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# THE DESCENT SPECTRUM AND PERTURBATIONS

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ABSTRACT. In the present paper we continue to study the descent spectrum of an operator on a Banach space. We obtain that a Banach space $X$ is finite-dimensional if and only if there exists a bounded operator $T$ on $X$ such that its commutant is formed by algebraic operators. We provide also an affirmative answer to a question of M.A. Kaashoek and D.C. Lay.

KEYWORDS: *Spectrum, descent, perturbation, semi-Fredholm.*

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## INTRODUCTION

Throughout this paper, $\mathcal{L}(X)$ will denote the algebra of all bounded linear operators on an infinite-dimensional complex Banach space $X$ and $\mathcal{K}(X)$ its ideal of compact operators. For an operator $T \in \mathcal{L}(X)$, let $\mathrm{N}(T)$ denote its kernel, $\mathrm{R}(T)$ its range, $\sigma(T)$ its spectrum and $\sigma_{\mathrm{su}}(T)$ its surjective spectrum. Also, for a subset $M$ of $X$, $\mathrm{Vect}\{M\}$ will denote the closed linear subspace generated by $M$.

An operator $T \in \mathcal{L}(X)$ is called *semi-Fredholm* if $\mathrm{R}(T)$ is closed and either $\dim \mathrm{N}(T)$ or $\mathrm{codim}\mathrm{R}(T)$ is finite. For such an operator the *index* is given by $\mathrm{ind}(T) = \dim \mathrm{N}(T) - \mathrm{codim}\mathrm{R}(T)$, and if it is finite then we say that $T$ is *Fredholm*.

Also from [13] we recall that for a bounded linear operator $T \in \mathcal{L}(X)$, the *ascent*, $\mathrm{a}(T)$, and the *descent*, $\mathrm{d}(T)$, are defined by $\mathrm{a}(T) = \inf\{n \geqslant 0 : \mathrm{N}(T^n) = \mathrm{N}(T^{n+1})\}$ and $\mathrm{d}(T) = \inf\{n \geqslant 0 : \mathrm{R}(T^n) = \mathrm{R}(T^{n+1})\}$, respectively; the infimum over the empty set is taken to be $\infty$. As shown in [13],

$$\text{(0.1)} \qquad \mathrm{d}(T) \text{ is finite } \Leftrightarrow \mathrm{R}(T) + \mathrm{N}(T^d) = X \text{ for some } d \geqslant 0,$$

and

$$\text{(0.2)} \qquad \mathrm{a}(T) \text{ is finite } \Leftrightarrow \mathrm{R}(T^d) \cap \mathrm{N}(T) = \{0\} \text{ for some } d \geqslant 0.$$

For $T \in \mathcal{L}(X)$, the *descent spectrum*, $\sigma_{\mathrm{desc}}(T)$, is defined as those complex numbers $\lambda$ for which $\mathrm{d}(T-\lambda)$ is not finite; the *descent resolvent set* is $\rho_{\mathrm{desc}}(T) = \mathbb{C} \setminus \sigma_{\mathrm{desc}}(T)$.

Evidently $\sigma_{\rm desc}(T) \subseteq \sigma(T)$ and $\sigma_{\rm desc}(T) = \sigma_{\rm desc}(LTL^{-1})$ for every invertible operator $L \in \mathcal{L}(X)$. Also we mention the following property that will be used in the rest of the paper: if $Y$ and $Z$ are two closed $T$-invariant subspaces such that $X = Y \oplus Z$ then $\sigma_{\rm desc}(T) = \sigma_{\rm desc}(T_{|Y}) \cup \sigma_{\rm desc}(T_{|Z})$.

The paper is organized as follows. In Section 2 we show that the descent spectrum is a compact subset of the spectrum, and for an operator $T \in \mathcal{L}(X)$, we prove that $\sigma_{\rm desc}(T)$ is empty precisely when $T$ is algebraic, that is, there exists a non-zero complex polynomial $p$ for which $p(T) = 0$. For a complex Banach algebra $\mathcal{A}$, the descent of an element $a$ is defined to be the descent of the corresponding left multiplication operator on $\mathcal{A}$; the main point of Section 3 is to establish that a complex Banach algebra $\mathcal{A}$ is algebraic if and only if the descent of each element $a$ in $\mathcal{A}$ is finite, which also is equivalent to the fact that the radical of $\mathcal{A}$ is formed by nilpotent elements and has finite codimension. On the other hand, a classical result of Kaashoek and Lay affirms that if $F$ is a bounded operator for which there exists a positive integer $n$ such that $F^n$ has finite rank, then for every $T \in \mathcal{L}(X)$ commuting with $F$, $T$ has finite descent if and only if $T + F$ has finite descent [5]. Therefore, they have conjectured that such operator $F$ can be characterized by the above perturbation property. In the last section we provide a positive answer to this question, and moreover we characterize the finiteness of $\dim X$ by the existence of an operator $T$ such that its commutant is algebraic. Also some perturbations results for semi-Fredholm operators of finite descent are given.

In a paper under preparation, equivalent results for the ascent and the essential ascent and descent will be provided.

## 1. DESCENT SPECTRUM

We begin this section by the following result which shows that an operator with finite descent is either surjective or 0 is an isolated point of its surjective spectrum.

PROPOSITION 1.1. *Let $T \in \mathcal{L}(X)$ be an operator with finite descent $d := \mathrm{d}(T)$, then there exists $\delta > 0$ such that for every $0 < |\lambda| < \delta$:*

(i) $\mathrm{d}(T - \lambda) = 0$;

(ii) $\dim \mathrm{N}(T - \lambda) = \dim(\mathrm{N}(T) \cap \mathrm{R}(T^d))$.

