J. OPERATOR THEORY
56:2(2006), 317–342



# MULTIDIMENSIONAL CAYLEY TRANSFORMS AND TUPLES OF UNBOUNDED OPERATORS

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*Communicated by Florian-Horia Vasilescu*

ABSTRACT. We generalize the Cayley transform to tuples of unbounded operators. To achieve this we introduce intrinsically defined objects, with spectrum in projective space, which admit an analytic functional calculus. We also provide an integral representation for this functional calculus.

KEYWORDS: *Cayley transform, functional calculus, Taylor spectrum, integral representation, projective space.*

MSC (2000): 47A13, 47A60, 32A26, 47A10, 47A65.

## INTRODUCTION

The Cayley transform, $a \mapsto (a+\mathrm{i})(a-\mathrm{i})^{-1}$, introduced by von Neumann in [13] induces a one-to-one correspondence between the self-adjoint operators and the unitary operators such that 1 is not in the point spectrum. More generally, one can consider any automorphism of $\widehat{\mathbb{C}}$ and apply it to an arbitrary closed operator provided that the point mapped to infinity is outside the point spectrum of the operator. In case this point is outside all of the spectrum then the image is a bounded operator. In this way we get an intrinsic object with spectrum in $\mathbb{CP}^1$ which for any choice of point at infinity, outside the point spectrum, and any linear coordinate gives rise to a closed operator. One possible generalization to higher dimensions, i.e., to tuples of operators, is to take the Cayley transform of each of the operators. This is possible if all the operators have nonempty resolvent sets, and if the operators commute in the strong sense we obtain a tuple of bounded commuting operators in this way. Vasilescu used this technique in [11] to prove spectral theorems for unbounded self-adjoint operators. In a more general setting this has recently been studied by Andersson and Sjöstrand in [2]. There is also a notion of Quaternionic Cayley transform introduced in [12] but we will not consider it here.

In this paper we are concerned with another generalization of the Cayley transform. We characterize the $n$-tuples of closed unbounded operators which by a projective transformation of $\mathbb{CP}^n$ can be mapped to tuples of bounded commuting operators. This is what we will call a *multidimensional Cayley transform*. One point to be made is that these tuples of unbounded operators may consist of operators with empty resolvent sets. The characterization is in terms of an algebraic relation linking the operators together and a commutation condition stronger but similar to the notion of permutability described in [5]. Tuples of closed unbounded operators satisfying these conditions will be called *affine operators*. We define a Taylor spectrum for the affine operators and we show that the spectral mapping property holds. In case all the operators making up the affine operator have resolvents we can also consider the tuple of one-dimensional Cayley transforms as mentioned above. This tuple has a well defined Taylor spectrum and as in e.g. [11] and [2] we can define a joint spectrum for the original tuple by claiming that the spectral mapping property should hold. We show that the spectrum we define is contained in this spectrum and that we have equality in the case of pairs of operators. To carry out our idea we introduce *projective operators*, an intrinsic object in $\mathbb{CP}^n$ with an invariant spectrum and admitting an analytic functional calculus. From the abstract point of view a projective operator is an $\mathscr{O}_{\mathbb{CP}^n}$-module as described in [4]; see also Section 6. More concretely, we realize projective operators as certain equivalence classes of $n+1$-tuples of bounded commuting operators. The spectrum for the projective operator can be described from the Taylor spectrum for a representative and via integral formulas inspired by [1] we can also describe the module structure from a representative of the equivalence class. In this paper we will only consider projective operators having a spectrum avoiding some hyperplane in $\mathbb{CP}^n$. In an affinization where we take such a hyperplane to be the hyperplane at infinity the projective operator corresponds to a tuple of bounded commuting operators and the module structure is Taylor's functional calculus.

The disposition of the paper is as follows.

In Section 1 we briefly review Taylor's functional calculus and we state the basic facts about one-dimensional Cayley transforms.

In Section 2 we define projective operators and study their fundamental properties.

In Section 3 we study the behavior of projective operators under various projections from $\mathbb{CP}^n$ to $\mathbb{C}^n$ and we define affine operators.

In Section 4 we define a Taylor spectrum for affine operators and relate it to some other existing definitions.

In Section 5 we summarize our results and interpret them on the affine level. In Section 6 we provide an integral representation for the analytic functional calculus obtained in Section 2.

## 1. PRELIMINARIES

If $a$ is an operator on some space $X$ then $\mathscr{D}(a)$ is the domain of definition for $a$ and $\mathscr{R}(a)$ is the range, i.e., the set of all $ax$ such that $x \in \mathscr{D}(a)$. The set of $x \in \mathscr{D}(a)$ such that $ax = 0$, the nullspace of $a$, is denoted $\mathscr{N}(a)$. If also $b$ is an operator on $X$ then $a \subseteq b$ means that the graph of $a$ is included in the graph of $b$ in $X \times X$. In particular $a = b$ means that $a$ and $b$ have the same domain of definition.

1.1. TAYLOR'S FUNCTIONAL CALCULUS. Let $X$ be a Banach space, $L(X)$ the algebra of bounded operators on $X$ and $E$ an $n$-dimensional complex vector space with a non-sense basis $\{e_1, \dots, e_n\}$. We write $\Lambda^k X$ for the tensor product $X \otimes \Lambda^k E$ of $X$ and the $k$th exterior product of $E$. Let $b = (b_1, \dots, b_n)$ be an $n$-tuple of bounded commuting operators on $X$. On $\Lambda X = \bigoplus\limits_0^n \Lambda^k X$ we have the natural operation of interior multiplication with the operator-valued co-vector $\sum\limits_1^n (z_j - b_j)e_j^*$. We denote this operation $\delta_{z-b}$. Since $b$ is commuting $\delta_{z-b} \circ \delta_{z-b} = 0$ and so we have the Koszul complex

$$0 \longleftarrow \Lambda^0 X \xleftarrow{\delta_{z-b}} \Lambda^1 X \xleftarrow{\delta_{z-b}} \cdots \xleftarrow{\delta_{z-b}} \Lambda^n X \longleftarrow 0, \tag{1.1}$$

or $K_\bullet(\delta_{z-b}, \Lambda^\bullet X)$ for short. The joint Taylor spectrum $\sigma(b)$ for $b$ is defined as the complement in $\mathbb{C}^n$ of the set of points $z$ such that $K_\bullet(\delta_{z-b}, \Lambda^\bullet X)$ is exact, [8]. Taylor's fundamental result in [8] and [9] is that the natural algebra homomorphism $\mathscr{O}(\mathbb{C}^n) \to L(X)$ given by $\sum\limits_\alpha c_\alpha z^\alpha \mapsto \sum\limits_\alpha c_\alpha b^\alpha$ extends to an algebra homomorphism $\mathscr{O}(\sigma(b)) \to L(X)$.

THEOREM 1.1 (Taylor, 1970). *There is an extension of the natural continuous algebra homomorphism $\mathscr{O}(\mathbb{C}^n) \to L(X)$ to a continuous algebra homomorphism*

$$f \mapsto f(b) : \mathscr{O}(U) \to L(X)$$

*for all open sets $U$ such that $\sigma(b) \subseteq U$. If $f = (f_1, \dots, f_m) \in \mathscr{O}(U, \mathbb{C}^m)$, then $f(\sigma(b)) = \sigma(f(b))$, where $f(b) = (f_1(b), \dots, f_n(b))$.*

The statement $f(\sigma(b)) = \sigma(f(b))$ will be referred to as the Spectral Mapping Theorem.

1.2. THE ONE-DIMENSIONAL CAYLEY TRANSFORM. Let $X$ be a Banach space and let $\mathscr{C}(X)$ be the set of closed, but not necessarily densely defined operators on $X$. For any linear operator $a$ on $X$ the spectrum of $a$, $\sigma(a)$, is the complement in $\mathbb{C}$ of the set of points $\lambda$ such that $\lambda - a$ is a bijection $\mathscr{D}(a) \to X$. The point spectrum, $\sigma_p(a) \subseteq \sigma(a)$, is the set of $\lambda \in \mathbb{C}$ such that $\lambda - a$ is not injective. For $a \in \mathscr{C}(X)$ we have by the Closed Graph Theorem that $\lambda \notin \sigma(a)$ if and only if $\lambda - a$ has a bounded inverse. We let $\widehat{\mathbb{C}}$ denote the extended complex plane $\mathbb{C} \cup \{\infty\}$ and we define the extended spectrum $\widehat{\sigma}(a)$ as $\sigma(a)$ if $a$ is bounded and $\sigma(a) \cup \{\infty\}$ if $a$ is not bounded.

