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SUBSCALAR OPERATORS AND GROWTH OF RESOLVENT

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ABSTRACT. We construct a Banach space bounded linear operator T which is not $\mathcal{E}(\mathbb{T})$-subscalar but $\|(T-z)^{-1}\| \leqslant (|z|-1)^{-1}$ for $|z|>1$ and $m(T-z) \geqslant \mathrm{const}\cdot(1-|z|)^3$ for $|z|<1$ (here m denotes the minimum modulus). This gives a negative answer to a variant of a problem of K.B. Laursen and M.M. Neumann. We also give a sufficient condition (in terms of growth of resolvent and of an analytic left inverse of $T-z$) implying that T is an $\mathcal{E}(\mathbb{T})$-subscalar operator. This condition is also necessary for Hilbert space operators.

KEYWORDS: *Subscalar operators, growth conditions, resolvents.*

MSC (2000): 47B40, 47A20, 47A10.

1. INTRODUCTION

Generalized scalar operators are those Banach spaces operators possessing a C^∞-functional calculus. To be more specific, let $\mathcal{E}(\mathbb{C})$ denote the usual Fréchet algebra of all C^∞-functions on $\mathbb{C}$ with the topology of uniform convergence of derivatives of all orders on compact subsets of $\mathbb{C}$. Let X be a complex Banach space. A bounded linear operator $S \in B(X)$ is said ([8]) to be an $\mathcal{E}(\mathbb{C})$-*scalar* (or *generalized scalar*) operator if there is a continuous algebra homomorphism $\Phi : \mathcal{E}(\mathbb{C}) \to B(X)$ for which $\Phi(1) = I$ and $\Phi(z) = S$. Here z denotes the identity function on $\mathbb{C}$. A bounded linear operator is $\mathcal{E}(\mathbb{C})$-*subscalar* if it is similar to the restriction of an $\mathcal{E}(\mathbb{C})$-scalar operator to one of its closed invariant subspaces. We refer to three books [8], [10] and [12] for more information on $\mathcal{E}(\mathbb{C})$-scalar and $\mathcal{E}(\mathbb{C})$-subscalar operators.

The following statements are known to be equivalent (see [8], [12]):

(i) S is $\mathcal{E}(\mathbb{T})$-*scalar*, i.e., it has a continuous functional calculus on the Fréchet algebra $\mathcal{E}(\mathbb{T})$ of C^∞-functions on the unit circle $\mathbb{T}$;

(ii) S is $\mathcal{E}(\mathbb{C})$-scalar with spectrum $\sigma(S)$ in the unit circle $\mathbb{T}$;

(iii) S is invertible, and there exist constants $C>0$, $p \geqslant 0$ and $q \geqslant 0$ such that

$$\|S^n\| \leqslant Cn^p \quad (n \in \mathbb{N}) \quad \text{and} \quad \|S^{-n}\| \leqslant Cn^q \quad (n \in \mathbb{N});$$

(iv) $\sigma(S) \subset \mathbb{T}$ and there exist constants $C > 0$, $p \geqslant 0$ and $q \geqslant 0$ such that

$$\|(S-z)^{-1}\| \leqslant C(|z|-1)^{-p} \quad (|z|>1) \quad \text{and} \quad \|(S-z)^{-1}\| \leqslant C(1-|z|)^{-q} \quad (|z|<1).$$

The distinction between the growth of norms of positive and negative powers (and the resolvent growth inside and outside unit disc) will become apparent later on.

For $T \in B(X)$ we denote

$$m(T) = \inf\{\|Tx\| : x \in X, \|x\| = 1\}.$$

This quantity is called the *minimum modulus* of T ([11]) or the *lower bound* of T ([12]). It is easy to see that $m(T) > 0$ if and only if $T \in B(X)$ is one-to-one and with closed range. For invertible operators S we have $m(S) = \|S^{-1}\|^{-1}$.