*Proof.* Let $T_{\rm o}$ be the restriction of $T$ to $\mathrm{R}(T^d)$. We define a new norm on $\mathrm{R}(T^d)$ by

$$|y| = \|y\| + \inf\{\|x\| : x \in X \text{ and } y = T^d x\}, \quad \text{for all } y \in \mathrm{R}(T^d).$$

It is easy to verify that $\mathrm{R}(T^d)$ equipped with this norm is a Banach space, and that $T_{\rm o}$ is a bounded surjection on $(\mathrm{R}(T^d), |\cdot|)$. Let $\delta > 0$ be such that for every $0 < |\lambda| < \delta$, $T_{\rm o} - \lambda$ is surjective, it follows then that $\mathrm{R}(T^d) = (T - \lambda)\mathrm{R}(T^d) \subseteq$

$\mathrm{R}(T-\lambda)$. On the other hand, observe that the following equality holds with no restriction on $T$:

$$\mathrm{R}(T-\lambda)+\mathrm{R}(T^n)=X \quad \text{for all } n\in\mathbb{N} \text{ and } \lambda\neq 0.$$

Indeed, let $n \geqslant 1$ and $\lambda \neq 0$, consider also the polynomials $p(z) = z-\lambda$ and $q(z) = z^n$. Since $p$ and $q$ have no common divisors then there exist two polynomials $u$ and $v$ such that $1 = p(z)u(z)+q(z)v(z)$ for all $z \in \mathbb{C}$. Hence $I = (T-\lambda)u(T)+T^n v(T)$ and so $X = \mathrm{R}(T-\lambda)+\mathrm{R}(T^n)$. Now, from this we obtain that $\mathrm{R}(T-\lambda) = X$, that is, $\mathrm{d}(T-\lambda) = 0$, for $0 < |\lambda| < \delta$. Also, since $\mathrm{N}(T-\lambda) \subseteq \mathrm{R}(T^d)$, we have that $\mathrm{N}(T-\lambda) = \mathrm{N}(T_o-\lambda)$. Thus, by the continuity of the index we get

$$\dim(\mathrm{N}(T)\cap\mathrm{R}(T^d)) = \dim\mathrm{N}(T_o) = \mathrm{ind}(T_o) = \mathrm{ind}(T_o-\lambda) = \dim\mathrm{N}(T-\lambda),$$

for all $0 < |\lambda| < \delta$, which completes the proof. ∎

REMARK 1.2. As consequence of Proposition 1.1 and the stability of the index, we mention that if $T$ is a semi-Fredholm operator with finite descent then $\mathrm{ind}(T) \geqslant 0$.

COROLLARY 1.3. *If $T \in \mathcal{L}(X)$, then $\sigma_{\mathrm{desc}}(T)$ is a compact subset of $\sigma(T)$.*

The spectral mapping theorem holds for the descent spectrum [10]:

THEOREM 1.4. *Let $T \in \mathcal{L}(X)$ and $f$ be an analytic function on an open neighbourhood of $\sigma(T)$, not identically constant in any connected component of its domain, then*

$$\sigma_{\mathrm{desc}}(f(T)) = f(\sigma_{\mathrm{desc}}(T)). \tag{1.1}$$

THEOREM 1.5. *If $T$ is a bounded operator on $X$, then*

$$\rho_{\mathrm{desc}}(T)\cap\partial\sigma(T) = \{\lambda\in\mathbb{C} : \lambda \text{ is a pole of the resolvent of } T\}. \tag{1.2}$$

*Moreover, the following assertions are equivalent:*

(i) $\sigma_{\mathrm{desc}}(T) = \emptyset$;
(ii) $\partial\sigma(T) \subseteq \rho_{\mathrm{desc}}(T)$;
(iii) *$T$ is algebraic.*

*Proof.* By Theorem 10.1 of [13], the poles of the resolvent of $T$ are contained in $\rho_{\mathrm{desc}}(T)\cap\partial\sigma(T)$. For the other inclusion, suppose $\lambda \in \rho_{\mathrm{desc}}(T)\cap\partial\sigma(T)$, then by Proposition 1.1, there exists a deleted connected neighbourhood $\Omega$ of $\lambda$ such that $T-\mu$ is surjective and $\dim\mathrm{N}(T-\mu) = \dim(\mathrm{N}(T-\lambda)\cap\mathrm{R}(T-\lambda)^d)$, where $d = \mathrm{d}(T)$ and $\mu \in \Omega$. But since $\lambda \in \partial\sigma(T)$, $\Omega\setminus\sigma(T)$ is non-empty, and hence it follows that $\mathrm{N}(T-\lambda)\cap\mathrm{R}(T-\lambda)^d = \{0\}$, which implies that $\mathrm{N}(T-\lambda)^d = \mathrm{N}(T-\lambda)^{d+1}$. Now, the ascent and the descent of $T-\lambda$ are finite, so that by Theorem 10.2 of [13], $\lambda$ is a pole of the resolvent of $T$.

(i) ⇒ (ii). Obvious.

(ii) ⇒ (iii). Suppose that $\partial\sigma(T) \subseteq \rho_{\mathrm{desc}}(T)$, then by the first assertion, $\partial\sigma(T)$ is the set of the poles of the resolvent of $T$. Consequently, $\sigma(T) = \partial\sigma(T)$ is a finite set of complex numbers $\{\lambda_i\}_1^n$, and (cf. Theorem 10.2 of [13])

$$X = \mathrm{R}(T-\lambda_i)^{d_i} \oplus \mathrm{N}(T-\lambda_i)^{d_i} \quad \text{for some integer } d_i \geqslant 1. \tag{1.3}$$

Consider the complex polynomial $p(\lambda) = \prod_{i=1}^{n}(\lambda-\lambda_i)^{d_i}$, we claim that $p(T) = 0$. Let $T_{\mathrm{o}}$ denote the restriction of $T$ to the closed subspace $M := \mathrm{R}(p(T)) = \bigcap_{i=1}^{n} \mathrm{R}(T-\lambda_i)^{d_i}$, evidently $\sigma(T_{\mathrm{o}}) \subseteq \sigma(T)$. Moreover, for each $i$,

$$\mathrm{N}(T_{\mathrm{o}}-\lambda_i) = \mathrm{N}(T-\lambda_i) \cap M \subseteq \mathrm{N}(T-\lambda_i)^{d_i} \cap \mathrm{R}(T-\lambda_i)^{d_i} = \{0\}, \tag{1.4}$$

and since $T-\lambda_i$ has finite descent, we have also

$$\begin{aligned}(T_{\mathrm{o}}-\lambda_i)M &= (T-\lambda_i)\prod_{j=1}^{n}(T-\lambda_j)^{d_j}X = \Big[\prod_{j=1,j\neq i}^{n}(T-\lambda_j)^{d_j}\Big](T-\lambda_i)^{d_i+1}X \\ &= \Big[\prod_{j=1,j\neq i}^{n}(T-\lambda_j)^{d_j}\Big](T-\lambda_i)^{d_i}X = \prod_{j=1}^{n}(T-\lambda_j)^{d_j}X = M.\end{aligned}$$

Therefore $T_{\mathrm{o}}-\lambda_i$ is invertible, for $1 \leqslant i \leqslant n$. This implies that $\sigma(T_{\mathrm{o}})$ is empty, hence $M = \{0\}$ and $T$ is algebraic.