Let $\phi$ be a projective, or Möbius transformation of $\widehat{\mathbb{C}}$. We claim that $\phi(a)$ has meaning as an element in $\mathscr{C}(X)$ if $\phi^{-1}(\infty) \notin \sigma_p(a)$. Given the projective transformation $\phi$ we let $M_\phi \in \mathrm{GL}(2, \mathbb{C})$ be the corresponding $2 \times 2$-matrix. If $M_\phi = \{m_{j,k}\}_{1 \leqslant j,k \leqslant 2}$ and $\phi^{-1}(\infty) \notin \sigma_p(a)$ we may put

$$\phi(a) = (m_{1,1}a + m_{1,2})(m_{2,1}a + m_{2,2})^{-1}. \tag{1.2}$$

The matrix $M_\phi$ acts naturally as a homeomorphism of $X \times X$ and it is straight forward to verify that $M_\phi \mathrm{Graph}(a) = \mathrm{Graph}(\phi(a))$ and hence $\phi(a)$ is closed if $a$ is. Moreover, it is not hard to see that $\phi(a)$ is bounded if and only if $\phi^{-1}(\infty) \notin \widehat{\sigma}(a)$. We conclude that the closed operators on $X$ which can be Cayley transformed to bounded operators are precisely those with a non-empty resolvent set. The spectral mapping property holds for these mappings, that is, for any closed operator $a$ on $X$ and projective transformation $\phi$ of $\widehat{\mathbb{C}}$ such that $\phi^{-1}(\infty) \notin \widehat{\sigma}_p(a)$ it holds that $\phi(\widehat{\sigma}(a)) = \widehat{\sigma}(\phi(a))$. For a more thorough treatment of the one-dimensional Cayley transform, see [11] and [7].

The preceding discussion suggests that the closed operator $a$ defines some invariant object on $\mathbb{CP}^1 = \widehat{\mathbb{C}}$ if $\infty \notin \sigma_p(a)$. In the canonical affine part of $\widehat{\mathbb{C}}$ this object becomes the operator $a$ and in some other affine part, corresponding to a Möbius transformation $\phi$ of the canonical one, it becomes $\phi(a)$ and has spectrum $\phi(\widehat{\sigma}(a))$.

## 2. PROJECTIVE OPERATORS AND ANALYTIC FUNCTIONAL CALCULUS

In analogy with the construction of projective space we consider an equivalence relation on a subset of the $n+1$-tuples of bounded commuting operators on a Banach space and define a *projective operator* as an equivalence class. We will see that a projective operator has a well defined invariant Taylor spectrum in $\mathbb{CP}^n$ and that it admits an analytic functional calculus.

DEFINITION 2.1. Let $b = (b_0, \dots, b_n)$ and $\widetilde{b} = (\widetilde{b}_0, \dots, \widetilde{b}_n)$ be tuples of bounded commuting operators on a Banach space $X$. We define $b \sim \widetilde{b}$ if there are finitely many bounded commuting tuples $b^j$, $j = 1, \dots, m$, such that $b^1 = b$ and $b^m = \widetilde{b}$ and for $j = 1, \dots, m-1$ we have $b^{j+1} = c_j b^j$ for some invertible $c_j \in (b^j)'$; the commutant of $b^j$.

LEMMA 2.2. *The relation $\sim$ of Definition 2.1 is an equivalence relation.*

*Proof.* We note that the relation $R$ on bounded commuting $n+1$-tuples defined by $bR\widetilde{b}$ if $\widetilde{b} = cb$ for some invertible $c \in (b)'$ is reflexive and symmetric. Reflexivity is obvious since $e \in (b)'$. It is symmetric because if $\widetilde{b} = cb$ for some invertible $c \in (b)'$ then $b = c^{-1}\widetilde{b}$ and letting $\widetilde{b} = (\widetilde{b}_0, \dots, \widetilde{b}_n)$ and $b = (b_0, \dots, b_n)$ we see that

$$c^{-1}\widetilde{b}_j = c^{-1}cb_j = b_j = b_j cc^{-1} = cb_j c^{-1} = \widetilde{b}_j c^{-1},$$

so $c^{-1} \in (\widetilde{b})'$. The relation $\sim$ is defined as the transitive closure of $R$ so it is by definition transitive and it inherits reflexivity and symmetry from $R$. ∎

REMARK 2.3. We will see later on, Remark 3.6, that for the tuples we will be interested in there is a simpler description of the relation $\sim$. For these tuples it will also turn out, see Remark 3.5, that even though $\sim$ is defined as the transitive closure of $R$, any two representatives for an equivalence class are not more than two steps from each other.

We denote the equivalence class containing $b$ by $[b]$ and we let $\pi$ denote the canonical mapping $\mathbb{C}^{n+1} \to \mathbb{CP}^n$.

PROPOSITION 2.4. *Let $b = (b_0, \dots, b_n)$ be a commuting tuple of bounded operators on $X$ and let $c \in (b)'$ be invertible. If $0 \notin \sigma(b)$ then $0 \notin \sigma(cb)$ and*

$$\pi\sigma(b_0, \dots, b_n) = \pi\sigma(cb_0, \dots, cb_n).$$

*Proof.* Define $\psi$ and $\phi : \mathbb{C}^{n+2} \to \mathbb{C}^{n+1}$ by

$$\begin{aligned} \psi(z, z_0, \dots, z_n) &= (zz_0, \dots, zz_n), \\ \phi(z, z_0, \dots, z_n) &= (z_0, \dots, z_n), \end{aligned}$$

respectively. The hyperplane in $\mathbb{C}^{n+2}$ orthogonal to the vector $(1, 0, \dots, 0)$ does not intersect $\sigma(c, b_0, \dots, b_n)$ since $c$ is invertible and we have

$$\sigma(c, b_0, \dots, b_n) \subseteq \sigma(c) \times \sigma(b_0, \dots, b_n) \tag{2.1}$$

according to [8]. Moreover from (2.1) and the assumption that $0 \notin \sigma(b)$ we see that $\sigma(c, b_0, \dots, b_n)$ also avoids the coordinate axis $(z, 0, \dots, 0)$. Hence we may take a neighborhood $U$ of $\sigma(c, b_0, \dots, b_n)$ such that $U$ does not intersect neither the hyperplane orthogonal to $(1, 0, \dots, 0)$ nor the coordinate axis $(z, 0, \dots, 0)$. Then the images $V_1$ and $V_2$ of $U$ under $\psi$ and $\phi$ respectively do not contain the origin and so the diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d, "\phi"'] & V_1 \arrow[d, "\pi"] \\ V_2 \arrow[r, "\pi"'] & \mathbb{CP}^n \end{tikzcd} \tag{2.2}$$

must commute. By the Spectral Mapping Theorem

$$\begin{aligned} \sigma(b_0, \dots, b_n) &= \sigma\, \phi(c, b_0, \dots, b_n) = \phi\, \sigma(c, b_0, \dots, b_n), \\ \sigma(cb_0, \dots, cb_n) &= \sigma\, \psi(c, b_0, \dots, b_n) = \psi\, \sigma(c, b_0, \dots, b_n), \end{aligned}$$

and since the diagram (2.2) commutes we conclude that $\pi\, \sigma(cb_0, \dots, cb_n)$ $= \pi\, \sigma(b_0, \dots, b_n)$. ∎

It follows immediately from this proposition that we have

COROLLARY 2.5. *Let $b \sim \widetilde{b}$ and assume $0 \notin \sigma(b)$. Then $0 \notin \sigma(\widetilde{b})$ and $\pi\sigma(b) = \pi\sigma(\widetilde{b})$.*

Hence if $0 \notin \sigma(b)$ then $0 \notin \sigma(\widetilde{b})$ for any $\widetilde{b} \in [b]$ and $\pi\sigma(b) = \pi\sigma(\widetilde{b})$ and so we can make the following definitions.

DEFINITION 2.6. Let $b$ be a commuting tuple of bounded operators on a Banach space $X$ such that $0 \notin \sigma(b)$. We define the *projective operator* $[b]$ as the equivalence class containing $b$.

DEFINITION 2.7. Let $[b]$ be a projective operator. The *spectrum*, $\sigma[b] \subseteq \mathbb{CP}^n$ of the projective operator $[b]$ is defined by

$$\sigma[b] = \pi\sigma(b).$$

We now construct the analytic functional calculus for the projective operators. The main theorem of this section is the following.