The main question we consider in this note is the problem of intrinsic characterizations of $\mathcal{E}(\mathbb{T})$-subscalar operators (i.e. operators similar to a restriction of an $\mathcal{E}(\mathbb{T})$-scalar operator to an invariant subspace). Compressions of $\mathcal{E}(\mathbb{T})$-scalar operators to invariant subspaces have been studied in [6].

Let $T \in B(X)$ be an $\mathcal{E}(\mathbb{T})$-subscalar operator. Using (iii) for the invertible extension of T we obtain the existence of constants $C > 0$, $p \geqslant 0$ and $q \geqslant 0$ such that:

$$\text{(P)} \qquad \|T^n\| \leqslant Cn^p \quad \text{and} \quad m(T^n)^{-1} \leqslant Cn^q.$$

It is natural to ask if the polynomial growth condition (P) above (in terms of norms and minimum moduli of iterates) characterizes $\mathcal{E}(\mathbb{T})$-subscalar operators (cf. Problem 6.1.15 of [12] and [9]). This problem was also discussed in [15], [18], [17], [16]. It was recently proved by the authors [5], [4] that $\mathcal{E}(\mathbb{T})$-subscalar operators are indeed characterized by the polynomial growth condition (P).

Using the resolvent condition (iv), it can be proved similarly that if $T \in B(X)$ is an $\mathcal{E}(\mathbb{T})$-subscalar operator then there exist constants $C > 0$, $p \geqslant 0$ and $q \geqslant 0$ such that

$$\text{(R)} \quad \|(T-z)^{-1}\| \leqslant \frac{C}{(|z|-1)^p} \quad (|z|>1) \quad \text{and} \quad m(T-z) \geqslant C(1-|z|)^q \quad (|z|<1).$$

Note that if T is $\mathcal{E}(\mathbb{T})$-subscalar then $\sigma_{\rm ap}(T)$, the approximate point spectrum of T given by

$$\sigma_{\rm ap}(T) = \{\lambda \in \mathbb{C} : \inf\{\|(T-\lambda)x\| : \|x\| = 1\} = 0\},$$

is included in the unit circle. Moreover, either $\sigma(T)$ is included in the unit circle (and so T is $\mathcal{E}(\mathbb{T})$-scalar) or $\sigma(T) = \overline{\mathbb{D}}$, the closed unit disc.

Again it is natural to ask if the condition (R) implies the $\mathcal{E}(\mathbb{T})$-subscalarity of T. This is a variant of the open Problem 6.1.14 in [12].

The aim of this note is to show that the answer to the above problem is negative: there is a Banach space operator T satisfying condition (R) (with suitable p and q) which is not $\mathcal{E}(\mathbb{T})$-subscalar. We also give a sufficient condition (in terms of growth of resolvent and of an analytic left inverse of $T-z$) implying that T is an $\mathcal{E}(\mathbb{T})$-subscalar operator. This condition is also necessary for Hilbert space operators.

We mention that a characterization of $\mathcal{E}(\mathbb{T})$-subscalar operators in terms of the growth of the local resolvent of the adjoint has been given by Didas [9]. We refer also to [14], [20], [19] for related papers considering conditions of type (P) or (R) (for small values of p and q) and studying the similarity of Hilbert space operators with unitary operators.

2. A COUNTEREXAMPLE

Recall that an equivalent definition of decomposable operators is the following: $T \in B(X)$ is *decomposable* if for every open cover $\mathbb{C} = U \cup V$, there are closed invariant (for T) subspaces Y and Z of X such that $X = Y + Z$ and $\sigma(T \mid Y) \subset U$, $\sigma(T \mid Z) \subset V$. We refer for instance to [8] and [12]. An operator $T \in B(X)$ has *Bishop's property* (β) if, for every open set $U \subset \mathbb{C}$, the operator T_U defined by $T_U(f)(z) = (T-z)f(z)$ on the set $\mathcal{O}(U, X)$ of holomorphic functions from U into X is injective and has closed range. According to a result by E. Albrecht and J. Eschmeier [1], $T \in B(X)$ is *subdecomposable* (i.e., T is similar to the restriction of a decomposable operator) if and only if T has Bishop's property (β).