(iii) ⇒ (i). Suppose that $T$ is algebraic and let $p(\lambda) = \prod_{i=1}^{n}(\lambda-\lambda_i)^{\alpha_i}$ be the minimal complex polynomial such that $p(T) = 0$. By the spectral mapping theorem, it follows that $\sigma(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. On the other hand, we have

$$X = \bigoplus_{i=1}^{n} \mathrm{N}(T-\lambda_i)^{\alpha_i}, \tag{1.5}$$

$$\mathrm{N}(T-\lambda_i)^{\alpha_i} \subseteq \mathrm{R}(T-\lambda_j) \quad \text{if} \quad i \neq j. \tag{1.6}$$

Therefore, for every $1 \leqslant i \leqslant n$,

$$\begin{aligned}\mathrm{R}(T-\lambda_i)^{\alpha_i} &= (T-\lambda_i)^{\alpha_i}\Big(\bigoplus_{1\leqslant j\leqslant n}\mathrm{N}(T-\lambda_i)^{\alpha_i}\Big) = (T-\lambda_i)^{\alpha_i}\Big(\bigoplus_{1\leqslant j\leqslant n, j\neq i}\mathrm{N}(T-\lambda_j)^{\alpha_j}\Big) \\ &\subseteq (T-\lambda_i)^{\alpha_i})(\mathrm{R}(T-\lambda_i)) = \mathrm{R}(T-\lambda_i)^{\alpha_i+1}.\end{aligned}$$

Consequently, $T-\lambda_i$ has finite descent for all $1 \leqslant i \leqslant n$. Thus $\sigma_{\mathrm{desc}}(T) = \emptyset$, and this completes the proof. ∎

COROLLARY 1.6. *If $T$ is a bounded operator on $X$, then we have*

$$\partial\sigma(T) \subseteq \sigma_{\mathrm{desc}}(T) \cup \{\textit{the poles of the resolvent of } T\}. \tag{1.7}$$

THEOREM 1.7. *If $T \in \mathcal{L}(X)$ and $\Omega$ is a connected component of $\rho_{\mathrm{desc}}(T)$, then*

$$\Omega \subset \sigma(T) \quad \textit{or} \quad \Omega \setminus E \subseteq \rho(T), \tag{1.8}$$

*where $E = \{\lambda \in \Omega : \lambda$ is a pole of the resolvent of $T\}$.*

*Proof.* Let $\Omega^r = \{\lambda \in \Omega : \mathrm{d}(T-\lambda) = 0\}$ and $\Omega^s = \{\lambda \in \Omega : 0 < \mathrm{d}(T-\lambda) < \infty\}$, then we have $\Omega = \Omega^r \cup \Omega^s$, and by Proposition 1.1, $\Omega^s$ is at most countable. Therefore $\Omega^r$ is connected, and if $\Omega \cap \rho(T)$ is non-empty then so is $\Omega^r \cap \rho(T)$, hence the continuity of the index ensures that $\mathrm{ind}(T-\lambda) = 0$ for all $\lambda \in \Omega^r$. But for $\lambda \in \Omega^r$, $T-\lambda$ is surjective, so it follows that $T-\lambda$ is invertible. Thus $\Omega^r \subseteq \rho(T)$. Consequently $\Omega^s$ is a set of isolated points in $\sigma(T)$ of finite descent, so

$$\Omega^s \subseteq \rho_{\mathrm{desc}}(T) \cap \partial\sigma(T) = \{\lambda \in \mathbb{C} : \text{pole of the resolvent of } T\}.$$

Finally, if we put $E = \Omega^s$, then $E = \{\lambda \in \Omega : \text{poles of the resolvent of } T\}$ and $\Omega \setminus E \subseteq \rho(T)$, as desired. ∎

COROLLARY 1.8. *If $T \in \mathcal{L}(X)$, then $\sigma_{\mathrm{desc}}(T)$ is at most countable if an only if $\sigma(T)$ is at most countable.*

*In this case we have $\sigma(T) = \sigma_{\mathrm{desc}}(T) \cup \{$the poles of the resolvent of $T\}$.*

*Proof.* Suppose that $\sigma_{\mathrm{desc}}(T)$ is at most countable, then $\rho_{\mathrm{desc}}(T)$ is connected, and since $\rho(T) \subseteq \rho_{\mathrm{desc}}(T)$, the previous theorem implies that $\rho_{\mathrm{desc}}(T) \setminus E \subseteq \rho(T)$ where $E$ is the set of the poles of the resolvent of $T$. Therefore $\sigma(T) = \sigma_{\mathrm{desc}}(T) \cup E$ is at most countable, which completes the proof. ∎

We recall that an operator $R \in \mathcal{L}(X)$ is said to be *Riesz* if $R - \lambda$ is Fredholm for every non-zero complex number $\lambda$.

Notice that in general, the fact that the descent spectrum of an operator $T$ is finite does not ensure that $\sigma(T)$ is finite. Indeed if we consider any Riesz operator $T$ with infinite spectrum, then every non-zero complex number of $\sigma(T)$ is a pole of the resolvent of $T$ (see [13]), that is, $\sigma_{\mathrm{desc}}(T) = \{0\}$.