THEOREM 2.8. *If $[b]$ is a projective operator, then there is a unique $\mathscr{O}(\sigma[b])$-module structure on $X$ given by*

$$\mathscr{O}(\sigma[b]) \times X \to X, \quad (f, x) \mapsto f([b])x$$

*and if $f = (f_1, \dots, f_m) \in \mathscr{O}(\sigma[b], \mathbb{C}^m)$ then $\sigma(f([b])) = f(\sigma[b])$ where $f([b]) = (f_1([b]), \dots, f_m([b]))$.*

*Proof.* We construct the module-structure as follows. Given some $f \in \mathscr{O}(\sigma[b])$ we consider the canonical lift $\widetilde{f}$ of $f$ to $\mathbb{C}^{n+1}$. Then $\widetilde{f}$ is holomorphic in a neighborhood of $\sigma(b)$ for any representative $b \in [b]$ and $\widetilde{f}$ is constant on the complex lines through the origin (with the origin deleted). From Taylor's analytic functional calculus we get for each $b \in [b]$ an operator $\widetilde{f}(b) \in L(X)$. We will see that in fact $\widetilde{f}(b)$ is independent of representative $b$ and our desired pairing $\mathscr{O}(\sigma[b]) \times X \to X$ will be $(f, x) \mapsto \widetilde{f}(b)x$ where $b$ is any representative of $[b]$.

Let $b \in [b]$ and let $c \in (b)'$ be invertible. Put $\widetilde{b} = cb$ and let $\phi$ and $\psi$ be the mappings defined in Proposition 2.4. Let $U_1$ be a neighborhood of $\sigma(b)$ in which $\widetilde{f}$ is holomorphic and let $V$ be a neighborhood of $\sigma(c)$ such that $\overline{D(0,r)} \cap V = \emptyset$ for some $0 < r < 1$. Since $c$ is invertible $0 \notin \sigma(c)$ and such a neighborhood exists. Let $U$ be the union over $\lambda \notin \overline{D(0,r)}$ of $\lambda U_1$. Then $\widetilde{f} \circ \phi$ and $\widetilde{f} \circ \psi$ are holomorphic in $V \times U$. Moreover since $r < 1$ we have $\sigma(b) \subseteq U$ and so $\sigma(c, b) \subseteq \sigma(c) \times \sigma(b) \subseteq V \times U$. Now since $\widetilde{f}$ is constant on the complex lines through the origin we have $\widetilde{f} \circ \phi \mid_{V \times U} = \widetilde{f} \circ \psi \mid_{V \times U}$ and we conclude from the composition rule that $\widetilde{f}(b) = \widetilde{f} \circ \phi \mid_{V \times U} (c, b) = \widetilde{f} \circ \psi \mid_{V \times U} (c, b) = \widetilde{f}(cb) = \widetilde{f}(\widetilde{b})$. It follows inductively that $\widetilde{f}(b) = \widetilde{f}(\widetilde{b})$ for any two representatives $b$ and $\widetilde{b}$ for $[b]$. Thus $f$ is well defined on $[b]$ and we write $f([b])$ for the operator $\widetilde{f}(b)$.

To prove the spectral mapping property we proceed as follows. Since $\widetilde{f}$ is constant on the complex lines through the origin, $\widetilde{f}(\sigma(b))$ only depends on $b \in [b]$

and so from Theorem 1.1 we get

$$f(\sigma[b]) = \widetilde{f}(\sigma(b)) = \sigma(\widetilde{f}(b)) = \sigma(f([b])).$$

Uniqueness follows from the spectral mapping property. See [4]. ∎

Let $M$ be a complex manifold and assume $f : U \supseteq \sigma[b] \to M$ is holomorphic. We obtain an $\mathscr{O}(M)$-module structure $\mathscr{M}$ on $X$ by

$$\mathscr{O}(M) \times X \to X, \quad (g, x) \mapsto g \circ f([b])x.$$

In [4] Eschmeier and Putinar define the spectrum $\sigma(M, \mathscr{M}) \subseteq M$ of the module $\mathscr{M}$ and show that the $\mathscr{O}(M)$-module structure extends uniquely to an $\mathscr{O}(\sigma(M, \mathscr{M}))$-module structure on $X$. Moreover they show a Spectral Mapping Theorem which in our case implies that

$$\sigma(M, \mathscr{M}) = f(\sigma[b]).$$

It is shown that if $M = \mathbb{C}^m$ we can realize the extended module structure as the analytic functional calculus for an $m$-tuple of commuting bounded operators $c$ on $X$ by choosing coordinates on $\mathbb{C}^m$ and that the spectrum of the abstract module is precisely $\sigma(c)$. The composition rule in Taylor's functional calculus is therefore built into the construction.

To stress the independence of coordinates in our study of projective operators we adopt an invariant notation. For a subset $M$ of $\mathbb{CP}^n$ we denote by $M^*$ the *dual complement* of $M$, that is

$$M^* = \{[\lambda] \in \mathbb{CP}^{n*} : \langle z, \lambda \rangle \neq 0 \ \forall [z] \in M\}.$$

Geometrically $M^*$ is the set of hyperplanes in $\mathbb{CP}^n$ which do not intersect $M$. The correspondence between hyperplanes in $\mathbb{CP}^n$ and points in $\mathbb{CP}^{n*}$ is the usual duality correspondence. To $[\lambda] \in \mathbb{CP}^{n*}$ we associate the hyperplane $\{[z] : \langle z, \lambda \rangle = 0\}$. We will not make any distinction between points in $\mathbb{CP}^{n*}$ and their corresponding hyperplanes and we will freely allow ourselves to speak about "the hyperplane $[\lambda]$" if $[\lambda] \in \mathbb{CP}^{n*}$.

LEMMA 2.9. *Let $[b]$ be a projective operator. Then*

$$\sigma[b]^* = \{[\lambda] \in \mathbb{CP}^{n*} : \langle b, \lambda \rangle \ \text{ is invertible}\}.$$

*Proof.* Since any two representatives of $[b]$ differ by an invertible operator we see that the statement in the lemma only depends on $[b]$. For the inclusion $\subseteq$ assume that $[\mu] \in \sigma[b]^*$. Then from the definition we have $\langle z, \mu \rangle \neq 0$ for all $[z] \in \sigma[b]$. Thus the function $z \mapsto 1/\langle z, \mu \rangle$ from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ is holomorphic in a neighborhood of $\sigma(b)$. Hence from the functional calculus we see that $\langle b, \mu \rangle$ is invertible.

For the other inclusion, assume that $\langle b, \mu \rangle$ is invertible. We shall show that $\sigma(b)$ does not intersect the hyperplane $[\mu]$. If $\mu = (1, 0, \ldots, 0)$ we have to show that if $b_0$ is invertible then $\sigma(b)$ does not intersect the hyperplane orthogonal to $(1, 0, \ldots, 0)$. But if $b_0$ is invertible then $0 \notin \sigma(b_0)$ and since

$$\sigma(b) \subseteq \sigma(b_0) \times \sigma(b_1, \ldots, b_n),$$

see [8], $\sigma(b)$ can not intersect the hyperplane in question. For the general case let $L$ be an invertible linear transformation sending $\mu$ to $(1, 0, \ldots, 0)$. By the Spectral Mapping Theorem, to show that $\sigma(b)$ does not intersect $\mu$ is equivalent to show that $\sigma(L^{*-1}b)$ does not intersect $L\mu = (1, 0, \ldots, 0)$. But the first component in $L^{*-1}b$ is $\langle L^{*-1}b, L\mu\rangle = \langle b, \mu\rangle$ which is invertible by assumption and so the lemma follows. ∎

REMARK 2.10. From now on we will always assume that $\sigma[b]$ avoids some hyperplane in $\mathbb{CP}^n$. We will do this because it will make it possible to realize the projective operator as an ordinary $n$-tuple of bounded operators. Since the objective of this paper is to construct multidimensional Cayley transforms of tuples of unbounded operators into tuples of bounded operators the assumption is natural.

If we fix some $[\widetilde{\lambda}] \in \sigma[b]^*$ then the function $[z] \mapsto \langle z, \widetilde{\lambda}\rangle / \langle z, \lambda\rangle$ is holomorphic in a neighborhood of $\sigma[b]$ if also $[\lambda] \in \sigma[b]^*$. Theorem 2.8 then implies that we get a holomorphic mapping from $\sigma[b]^*$ to the algebra generated by $b$ for $b \in [b]$ given by $[\lambda] \mapsto \langle b, \widetilde{\lambda}\rangle / \langle b, \lambda\rangle$. This is the *Fantappiè transform* of the $L(X)$-valued analytic functional $\mathscr{O}(\sigma[b]) \to L(X)$, $f \mapsto f([b])$ given by Theorem 2.8.

## 3. AFFINE OPERATORS

We extend the Fantappiè transform to a larger set $\sigma[b]^*_{\mathrm{adm}}$, called the set of admissible hyperplanes, and get instead a $\mathscr{C}(X)$-valued mapping. We will define *affine operators* to be tuples of closed operators with certain commutation properties. We will show that affine operators are precisely the tuples obtained by projecting a projective operator from an admissible hyperplane. In order to keep track of the various domains of definition that turn up we start with some technical results.

In what follows we will often make implicit use of the following easily checked fact.

