma)}{(2a)^2}^2 + y^2 = 1.$$

Combining these equations together with the restriction on γ implies that

$$\begin{aligned} &\mathrm{Spec}(H) \cap \left\{z \in \mathbb{C} : \frac{1}{4a^2+1}\gamma < |x| \leqslant \gamma + 2a\right\} \\ &\subseteq \left\{z \in \mathbb{C} : \frac{1}{4a^2+1}\gamma < |x| \leqslant \gamma + 2a \text{ and } |y| \leqslant \sqrt{1 - \frac{(x-\gamma)^2}{(2a)^2}}\right\}. \end{aligned} \tag{3.11}$$

Now, in light of Theorem 3.2 this inclusion together with (3.10) renders the set in the statement of the theorem. ∎

Let us now turn to the case when $\gamma > \frac{4a^2+1}{2a}$ where we are able to completely determine the spectrum of H.

THEOREM 3.11. *If $\gamma > \frac{4a^2+1}{2a}$ and $|v_n| = \gamma > 0$ for all n, then*

$$\mathrm{Spec}(H) = \mathrm{conv}(\mathrm{Spec}(H_0) + \gamma) \cup \mathrm{conv}(\mathrm{Spec}(H_0) - \gamma).$$

Proof. As we have already noted, the ellipse that defines $\mathrm{Spec}(H_0) + \gamma$ coincides with that described by $\frac{(x-\gamma)^2}{(2a)^2} + y^2 = 1$ so in order to prove the statement of the theorem we will proceed in the same manner as before. Consider the family of hyperbolae $F(x,y,s) = (x-s)^2 - y^2 + 1 - \frac{1}{4a^2+1}(\gamma - s)^2$. We note that if we let $0 < s < \gamma$ then the parametric equations of the envelope given by

$$x(s) = \frac{1}{4a^2+1}(\gamma + 4a^2 s), \quad y(s) = \pm\frac{1}{4a^2+1}\sqrt{(4a^2+1)^2 - 4a^2(\gamma - s)^2}$$

are not well defined, however, this which would appear to be a problem is the reason behind the strength of our result in this case.

The function $y(s)$ is well defined for $\gamma - \frac{4a^2+1}{2a} \leqslant s \leqslant \gamma$ and for this range of s, $x(s)$ ranges from $\frac{1}{4a^2+1}\left(\gamma + 4a^2\left(\gamma - \frac{4a^2+1}{2a}\right)\right) = \gamma - 2a$ to γ, thus obtaining the entire left half of the ellipse as the envelope for the family F. This, together with the symmetry of the problem, implies that

$$\mathrm{Spec}(H) \subseteq \mathrm{conv}(\mathrm{Spec}(H_0) + \gamma) \cup \mathrm{conv}(\mathrm{Spec}(H_0) - \gamma).$$

The other inclusion is precisely the result proved in Theorem 3.2. ∎

Let us now consider more general potentials in the definition of H. So far we have dealt with the case when $v_n = \pm\gamma \in \mathbb{R}^+$ and we will look now at the case when the potential takes values in an interval (as done in Theorem 3.1) or a more general compact set contained in $\mathbb{R}$. It is important to note that the values v_n at each point will continue to be independent and identically distributed according to a probability law with compact support $M \subset \mathbb{R}$.

We derive the following simple result from Theorem 3.1.

COROLLARY 3.12. *If* $M = [-\gamma, \gamma]$, *then* $[-\gamma, \gamma] + \mathrm{i}$ *and* $[-\gamma, \gamma] - \mathrm{i}$ *are contained in* $\mathrm{Spec}(H)$.

Proof. This is proved by Theorem 3.1. ∎

Now, if M is an arbitrary compact set, we have the following result.

THEOREM 3.13. *Let M be any compact set in $\mathbb{R}$. Then*

$$m \pm \mathrm{i} \in \mathrm{Spec}(H) \quad \textit{if and only if } m \in M.$$

Furthermore, if M has any gaps of length $> 4a$ then the spectrum of H is disconnected.

Proof. If $m \in M$ then the ellipse given by $m + (\alpha + \beta)\cos\theta + \mathrm{i}\sin\theta$ for $\theta \in [0, 2\pi]$ is contained in $\mathrm{Spec}(H)$ by equation (3.5), and hence it follows that the points $m + \mathrm{i}$ and $m - \mathrm{i}$ (obtained when $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ respectively) are contained in the spectrum of H.

On the other hand, if $m \pm \mathrm{i} \in \mathrm{Spec}(H)$, then equation (3.5) implies that $m \in \mathrm{conv}(M)$. We will assume that $M \subset \mathrm{conv}(M)$ as otherwise the result follows from a translation argument and Corollary 3.12. If we suppose that $m \notin M$ then following the arguments used in Corollary 3.6 we can determine a hyperbolic region which contains the points $m \pm \mathrm{i}$ and which cannot lie in $\mathrm{Spec}(H)$.

Finally, to prove the last statement of the theorem we note that Theorem 2.4 implies that

$$\mathrm{Spec}(H) \subseteq \bigcup_{m \in M} B(m, 2a). \quad ∎$$

COROLLARY 3.14. *Let M be any compact set in $\mathbb{R}$ and let k be the number of gaps in M bigger than $4a$; then $\mathrm{Spec}(H)$ has at least $k + 1$ components.*

Proof. The proof follows from the last statement of Theorem 3.13. ∎

3.2. THE CASE $Hf_n = -\beta f_{n-1} + v_n f_n + \alpha f_{n+1}$. In this subsection we will again assume that $0 < \beta < \alpha$ and we will suppose that H has been scaled by a suitable constant so that $\alpha + \beta = 1$. We rewrite H as before as

$$H = A + \mathrm{i}B = (C + V) + \mathrm{i}B = H_0 + V$$

where

$$\begin{aligned} Af_n &= af_{n-1} + v_n f_n + af_{n+1}, & Bf_n &= bf_{n-1} - bf_{n+1}, \\ Cf_n &= af_{n-1} + af_{n+1}, & Vf_n &= v_n f_n, \end{aligned}$$

with $a = \frac{1}{2}(\alpha - \beta)$ and $b = \frac{\mathrm{i}}{2}$. It is clear that the identities stated for the analogous operators at the start of Subsection 3.1 hold in this case as well and the following inclusion is satisfied:

$$\mathrm{Spec}(H) \subseteq \{z = x + \mathrm{i}y \in \mathbb{C} : |y| \leqslant 1\}.$$

Many of the results obtained in Subsection 3.1 do not depend on taking $-\beta$ or β as coefficient of f_{n-1} in the definition of H. In fact, we have the following results, due to Davies [8], whose proofs we omit as this change in coefficients bears no relation to the arguments of the proof. We consider the case when M consists solely of two points: $v_n = \pm\gamma$ for all n. Then

$$\mathrm{Spec}(H_0) = \{(\alpha - \beta)\cos\theta + \mathrm{i}\sin\theta : \theta \in [0, 2\pi]\}. \tag{3.12}$$

We also have that

$$\mathrm{Spec}(H_0) + \{\pm\gamma\} \subseteq \mathrm{Spec}(H) \subseteq \mathrm{conv}(\mathrm{Spec}(H_0)) + [-\gamma, \gamma], \tag{3.13}$$

$$\mathrm{Spec}(H) \subseteq B(\gamma, 1) \cup B(-\gamma, 1). \tag{3.14}$$

Furthermore, if $|\gamma| > 1$ then

$$\begin{aligned} &\mathrm{conv}(\mathrm{Spec}(H_0) + \gamma) \cup \mathrm{conv}(\mathrm{Spec}(H_0) - \gamma) \\ &\qquad\qquad \subseteq \mathrm{Spec}(H) \subseteq \mathrm{conv}(\mathrm{Spec}(H_0)) + [-\gamma, \gamma]. \end{aligned} \tag{3.15}$$

Now, following the same procedure as in Theorem 3.5 and Corollary 3.6 we have the following result.