EXAMPLE 2.1. On the Banach space $X = c_0$, there exists an operator $T \in B(X)$ such that:

(i) $\|T\| \leqslant 1$, $\sigma_{\mathrm{ap}}(T) = \mathbb{T}$ and $\sigma(T) = \overline{\mathbb{D}}$;

(ii) $\|(T-z)^{-1}\| \leqslant (|z|-1)^{-1}$ $(|z| > 1)$;

(iii) there is a constant $C > 0$ such that

$$m(T-z) \geqslant C(1-|z|)^3 \quad (z \in \mathbb{D});$$

(iv) T is not $\mathcal{E}(\mathbb{T})$-subscalar;

(v) T has Bishop's property (β).

Proof. Let $X = c_0$ be the Banach space of all complex sequences converging to zero endowed with the supremum norm. We denote its standard basis by $e_1, e_2, \ldots$. For $n \geqslant 1$ let

$$w_n = \mathrm{e}^{\ln^2(n+2) - \ln^2(n+3)}.$$

Let $T \in B(X)$ be the weighted shift defined by $Te_n = w_n e_{n+1}$ $(n \geqslant 1)$.

The proof of the properties of Example 2.1 will be obtained in several steps.

We first remark that $0 < w_n < 1$ for all n.

Claim 1. (w_n) is an increasing sequence and $\lim_{n\to\infty} w_n = 1$.

Proof. For each $n \geqslant 1$ there exists $x = x(n)$ such that $n+2 \leqslant x \leqslant n+3$ and

$$\ln^2(n+2) - \ln^2(n+3) = -2\frac{\ln x}{x}.$$

The function $g(x) = -2\frac{\ln x}{x}$ is increasing since

$$g'(x) = -2\frac{1-\ln x}{x^2} > 0 \quad (x > \mathrm{e}).$$

Therefore $(\ln^2(n+2) - \ln^2(n+3))$ is an increasing sequence for $n \geqslant 1$ and

$$\lim_{n\to\infty}(\ln^2(n+2) - \ln^2(n+3)) = 0.$$

Hence (w_n) is an increasing sequence and $\lim_{n\to\infty} w_n = 1$. ∎

The previous claim implies that $\|T\| \leqslant 1$. Therefore, for $|z| > 1$, we have

$$\|(T-z)^{-1}\| = \Big\| -\frac{1}{z}\sum_{n\geqslant 0}\frac{1}{z^n}T^n\Big\| \leqslant \frac{1}{|z|-1}.$$

This proves (ii).

For $n \geqslant 1$ we have $T^n e_k = w_k w_{k+1}\cdots w_{k+n-1}e_{k+n}$ $(k \geqslant 1)$, and so

$$\begin{aligned} m(T^n) &= \inf_k w_k \cdots w_{k+n-1} = w_1\cdots w_n = \mathrm{e}^{\ln^2 3 - \ln^2 4}\mathrm{e}^{\ln^2 4 - \ln^2 5}\cdots \mathrm{e}^{\ln^2(n+2)-\ln^2(n+3)} \\ &= \mathrm{e}^{\ln^2 3 - \ln^2(n+3)} = \frac{3^{\ln 3}}{(n+3)^{\ln(n+3)}}. \end{aligned}$$

Therefore T does not satisfy condition (P), and so T is not $\mathcal{E}(\mathbb{T})$-subscalar. This proves (iv).

We also have $\lim_{n\to\infty} m(T^n)^{1/n} = 1$. Therefore (see [13])

$$\sigma_{\mathrm{ap}}(T) \subset \{z : |z| = 1\}.$$

Since the spectrum of a weighted shift is circularly symmetric, we have in fact $\sigma_{\mathrm{ap}}(T) = \{z : |z| = 1\}$. But $\partial\sigma(T) \subset \sigma_{\mathrm{ap}}(T) \subset \sigma(T)$ and thus $\sigma(T)$ is either equal to $\overline{\mathbb{D}}$ or contained in $\mathbb{T}$. Since T is not invertible we have $\sigma(T) = \overline{\mathbb{D}}$. This completes the proof of (i).