PROPOSITION 3.1. *Let $a$ be any closed operator on $X$ and let $b$ be bounded. Then, the operator $ab$ with domain $\mathscr{D}(ab) = \{x \in X : bx \in \mathscr{D}(a)\}$ is closed.*

The following lemma generalizes the fact that if $b$ and $c$ are bounded operators and $b$ is invertible, then $bc = cb$ if and only if $b^{-1}c = cb^{-1}$.

LEMMA 3.2. *Let $b$ and $c$ be bounded operators on $X$ and assume that $b$ is injective. Then $bc = cb$ if and only if $cb^{-1} \subseteq b^{-1}c$. If this condition is fulfilled and in addition $c$ is invertible then actually $cb^{-1} = b^{-1}c$.*

*Proof.* Assume that $bc = cb$ and let $x \in \mathscr{D}(cb^{-1}) = \mathscr{D}(b^{-1})$. Then $x = by$ for some $y \in X$. Since $b$ and $c$ commute we get $cx = cby = bcy$ and so we must have $cx \in \mathscr{D}(b^{-1})$. Hence, $\mathscr{D}(cb^{-1}) \subseteq \mathscr{D}(b^{-1}c)$ and

$$cb^{-1}x = cb^{-1}by = cy = b^{-1}bcy = b^{-1}cby = b^{-1}cx.$$

It follows that $cb^{-1} \subseteq b^{-1}c$. Conversely assume $cb^{-1} \subseteq b^{-1}c$. Then if $x \in \mathscr{D}(b^{-1})$ we have $cx \in \mathscr{D}(b^{-1})$ and so $b^{-1}cb \in L(X)$. By assumption $b^{-1}cb \supseteq cb^{-1}b = c$ and because $c \in L(X)$ we must have equality. Multiplying by $b$ from the left we obtain $cb = bc$.

For the last statement assume $c$ is invertible and commutes with $b$. Then $c^{-1}$ also commutes with $b$. To show $cb^{-1} = b^{-1}c$ it is enough to show $\mathscr{D}(b^{-1}c) \subseteq \mathscr{D}(cb^{-1})$ by the proof this far. Take $x \in \mathscr{D}(b^{-1}c)$, i.e., such that $cx \in \mathscr{D}(b^{-1})$. Then $cx = by$ for some $y \in X$. We get $x = c^{-1}by = bc^{-1}y$ and so $x \in \mathscr{D}(b^{-1}) = \mathscr{D}(cb^{-1})$. ∎

The next lemma and the remarks following it shed some light on the equivalence classes $[b]$.

LEMMA 3.3. *Let $[b]$ be a projective operator and assume that $[\lambda] \in \sigma[b]^*$. Then there is a representative $b'$ for $[b]$ such that*

$$\langle b', \lambda \rangle = \sum_{0}^{n} \lambda_j b'_j = e.$$

*Proof.* If $\lambda \in \sigma[b]^*$ Lemma 2.9 says that $B = \langle b, \lambda \rangle$ is invertible. Then clearly $[B^{-1}b] = [b]$ and $b' = B^{-1}b$ is the desired representative. ∎

REMARK 3.4. There is no loss of generality in assuming that $\lambda_0 \neq 0$ because we may perturbate $[\lambda]$ a little and still belong to $\sigma[b]^*$.

REMARK 3.5. We have defined the equivalence relation on commuting tuples as the transitive closure of a symmetric and reflexive relation $R$. The proof of Lemma 3.3 shows that given a class $[b]$ such that $\sigma[b]^*$ is nonempty, any representative is not more than one step from the representative $b'$ with $\langle b', \lambda \rangle = e$. Hence if $b$ and $\widetilde{b}$ are any two representatives for $[b]$ then they are not more then two steps from each other.

REMARK 3.6. Lemma 3.3 also enables us to to give an alternative description of the equivalence relation $\sim$ if we restrict ourselves to look at commuting $n+1$-tuples of operators with the additional property that their spectrum avoid some hyperplane through the origin in $\mathbb{C}^{n+1}$. In fact for such tuples, $b$ and $\widetilde{b}$, we have $b \sim \widetilde{b}$ if and only if $\widetilde{b} = cb$ for some invertible $c$. The only if part is clear. Conversely assume that $\widetilde{b} = cb$ for some invertible $c$. The assumption on the spectrum for $\widetilde{b}$ says precisely that $\sigma[\widetilde{b}]^*$ is nonempty and so from Lemma 3.3 we see that we may assume that $\langle \widetilde{b}, \lambda \rangle = e$ for some $[\lambda]$. Hence $\langle b, \lambda \rangle = c^{-1}$ so $c \in (b)'$ and therefore $[b] = [\widetilde{b}]$.

DEFINITION 3.7. Let $[b]$ be a projective operator. We define $\sigma[b]^*_{\mathrm{adm}}$, the set of *admissible hyperplanes* for $[b]$, by saying that $[\alpha] \in \sigma[b]^*_{\mathrm{adm}}$ if $\langle b, \alpha\rangle\langle b, \lambda\rangle^{-1}$ is injective, where $[\lambda]$ is some hyperplane in $\sigma[b]^*$.

The definition clearly does not depend on the representative $b$ for $[b]$ and also not on the choice of $[\lambda]$ because if $[\widetilde{\lambda}] \in \sigma[b]^*$ is some other choice, then $\langle b, \alpha\rangle\langle b, \widetilde{\lambda}\rangle^{-1} = \langle b, \alpha\rangle\langle b, \lambda\rangle^{-1}\langle b, \lambda\rangle\langle b, \widetilde{\lambda}\rangle^{-1}$ and $\langle b, \lambda\rangle\langle b, \widetilde{\lambda}\rangle^{-1}$ is invertible by the functional calculus.

REMARK 3.8. Observe that $\sigma[b]^*_{\mathrm{adm}}$ is not defined as the dual complement of some set in $\mathbb{CP}^n$. It is defined directly as a subset of $\mathbb{CP}^{n*}$. However, in the one variable case $\sigma[b]^*_{\mathrm{adm}}$ corresponds to the point spectrum in the following sense. If $[\lambda] \in \sigma[b]^* \subseteq \mathbb{CP}^1$ and $P_\lambda$ a projection from the hyperplane (point) $[\lambda]$ onto $\mathbb{C}$ then

$$\sigma[b]^*_{\mathrm{adm}} = (P_\lambda^{-1}\sigma_p(P_\lambda([b])))^*.$$

PROPOSITION 3.9. *Let $[b]$ be a projective operator and let $[\lambda] \in \sigma[b]^*$ and $[\alpha] \in \sigma[b]^*_{\mathrm{adm}}$. Then $\langle b, \alpha\rangle^{-1}\langle b, \lambda\rangle$ is a closed operator which does not depend on the particular representative $b \in [b]$. Moreover $\langle b, \alpha\rangle^{-1}\langle b, \lambda\rangle = \langle b, \lambda\rangle\langle b, \alpha\rangle^{-1}$ and we denote this operator $\langle b, \lambda\rangle/\langle b, \alpha\rangle$. Its domain of definition, $\mathscr{D}_\alpha := \mathscr{D}(\langle b, \lambda\rangle/\langle b, \alpha\rangle)$, does not depend on the choice of $[\lambda] \in \sigma[b]^*$. Finally if $[\beta_1], \dots, [\beta_n]$ are any points such that $[\alpha], [\beta_1], \dots, [\beta_n]$ are in general position then*

$$\mathscr{D}_\alpha = \bigcap_{j=1}^{n} \mathscr{D}(\langle b, \alpha\rangle^{-1}\langle b, \beta_j\rangle).$$

*Proof.* It is clear that $\langle b, \alpha\rangle^{-1}\langle b, \lambda\rangle$ is a closed linear operator on $X$. Since $[\lambda] \in \sigma[b]^*$ we have that $\langle b, \lambda\rangle$ is invertible and so it follows from Lemma 3.2 that $\langle b, \alpha\rangle^{-1}\langle b, \lambda\rangle = \langle b, \lambda\rangle\langle b, \alpha\rangle^{-1}$. From this we immediately obtain that

$$\langle b, \lambda\rangle/\langle b, \alpha\rangle = (\langle b, \alpha\rangle/\langle b, \lambda\rangle)^{-1} \tag{3.1}$$

in the set theoretical sense and hence $\langle b, \lambda\rangle/\langle b, \alpha\rangle$ does not depend on the representative $b \in [b]$ since the right hand side of (3.1) does not. Moreover, since $\mathscr{D}(\langle b, \lambda\rangle\langle b, \alpha\rangle^{-1}) = \mathscr{D}(\langle b, \widetilde{\lambda}\rangle\langle b, \alpha\rangle^{-1})$ for any other $[\widetilde{\lambda}] \in \sigma[b]^*$ the domain $\mathscr{D}_\alpha$ can not depend on the choice of $[\lambda] \in \sigma[b]^*$. For the last statement we first assume that $[\alpha] = [1, 0, \dots, 0]$ and $[\beta_j] = [0, \dots, 1, \dots, 0]$ where the 1 is in the $j$th position. Then what we have to show is that $\mathscr{D}(\langle b, \lambda\rangle/b_0) = \bigcap_1^n \mathscr{D}(b_0^{-1}b_j)$. But from Lemma 3.2 we see that $\mathscr{D}(b_0^{-1}b_j) \supseteq \mathscr{D}(b_j b_0^{-1}) = \mathscr{D}(\langle b, \lambda\rangle b_0^{-1})$ and so $\mathscr{D}(\langle b, \lambda\rangle/b_0) \subseteq \bigcap_1^n \mathscr{D}(b_0^{-1}b_j)$. On the other hand $\bigcap_1^n \mathscr{D}(b_0^{-1}b_j) \subseteq \mathscr{D}(b_0^{-1}\langle b, \lambda\rangle)$ so we are done. We reduce the general case to this one by considering the projective transformation $P$ defined by $[z] \mapsto [\langle z, \alpha\rangle, \langle z, \beta_1\rangle, \dots, \langle z, \beta_n\rangle]$. Then $P^{*-1}[\alpha] = [1, 0, \dots, 0]$ and