THEOREM 3.15. *The spectrum of H satisfies*

$$\mathrm{Spec}(H) \subseteq \{z \in \mathbb{C} : |y| \leqslant 1\} \setminus \Big\{z \in \mathbb{C} : x^2 - y^2 < -1 + \frac{1}{4a^2 + 1}\gamma^2\Big\}.$$

Proof. The proof is identical to the proofs of the results cited above. ∎

In a similar fashion we also have the following results (see Subsection 3.1 for the proofs).

THEOREM 3.16. *Let* $\gamma < \sqrt{4a^2+1}$. *It follows that* $\mathrm{Spec}(H)$ *does not intersect the set*

$$\left\{z \in \mathbb{C} : \frac{1}{4a^2+1}\gamma \leqslant x \leqslant \gamma \text{ and } \sqrt{1 - \frac{(x-\gamma)^2}{(2a)^2}} \leqslant |y| \leqslant 1\right\}$$
$$\cup \left\{z \in \mathbb{C} : -\gamma \leqslant x \leqslant -\frac{1}{4a^2+1}\gamma \text{ and } \sqrt{1 - \frac{(x+\gamma)^2}{(2a)^2}} \leqslant |y| \leqslant 1\right\}.$$

COROLLARY 3.17. *If* $\gamma < 2a$ *then* $\mathrm{Spec}(H)$ *is contained in the set*

$$\left\{z \in \mathbb{C} : 0 \leqslant |x| \leqslant \frac{1}{4a^2+1}\gamma \text{ and } |y| \leqslant \sqrt{x^2 + 1 - \frac{1}{4a^2+1}\gamma^2}\right\}$$
$$\cup \left\{z \in \mathbb{C} : \frac{1}{4a^2+1}\gamma < |x| \leqslant \gamma + 2a \text{ and } |y| \leqslant \sqrt{1 - \frac{(x-\gamma)^2}{(2a)^2}}\right\}.$$

COROLLARY 3.18. *If* $2a \leqslant \gamma < \sqrt{4a^2+1}$ *then* $\mathrm{Spec}(H)$ *is contained in the set*

$$\left\{z \in \mathbb{C} : \gamma - 2a \leqslant |x| \leqslant \frac{1}{4a^2+1}\gamma \text{ and } |y| \leqslant \sqrt{x^2 + 1 - \frac{1}{4a^2+1}\gamma^2}\right\}$$
$$\cup \left\{z \in \mathbb{C} : \frac{1}{4a^2+1}\gamma < |x| \leqslant \gamma + 2a \text{ and } |y| \leqslant \sqrt{1 - \frac{(x-\gamma)^2}{(2a)^2}}\right\}.$$

THEOREM 3.19. *If* $\sqrt{4a^2+1} \leqslant \gamma \leqslant \frac{4a^2+1}{2a}$ *then* $\mathrm{Spec}(H)$ *is contained in*

$$\left\{z \in \mathbb{C} : \gamma - 2a \leqslant |x| \leqslant \frac{1}{4a^2+1}\gamma \text{ and } |y| \leqslant \sqrt{x^2 + 1 - \frac{1}{4a^2+1}\gamma^2}\right\}$$
$$\cup \left\{z \in \mathbb{C} : \frac{1}{4a^2+1}\gamma < |x| \leqslant \gamma + 2a \text{ and } |y| \leqslant \sqrt{1 - \frac{(x-\gamma)^2}{(2a)^2}}\right\}.$$

And finally we have the following result which completely determines the spectrum of H for a range of values of γ.

THEOREM 3.20. *If* $\gamma > \frac{4a^2+1}{2a}$ *then*

$$\mathrm{Spec}(H) = \mathrm{conv}(\mathrm{Spec}(H_0) + \gamma) \cup \mathrm{conv}(\mathrm{Spec}(H_0) - \gamma).$$

These results, which are also true in the case when H is given by (3.2), allow us to see the similarities between the operators studied in Subsection 3.1 and the present section. However, the following results show us that there is an intrinsic difference between these operators. The theorems we present below do not provide any information in the case of (3.2) when studying outer bounds for $\mathrm{Spec}(H)$ but are of great importance here.

We follow a method introduced by Davies in [7] and extended in [8]. It allows us to make a more precise determination of the spectrum of H by looking at certain sets that can be proved to be contained in $\mathrm{Spec}(H)$. We need the following definitions:

Let $f \in l^2(\mathbb{Z})$ with $\|f\| = 1$. We define

$$\mathrm{var}(f) = \langle Q^2 f, f\rangle - \langle Qf, f\rangle^2$$

where Q is given by $Qf(x) = xf(x)$. We say that $\lambda \in \sigma_{\mathrm{loc}}(H)$ (the localized spectrum of H) if there exist k and a sequence $f_n \in l^2(\mathbb{Z})$ such that $\|f_n\| = 1$ for all n, $\mathrm{var}(f_n) \leqslant k$ for all n and $\lim_{n\to\infty} \|Hf_n - \lambda f_n\| = 0$.

Now, given any finite sequence $u = (u_0, \ldots, u_{n-1})$ of γ's and $-\gamma$'s, let W_u be the periodic potential such that $W_{u,m} = u_r$ if $m = r \mod n$. Then, if $f \in l^2(\mathbb{Z})$ and $w_m = (f_{m-1}, f_m)$ the eigenvalue problem given by

$$-\beta f_{m-1} + W_{u,m} f_m + \alpha f_{m+1} = \lambda f_m \tag{3.16}$$

may be written as

$$w_{m+1} = w_m A_m$$

where

$$A_m = \begin{pmatrix} 0 & \frac{\beta}{\alpha} \\ 1 & \frac{1}{\alpha}(\lambda - W_{u,m}) \end{pmatrix}.$$

Hence, $w_m = w_0 A_0 A_1 \cdots A_{m-1}$ and in general, for any $r \in \mathbb{Z}$

$$w_{n(r+1)} = w_{nr} A_0 A_1 \cdots A_{n-1}.$$

Now, let $B = A_0 A_1 \cdots A_{n-1}$, it follows that $\det B = \left(-\frac{\beta}{\alpha}\right)^n$, that is $\det B < 1$.

The following result is mainly due to Davies. A different eigenvalue equation is considered in [8], however the proof works exactly as written out and we thus omit it here.