Note also that

$$\sum_{n\geqslant 1}\frac{|\ln m(T^n)|}{n^2} < \infty,$$

so T satisfies the Beurling-type condition (B) (cf. Section 4 of [5]). Consequently, T has Bishop's property (β) (see Theorem 4.5 of [5]).

We prove now (iii).

Claim 2. $\lim_{n\to\infty}\frac{(1-w_n)^3}{w_{n+1}-w_n} = 0.$

Proof. Let $n \in \mathbb{N}$. Then there is an $x = x(n)$, $n+2 \leqslant x \leqslant n+3$, such that

$$\begin{aligned} w_{n+1} - w_n &= \mathrm{e}^{\ln^2(n+3)-\ln^2(n+4)} - \mathrm{e}^{\ln^2(n+2)-\ln^2(n+3)} \\ &= \mathrm{e}^{\ln^2 x - \ln^2(x+1)}\Big(\frac{2\ln x}{x} - \frac{2\ln(x+1)}{x+1}\Big) \end{aligned}$$

and there is a $y = y(n)$, $x \leqslant y \leqslant x+1$ (i.e., $n+2 \leqslant y \leqslant n+4$) such that

$$w_{n+1} - w_n = -2\mathrm{e}^{\ln^2 x - \ln^2(x+1)} \frac{1-\ln y}{y^2}.$$

Similarly, there is an $x' = x'(n)$, $n+2 \leqslant x' \leqslant n+3$, such that

$$\ln^2(n+2) - \ln^2(n+3) = -\frac{2\ln x'}{x'}.$$

We have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{(1-w_n)^3}{w_{n+1}-w_n} \\
&= \lim_{n\to\infty} \frac{\left(\frac{1-\mathrm{e}^{\ln^2(n+2)-\ln^2(n+3)}}{\ln^2(n+2)-\ln^2(n+3)}\right)^3 (\ln^2(n+2)-\ln^2(n+3))^3}{-2\mathrm{e}^{\ln^2 x - \ln^2(x+1)} \frac{1-\ln y}{y^2}} \\
&= (-1)^3\Big(-\frac{1}{2}\Big) \lim_{n\to\infty} \frac{(\ln^2(n+2)-\ln^2(n+3))^3}{\frac{1-\ln y}{y^2}} = \frac{1}{2}\lim_{n\to\infty} \frac{\left(\frac{-2\ln x'}{x'}\right)^3}{\frac{1-\ln y}{y^2}} \\
&= -4 \lim_{n\to\infty} \frac{y^2}{x'^2} \cdot \lim_{n\to\infty} \frac{\ln^3 x'}{x'(1-\ln y)} = 0. \quad \blacksquare
\end{aligned}$$

Claim 3. There is an $r > 0$ such that $m(T-z) \geqslant (1-|z|)^3$ for all $z \in \mathbb{D}$, $|z| \geqslant r$.

Proof. Find n_0 such that

$$\frac{(1-w_n)^3}{w_{n+1}-w_n} < \frac{1}{16}$$

for all $n \geqslant n_0$. Find r, $\frac{1}{2} \leqslant r < 1$, such that $r - (1-r)^3 > w_{n_0}$.