$P^{*-1}[\beta_j] = [0,\dots,1,\dots,0]$. We want to show the equality

$$\mathscr{D}(\langle b,\lambda\rangle/\langle b,\alpha\rangle) = \bigcap_{j=1}^{n} \mathscr{D}(\langle b,\alpha\rangle^{-1}\langle b,\beta_j\rangle)$$

but this is equivalent to

$$\mathscr{D}(\langle Pb,P^{*-1}\lambda\rangle/\langle Pb,P^{*-1}\alpha\rangle) = \bigcap_{j=1}^{n} \mathscr{D}(\langle Pb,P^{*-1}\alpha\rangle^{-1}\langle Pb,P^{*-1}\beta_j\rangle).$$

Hence, the proposition follows from the special case above. ∎

REMARK 3.10. We saw in the proof that there was no loss of generality in assuming that the hyperplanes were of a special kind because we could reduce to this case by a projective transformation of $\mathbb{CP}^n$. In order to simplify calculations in the proofs below we will often make such assumptions and it is supposed to be understood that there is no loss of generality in doing it.

Let us fix an $[\alpha] \in \sigma[b]^*_{\mathrm{adm}}$ and $[\beta_1],\dots,[\beta_n] \in \mathbb{CP}^{n*}$ such that $[\alpha],[\beta_1],\dots,[\beta_n]$ are in general position. We denote the closed operator $\langle b,\alpha\rangle^{-1}\langle b,\beta_j\rangle$ by $a_j$.

PROPOSITION 3.11. *With the hypothesis of the preceding proposition, if $x \in \mathscr{D}(a_j)\cap\mathscr{D}(a_k)$ then the following conditions are equivalent:*

$$a_j x \in \mathscr{D}(a_k),$$
$$a_k x \in \mathscr{D}(a_j).$$

*If any of these conditions are satisfied then also $a_k a_j x = a_j a_k x$.*

*Proof.* We may assume $[\alpha] = [1,0,\dots,0]$ and $[\beta_j] = [0,\dots,1,\dots,0]$ and hence $a_j = b_0^{-1}b_j$. Suppose $x \in \mathscr{D}(b_0^{-1}b_j)\cap\mathscr{D}(b_0^{-1}b_k)$. Then from Lemma 3.2 we get $b_k b_0^{-1} b_j x = b_j b_0^{-1} b_k x$. Hence $b_0^{-1}b_j x \in \mathscr{D}(b_0^{-1}b_k)$ precisely when $b_0^{-1}b_k x \in \mathscr{D}(b_0^{-1}b_j)$ and $b_0^{-1}b_k b_0^{-1}b_j x = b_0^{-1}b_j b_0^{-1}b_k x$. ∎

DEFINITION 3.12. A tuple $(a_1,\dots,a_n)$ of closed operators on $X$ is called an *affine operator* if

(i) there exists a $[\lambda] \in \mathbb{CP}^n$ such that the operator

$$a_0 := \lambda_0 + \sum_1^n \lambda_j a_j$$

with domain $\mathscr{D}(a_0) = \bigcap_1^n \mathscr{D}(a_j)$ is closed, injective and surjective;

(ii) the operators $a_0,a_1,\dots,a_n$ satisfy the following commutation conditions; if $x \in \mathscr{D}(a_j)\cap\mathscr{D}(a_j a_k)$ then $x \in \mathscr{D}(a_k a_j)$ and $a_j a_k x = a_k a_j x$ for $j,k = 0,1,\dots,n$.

REMARK 3.13. In the one variable case Definition 3.12 just means that $\sigma(a)$ is not all of $\mathbb{C}$. In fact, if $\lambda_1 \neq 0$ then $\lambda_0 + \lambda_1 a$ is injective and surjective if and only if $-\lambda_0/\lambda_1 \notin \sigma(a)$ and if $\lambda_1 = 0$ then $\mathscr{D}(a) = X$, i.e., $a$ is bounded and therefore

$\sigma(a) \neq \mathbb{C}$. The commutation conditions are clearly satisfied in the one variable case and so from Section 1 we see that a closed operator is affine if and only if it can be Cayley transformed to a bounded operator.

REMARK 3.14. Morally, what condition (i) should mean is that no matter how we may define the spectrum of $a$, the hyperplane $[\lambda]$ should avoid its closure in $\mathbb{CP}^n$. For instance if $[\lambda] = [1, 0, \ldots, 0]$, that is the spectrum of $a$ does not intersect the hyperplane at infinity, then one should expect that all the $a_j$ are bounded. In fact if $[1, 0, \ldots, 0]$ works as $[\lambda]$ in Definition 3.12 then condition (i) says that the domain of the identity is $\bigcap_1^n \mathscr{D}(a_j)$, that is $\mathscr{D}(a_j) = X$ for all $j$ and so all the $a_j$ are bounded by the Closed Graph Theorem.

REMARK 3.15. We do not demand that each $a_j$ has a non-empty resolvent set. We will see in Example 3.18 that there are affine operators such that some of the components have all of $\mathbb{CP}^1$ as spectrum.

REMARK 3.16. Condition (ii) of Definition 3.12 implies that affine operators are permutable multioperators in the sense of [5]. It also implies that the operators $a_1, \ldots, a_n$ commute with the bounded operator $a_0^{-1}$ in the sense that $a_0^{-1}a_j \subseteq a_j a_0^{-1}$. In fact, let $x \in \mathscr{D}(a_j)$. Then clearly $a_0^{-1}x \in \mathscr{D}(a_j) \cap \mathscr{D}(a_j a_0)$ and so condition (ii) implies that $a_0^{-1}x \in \mathscr{D}(a_0 a_j)$ and $a_0 a_j a_0^{-1}x = a_j a_0 a_0^{-1}x = a_j x$. Hence $a_j a_0^{-1}x = a_0^{-1}a_j x$ for all $x \in \mathscr{D}(a_j)$. It will follow that if all $a_j$ have resolvents then these commute, see Corollary 3.19.

The operators we get when we project a projective operator from an admissible hyperplane are affine, and these are the only affine operators as we now show.

THEOREM 3.17. *A tuple $a = (a_1, \ldots, a_n)$ of closed operators on $X$ is affine if and only if there is a projective operator $[b]$ with $\sigma[b]^*$ nonempty, an $[\alpha] \in \sigma[b]^*_{\mathrm{adm}}$ and $[\beta_1], \ldots, [\beta_n] \in \mathbb{CP}^{n*}$ in general position together with $[\alpha]$, such that*

$$a_j = \langle b, \alpha\rangle^{-1}\langle b, \beta_j\rangle, \quad j = 1, \ldots, n.$$