An operator $T \in \mathcal{L}(X)$ is called *meromorphic* if the non-zero points of its spectrum are poles of the resolvent of $T$. It is a classical fact that every compact, or more generally, Riesz operator is meromorphic.

COROLLARY 1.9. *If $T$ is a bounded operator on $X$, then*

$$T \text{ is meromorphic } \Leftrightarrow \sigma_{\mathrm{desc}}(T) \subseteq \{0\}. \tag{1.9}$$

For $T \in \mathcal{L}(X)$, let $L_T : \mathcal{L}(X) \to \mathcal{L}(X)$ denote the corresponding multiplication operator given by $L_T(S) := TS$. If $\mathrm{d}(L_T)$ is finite then so is $\mathrm{d}(T)$. Indeed, suppose that $T^d\mathcal{L}(X) = T^{d+1}\mathcal{L}(X)$ for some integer $d$, then there exists $S \in \mathcal{L}(X)$ such that $T^d = T^{d+1}S$, and hence we obtain $\mathrm{R}(T^d) \subseteq \mathrm{R}(T^{d+1})$. Thus $T$ has finite descent.

COROLLARY 1.10. *Let $X$ be a Banach space, the following assertions are equivalent:*

(i) *$X$ is finite-dimensional;*
(ii) $\mathrm{d}(L_T)$ *is finite for all $T \in \mathcal{L}(X)$;*
(iii) $\mathrm{d}(T)$ *is finite for all $T \in \mathcal{L}(X)$;*
(iv) $\sigma_{\mathrm{desc}}(T) = \emptyset$ *for all $T \in \mathcal{L}(X)$;*

(v) $\mathcal{L}(X)$ *is algebraic.*

*Proof.* The implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) are obvious. Also by Theorem 1.5, we get that (iv) entails (v). For (iv) ⇒ (i), we use Theorem 5.4.2 of [1]. ∎

## 2. DESCENT IN BANACH ALGEBRAS

Throughout this section, $\mathcal{A}$ will denote a complex Banach algebra with unit and $\mathrm{Rad}(\mathcal{A})$ its (Jacobson) radical. For every $a \in \mathcal{A}$, the left multiplication operator $L_a$ is given by $L_a(x) = ax$ for all $x \in \mathcal{A}$. By definition the *descent* of an element $a \in \mathcal{A}$ is $\mathrm{d}(a) := \mathrm{d}(L_a)$, and the descent spectrum of $a$ is the set $\sigma_{\mathrm{desc}}(a) := \{\lambda \in \mathbb{C} : \mathrm{d}(a - \lambda) = \infty\}$.

REMARK 2.1. (i) An element $a \in \mathcal{A}$ has finite descent if and only if there exists a positive integer $n$ such that $a$ is right-invertible modulo $\mathrm{N}(L_a^n)$. Indeed, $n := \mathrm{d}(a)$ finite means precisely that $a^n\mathcal{A} = a^{n+1}\mathcal{A}$, that is, there exists $b \in \mathcal{A}$ for which $a^n = a^{n+1}b$, i.e, there exists $b \in \mathcal{A}$ such that $1 - ab \in \mathrm{N}(L_a^n)$.

(ii) $\mathrm{Rad}(\mathcal{A}) \cap \{a \in \mathcal{A} : \mathrm{d}(a) \text{ is finite }\} \subseteq \mathcal{N}(\mathcal{A})$, where $\mathcal{N}(\mathcal{A})$ is the set of nilpotent elements of $\mathcal{A}$. In fact, if $a \in \mathrm{Rad}(\mathcal{A})$ and $n := \mathrm{d}(a)$ is finite, then there exists $b \in \mathcal{A}$ such that $a^n(1 - ab) = 0$, and since $1 - ab$ is invertible, we get that $a$ is nilpotent.

THEOREM 2.2. *Let $\mathcal{A}$ be a Banach algebra. The following assertions are equivalent :*

(i) $\dim(\mathcal{A}/\mathrm{Rad}\mathcal{A})$ *is finite and* $\mathrm{Rad}\mathcal{A}$ *is a nil ideal (i.e.* $\mathrm{Rad}\mathcal{A} \subseteq \mathcal{N}(\mathcal{A})$*);*

(ii) $\mathrm{d}(a)$ *is finite for every* $a \in \mathcal{A}$*;*

(iii) $\sigma_{\mathrm{desc}}(a) = \emptyset$ *for every* $a \in \mathcal{A}$*;*

(iv) $\sigma_{\mathrm{desc}}(a) = \emptyset$ *for every $a$ in a non-empty open subset $U$ of* $\mathcal{A}$*;*

(v) $\mathcal{A}$ *is algebraic.*

*Proof.* (i) ⇒ (ii). If $a \in \mathcal{A}$, and since $\mathcal{A}/\mathrm{Rad}\mathcal{A}$ is a finite-dimensional algebra, there exists a non-zero complex polynomial $p$ such that $p(a + \mathrm{Rad}\mathcal{A}) = 0$. It follows then that $p(a)$ belongs to the nil ideal $\mathrm{Rad}\mathcal{A}$, and hence $p(a)^n = 0$ for some positive integer $n$, which proves that $a$ is algebraic.

The implications (ii) ⇒ (iii) ⇒ (iv) and the equivalence (v) ⇔ (iii) are obvious.

(iv) ⇒ (i). Since for $a \in U$, $\sigma_{\mathrm{desc}}(L_a) = \sigma_{\mathrm{desc}}(a) = \emptyset$, Theorem 1.5 implies that there exists a non-zero complex polynomial $p$ for which $p(L_a) = 0$, that is, $p(a) = 0$. Therefore by Theorem 5.4.2 of [1], $\dim \mathcal{A}/\mathrm{Rad}\mathcal{A}$ is finite. Moreover, if $b \in \mathrm{Rad}\mathcal{A}$, then $b$ is quasi-nilpotent and algebraic, and hence nilpotent. ∎

REMARK 2.3. Note that if $\mathrm{Rad}\mathcal{A}$ is finite-dimensional, the above assertions (i)–(v) are equivalent to $\mathcal{A}$ being finite-dimensional.