Suppose on the contrary that there is a $\lambda \in \mathbb{D}$, $|\lambda| \geqslant r$ such that

$$m(T-\lambda) < (1-|\lambda|)^3.$$

Thus there exists $x = (x_i) \in X$ with $\|x\| = \max_i |x_i| = 1$ and $\|(T-\lambda)x\| < (1-|\lambda|)^3$. Since $(T-\lambda)x = (-\lambda x_1, w_1x_1 - \lambda x_2, w_2x_2 - \lambda x_3, \ldots)$, we have $|\lambda||x_1| < (1-|\lambda|)^3$ and $\sup_i |w_ix_i - \lambda x_{i+1}| < (1-|\lambda|)^3$. Without loss of generality we may assume that $\lambda > 0$ and $x_i > 0$ for all $i \geqslant 1$. Indeed, replace λ by $|\lambda|$ and x_i by $|x_i|$ $(i \geqslant 1)$. We have

$$\sup_i |w_i|x_i| - |\lambda||x_{i+1}|| \leqslant \sup_i |w_ix_i - \lambda x_{i+1}| < (1-|\lambda|)^3.$$

Thus we may assume that there are r, μ with $\frac{1}{2} \leqslant r < \mu < 1$ and $u = (u_i) \in X$ with $u_i \geqslant 0$ $(i \in \mathbb{N})$, $\|u\| = \max_i u_i = 1$ and

$$\mu u_1 < (1-\mu)^3, \quad \sup_i |w_iu_i - \mu u_{i+1}| < (1-\mu)^3. \tag{2.1}$$

We show that this is not possible. Write for short $a = (1-\mu)^3$. Let $m \in \mathbb{N}$ satisfy $u_m = 1$ and $u_j < 1$ for all $j < m$. We have $u_1 < \frac{(1-\mu)^3}{\mu} < 1$. Thus $m \geqslant 2$.

We show that $w_{m-1} \geqslant \mu - a$. Suppose on the contrary that $w_{m-1} < \mu - a$. By (2.1), we have

$$\begin{aligned} a &> |w_{m-1}u_{m-1} - \mu u_m| \geqslant \mu u_m - w_{m-1}u_{m-1} \geqslant \mu - (\mu - a)u_{m-1} \\ &= (\mu - a)(1 - u_{m-1}) + a \geqslant a, \end{aligned}$$

a contradiction. Hence

$$w_{m-1} \geqslant \mu - a. \tag{2.2}$$

We show now that $w_m \geqslant \mu + a$. Suppose on the contrary that $w_m < \mu + a$. Then $w_m - w_{m-1} \leqslant 2a$ and $1 - w_{m-1} \geqslant 1 - w_m \geqslant 1 - \mu - a$. Therefore we have

$$\frac{(1-w_m)^3}{w_m - w_{m-1}} \geqslant \frac{(1-\mu-a)^3}{2a} = \frac{(1-\mu-(1-\mu)^3)^3}{2(1-\mu)^3} \geqslant \frac{1}{16},$$

since $\mu \geqslant \frac{1}{2}$ and $(1-\mu) - (1-\mu)^3 = (1-\mu)\mu(2-\mu) \geqslant \frac{1}{2}(1-\mu)$. Thus $m - 1 < n_0$, and so

$$\mu - a \geqslant r - (1-r)^3 > w_{n_0} \geqslant w_{m-1},$$

a contradiction with (2.2). Hence

$$w_m \geqslant \mu + a. \tag{2.3}$$

Since $|w_m u_m - \mu u_{m+1}| < a$, we have $\mu u_{m+1} > w_m - a$, and so

$$u_{m+1} > \frac{w_m - a}{\mu} \geqslant 1,$$

a contradiction with the assumption that $\|u\| = 1$.

Hence $m(T-z) \geqslant (1-|z|)^3$ for all $z \in \mathbb{D}$ with $|z| \geqslant r$. ∎

Since $m(T-z) > 0$ for all $z \in \mathbb{D}$ and the function

$$z \mapsto \frac{m(T-z)}{(1-|z|)^3}$$

is continuous on $\mathbb{D}$, there is a constant $C > 0$ such that $m(T-z) \geqslant C(1-|z|)^3$ for all $z \in \mathbb{D}$.

The proof of Example 2.1 is now complete. ∎

REMARKS 2.2. (i) Another proof of Bishop's property (β) for T can be given using 1.7.1 of [12].