*Proof.* We may assume that $\alpha = [1, 0, \ldots, 0]$, $\beta_j = [0, \ldots, 0, 1, 0, \ldots, 0]$ where 1 is in the $j$th place. First assume that $a_j = \langle b, \alpha\rangle^{-1}\langle b, \beta_j\rangle$, $j = 1, \ldots, n$ for some projective operator $[b]$, that is $a_j = b_0^{-1}b_j$. Let $[\lambda] \in \sigma[b]^*$ so that $B = \langle b, \lambda\rangle$ is invertible. From Proposition 3.2 we get that $b_0^{-1}B = Bb_0^{-1}$ and so we see that

$$a_0 := b_0^{-1}B = b_0^{-1}\sum_0^n \lambda_j b_j = \sum_0^n \lambda_j b_0^{-1}b_j = \lambda_0 + \sum_1^n \lambda_j a_j$$

has domain $\mathscr{D}(a_0) = \mathscr{D}(b_0^{-1}) = \bigcap_1^n \mathscr{D}(a_j)$ by Proposition 3.9, is closed, injective and surjective. Hence $a$ satisfies condition (i) in Definition 3.12. Moreover Proposition 3.11 implies that if $x \in \mathscr{D}(a_j) \cap \mathscr{D}(a_j a_k)$ then $x \in \mathscr{D}(a_k a_j)$ and $a_j a_k x = a_k a_j x$

for $j,k = 1,\dots,n$. To see that this is also satisfied for $j = 0$ and $k = 0$ respectively we first assume that $x \in \mathscr{D}(a_0) \cap \mathscr{D}(a_0 a_k)$. Then since $x \in \mathscr{D}(a_0 a_k)$ we have that $b_0^{-1} b_k x \in \mathscr{D}(b_0^{-1})$ and since also $x \in \mathscr{D}(b_0^{-1})$ Lemma 3.2 implies that $b_0^{-1} b_k x = b_k b_0^{-1} x$. Hence $b_k b_0^{-1} x \in \mathscr{D}(b_0^{-1})$, that is $x \in \mathscr{D}(a_k a_0)$, and $b_0^{-1} b_0^{-1} b_k x = b_0^{-1} b_k b_0^{-1} x$ that is $a_0 a_k x = a_k a_0 x$. Now assume that $x \in \mathscr{D}(a_j) \cap \mathscr{D}(a_j a_0)$ which just means that $x \in \mathscr{D}(b_0^{-1})$ and $b_j b_0^{-1} x \in \mathscr{D}(b_0^{-1})$. From Lemma 3.2 we see that $b_j b_0^{-1} x = b_0^{-1} b_j x$ so $b_0^{-1} b_j x \in \mathscr{D}(b_0^{-1})$ and $b_0^{-1} b_0^{-1} b_j x = b_0^{-1} b_j b_0^{-1} x$. Hence $x \in \mathscr{D}(a_0 a_j)$ and $a_0 a_j x = a_j a_0 x$ so $a$ also satisfies condition (ii) and thus $a$ is affine.

Conversely assume that $a$ is affine and take $[\lambda] \in \mathbb{CP}^n$ such that the operator $a_0 = \lambda_0 + \sum\limits_1^n \lambda_j a_j$ satisfies the requirements of condition (i) in Definition 3.12. Then

$$b_0 := \Big(\lambda_0 + \sum_1^n \lambda_j a_j\Big)^{-1}, \quad b_j := a_j \Big(\lambda_0 + \sum_1^n \lambda_j a_j\Big)^{-1} \quad j = 1,\dots,n$$

are bounded operators by the Closed Graph Theorem. We claim that $b = (b_0,\dots, b_n)$ is commutative, that $\langle b,\lambda\rangle$ is invertible and that $a_j = b_0^{-1} b_j$. We start by showing commutativity. In Remark 3.16 we saw that it followed from condition (ii) that $a_0^{-1} a_j \subseteq a_j a_0^{-1}$, that is $b_0 a_j \subseteq a_j b_0$ for $j = 1,\dots,n$. Hence for any $x \in X$ we have $a_k b_0^2 x = b_0 a_k b_0 x \in \bigcap\limits_1^n \mathscr{D}(a_j)$. So we see from condition (ii) that for any $x \in X$ we have $a_l b_0 a_k b_0 x = a_l a_k b_0^2 x = a_k a_l b_0^2 x = a_k b_0 a_l b_0 x$. Thus $b$ is commutative. To see that $a_k = b_0^{-1} b_k$ we assume $x \in \mathscr{D}(a_k)$. Then condition (ii), via Remark 3.16, implies that $a_k b_0 x = b_0 a_k x \in \bigcap\limits_1^n \mathscr{D}(a_j)$. Hence $a_l a_k b_0 x = a_k a_l b_0 x$ for all $l$ by condition (ii), and we obtain $b_0^{-1} a_k b_0 x = a_k x$. Thus $a_k \subseteq b_0^{-1} b_k$. To show equality it suffices to show $\mathscr{D}(b_0^{-1} b_k) \subseteq \mathscr{D}(a_k)$. Therefore assume $x \in \mathscr{D}(b_0^{-1} b_k)$, that is $a_k b_0 x \in \mathscr{D}(b_0^{-1})$ and so, again by condition (ii), we have $a_l a_k b_0 x = a_k a_l b_0 x$. Hence $a_l b_0 x \in \mathscr{D}(a_k)$ for all $l$ and this gives us $x = b_0^{-1} b_0 x \in \mathscr{D}(a_k)$. Finally we observe that

$$\langle b,\lambda\rangle = \sum_0^n \lambda_j b_j = \lambda_0 b_0 + \sum_1^n \lambda_j a_j b_0 = \Big(\lambda_0 + \sum_1^n \lambda_j a_j\Big) b_0 = e.$$

Hence $\sigma(b)$ avoids the hyperplane $\lambda$ through the origin in $\mathbb{C}^{n+1}$ and hence $[b]$ is a projective operator with $\sigma[b]^*$ nonempty. ∎

EXAMPLE 3.18. Let $K$ be the compact subset of $\mathbb{C}^3$ defined by

$$K = \{(1,z_1,0) : |z_1| \leqslant 1\} \cup \{(1/z_1, 1, 1/z_1) : |z_1| \geqslant 1\} \cup \{(0,1,0)\}.$$

Let $X = C(K)$ be the Banach space of continuous functions on $K$ and let $b_j$ denote the operator on $X$ of multiplication with the coordinate function $z_j$, $j = 0,1,2$. Then $b = (b_0,b_1,b_2)$ defines a projective operator $[b]$ and $\sigma[b] = \pi(K)$, the projection of $K$ on $\mathbb{CP}^2$. Moreover, one checks that the hyperplane $[2,1,-3/2]$ avoids $\sigma[b]$. Clearly $b_0$ is injective and so the hyperplane $[1,0,0]$ is admissible. We get the affine operator $(a_1, a_2) = (b_0^{-1}b_1, b_0^{-1}b_2)$. We claim that $\sigma(a_1) = \mathbb{C}$. Let $w \in \mathbb{C}$ be arbitrary and take a point $(z_0, z_1, z_2) \in K$ such that $z_1/z_0 = w$. If $f \in C(K)$ is such that $f(z_0, z_1, z_2) \neq 0$ then $f$ is not in the range of $w - a_1$ and therefore $w \in \sigma(a_1)$.

COROLLARY 3.19. *If $(a_1, \dots, a_n)$ is affine and each $a_j$ has resolvents then these commute.*

*Proof.* Let $[b]$ be a projective operator such that $a_j = b_0^{-1}b_j$. We consider the case when each $a_j$ has a bounded inverse. The general case is completely analogous. We first check that $a_j^{-1} = b_j^{-1}b_0$. Actually, $b_j$ has to be injective since otherwise $b_jx = 0$ for some $x \neq 0$, but then $x \in \mathscr{D}(a_j)$ and $a_jx = 0$ which is impossible. Also, $b_j$ has to be surjective onto $\mathscr{D}(b_0^{-1})$ and hence $\mathscr{R}(b_0) \subseteq \mathscr{D}(b_j^{-1})$. The closed operator $b_j^{-1}b_0$ therefore has to be bounded. It follows that $b_0^{-1}b_jb_j^{-1}b_0$ is the identity on $X$ and that $b_j^{-1}b_0b_0^{-1}b_j$ is the identity on $\mathscr{D}(a_j)$ and so $a_j^{-1} = b_j^{-1}b_0$. Now we use Lemma 3.2 to see that $a_j^{-1}$ and $a_k^{-1}$ commute. Let $y = a_j^{-1}a_k^{-1}x = b_j^{-1}b_0b_k^{-1}b_0x = b_j^{-1}b_k^{-1}b_0^2x$. Then $b_0^2x = b_kb_jy = b_jb_ky$ and hence $y = b_k^{-1}b_0b_j^{-1}b_0x = a_k^{-1}a_j^{-1}x$ since $\mathscr{R}(b_0) \subseteq \mathscr{D}(b_j^{-1})$. ∎

COROLLARY 3.20. *If $(a_1, \dots, a_n)$ is affine then affine combinations of the $a_j$ are closable.*

*Proof.* To any affine map of $\mathbb{C}^n$ corresponds a projective transformation of $\mathbb{CP}^n$. Substituting a projective operator, representing $(a_1, \dots, a_n)$, into this map and projecting the result back to $\mathbb{C}^n$ we obtain a closed extension of the affine combination. ∎

The correspondence between affine and projective operators is one-to-one in the following sense.