We mention that in the setting of Hilbert space, the descent of $T$ as element in the Banach algebra $\mathcal{L}(H)$, is finite if and only if the descent of $T$ is finite. In fact, if $d := \mathrm{d}(T) < \infty$ then $\mathrm{R}(T^d) = \mathrm{R}(T^{d+1})$, and therefore there exists $S \in \mathcal{L}(H)$ such that $T^d = T^{d+1}S$ [4]. Consequently, $\mathrm{d}(L_T)$ is finite.

For a Banach algebra $\mathcal{A}$, one can define the descent of an element $a \in \mathcal{A}$ to be the descent of the right multiplication operator $R_a$ given by $R_a(x) = xa$, evidently Theorem 2.2 holds also for this definition. However, we note that for $T \in \mathcal{L}(X)$, there is no relation that lies the descent of $T$ as an operator and the descent of $T$ as element of the algebra $\mathcal{L}(X)$. In fact, if we consider the unilateral right shift operator $T$ defined on the Hilbert space $\ell^2(\mathbb{N})$ by $T(x_1, x_2, \ldots) = (0, x_1, \ldots)$, then $\mathrm{d}(R_T) = \mathrm{d}(L_{T^*}) = \mathrm{d}(T^*) = 0$ and $\mathrm{d}(T) = \infty$.

Let $\mathcal{K}(H)$ denote the ideal of compact operators on $H$ and $\pi$ the canonical surjection from $\mathcal{L}(H)$ to the Calkin algebra $\mathcal{C}(H) := \mathcal{L}(H)/\mathcal{K}(H)$. For $T \in \mathcal{L}(H)$, $\mathrm{d}(T)$ is finite implies that $\mathrm{d}(\pi(T))$ is finite. Indeed, there exists $S \in \mathcal{L}(H)$ such that $T^d = T^{d+1}S$ where $d = \mathrm{d}(T)$. Hence $\pi(T)^d = \pi(T)^{d+1}\pi(S)$ and so $\mathrm{d}(\pi(T))$ is finite. Now if we define the *essential descent spectrum* of $T \in \mathcal{L}(H)$ by $\sigma^{\mathrm{e}}_{\mathrm{desc}}(T) := \sigma_{\mathrm{desc}}(\pi(T))$, then it follows that $\sigma^{\mathrm{e}}_{\mathrm{desc}}(T) = \sigma^{\mathrm{e}}_{\mathrm{desc}}(T+K) \subseteq \sigma_{\mathrm{desc}}(T+K)$ for every $K \in \mathcal{K}(X)$, and consequently

$$\sigma^{\mathrm{e}}_{\mathrm{desc}}(T) \subseteq \bigcap_{K \in \mathcal{K}(X)} \sigma_{\mathrm{desc}}(T+K).$$

Natural questions can be asked:

1. Is the above inclusion an equality ?

2. Does there exist a compact operator $K$ such that $\sigma^{\mathrm{e}}_{\mathrm{desc}}(T) = \sigma_{\mathrm{desc}}(T+K)$?

In the general context, the answers to these questions are negatives. Consider the unilateral right shift operator $T$. Because $T + K - \lambda$ is a Fredholm operator with non positive index, for every $|\lambda| < 1$ and every compact $K$, then it follows that $\sigma_{\mathrm{desc}}(T+K)$ contains the closed unit disk. However, for $|\lambda| < 1$, $\pi(T-\lambda)$ is invertible, and therefore $\sigma^{\mathrm{e}}_{\mathrm{desc}}(T)$ is contained in the unit circle.

**Question 1.** Let $T \in \mathcal{L}(X)$ and denote by $\rho^{-}_{\mathrm{SF}}(T)$ the set of complex numbers $\lambda$ for which $T - \lambda$ is semi-Fredholm of non positive index. Does it follow that

$$\sigma^{\mathrm{e}}_{\mathrm{desc}}(T) \cup \rho^{-}_{\mathrm{SF}}(T) = \bigcap_{K \in \mathcal{K}(X)} \sigma_{\mathrm{desc}}(T+K) \ ? \tag{2.1}$$

## 3. THE DESCENT SPECTRUM AND PERTURBATIONS

In Theorem 2.2 of [5] it was shown by M. Kaashoek and D. Lay that if $F$ is a bounded operator on $X$ for which there exists a positive integer $n$ such that $F^n$ is of finite rank, then

$$\sigma_{\mathrm{desc}}(T+F) = \sigma_{\mathrm{desc}}(T) \quad \text{for every operator } T \in \mathcal{L}(X) \text{ commuting with } F. \tag{3.1}$$

In the same paper, they have conjectured that such operator $F$ can be characterized by (4.1). The following theorem gives a positive answer to this question.

THEOREM 3.1. *If $F \in \mathcal{L}(X)$, then the following assertions are equivalent:*
(i) $\sigma_{\text{desc}}(T+F) = \sigma_{\text{desc}}(T)$ *for every* $T \in \mathcal{L}(X)$ *such that* $TF = FT$*;*
(ii) *there exists* $n \in \mathbb{N}$ *for which* $F^n$ *is of finite rank.*

Before giving the proof of this theorem, we establish some preliminary results.

LEMMA 3.2. *Let $N \in \mathcal{L}(X)$ be an infinite-rank operator such that $N^2 = 0$, then there exists a compact operator $K \in \mathcal{L}(X)$ such that $NK$ is non-algebraic.*