(ii) The fact that T has Beurling-type property (B) implies by Theorem 4.5 of [5] that there exists a Banach space Y containing c_0 and an invertible operator $S \in B(Y)$ such that $T = S_{|X}$ and S satisfies

$$\sum_{n=-\infty}^{\infty} \frac{\log \|S^n\|}{1+n^2} < \infty.$$

Note that this condition implies ([8]) that S is decomposable.

(iii) We don't know if the weighted shift T on the Hilbert space $\ell_2 = \ell_2(\mathbb{N})$ given by

$$Te_n = \exp(\ln^2(n+2) - \ln^2(n+3))e_{n+1} \quad (n \geqslant 1)$$

is a hilbertian counterexample to the variant of Laursen-Neumann problem.

THEOREM 2.3. *Let X be a separable Banach space containing (an isomorphic copy of) c_0. Then there exist $R \in B(X)$ and a constant $C > 0$ such that:*

(i) $\sigma(R) = \overline{\mathbb{D}}$;
(ii) $\|(R-z)^{-1}\| \leqslant C(|z|-1)^{-1}$ $(|z| > 1)$;
(iii) $m(R-z) \geqslant C(1-|z|)^3$ $(z \in \mathbb{D})$;
(iv) *R is not $\mathcal{E}(\mathbb{T})$-subscalar;*
(v) *R has Bishop's property (β).*

Proof. According to a result due to A. Sobczyk (see [7]), if X is a separable Banach space containing an isomorphic copy of c_0, then X contains a subspace Y, isomorphic to c_0, which is complemented in X. We consider the operator R on X equal to the operator of Example 2.1 on Y and equal to the identity on its complement. Then R satisfies all the requirements because of the properties of T. ∎

3. SUFFICIENT CONDITIONS

We begin with the following sufficient condition.

PROPOSITION 3.1. *Let $T \in B(X)$ be a Banach space operator satisfying*

$$\|(T-z)^{-1}\| \leqslant C(|z|-1)^{-p} \quad (|z| > 1),$$

for some fixed constants $C > 0$ and $p \geqslant 0$. Suppose that there are $q \geqslant 0$ and an analytically dependent left inverse function $L : \mathbb{D} \to B(X)$ such that $L(z)(T-z) = I$ and

$$\|L(z)\| \leqslant C(1-|z|)^{-q} \quad (z \in \mathbb{D}).$$

Then T is $\mathcal{E}(\mathbb{T})$-subscalar.

We note that the growth condition on the analytically dependent left inverse function L implies that

$$\|x\| = \|L(z)(T-z)x\| \leqslant C(1-|z|)^{-q}\|(T-z)x\|;$$

hence

$$m(T-z) \geqslant C^{-1}(1-|z|)^q.$$

We also note that if $T-z$ is left invertible for each $z \in \mathbb{D}$, then there is an analytically dependent left inverse function on $\mathbb{D}$ (see [3], [2]).

Proof of Proposition 3.1. A proof of this result can be given using Didas' criterion [9] in terms of local resolvent of the adjoint of T. We give here a different proof.

It is a classical result (see Theorem 1.5.12 of [12]) that the resolvent growth condition outside the closed unit disc implies a polynomial growth condition for the powers of T: there is a constant $c > 0$ such that

$$\|T^n\| \leqslant cn^p \quad (n \in \mathbb{N}).$$

Write $L(z) = \sum_{i=0}^{\infty} L_i z^i$ $(z \in \mathbb{D})$, with $L_i \in B(X)$. Let $0 < r < 1$. By the Cauchy formula, for each $n \in \mathbb{N}$ we have

$$\|L_n\| \leqslant \frac{\max\{\|L(z)\| : |z| \leqslant r\}}{r^n} \leqslant \frac{C}{r^n(1-r)^q}.$$

In particular, for $r = n/(n+q)$ (where the function $r \mapsto r^{-n}(1-r)^{-q}$ attains the minimum) we obtain $\|L_n\| \leqslant C\big(\frac{n}{n+q}\big)^{-n}\big(1 - \frac{n}{n+q}\big)^{-q}$. We have $\lim_{n\to\infty} \big(\frac{n}{n+q}\big)^{-n} = \lim_{n\to\infty} \big(1 + \frac{q}{n}\big)^n = \mathrm{e}^q$. Further, for $n \geqslant q$ we have $\big(1 - \frac{n}{n+q}\big)^{-q} = \big(\frac{n+q}{q}\big)^q \leqslant \big(\frac{2n}{q}\big)^q$. Thus there is a constant $K > 0$ such that $\|L_n\| \leqslant K \cdot n^q$ for all n.