THEOREM 3.21. *Let E^n denote the set $\{\mathrm{e}^{\mathrm{i}\theta} + \left(-\frac{\beta}{\alpha}\right)^n \mathrm{e}^{-\mathrm{i}\theta} : \theta \in [0, 2\pi]\}$ and*

$$E_u = \{\lambda : \mathrm{tr}(A_0 A_1 \cdots A_{n-1}) \in E^n\}.$$

Then E_u is closed and bounded and $E_u \subseteq \mathrm{Spec}(H)$.

We can now state the following two results which are also due to Davies and which will be of great importance in our analysis of $\mathrm{Spec}(H)$. See [8] for the proofs; we note that given our assumption that $\alpha > \beta$ the proofs need no amendment to work for the results as stated below. (The case $\alpha < \beta$ can be proved analogously as having $\det B > 1$ would simply interchange the roles of the exponentially increasing/decreasing arguments used in the proof).

LEMMA 3.22. *Let $u = (u_0, \ldots, u_{n-1})$ be a vector in $\{\pm\gamma\}^n$. Then*

$$I_u := \{\lambda : \mathrm{tr}(A_0 A_1 \cdots A_{n-1}) \in \mathrm{int}(E^n)\},\ O_u := \{\lambda : \mathrm{tr}(A_0 A_1 \cdots A_{n-1}) \in \mathrm{ext}(E^n)\}$$

and E_u are disjoint sets and cover the complex plane.

THEOREM 3.23. *Let $n, p \in \mathbb{N}$. If $u \in \{\pm\gamma\}^n$ and $u' \in \{\pm\gamma\}^p$ then $I_u \cap O_{u'} \subseteq \sigma_{\mathrm{loc}}(H)$.*

Let us consider first the case $n = 1$ in Theorem 3.21. There are only two possibilities for u, namely, $u = \pm\gamma$. It is straightforward to see that we obtain no new information from this case, that is, the results yielded by considering $n = 1$ simply state that

$$E_\gamma \cup E_{-\gamma} \subseteq \mathrm{Spec}(H)$$

which is an equivalent statement to that made by the first inclusion in equation (3.13).

The case $n = 2$ however, is much more interesting and it will allow us to study the case $n = 2, p = 1$ in Theorem 3.23.

THEOREM 3.24. *If $u = (\gamma, -\gamma)$, then E_u consists of the curves described by the roots of the polynomial (in λ) given by*

$$\lambda^2 - \gamma^2 + 2\alpha\beta - \alpha^2 \mathrm{e}^{\mathrm{i}\theta} - \beta^2 \mathrm{e}^{-\mathrm{i}\theta} = 0,$$

as θ varies between 0 and 2π.

Moreover, if $\gamma < 1$ then the roots describe a single simple closed symmetric curve, whereas if $\gamma > 1$ then two simple closed symmetric curves are obtained. If $\gamma = 1$ the roots of the polynomial vary along two symmetric loops that intersect at the origin.

Proof. Let $u = (\gamma, -\gamma)$ and let $B = A_0 A_1$. Then

$$B = \begin{pmatrix} 0 & \frac{\beta}{\alpha} \\ 1 & \frac{1}{\alpha}(\lambda - \gamma) \end{pmatrix} \begin{pmatrix} 0 & \frac{\beta}{\alpha} \\ 1 & \frac{1}{\alpha}(\lambda + \gamma) \end{pmatrix} = \begin{pmatrix} \frac{\beta}{\alpha} & \frac{\beta}{\alpha^2}(\lambda + \gamma) \\ \frac{1}{\alpha}(\lambda - \gamma) & \frac{\lambda^2 - \gamma^2 + \alpha\beta}{\alpha^2} \end{pmatrix},$$

and hence, $\det B = \frac{\beta^2}{\alpha^2}$ and $\mathrm{tr} B = \frac{\lambda^2 - \gamma^2 + 2\alpha\beta}{\alpha^2}$.

Now, $E^2 = \{\mathrm{e}^{\mathrm{i}\theta} + \left(\frac{\beta}{\alpha}\right)^2 \mathrm{e}^{-\mathrm{i}\theta} : \theta \in [0, 2\pi]\}$ and

$$E_{(\gamma,-\gamma)} = \left\{\lambda : \frac{\lambda^2 - \gamma^2 + 2\alpha\beta}{\alpha^2} \in E^2\right\}.$$

That is, $E_{(\gamma,-\gamma)}$ is the set of $\lambda \in \mathbb{C}$ such that

$$\lambda^2 - \gamma^2 + 2\alpha\beta = \alpha^2 \mathrm{e}^{\mathrm{i}\theta} + \beta^2 \mathrm{e}^{-\mathrm{i}\theta} \tag{3.17}$$

for $\theta \in [0, 2\pi]$. This last equation can be solved explicitly for λ and it is quite simple to see that there are three qualitatively different cases, namely $\gamma < 1$, $\gamma = 1$ and $\gamma > 1$. Let us consider first the case when $\gamma < 1$. If $\theta = 0$ then equation (3.17) becomes $\lambda^2 - \gamma^2 + 2\alpha\beta - \alpha^2 - \beta^2 = 0$, and hence, λ is real. As θ grows then λ takes complex values until $\theta = \pi$ when $\lambda = \pm\sqrt{-1 + \gamma^2}$ which lies on the imaginary axis; and we note that this value coincides with the intersection of the balls given in equation (3.14), that is $B(\gamma, 1)$ and $B(-\gamma, 1)$. As θ varies between π and 2π a symmetric curve is obtained and hence a single closed curve is obtained.

If $\gamma = 1$ there is a crucial difference with the case described above: at the point when $\theta = \pi$, $\lambda = \pm\sqrt{-1+\gamma^2} = 0$ and hence the curve described by the roots in this case consists of two loops that are not disjoint but that intersect solely at the origin. Finally, if $\gamma > 1$ the value of λ when $\theta = \pi$ is real and nonzero and hence as the roots vary they do so along two distinct (but symmetric) loops. ∎

If we take $u = (\gamma, -\gamma)$ and $u' = (\gamma)$ in Theorem 3.23, then I_u is the interior of the curves of Theorem 3.24 and $O_{u'}$ is the exterior of the union of two ellipses. By Theorems 3.21 and 3.23, the curve(s) E_u and the regions $I_u \cap O_{u'}$ are contained in $\mathrm{Spec}(H)$. The three examples in Figure 4 show the new regions that have been proved to be contained in $\mathrm{Spec}(H)$ by Theorem 3.23 (case $n = 2$, $p = 1$).

REMARK. This result greatly contrasts with the results that would be obtained from this theorem in Subsection 3.1 where the curve obtained for the case $n = 2$ never lies outside the curves obtained for the case $n = 1$ and thus has been omitted from our study as it cannot be used to determine whether any portion of the spectrum lies outside the curves obtained for the case $n = 1$.

We know from our previous results (namely, equation (3.14)) that 0 can only be in the spectrum of H if $\gamma \leqslant 1$, but we can now conclude the following:

COROLLARY 3.25. *If* $\alpha - \beta \leqslant \gamma \leqslant 1$ *then* $0 \in \mathrm{Spec}(H)$.