*Proof.* Let $x_1$ be such that $Nx_1 \neq 0$ then $\{x_1, Nx_1\}$ is linearly independent. Write $X = \text{Vect}\{x_1, Nx_1\} \oplus X_1$ and let $f_1$ be the linear form given by $f_1(x_1) = f_1(Nx_1) = 1$ and $f_1 = 0$ on $X_1$. Because $N$ is of infinite rank, we can choose $x_2 \in X_1$ such that $Nx_2$ is non-zero and belongs to $X_1$. Analogously, we decompose $X_1 = \text{Vect}\{x_2, Nx_2\} \oplus X_2$, and we define $f_2$ by $f_2(x_2) = f_2(Nx_2) = 1$ and $f_2 = 0$ on $\text{Vect}\{x_1, Nx_1\} \oplus X_2$. By repeating the same argument, we construct a countable sets of vectors $\{x_1, x_2, \ldots\}$ and continuous linear forms $\{f_1, f_2, \ldots\}$ such that $\{x_n, Nx_n : n \geqslant 1\}$ consists of linearly independent vectors and $f_i(x_j) = f_i(Nx_j) = \delta_{ij}$. Now, consider the compact operator $K := \sum \alpha_i x_i \otimes f_i$ where $\alpha_i$ are a distinct complex numbers for which $\sum\limits_{i=1}^{+\infty} |\alpha_i| \|x_i\| \|f_i\|$ is finite. It follows then that $NK = \sum \alpha_i Nx_i \otimes f_i$ is compact and $\sigma(NK) = \{0\} \cup \{\alpha_n : n \geqslant 1\}$. In particular $NK$ is non-algebraic. ∎

Let $N$ be a nilpotent operator and $n$ be a positive integer such that $N^n = 0$, then for every $X \in \mathcal{L}(X)$, the operator $S := \sum\limits_{i=1}^{n} N^{i-1} X N^{n-i}$ commutes with $N$; see [2].

PROPOSITION 3.3. *The commutant of every bounded operator on an infinite-dimensional complex Banach space contains a non-algebraic operator.*

*Proof.* Without loss of generality we may suppose that $T$ is algebraic. Let $\sigma(T) := \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ then we can decompose $X$ as follows

$$X = X_1 \oplus X_2 \oplus \cdots \oplus X_n \tag{3.2}$$

where the subspaces $X_i$ are invariant by $T$ and the restriction of $T - \lambda_i$ to $X_i$ is nilpotent. Since $X$ has infinite dimension, there exists $i$ such that $\dim X_i$ is infinite. Therefore, it suffices to prove that every nilpotent operator on an infinite-dimensional Banach space contains a non-algebraic operator in its commutant.

Suppose that $T$ is nilpotent and let $n \geqslant 2$ for which $T^n = 0$ and $T^{n-1} \neq 0$. If $T^{n-1}$ is of infinite rank, then by Lemma 3.2 there exists a compact operator $K \in \mathcal{L}(X)$ such that $T^{n-1}K$ is non-algebraic. Let $R := \sum\limits_{i=1}^{n-1} T^{i-1} K T^{n-i}$ and

$S := R + T^{n-1}K$ then $TS = ST$. Moreover, because $T^{n-1}K$, $S$ are compact and $RT^{n-1}K = 0$, we get that $\sigma(T^{n-1}K) \subseteq \sigma(S)$. Consequently $S$ is non-algebraic. Now, suppose that $\mathrm{R}(T^{n-1})$ has finite dimension, and consider an arbitrary associated basis $\{T^{n-1}x_1, T^{n-1}x_2, \ldots, T^{n-1}x_k\}$. We show easily that $\{T^p x_j : 0 \leqslant p \leqslant n-1 \text{ and } 1 \leqslant j \leqslant k\}$ consists of linearly independent vectors. Hence, there exists a finite family of continuous linear forms $\{f_j\}_{j=1}^k$ such that

$$f_j(T^{n-1}x_j) = 1 \quad \text{and} \quad f_j(T^p x_r) = 0 \quad \text{if } r \neq j \text{ or } (r = j \text{ and } p \neq n-1). \tag{3.3}$$

If we let $V := \sum_{j=1}^{k} \sum_{p=1}^{n} T^{p-1}(x_j \otimes f_j)T^{n-p}$, then it follows that $V$ is a finite-rank projection commuting with $T$ and $\mathrm{R}(T^{n-1}) \subseteq \mathrm{R}(V)$, consequently $T^{n-1}_{|\mathrm{N}(V)} = 0$. By repeating successively the same argument, we obtain that $X = Y \oplus Z$ where $Y$ and $Z$ are $T$-invariant, $\dim Y$ is finite, $T^h_{|Z} = 0$ and $\mathrm{R}(T^{h-1}_{|Z})$ is of infinite dimension for some $h \geqslant 1$. If $h > 1$ then the above argument provides a non-algebraic operator $S$ on $Z$ that commutes with $T_{|Z}$. Consequently, $0 \oplus S$ is non-algebraic and commutes with $T$. To complete the proof, we may suppose $h = 1$, that is, $T_{|Z} = 0$ and $T$ has finite-rank. Consider an arbitrary non-algebraic operator $S$ on $Z$, then we have that $0 \oplus S$ is non-algebraic and commutes with $T$. ∎

*Proof of Theorem* 3.1. (ii) ⇒ (i). See [5].

(i) ⇒ (ii). By taking $T = 0$ we obtain that $\sigma_{\mathrm{desc}}(F)$ is empty, and hence $F$ is algebraic. Therefore

$$X = X_1 \oplus X_2 \oplus \cdots \oplus X_n \tag{3.4}$$

where $\sigma(F) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and the restriction of $F - \lambda_i$ to $X_i$ is nilpotent for every $1 \leqslant i \leqslant n$. We claim that if $\lambda_i \neq 0$, $\dim X_i$ is finite. Suppose to the contrary that $\lambda_i \neq 0$ and $X_i$ is infinite dimensional. By Proposition 3.3, there exists a non-algebraic operator $S_i$ on $X_i$ commuting with the restriction $F_i$ of $F$ to this space. Let $S$ denote the extension of $S_i$ to $X$ given by $S = 0$ on each $X_j$ such that $j \neq i$, obviously $SF = FS$ and so $\sigma_{\mathrm{desc}}(S + F) = \sigma_{\mathrm{desc}}(S)$ by hypothesis. On the other hand, since $\sigma_{\mathrm{desc}}(S) = \sigma_{\mathrm{desc}}(S_i)$ and $\sigma_{\mathrm{desc}}(S + F) = \sigma_{\mathrm{desc}}(S_i + F_i)$, we obtain that $\sigma_{\mathrm{desc}}(S_i) = \sigma_{\mathrm{desc}}(S_i + F_i) = \sigma_{\mathrm{desc}}(S_i + \lambda_i)$ because $F_i - \lambda_i$ is nilpotent. Choose an arbitrary complex number $\alpha \in \sigma_{\mathrm{desc}}(S) \neq \emptyset$, it follows that $k\lambda_i + \alpha \in \sigma_{\mathrm{desc}}(S)$ for every positive integer $k$, which implies that $\lambda_i = 0$, the desired contradiction. ∎

We shall denote by $\mathcal{A}(X)$ the set of algebraic operators on $X$, and by $\{T\}'$ the commutant of $T \in \mathcal{L}(X)$. The following corollary follows immediately from Proposition 3.3.