We have

$$I = L(z)(T-z) = \sum_{i=0}^{\infty} L_i z^i (T-z) = L_0 T + \sum_{i=1}^{\infty} z^i (L_i T - L_{i-1})$$

for all $z \in \mathbb{D}$. Thus $L_0 T = I$ and $L_i T = L_{i-1}$ for all $i \geqslant 1$. Hence

$$L_n T^{n+1} = L_{n-1} T^n = \cdots = L_0 T = I.$$

Let $x \in X$, $\|x\| = 1$. Then

$$1 = \|x\| = \|L_{n-1} T^n x\| \leqslant \|L_{n-1}\| \cdot \|T^n x\|.$$

Thus $\|T^n x\| \geqslant \|L_{n-1}\|^{-1}$, and so for some constant K' we have $m(T^n) \geqslant K' n^{-q}$ for all n. Hence T is $\mathcal{E}(\mathbb{T})$-subscalar by Theorem 4.1 of [5]. ∎

The next result gives an intrinsic characterization of $\mathcal{E}(\mathbb{T})$-subscalar operators on Hilbert spaces.

THEOREM 3.2. *Let H be a Hilbert space and $T \in B(H)$. Then T is $\mathcal{E}(\mathbb{T})$-subscalar if and only if there are constants $C > 0$, $p \geqslant 0$, $q \geqslant 0$ and an analytic operator-valued function $L : \mathbb{D} \to B(H)$ such that:*

(i) $\|(T-z)^{-1}\| \leqslant C(|z|-1)^{-p}$ $(|z| > 1)$;
(ii) $L(z)(T-z) = I$ $(|z| < 1)$;
(iii) $\|L(z)\| \leqslant C(1-|z|)^{-q}$ $(|z| < 1)$.

Proof. Suppose that T is a Hilbert space $\mathcal{E}(\mathbb{T})$-subscalar operator. According to Theorem 4.1 of [5], there are a Hilbert space K, constants $C' > 0$, $s \geqslant 0$ and an $\mathcal{E}(\mathbb{T})$-scalar extension $S \in B(K)$ such that $\sigma(S) = \sigma_{\mathrm{ap}}(T) \subset \mathbb{T}$ and

$$\|S^m\| \leqslant C'|m|^s \quad (m \in \mathbb{Z}, m \neq 0).$$

It is known ([12], 1.5.12) that the power growth estimate $\|S^m\| \leqslant C'|m|^s$ implies that $\|(S-z)^{-1}\| \leqslant C||z|-1|^{-s-1}$ $(|z| \neq 1)$ for a suitable constant $C > 0$. This implies

$$\|(T-z)^{-1}\| \leqslant C(|z|-1)^{-s-1} \quad (|z| > 1).$$

We define $L : \mathbb{D} \mapsto B(H)$ by

$$L(z)x = P_H(S-z)^{-1}x \quad (z \in \mathbb{D}, x \in H),$$

where $P_H \in B(K)$ is the orthogonal projection onto H.

Then L is analytic and we have

$$\|L(z)\| \leqslant \|(S-z)^{-1}\| \leqslant C(1-|z|)^{-s-1} \quad (|z| < 1).$$

The equality $L(z)(T-z) = I$ on $\mathbb{D}$ follows from the equalities $(S-z)^{-1}(S-z) = I$ and $S_{|H} = T$.

The second implication follows from Proposition 3.1. ∎

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