THEOREM 3.21. *Fix $[\alpha], [\beta_1], \dots, [\beta_n] \in \mathbb{CP}^n$ in general position. Then to any affine operator $a = (a_1, \dots, a_n)$ corresponds a unique projective operator $[b]$ with non-empty $\sigma[b]^*$ and with $[\alpha] \in \sigma[b]^*_{\mathrm{adm}}$ such that $a_j = \langle b, \alpha\rangle^{-1}\langle b, \beta_j\rangle$ for $j = 1, \dots, n$.*

*Proof.* The existence of a projective operator $[b]$ and $[\widetilde{\alpha}]$ and $[\widetilde{\beta}_j]$, $j = 1, \dots, n$, in general position such that $a_j = \langle b, \widetilde{\alpha}\rangle^{-1}\langle b, \widetilde{\beta}_j\rangle^{-1}$ is part of Theorem 3.17. Let $L$ be an invertible projective transformation sending $[\widetilde{\alpha}]$ to $[\alpha]$ and $[\widetilde{\beta}_j]$ to $[\beta_j]$. Then $L^{*-1}[b]$ is a projective operator with $a_j = \langle L^{*-1}b, \alpha\rangle^{-1}\langle L^{*-1}b, \beta_j\rangle^{-1}$. For uniqueness we assume that $\alpha = [1, 0, \dots, 0]$, $\beta_j = [0, \dots, 0, 1, 0, \dots, 0]$ and that $[b]$ and $[\widetilde{b}]$ are two projective operators corresponding to $a$, i.e., we assume that $b_0^{-1}b_j = a_j = \widetilde{b}_0^{-1}b_j$, $j = 1, \dots, n$. We may also assume that $b$ is the representative for $[b]$ such that $e = \langle b, \lambda\rangle$ by Lemma 3.3. We show that $[b] = [\widetilde{b}]$. From Proposition 3.9 we get

$$\mathscr{D}(b_0^{-1}) = \bigcap \mathscr{D}(b_0^{-1}b_j) = \bigcap \mathscr{D}(\widetilde{b}_0^{-1}\widetilde{b}_j) = \mathscr{D}(\widetilde{b}_0^{-1}).$$

Hence $c := \widetilde{b}_0^{-1} b_0$ is an invertible bounded operator. Moreover from Lemma 3.2 and the assumption we see that $\widetilde{b}_j c = \widetilde{b}_j \widetilde{b}_0^{-1} b_0 = \widetilde{b}_0^{-1} \widetilde{b}_j b_0 = b_0^{-1} b_j b_0 = b_j$ and so $b = \widetilde{b} c$. It remains to show that $c \in (\widetilde{b})'$. But $e = \langle b, \lambda \rangle = \sum_0^n \lambda_j b_j$ so $c^{-1} = \sum_0^n \lambda_j \widetilde{b}_j$ and hence $c \in (\widetilde{b})'$. ∎

DEFINITION 3.22. Let $[\alpha], [\beta_1], \ldots, [\beta_n] \in \mathbb{CP}^n$ be fixed in general position. We define $\rho_{\alpha,\beta}$ to be the mapping

$$[z] \mapsto (\langle z, \alpha \rangle^{-1} \langle z, \beta_1 \rangle, \ldots, \langle z, \alpha \rangle^{-1} \langle z, \beta_n \rangle).$$

The one-to-one correspondence can now be stated by saying that the mapping $\rho_{\alpha,\beta} : \{[b] : \sigma[b]^* \neq \emptyset,\ [\alpha] \in \sigma[b]^*_{\mathrm{adm}}\} \to \{a : a \text{ is affine}\}$ is one-to-one and onto.

## 4. SPECTRA OF AFFINE OPERATORS

We define the spectrum of an affine operator $a$, corresponding to a projective operator $[b]$ via $\rho_{\alpha,\beta}([b]) = a$, and show that $\rho_{\alpha,\beta}(\sigma[b]) = \sigma(a)$. Throughout this section we will assume that $\alpha = [1, 0, \ldots, 0]$ and $\beta_j = [0, \ldots, 1, \ldots, 0]$ in the proofs.

Let $a = (a_1, \ldots, a_n)$ be an affine operator. For $z \in \mathbb{C}^n$ we let $\delta_{z-a}$ denote interior multiplication with $\sum_1^n (z_j - a_j) e_j^*$ and the domain of definition, $\mathscr{D}(\delta_{z-a})$, for this operator is all forms with coefficients in $\bigcap_1^n \mathscr{D}(a_j)$. See Section 1 for notation.

DEFINITION 4.1. Let $a = (a_1, \ldots, a_n)$ be an affine operator. We define $\sigma(a) \subseteq \mathbb{C}^n$ by specifying its complement: $z \notin \sigma(a)$ if and only if for any $k$-form $f^k \in \mathscr{N}(\delta_{z-a})$ it exists a $k+1$-form $f^{k+1}$ with coefficients in $\bigcap_{j,k=1}^n \mathscr{D}(a_j a_k)$ such that $f^k = \delta_{z-a} f^{k+1}$.

REMARK 4.2. Affine operators are permutable multioperators and as such they also have a joint Ionaşcu-Vasilescu spectrum, [5]. If all components in an affine operator have resolvents we can also consider the joint spectrum associated to an iterated one-dimensional Cayley transform as in [11] and [2]. It is shown in [5] that this spectrum equals the Ionaşcu-Vasilescu spectrum in this case. We will see in Theorem 4.5 that our spectrum is contained in the spectrum obtained by an iterative one-dimensional Cayley transform in case both spectra are defined.

We denote the set of all forms with coefficients in $\bigcap_{j,k=1}^n \mathscr{D}(a_j a_k)$ by $\mathscr{D}^2$.

LEMMA 4.3. *Let $[b]$ be a projective operator and assume that $[1, 0, \ldots, 0]$ is an admissible hyperplane and that $\sigma[b]^*$ is nonempty. Put $b' = (b_1, \ldots, b_n)$ and let $a =$*

$(b_0^{-1}b_1,\ldots,b_0^{-1}b_n)$. *Then* $K_\bullet(\delta_{b'},X)$ *is exact if and only if for any* $f^k \in \mathcal{N}(\delta_a)$ *there exists an* $f^{k+1}$ *with coefficients in* $\mathcal{D}(b_0^{-2}) = \mathcal{R}(b_0^2)$ *such that* $f^k = \delta_a f^{k+1}$.

*Proof.* Note that $\mathcal{D}(b_0^{-1}) = \bigcap_1^n \mathcal{D}(b_0^{-1}b_j)$ by Proposition 3.9. Assume that $K_\bullet(\delta_{b'},X)$ is exact and let $f^k \in \mathcal{N}(\delta_a)$. Then $\delta_{b'}b_0^{-1}f^k = 0$ and so there is an $\widetilde{f}^{k+1}$ such that $b_0^{-1}f^k = \delta_{b'}\widetilde{f}^{k+1}$. But then $f^k = \delta_{b'}b_0\widetilde{f}^{k+1} = \delta_a b_0^2\widetilde{f}^{k+1}$. Thus $f^{k+1} := b_0^2\widetilde{f}^{k+1}$ has coefficients in $\mathcal{D}(b_0^{-2})$ and $f^k = \delta_a f^{k+1}$.

Now assume that if $f^k \in \mathcal{N}(\delta_a)$ it exists an $f^{k+1}$ with coefficients in $\mathcal{D}(b_0^{-2})$ such that $f^k = \delta_a f^{k+1}$. If $\delta_{b'}\widetilde{f}^k = 0$ then clearly $b_0\widetilde{f}^k \in \mathcal{N}(\delta_a)$ and so there is an $\widetilde{f}^{k+1}$ with coefficients in $\mathcal{D}(b_0^{-2})$ such that $b_0\widetilde{f}^k = \delta_a\widetilde{f}^{k+1} = \delta_{b'}b_0^{-1}\widetilde{f}^{k+1}$. Hence $\widetilde{f}^k = \delta_{b'}b_0^{-2}\widetilde{f}^{k+1}$ and so $K_\bullet(\delta_{b'},X)$ is exact. ∎

THEOREM 4.4. *Let a be an affine operator and let* $[b]$ *be a projective operator with nonempty* $\sigma[b]^*$. *If* $[\alpha] \in \sigma[b]^*_{\mathrm{adm}}$ *has the property that* $a = \rho_{\alpha,\beta}([b])$ *then* $\sigma(a) = \rho_{\alpha,\beta}(\sigma[b])$.