COROLLARY 3.4. *If $X$ is a complex Banach space, then the following assertions are equivalent:*

(i) *$X$ is finite-dimensional;*

(ii) *$\{T\}' \subseteq \mathcal{A}(X)$ for every $T \in \mathcal{L}(X)$;*

(iii) *there exists $T \in \mathcal{L}(X)$ such that $\{T\}' \subseteq \mathcal{A}(X)$;*

(iv) *there exists a nilpotent operator* $N \in \mathcal{L}(X)$ *such that* $\{N\}' \subseteq \mathcal{A}(X)$.

REMARK 3.5. Notice that in the case when

$$\dim X < \infty \Leftrightarrow \{T\}' \subseteq \mathcal{A}(X) \quad \text{for every } T \in \mathcal{L}(X), \tag{3.5}$$

we have $\dim X = \mathrm{Sup}\{\mathrm{d}^{\mathrm{o}}P : P \in \mathcal{P}_T\}$ where $\mathcal{P}_T$ denotes the set of complex polynomials $P$ for which there exists $S \in \{T\}'$ such that $P$ is the minimal polynomial satisfying $P(S) = 0$. Indeed it follows from the simple fact that for every nilpotent operator $N$ on a finite-dimensional space $Y$ there exists an operator $S \in \{N\}'$ and a minimal complex polynomial $P$ of degree $\dim Y$ such that $P(S) = 0$.

Corollary 3.4 suggests the following question:

**Question 2.** Let $\mathcal{A}$ be a complex semi-simple Banach algebras. Does we have an equivalence between the following assertions:
(i) $\mathcal{A}$ is finite-dimensional;
(ii) there exists $a \in \mathcal{A}$ such that its commutant is formed by algebraic elements.

The descent spectrum does not remain invariant under arbitrary finite-rank perturbation, (cf. [10]). However, for algebraic operators we have:

PROPOSITION 3.6. *Let* $T \in \mathcal{L}(X)$ *be algebraic and* $F$ *be a finite-rank operator, then* $T + F$ *is algebraic.*

*Proof.* Let $p(z) = \sum_{k=0}^{n} \alpha_k z^k$ be a non-zero complex polynomial such that $p(T) = 0$. Then we have

$$p(T+F) = p(T+F) - p(T) = \sum_{k=0}^{n} \alpha_k[(T+F)^k - T^k]. \tag{3.6}$$

Moreover, it is easy to verify that for each $k$, $(T+F)^k - T^k$ has finite rank. Therefore, $p(T+F)$ has finite rank. Thus $p(T+F)$ is algebraic, and hence so is $T+F$. ∎

Let $T$ be a bounded operator on $X$. According to Kaashoek and Lay [5], $\sigma_{\mathrm{desc}}(T)$ is stable under commuting finite-rank perturbations. We also notice that the *semi-Fredholm spectrum* of $T$, the set $\sigma_{\mathrm{SF}}(T)$ of complex numbers $\lambda$ such that $T - \lambda$ is not semi-Fredholm, is stable under the same perturbations (see [3]). V. Rakočević showed more in [12] that the union of the descent and the semi-Fredholm spectrum, $\sigma^{\mathrm{d}}_{\mathrm{SF}}(T) := \sigma_{\mathrm{SF}}(T) \cup \sigma_{\mathrm{desc}}(T)$, is the largest subset of the surjective spectrum remaining invariant under any commuting compact perturbation (or more generally, commuting Riesz perturbation).

For an operator $T$, we denote by $\Pi(T)$ the set of all isolated points $\lambda$ of $\sigma_{\mathrm{su}}(T)$ for which $T - \lambda$ is semi-Fredholm.

PROPOSITION 3.7. *Let* $T$ *be a bounded operator on* $X$, *we have*

$$\sigma^{\mathrm{d}}_{\mathrm{SF}}(T) = \sigma_{\mathrm{su}}(T) \setminus \Pi(T). \tag{3.7}$$

*Proof.* From the proof of Proposition 2.1, we conclude that if $\lambda \notin \sigma^{\mathrm{d}}_{\mathrm{SF}}(T)$ then either $T-\lambda$ is surjective or $\lambda$ is an isolated point of the surjective spectrum, which establish $\sigma_{\mathrm{su}}(T) \setminus \Pi(T) \subseteq \sigma^{\mathrm{d}}_{\mathrm{SF}}(T)$. For the other inclusion, let $\lambda \in \Pi(T)$, then $T-\lambda$ is semi-Fredholm, and by the Kato decomposition (cf. [6]), there exist two closed $T$-invariant subspaces $X_1$, $X_2$ such that $X = X_1 \oplus X_2$, $T_{|X_1} - \lambda$ is nilpotent and $T_{|X_2} - \lambda$ is semi-regular (i.e, $\mathrm{R}(T)$ is closed and $\mathrm{N}(T^n) \subseteq \mathrm{R}(T)$ for all integers $n \in \mathbb{N}$, see [9], [11]). Now, because $\lambda$ is an isolated point in $\sigma_{\mathrm{su}}(T)$, there exists $\delta > 0$ such that for every $0 < |\mu - \lambda| < \delta$, $T-\mu$ is surjective. Therefore, for $0 < |\mu-\lambda| < \delta$, $T_{|X_2} - \mu$ is surjective, and hence so is $T_{|X_2} - \lambda$ [9]. Finally, since $T_1 - \lambda$ is nilpotent, we obtain that $T-\lambda$ has finite descent, which completes the proof. ∎

We denote by $\mathcal{F}(X)$ the set of all finite-rank operators, and by $\mathcal{P}_{\mathrm{f}}$ the set of all projections with finite-dimensional null space. The restriction of an operator $T \in \mathcal{L}(X)$ to the range of $Q$, where $Q \in \mathcal{P}_{\mathrm{f}}$ and $TQ = QT$, is denoted by $T_Q$.