And in fact, using the results used in [8] to obtain Corollary 15 there, it follows that

COROLLARY 3.26. $0 \in \mathrm{Spec}(H) \iff \alpha - \beta \leqslant \gamma \leqslant 1$.

The following two theorems give the polynomials whose roots determine the desired curves in the cases $n = 3$ and $n = 4$ of Theorem 3.21. However, their study becomes quite complicated and it is perhaps only practical to study these cases numerically.

THEOREM 3.27. *Let* $u_1 = (\gamma, \gamma, -\gamma)$ *and* $u_2 = (-\gamma, -\gamma, \gamma)$. *Then* E_{u_1} *and* E_{u_2} *are the curves traced out by the roots of the polynomials*

$$\lambda^3 - \gamma\lambda^2 + [3\alpha\beta - \gamma^2]\lambda - \alpha\beta\gamma + \gamma^3 - \alpha^3 e^{i\theta} - \beta^3 e^{-i\theta} = 0,$$
$$\lambda^3 + \gamma\lambda^2 + [3\alpha\beta - \gamma^2]\lambda + \alpha\beta\gamma - \gamma^3 - \alpha^3 e^{i\theta} - \beta^3 e^{-i\theta} = 0,$$

respectively, as θ *varies over* $[0, 2\pi]$.

Proof. The proof follows from considering the matrices defined by u_1 and u_2 using Theorem 3.21. Namely,

$$B = \begin{pmatrix} 0 & \frac{\beta}{\alpha} \\ 1 & \frac{1}{\alpha}(\lambda - \gamma) \end{pmatrix} \begin{pmatrix} 0 & \frac{\beta}{\alpha} \\ 1 & \frac{1}{\alpha}(\lambda - \gamma) \end{pmatrix} \begin{pmatrix} 0 & \frac{\beta}{\alpha} \\ 1 & \frac{1}{\alpha}(\lambda + \gamma) \end{pmatrix},$$
$$B' = \begin{pmatrix} 0 & \frac{\beta}{\alpha} \\ 1 & \frac{1}{\alpha}(\lambda + \gamma) \end{pmatrix} \begin{pmatrix} 0 & \frac{\beta}{\alpha} \\ 1 & \frac{1}{\alpha}(\lambda + \gamma) \end{pmatrix} \begin{pmatrix} 0 & \frac{\beta}{\alpha} \\ 1 & \frac{1}{\alpha}(\lambda - \gamma) \end{pmatrix}. \quad ∎$$

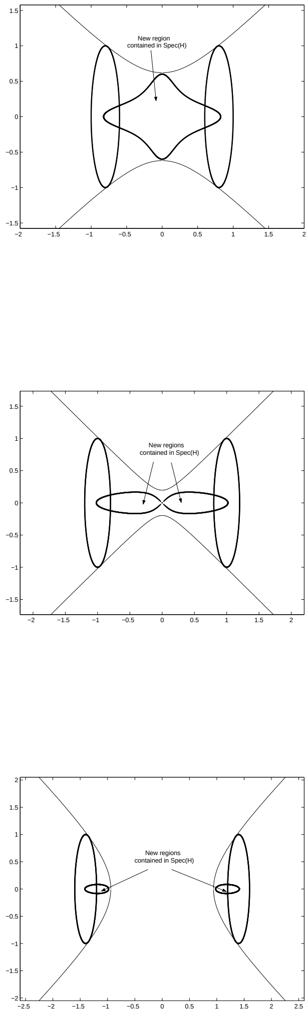

FIGURE 4. In these cases $\alpha = \frac{3}{5}, \beta = \frac{2}{5}$ and γ takes values $0.8, 1$ and 1.4 respectively.

THEOREM 3.28. *Let $\theta \in [0, 2\pi]$. The roots of the following equations are contained in* $\mathrm{Spec}(H)$*:*

$$\lambda^4 + [4\alpha\beta - 2\gamma^2]\lambda^2 + \gamma^4 + 2\alpha^2\beta^2 - \alpha^4 e^{i\theta} - \beta^4 e^{i\theta} = 0,$$
$$\lambda^4 - 2\gamma\lambda^3 + 4\alpha\beta\lambda^2 + [2\gamma^3 - 4\alpha\beta\gamma]\lambda - \gamma^4 + 2\alpha^2\beta^2 - \alpha^4 e^{i\theta} - \beta^4 e^{i\theta} = 0,$$
$$\lambda^4 + 2\gamma\lambda^3 + 4\alpha\beta\lambda^2 + [4\alpha\beta\gamma - 2\gamma^3]\lambda - \gamma^4 + 2\alpha^2\beta^2 - \alpha^4 e^{i\theta} - \beta^4 e^{i\theta} = 0.$$

Proof. The proof follows by considering the three vectors given by $u_1 = (\gamma, \gamma, -\gamma, -\gamma)$, $u_2 = (\gamma, \gamma, \gamma, -\gamma)$ and $u_3 = (-\gamma, -\gamma, -\gamma, \gamma)$. ∎

We note that the first equation in the statement of Theorem 3.28 above can be solved explicitly and we thus have the following corollary.

COROLLARY 3.29. *The curves described by*

$$\pm\sqrt{\gamma^2 - 2\alpha\beta \pm \sqrt{4\alpha^2\beta^2 - 2\gamma^2 + \alpha^4 e^{i\theta} + \beta^4 e^{-i\theta}}}$$

as θ varies in $[0, 2\pi]$ lie in $\operatorname{Spec}(H)$.

REMARK. Theorem 3.23 establishes that the regions determined by the intersection of the exterior of any one of the curves determined by the previous two theorems with the interior of another such curve is contained in the spectrum of H.

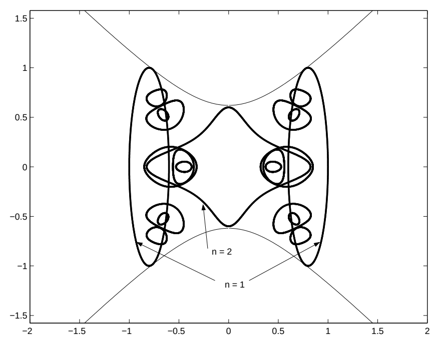

FIGURE 5. The curves shown in bold are contained in $\operatorname{Spec}(H)$ and are obtained from cases $n = 1$ to 4 of Theorem 3.21. The curves obtained from the cases $n = 1$ and $n = 2$ are shown explicitly and the remaining curves belong to the cases $n = 3, 4$. For this particular example $\alpha = \frac{3}{5}, \beta = \frac{2}{5}, \gamma = \frac{4}{5}$. Theorem 3.23 tells us that the region that lies outside any of these curves and inside another is also contained in $\operatorname{Spec}(H)$.

Figure 5 shows an example of these curves and the regions they determine for the case $\alpha = \frac{3}{5}, \beta = \frac{2}{5}, \gamma = \frac{4}{5}$.

Figure 6 contains the same graphical information as Figure 5 but it also shows the eigenvalues obtained for 100 finite 100×100 matrices which we know

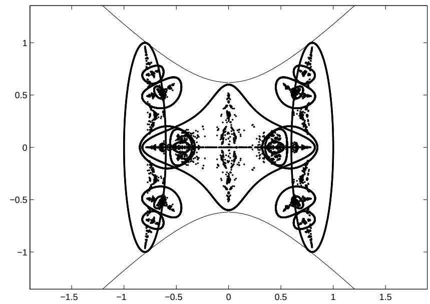

FIGURE 6. This figure shows the same curves as Figure 5 together with the eigenvalues from the finite matrix with periodic boundary conditions.

are contained in the spectrum of H. The high coincidence between these points and the curves obtained is clear.

3.3. A SPECIAL CASE. In this subsection we will consider the particular case of the operator H when $\alpha = \beta$, namely

$$Hf_n = -\frac{1}{2}f_{n-1} + v_n f_n + \frac{1}{2}f_{n+1}. \tag{3.18}$$

We will again commence by considering the case when $v_n = \pm\gamma \in \mathbb{R} \setminus \{0\}$ for all n. Keeping the same notation as before we have

$$\mathrm{Spec}(H_0) = \left\{\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta} - \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta} : \theta \in [0, 2\pi]\right\} = \{\mathrm{i}\sin(\theta) : \theta \in [0, 2\pi]\},$$

that is, $\mathrm{Spec}(H_0)$ is the interval $[-\mathrm{i}, \mathrm{i}]$. Hence, it follows from (3.5) that

$$[-\mathrm{i}, \mathrm{i}] \pm \gamma \subseteq \mathrm{Spec}(H) \subseteq \{z \in \mathbb{C} : |x| \leqslant \gamma \text{ and } |y| \leqslant 1\}. \tag{3.19}$$

We again rewrite H as $A + \mathrm{i}B$ and in this case $a = 0$ and $b = \frac{\mathrm{i}}{2}$, hence, it is immediate that $\|A\| = \gamma$ and $\|B\| = 1$, and in fact we have the following identity for B^2:

$$B^2 = I - \frac{1}{4}(L + R)^2 \tag{3.20}$$

where L and R continue to denote the left and right shift operators.

We now follow the method described in the previous sections using the second order numerical range to obtain more information about $\mathrm{Spec}(H)$.

THEOREM 3.30. *$\mathrm{Re}(H^2) \geqslant (-1+\gamma^2)I$ and hence, the spectrum of H is contained in the set $\{z : \mathrm{Re}(z^2) \geqslant -1+\gamma^2\}$.*

Proof. See Theorem 3.5. ∎

COROLLARY 3.31. *The spectrum of H satisfies*

$$\mathrm{Spec}(H) \subseteq \{z \in \mathbb{C} : |x| \leqslant \gamma \text{ and } |y| \leqslant 1\} \setminus \{z \in \mathbb{C} : x^2 - y^2 < -1+\gamma^2\}.$$

The shape of the hyperbola $x^2 - y^2 = -1+\gamma^2$ clearly depends on the value of γ. For $\gamma < 1$ the hyperbola opens about the imaginary axis and has vertices at $\pm \mathrm{i}\sqrt{1-\gamma^2}$, if $\gamma = 1$ it degenerates into two orthogonal lines through the origin and if $\gamma > 1$ the hyperbola opens to the left and to the right and has vertices at $\pm\sqrt{\gamma^2-1}$. It is worth noting that regardless of the value of γ the hyperbola intersects the intervals $[-\mathrm{i},\mathrm{i}]-\gamma$ and $[-\mathrm{i},\mathrm{i}]+\gamma$ at the points $(-\gamma,\pm\mathrm{i})$ and $(\gamma,\pm\mathrm{i})$ respectively, and we have thus obtained outer bounds for $\mathrm{Spec}(H)$. Figure 7 shows three specific examples ($\gamma = 0.75, \gamma = 1$ and $\gamma = 1.5$) of these 3 qualitatively different cases together with the results that Theorem 3.33 will provide.

We note that in this case working out the envelope of the family of hyperbolae given by

$$F(x,y,s) = (x-s)^2 - y^2 + 1 - (\gamma - s)^2 \tag{3.21}$$

is not a useful technique as the envelope consists solely of four points, $(\pm\gamma,\pm\mathrm{i})$.

We thus turn to the method of periodic potentials introduced in the last section. It is important to note that the added symmetry of our operator in this case makes many of the calculations much more accessible and many detailed results can be obtained, however it is precisely the added symmetry of the problem that causes the simple closed curves which appeared in the previous section as solutions to the polynomials to collapse into arcs and hence stops us from being able to apply Theorem 3.23 to the operator defined in (3.18).

We have

$$A_m = \begin{pmatrix} 0 & 1 \\ 1 & 2(\lambda - W_{u,m}) \end{pmatrix}$$

and if $f \in l^2(\mathbb{Z})$ and $w_m = (f_{m-1}, f_m)$ the equation

$$-\frac{1}{2} f_{m-1} + W_{u,m} f_m + \frac{1}{2} f_{m+1} = \lambda f_m$$

may be rewritten as

$$w_{m+1} = w_m A_m$$

as before. It is easy to see that $\det A_0 A_1 \cdots A_{n-1} = (-1)^n$ and Theorem 3.21 may be rewritten as the following result.

THEOREM 3.32. *Let $E^n = \{\mathrm{e}^{\mathrm{i}\theta} + (-1)^n \mathrm{e}^{-\mathrm{i}\theta} : \theta \in [0, 2\pi]\}$, and let*

$$E_u = \{\lambda : \mathrm{tr}(A_0 A_1 \cdots A_{n-1}) \in E^n\}.$$

Then E_u is closed and bounded and $E_u \subseteq \mathrm{Spec}(H)$.

The case $n = 1$ coincides with the first inclusion of (3.19). The case $n = 2$ however, produces new information. In fact, we have the following theorem.

THEOREM 3.33. *The spectrum of H satisfies one of the following inclusions depending on the value of γ:*

$$[-\gamma,\gamma] \cup \Big[-\mathrm{i}\sqrt{1-\gamma^2}, \mathrm{i}\sqrt{1-\gamma^2}\Big] \subseteq \mathrm{Spec}(H) \quad \text{if } \gamma < 1, \tag{3.22}$$

$$[-\gamma,\gamma] \subseteq \mathrm{Spec}(H) \quad \text{if } \gamma = 1, \tag{3.23}$$

$$\Big[-\gamma, -\sqrt{\gamma^2-1}\Big] \cup \Big[\sqrt{\gamma^2-1}, \gamma\Big] \subseteq \mathrm{Spec}(H) \quad \text{if } \gamma > 1. \tag{3.24}$$

Figure 7 shows particular examples of each of these three cases together with information obtained from Corollary 3.31 .

Proof. Let $u = (\gamma, -\gamma)$. It is worthwhile to note that the cases (γ,γ) and $(-\gamma,-\gamma)$ produce the same results as the case $n = 1$ for obvious reasons and will hence not be considered here. Let $B = A_0A_1$. We have that

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 2(\lambda-\gamma) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2(\lambda+\gamma) \end{pmatrix} = \begin{pmatrix} 1 & 2(\lambda+\gamma) \\ 2(\lambda-\gamma) & 1+4(\lambda^2-\gamma^2) \end{pmatrix},$$

and hence, $\det B = 1$ and $\mathrm{tr}B = 2 + 4(\lambda^2 - \gamma^2)$.