PROPOSITION 3.8. *If $T \in \mathcal{L}(X)$, then the following assertions are equivalent:*
(i) *there exists $Q \in \mathcal{P}_{\mathrm{f}}$ such that $TQ = QT$ and $\sigma^{\mathrm{d}}_{\mathrm{SF}}(T) = \sigma_{\mathrm{su}}(T_Q)$;*
(ii) *there exists $F \in \mathcal{F}(X)$ such that $TF = FT$ and $\sigma^{\mathrm{d}}_{\mathrm{SF}}(T) = \sigma_{\mathrm{su}}(T+F)$;*
(iii) *$\Pi(T)$ is finite.*

*Proof.* (i) $\Rightarrow$ (ii). Let $Q \in \mathcal{P}_{\mathrm{f}}$ be such that $QT = TQ$, $\mathrm{N}(Q)$ is finite-dimensional and $\sigma^{\mathrm{d}}_{\mathrm{SF}}(T) = \sigma_{\mathrm{su}}(T_Q)$. In particular $\sigma_{\mathrm{su}}(T_{|\mathrm{N}(Q)})$ is a finite set $\{\lambda_i\}_{i=1}^{n}$ and $\mathrm{N}(Q) = N_1 \oplus N_2 \oplus \cdots \oplus N_n$ where $N_i$ is invariant by $T$ and $\sigma(T_{|N_i}) = \{\lambda_i\}$. Now for each $1 \leqslant i \leqslant n$, let $\alpha_i$ be a complex number such that $\lambda_i - \alpha_i \in \sigma_{\mathrm{su}}(T_{|\mathrm{R}(Q)})$. Consider the finite-rank operator defined by $F_{|N_i} = \alpha_i I_{|N_i}$, $1 \leqslant i \leqslant n$, and $F_{|\mathrm{R}(Q)} = 0$. Then it is clear that $FT = TF$ and

$$\sigma_{\mathrm{su}}(T+F) = \{\lambda_i - \alpha_i\}_{i=1}^{n} \cup \sigma_{\mathrm{su}}(T_{|\mathrm{R}(Q)}) = \sigma_{\mathrm{su}}(T_{|\mathrm{R}(Q)}) = \sigma^{\mathrm{d}}_{\mathrm{SF}}(T).$$

(ii) $\Rightarrow$ (iii). Let $F$ be a finite-rank operator commuting with $T$ and for which $\sigma^{\mathrm{d}}_{\mathrm{SF}}(T) = \sigma_{\mathrm{su}}(T+F)$. Because the spectrum of $F$ is finite, the spectral decomposition provides two closed subspaces $Y_1$, $Y_2$ invariant by $T$ and $F$ for which $X = Y_1 \oplus Y_2$, $\sigma(F_{|Y_1}) = \{0\}$ and $F_{|Y_2}$ is invertible. Since $F_{|Y_2}$ is a finite-rank operator, $Y_2$ is finite-dimensional. We claim that $\Pi(T)$ is contained in the finite set $\sigma(T_{|Y_2})$. Assume to the contrary that there exists $\lambda \in \Pi(T) \setminus \sigma(T_{|Y_2})$; then, in particular $T-\lambda$ is not surjective. Moreover, because $\Pi(T) \cap \sigma_{\mathrm{su}}(T+F) = \Pi(T) \cap \sigma^{\mathrm{d}}_{\mathrm{SF}}(T) = \emptyset$, $T+F-\lambda$ is surjective and hence so is $(T+F)_{|Y_1} - \lambda$. But, $F_{|Y_1}$ is quasi-nilpotent, so we see that $T_{|Y_1} - \lambda$ is surjective. Finally, $T_{|Y_2} - \lambda$ is invertible, therefore $T-\lambda$ is surjective, the desired contradiction.

(iii) $\Rightarrow$ (i). Suppose that $\Pi(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. As in the proof of the previous proposition, we have the following decomposition $X = X_1 \oplus Z_1$, where $\dim X_1$ is finite, $T_{|X_1} - \lambda_1$ is nilpotent and $T_{|Z_1} - \lambda_1$ is surjective; consequently $\Pi(T_{|Z_1}) = \{\lambda_2, \ldots, \lambda_n\}$. By using successively the same argument, we obtain that $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n \oplus Z$, where the spaces $X_i$ are finite-dimensional, invariant by $T$ and $\sigma_{\mathrm{su}}(T_{|Z}) = \sigma_{\mathrm{su}}(T) \setminus \Pi(T) = \sigma^{\mathrm{d}}_{\mathrm{SF}}(T)$. Therefore, if we let $Q$ be the projection on $Z$ with respect to the above decomposition, then it follows that $QT = TQ$ and $\sigma_{\mathrm{su}}(T_Q) = \sigma^{\mathrm{d}}_{\mathrm{SF}}(T)$. ∎

REMARK 3.9. Let $T$ be a bounded operator on $X$. As mentioned above we have

$$\sigma^{\mathrm{d}}_{\mathrm{SF}}(T) = \bigcap_{R \in \mathcal{R}(X), RT=TR} \sigma_{\mathrm{su}}(T+R), \tag{3.8}$$

where $\mathcal{R}(X)$ denote the set of Riesz operators. Also, J. Zemánek has established in [14] that $\sigma^{\mathrm{d}}_{\mathrm{SF}}(T)$ can be obtained as the intersection of all surjective spectra of $T_Q$, the intersection being taken over all $Q \in \mathcal{P}_{\mathrm{f}}$ such that $QT = TQ$.

**Question 3.** Given $T \in \mathcal{L}(X)$, does exist a Riesz operator $R$ such that $TR = RT$ and $\sigma^{\mathrm{d}}_{\mathrm{SF}}(T) = \sigma_{\mathrm{su}}(T+R)$ ?

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