Now, $E^2 = \{e^{i\theta} + e^{-i\theta} : \theta \in [0, 2\pi]\}$ and

$$E_{(\gamma,-\gamma)} = \{\lambda : 2 + 4(\lambda^2 - \gamma^2) \in E^2\}.$$

That is, $E_{(\gamma,-\gamma)}$ is the set of $\lambda \in \mathbb{C}$ such that

$$2 + 4(\lambda^2 - \gamma^2) = e^{i\theta} + e^{-i\theta}$$

for $\theta \in [0, 2\pi]$, which can be simplified and written as

$$\lambda^2 = \frac{1}{2}(\cos\theta - 1) + \gamma^2. \tag{3.25}$$

This last equation can be solved explicitly and it is quite simple to see that there are three different cases, namely $\gamma < 1$, $\gamma = 1$ and $\gamma > 1$.

If $\gamma < 1$, then $0 < \gamma^2 < 1$ as γ is positive, and

$$-1 < \lambda^2 = \frac{1}{2}(\cos\theta - 1) + \gamma^2 < 1.$$

Now, as θ varies in the interval $[0, 2\pi]$ we obtain the following results: If $\theta = 0$ then $\lambda = \pm\gamma$, and as θ begins to increase $\frac{1}{2}(\cos\theta - 1) + \gamma^2$ remains positive and hence, λ^2 has two real roots. When $\theta = \arccos(1 - 2\gamma^2)$ (which is well defined as $-1 < (1 - 2\gamma^2) < 1$), $\lambda = 0$ and for $\arccos(1 - 2\gamma^2) < \theta < \pi$, λ^2 has two purely imaginary roots whose moduli increase as θ continues to increase. If $\theta = \pi$ then $\lambda = \pm i\sqrt{1-\gamma^2}$ and as θ varies from π to 2π the same roots are obtained in reverse order. We observe that the roots obtained when $\theta = \pi$ coincide with the vertices of the hyperbola obtained in Theorem 3.30.

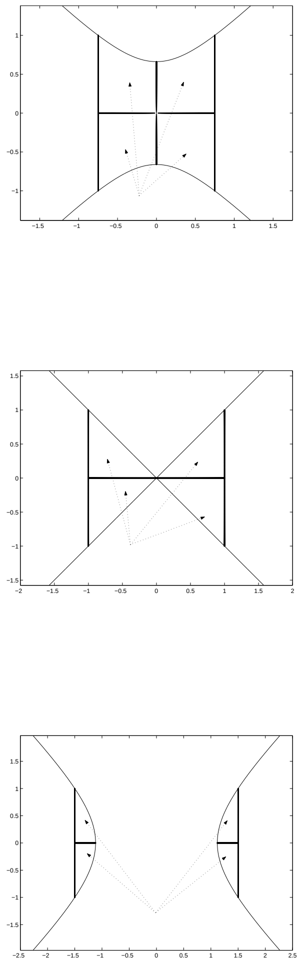

FIGURE 7. For each value of γ (0.75, 1 and 1.5 respectively), $\mathrm{Spec}(H)$ is contained in the regions pointed to by arrows and contains the bold lines.

Summarizing these results, we have that

$$[-\gamma,\gamma]\cup\Big[-\mathrm{i}\sqrt{1-\gamma^2},\mathrm{i}\sqrt{1-\gamma^2}\Big]\subseteq\mathrm{Spec}(H)\quad\text{if } \gamma<1$$

and hence, equation (3.22) in the statement of the theorem is satisfied.

If $\gamma=1$ then $0\leqslant\lambda^2=\frac{1}{2}(\cos\theta-1)+\gamma^2\leqslant 1$, as θ varies in $[0,2\pi]$ the roots fill up the real axis between -1 and 1. That is,

$$[-\gamma,\gamma]\subseteq\mathrm{Spec}(H)\quad\text{if } \gamma=1\ .$$

And finally, to prove equation (3.24), consider $\gamma > 1$. It follows that

$$0 < -1 + \gamma^2 \leqslant \lambda^2 = \frac{1}{2}(\cos\theta - 1) + \gamma^2 \leqslant \gamma^2.$$

Hence, as θ varies the roots oscillate between $\sqrt{-1+\gamma^2}$ and γ and the symmetric negative interval:

$$\Big[-\gamma, -\sqrt{\gamma^2 - 1}\Big] \cup \Big[\sqrt{\gamma^2 - 1}, \gamma\Big] \subseteq \mathrm{Spec}(H) \quad \text{if } \gamma > 1\,.$$

And hence the result follows as required. ∎

The case $n = 3$ of Theorem 3.32 produces polynomials of degree 3 (as the cases $n = 1, 2$ produced polynomials of degree 1 and 2 respectively) so an in depth analysis of the roots of the polynomial in general can be a very complicated task, however the added symmetry of our operator in this case ($\alpha = \beta = \frac{1}{2}$) makes the analysis somewhat simpler than it would be in more general cases, and in fact we have the following result.

THEOREM 3.34. *The set of complex roots of the following two equations obtained as θ varies in the interval $[0, 2\pi]$, are contained in* $\mathrm{Spec}(H)$*:*

$$4\lambda^3 - 4\gamma\lambda^2 + (3 - 4\gamma^2)\lambda + 4\gamma^3 - \gamma = \mathrm{i}\sin\theta \tag{3.26}$$

$$4\lambda^3 + 4\gamma\lambda^2 + (3 - 4\gamma^2)\lambda - 4\gamma^3 + \gamma = \mathrm{i}\sin\theta. \tag{3.27}$$

Moreover, this set is made up of six arcs whose end points are obtained when $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.

Proof. The proof follows as before by considering $u_1 = (\gamma, \gamma, -\gamma)$ and $u_2 = (-\gamma, -\gamma, \gamma)$. ∎

The case $n = 4$ will be the last case we look at in detail. For $n \geqslant 5$ Theorem 3.32 is obviously still valid but it is perhaps only feasible to carry out the analysis numerically as one encounters several polynomials of degree $n \geqslant 5$. However we will make some general remarks about the type of information that can be expected in all cases and what can be said about the spectrum of H in each case.

THEOREM 3.35. *The roots of the following three equations, obtained as θ varies in $[0, 2\pi]$, are contained in* $\mathrm{Spec}(H)$*:*

$$8\lambda^4 + (8 - 16\gamma^2)\lambda^2 + 8\gamma^4 + 1 = \cos\theta, \tag{3.28}$$

$$8\lambda^4 - 16\gamma\lambda^3 + 8\lambda^2 + (16\gamma^3 - 8\gamma)\lambda - 8\gamma^4 + 1 = \cos\theta, \tag{3.29}$$

$$8\lambda^4 + 16\gamma\lambda^3 + 8\lambda^2 + (8\gamma - 16\gamma^3)\lambda - 8\gamma^4 + 1 = \cos\theta. \tag{3.30}$$

Furthermore, the solutions to these equations lie on simple arcs and in fact the solutions to equation (3.28) *lie on two parabola arcs for $\gamma \geqslant \frac{1}{2}$, and on two parabola arcs and the imaginary axis for $\gamma < \frac{1}{2}$.*

Proof. Three cases need to be considered:

$$u_1 = (\gamma, \gamma, -\gamma, -\gamma), \quad u_2 = (\gamma, \gamma, \gamma, -\gamma), \quad u_3 = (-\gamma, -\gamma, -\gamma, \gamma).$$

Let $u_1 = (\gamma, \gamma, -\gamma, -\gamma)$ and let $B = A_0A_1A_2A_3$. Then

$$\begin{aligned} B &= \begin{pmatrix} 0 & 1 \\ 1 & 2(\lambda-\gamma) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2(\lambda-\gamma) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2(\lambda+\gamma) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 2(\lambda+\gamma) \end{pmatrix} \\ &= \begin{pmatrix} 4\lambda^2+1-4\gamma^2 & 8\lambda^3+8\gamma\lambda^2+(4-8\gamma^2)\lambda-8\gamma^3 \\ 8\lambda^3-8\gamma\lambda^2+(4-8\gamma^2)\lambda+8\gamma^3 & 16\lambda^4+(12-32\gamma^2)\lambda^2+16\gamma^4+4\gamma^2+1 \end{pmatrix} \end{aligned}$$

and hence, $E^4 = \{e^{i\theta} + e^{-i\theta} : \theta \in [0, 2\pi]\}$.

It follows that $E_{(\gamma,\gamma,-\gamma,-\gamma)}$ is the set of complex λ such that

$$16\lambda^4 + 16\lambda^2 - 32\gamma^2\lambda^2 + 16\gamma^4 + 2 = 2\cos\theta$$

and hence (3.28) is obtained. This equation can be solved explicitly for λ^2 and hence for λ obtaining the following in terms of θ:

$$\lambda = \pm\frac{1}{2}\sqrt{-2+4\gamma^2+\sqrt{2-16\gamma^2+2\cos\theta}},$$
$$\lambda = \pm\frac{1}{2}\sqrt{-2+4\gamma^2-\sqrt{2-16\gamma^2+2\cos\theta}}.$$

If $\gamma = \frac{1}{2}$ then as θ varies the roots of the above equations lie on two parabola arcs. Namely, the parabolic segment with vertex $\frac{1}{2}i$ (point obtained when $\theta = 0$) and with end points $\frac{1}{2}\sqrt{-1+2i}$ and $-\frac{1}{2}\sqrt{-1-2i}$ (obtained when $\theta = \pi$), and the symmetric arc reflected on the real axis.

If $\gamma > \frac{1}{2}$ the roots of these equations determine two parabolas as well but only a portion of these arcs is contained in the spectrum of H; the sections that lie away from the imaginary axis and whose end points are obtained when $\theta = 0$ and $\theta = \pi$. More precisely, the segments contained between $\frac{1}{2}\sqrt{-2+4\gamma^2+\sqrt{4-16\gamma^2}}$ and $\frac{1}{2}\sqrt{-2+4\gamma^2+\sqrt{-16\gamma^2}}$ in the first quadrant, and in the second quadrant between $-\frac{1}{2}\sqrt{-2+4\gamma^2-\sqrt{4-16\gamma^2}}$ and $-\frac{1}{2}\sqrt{-2+4\gamma^2-\sqrt{-16\gamma^2}}$, together with the symmetric segments obtained by reflection on the real axis.

Finally, for $\gamma < \frac{1}{2}$ the roots of equation (3.28) not only lie on parabola arcs as mentioned above but along the imaginary axis. When $\theta = 0$, both

$$\pm\frac{1}{2}\sqrt{-2+4\gamma^2+\sqrt{4-16\gamma^2}} \text{ and } \pm\frac{1}{2}\sqrt{-2+4\gamma^2-\sqrt{4-16\gamma^2}}$$

are purely imaginary and these values determine the extreme points of the interval on the imaginary axis on which the roots of equation (3.28) lie. As θ grows the roots continue to move along the imaginary axis until they reach the vertices of the parabolic segments described in the previous two cases and then, when $\theta = \pi$ the end points of these segments are obtained. As θ then varies from π to 2π the same path is described in inverse order.

Equations (3.29) and (3.30) can be obtained analogously by considering $u_2 = (-\gamma, -\gamma, -\gamma, \gamma)$ and $u_3 = (\gamma, \gamma, \gamma, -\gamma)$ respectively.

Thus u_2 is associated with the matrix B' whose entries are given as follows:

$$B'_{11} = 4\lambda^2 - 8\gamma\lambda + 4\gamma^2 + 1,$$
$$B'_{12} = 8\lambda^3 - 8\gamma\lambda^2 + (4 - 8\gamma^2)\lambda + 8\gamma^3,$$
$$B'_{21} = 8\lambda^3 - 24\gamma\lambda^2 + (4 + 24\gamma^2)\lambda - 8\gamma^3 - 4\gamma,$$
$$B'_{22} = 16\lambda^4 - 32\gamma\lambda^3 + 12\lambda^2 + (32\gamma^3 - 8\gamma)\lambda - 16\gamma^4 - 4\gamma^2 + 1,$$

and thus, $\det B' = 1$ and $\mathrm{tr} B'$ is given by

$$16\lambda^4 - 32\gamma\lambda^3 + 16\lambda^2 + (32\gamma^3 - 16\gamma)\lambda - 16\gamma^4 + 2.$$

This in turn implies that $E_{(\gamma,\gamma,\gamma,-\gamma)}$ is the set of $\lambda \in \mathbb{C}$ such that

$$8\lambda^4 - 16\gamma\lambda^3 + 8\lambda^2 + (16\gamma^3 - 8\gamma)\lambda - 8\gamma^4 + 1 = \cos\theta$$

and similarly $E_{(-\gamma,-\gamma,-\gamma,\gamma)}$ is the set of λ such that

$$8\lambda^4 + 16\gamma\lambda^3 + 8\lambda^2 + (8\gamma - 16\gamma^3)\lambda - 8\gamma^4 + 1 = \cos\theta.$$

These equations are more complicated to work with than equation (3.28) but it is still possible to analyze them to some extent. The value $\gamma = \frac{1}{\sqrt{2}}$ is a critical point as qualitatively different results are obtained if $\gamma > \frac{1}{\sqrt{2}}$ or if $\gamma < \frac{1}{\sqrt{2}}$.

If $\gamma = \frac{1}{\sqrt{2}}$ the set of roots of equations (3.29) and (3.30) contain the interval $[-\gamma, -\gamma]$ together with four additional arcs, one in each quadrant. We will describe the curve in the first quadrant here and then appeal to the symmetry of the problem to obtain the remaining three. Its end points are obtained when $\theta = 0$ and $\theta = \pi$. The curve starts at the point $\gamma + \frac{\sqrt{2}}{2}\mathrm{i}$ and moves 'upwards' towards the hyperbola described by $x^2 - y^2 = -1 + \gamma^2$ but without actually reaching it.

If $\gamma > \frac{1}{\sqrt{2}}$, each of the curves in the four quadrants remains the same, however the roots along the real axis no longer cover the entire interval between $-\gamma$ and γ, in other words, a gap occurs around the origin. However, the analysis of this gap is not crucial to our problem as the case $n = 2$ of Theorem 3.32 determines exactly what portion of the coordinate axes is contained in the spectrum of H.

If $\gamma < \frac{1}{\sqrt{2}}$ the roots move along the curves described in the case when $\gamma = \frac{1}{\sqrt{2}}$ but also along two other curves. These curves share their end points (which lie on the imaginary axis and on the parabola described in the analysis of equation (3.28)) and form a loop around the origin. ∎

Figure 8 shows the information obtained from all cases studied thus far ($n = 1$ to 4) in the particular instance when $\gamma = 0.9$. We note that all the curves that have been obtained from Theorem 3.32 in this manner are not closed curves. This follows from the intrinsic symmetry of the operator we are considering. In each

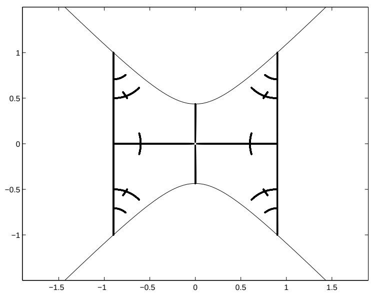

FIGURE 8. The thick curves shown are contained in Spec(H), they are obtained from cases $n = 1$ to 4 of Theorem 3.32 in the case when $\gamma = 0.9$

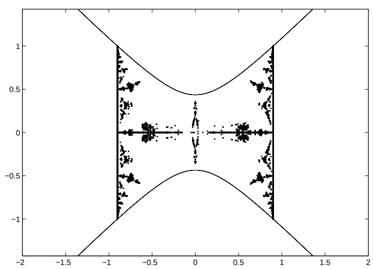

FIGURE 9. Eigenvalues of a hundred 100×100 finite matrices with periodic boundary conditions when $\gamma = 0.9$.

case a polynomial of degree n is obtained and this is set equal to either $\mathrm{i}\sin\theta$ or $\cos\theta$ depending on whether the degree of the polynomial is odd or even and hence closed curves will not be obtained as inverse images of a segment under a polynomial mapping. The proof is straightforward. (See [1] for instance.) This is important in retrospect when we consider the information that was obtained in the previous section when this operator symmetry was not present.

Finally, Figure 9 shows the eigenvalues obtained numerically for the finite case when $\gamma = 0.9$. In fact, it is data obtained from 100 runs of calculating the eigenvalues of 100×100 matrices with periodic boundary conditions. The high degree of correspondence between this data and that shown in Figure 8 is clear.

It is known that the eigenvalues are contained in the spectrum of the infinite volume operator and our example reveals a high degree of correspondence between these finite volume spectra and the curves we have studied.

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REFERENCES

[1] L.V. AHLFORS, *Complex Analysis*, McGraw-Hill Inc., New York 1979.

[2] R. BEALS, *Topics in Operator Theory*, University of Chicago Press, Chicago 1971.

[3] A. BÖTTCHER, M. EMBREE, V.I. SOKOLOV, Infinite Toeplitz and Laurent matrices with localized impurities, *Linear Algebra Appl.* **343/344**(2002), 101–118.

[4] A. BÖTTCHER, M. EMBREE, V.I. SOKOLOV, The spectra of large Toeplitz band matrices with a randomly perturbed entry, *Math. Comp.* **72**(2003), 1329–1348.

[5] J. BURKE, A. GREENBAUM, Some equivalent characterizations of the polynomial numerical hull of degree k, preprint, 2004.

[6] H.L. CYCON, R.G. FROESE, W. KIRSCH, B. SIMON, *Schrödinger Operators*, Springer-Verlag, New York 1987.

[7] E.B. DAVIES, Spectral properties of random non-self-adjoint matrices and operators, *Proc. Roy. Soc. London Ser. A* **457**(2001), 191–206.

[8] E.B. DAVIES, Spectral theory of pseudo-ergodic operators, *Comm. Math. Phys.* **216**(2001), 687–704.

[9] E.B. DAVIES Spectral bounds using higher order numerical ranges, *LMS J. Comput. Math.* **8**(2005), 17–45.

[10] E.B. DAVIES, B. SIMON, L^1 properties of intrinsic Schrödinger operators, *J. Funct. Anal.* **65**(1986), 126–146.

[11] I.Y. GOLDSHEID, B.A. KHORUZHENKO, Distribution of eigenvalues in non-hermitian Anderson Model, *Phys. Rev. Lett.* **80**(1998), 2897–2901.

[12] I.Y. GOLDSHEID, B.A. KHORUZHENKO, Eigenvalue curves of asymmetric tridiagonal random matrices, *Electron. J. Probab.* **5**(2000), 1–28.

[13] I.Y. GOLDSHEID, B.A. KHORUZHENKO, Regular spacings of complex eigenvalues in the one-dimensional non-hermitian Anderson Model, *Comm. Math. Phys.* **238**(2003), 505–524.

[14] K.E. GUSTAFSON, D.K.M. RAO, *Numerical Range. The Field of Values of Linear Operators and Matrices*, Springer-Verlag, New York 1997.

[15] N. HATANO, D.R. NELSON, Localization transitions in non-hermitian quantum mechanics, *Phys. Rev. Lett.* **77**(1996), 570–573.

[16] N. HATANO, D.R. NELSON, Vortex pinning and non-hermitian quantum mechanics, *Phys. Rev. B* **56**(1997), 8651–8673.

[17] N. HATANO, D.R. NELSON, Non-hermitian localization and eigenfunctions, *Phys. Rev. B* **58**(1998), 8384–8390.

[18] P. LANCASTER, *The Theory of Matrices*, Academic Press, New York 1969.

[19] H. LANGER, C. TRETTER, Spectral decomposition of some non-self-adjoint block operator matrices, *J. Operator Theory* **39**(1998), 339–359.

[20] H. LANGER, A. MARKUS, V. MATSAEV, C. TRETTER, A new concept for block operator matrices: the quadratical numerical range, *Linear Algebra Appl.* **330**(2001), 89–112.

[21] E.H. LOCKWOOD, *A Book of Curves*, Cambridge Univ. Press, Cambridge 1961.

[22] C. MARTINEZ, *Spectral properties of tridiagonal operators*, Ph.D. Dissertation, King's College London, London 2005.

[23] D.R. NELSON, N.M. SHNERB, Non-hermitian localization and population biology, *Phys. Rev. E* **58**(1998), 1383–1403.

[24] O. NEVANLINNA, *Convergence of Iterations for Linear Equations*, Birkhäuser, Basel 1993.

[25] J.M. ORTEGA, *Matrix Theory*, Plenum Press, New York 1987.

[26] M.H. STONE, *Linear Transformations in Hilbert Space and their Applications to Analysis*, Amer. Math. Soc. Colloq. Publ., vol. 15, Amer. Math. Soc., Providence, R.I. 1932.

[27] L.N. TREFETHEN, M. CONTEDINI, M. EMBREE, Spectra, pseudospectra and localization for random bidiagonal matrices, *Comm. Pure Appl. Math.* **54**(2001), 595–623.

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