*Proof.* Under our assumptions on $[\alpha]$ and $[\beta]$ we have that $\rho_{\alpha,\beta}$ is the mapping $[z] \mapsto (z_1/z_0,\ldots,z_n/z_0)$. We will show that $[1,0,\ldots,0] \notin \sigma[b]$ if and only if $0 \notin \sigma(a)$. By the Spectral Mapping Theorem we get that the line through the origin and $(1,0,\ldots,0)$ in $\mathbb{C}^{n+1}$ does not intersect $\sigma(b)$ if and only if $0 \notin \sigma(b_1,\ldots,b_n)$. Thus what we have to show is that $0 \notin \sigma(b_1,\ldots,b_n)$ if and only if $0 \notin \sigma(a)$. But this is exactly the statement in Lemma 4.3 and so the only thing left in order to prove the theorem is to check that $\mathcal{D}(b_0^{-2}) = \bigcap_{j,k=1}^n \mathcal{D}(a_ja_k)$. Since $a_ja_k = b_0^{-1}b_jb_0^{-1}b_k \supseteq b_jb_kb_0^{-2}$ the inclusion $\subseteq$ is clear. Conversely, assume $x \in \bigcap_{j,k=1}^n \mathcal{D}(a_ja_k)$. Then, at least $x \in \bigcap_j^n \mathcal{D}(a_j) = \mathcal{D}(b_0^{-1})$ by Proposition 3.9. Thus $x = b_0y$ for some $y$. The assumption on $x$ now implies that $b_ky = b_0^{-1}b_kx \in \bigcap_j^n \mathcal{D}(a_j) = \mathcal{D}(b_0^{-1})$ for $k = 1,\ldots,n$. Since we may assume that $e = \sum_0^n \lambda_kb_k$ we get $y = \sum_0^n \lambda_kb_ky \in \mathcal{D}(b_0^{-1})$. Thus $x = b_0y \in \mathcal{D}(b_0^{-2})$ and we are done. ∎

Theorem 3.21 implies that to an affine operator $a$ we have a unique projective operator $[b]$ such that $a = \rho_{\alpha,\beta}([b])$ for some fixed choice of $[\alpha],[\beta_1],\ldots,[\beta_n]$ in general position. So applying Theorem 4.4 we see that $\sigma(a)$ has a well defined, invariant and closed extension $\widehat{\sigma}(a) \subseteq \mathbb{CP}^n$ defined by

$$\widehat{\sigma}(a) = \sigma[b].$$

Now suppose that $a = (a_1,\ldots,a_2)$ is affine and assume in addition that each $a_j$ has a resolvent. As we have seen, Example 3.18, affine operators need not have this property but may of course have it, see Example 4.7 below. After an affine transformation we may assume that each $a_j$ has a bounded inverse $a_j^{-1} \in L(X)$. Then as in e.g. [11] and [2] we can define the spectrum for $a$ as the inverse image of the spectrum of $(a_1^{-1}, \dots, a_n^{-1})$ under the mapping $(z_1, \dots, z_n) \mapsto (1/z_1, \dots, 1/z_n)$. We will denote this spectrum by $\widetilde{\sigma}(a)$.

THEOREM 4.5. *Let $a = (a_1, \dots, a_n)$ be an affine operator and assume that each $a_j$ has a resolvent. Then $\sigma(a) \subseteq \widetilde{\sigma}(a)$ and in the case $n = 2$ we have equality.*

Since $(a_1, \dots, a_n)$ is affine there exists a unique projective operator $[b] = [b_0, \dots, b_n]$ such that $a_j = b_0^{-1} b_j$ for $j = 1, \dots, n$. Before we prove Theorem 4.5 we prove a lemma.

LEMMA 4.6. *A point $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is outside $\sigma(a)$ if and only if $0 \in \mathbb{C}^n$ is outside $\sigma(b_1 - \lambda_1 b_0, \dots, b_n - \lambda_n b_0)$.*

*Proof.* First note that from Theorem 4.4, $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is outside $\sigma(a)$ if and only if $[1, \lambda_1, \dots, \lambda_n] \in \mathbb{CP}^n$ is outside $\sigma[b]$ which in turn precisely means that the line through the origin and $(1, \lambda_1, \dots, \lambda_n)$ in $\mathbb{C}^{n+1}$ does not intersect $\sigma(b_0, \dots, b_n)$. The lemma now follows by applying the Spectral Mapping Theorem to the mapping $(z_0, \dots, z_n) \mapsto (z_1 - \lambda_1 z_0, \dots, z_n - \lambda_n z_0)$. ∎

*Proof of Theorem* 4.5. We may assume without loss of generality that $0 \notin \sigma(a_j)$ for $j = 1, \dots, n$, i.e., that each $a_j^{-1}$ is bounded. It follows from the proof of Corollary 3.19 that $a_j^{-1} = b_j^{-1} b_0$ and thus in particular that $\mathscr{D}(b_0^{-1}) \subseteq \mathscr{D}(b_j^{-1})$ for all $j$. Let $(\lambda_1, \dots, \lambda_n)$ be any point in $\mathbb{C}^n$. We define two boundary operators $\delta$ and $\widetilde{\delta}$ on $\Lambda X = \bigoplus_0^n \Lambda^j X$ by letting $\delta$ and $\widetilde{\delta}$ be interior multiplication with $\sum_1^n (b_j - \lambda_j b_0) e_j^*$ and $\sum_{j \in I_0} e_j^* + \sum_{j \in I_1} (1/\lambda_j - b_j^{-1} b_0) e_j^*$ respectively, where $I_0$ and $I_1$ are the set of indices with $\lambda_j = 0$ and $\lambda_j \neq 0$ respectively. By the previous lemma, $(\lambda_1, \dots, \lambda_n)$ is outside $\sigma(a_1, \dots, a_n)$ if and only if the complex $K_\bullet(\delta, \Lambda^\bullet X)$ is exact. If some $\lambda_j = 0$ then we are automatically outside $\widetilde{\sigma}(a_1, \dots, a_n)$ and if all $\lambda_j$ are non zero we are outside this spectrum if and only if the complex $K_\bullet(\widetilde{\delta}, \Lambda^\bullet X)$ is exact. We now define morphisms of complexes $\Psi : K_\bullet(\delta, \Lambda^\bullet X) \to K_\bullet(\widetilde{\delta}, \Lambda^\bullet X)$ and $\Phi : K_\bullet(\widetilde{\delta}, \Lambda^\bullet X) \to K_\bullet(\delta, \Lambda^\bullet X)$. Let $\mathscr{A}$ be a commutative unital subalgebra of $L(X)$ containing $b_0, b_j, b_j^{-1} b_0$, $j = 1, \dots, n$. Note that $\delta$ and $\widetilde{\delta}$ are natural mappings $E \to \mathscr{A}$ extended as anti-derivations to $\mathscr{A} \otimes \Lambda E \to \mathscr{A} \otimes \Lambda E$. We define the injective mapping $\Psi : \mathscr{A} \otimes \Lambda E \to \mathscr{A} \otimes \Lambda E$ inductively by setting $\Psi(e_j) = \lambda_j b_j e_j$ if $\lambda_j \neq 0$, $\Psi(e_j) = b_j e_j$ if $\lambda_j = 0$, and $\Psi(f \wedge g) = \Psi(f) \wedge \Psi(g)$. The operator $b_1 \cdots b_n$ is naturally a mapping $\mathscr{A} \otimes \Lambda E \to \mathscr{A} \otimes \Lambda E$ and if we are in the image of this mapping we are in the image of $\Psi$. We may thus define $\Phi := \Psi^{-1} \lambda_1 \cdots \lambda_n b_1 \cdots b_n$ and get a mapping $\Phi : \mathscr{A} \otimes \Lambda E \to \mathscr{A} \otimes \Lambda E$. It is straight forward to check that $\widetilde{\delta} \Psi = \Psi \delta$ and $\delta \Phi = \Phi \widetilde{\delta}$ as mappings $\mathscr{A} \otimes \Lambda E \to \mathscr{A} \otimes \Lambda E$ and so we obtain our morphisms of complexes $\Psi : K_\bullet(\delta, \Lambda^\bullet X) \to K_\bullet(\widetilde{\delta}, \Lambda^\bullet X)$ and $\Phi : K_\bullet(\widetilde{\delta}, \Lambda^\bullet X) \to K_\bullet(\delta, \Lambda^\bullet X)$. We now claim that the quotient complex

$$0 \longleftarrow \Lambda^0 X/\Lambda^0\mathscr{R}(b_0) \longleftarrow \cdots \longleftarrow \Lambda^n X/\Lambda^n\mathscr{R}(b_0) \longleftarrow 0 \tag{4.1}$$

with boundary operator $\delta$ is exact. Indeed, since $a$ is affine there is some point $\mu = (\mu_1, \dots, \mu_n)$ outside $\sigma(a)$. Let $\delta_\mu$ be interior multiplication with $\sum_1^n (b_j - \mu_j b_0)e_j^*$. From the previous lemma we know that $K_\bullet(\delta_\mu, \Lambda^\bullet X)$ is exact. Since $\delta$ and $\delta_\mu$ are equal modulo elements with coefficients in $\mathscr{R}(b_0)$ it suffices to see that (4.1) is exact with $\delta_\mu$ as boundary operator. Assume that $\delta_\mu(f^k) \in \Lambda^{k-1}\mathscr{R}(b_0)$, i.e., $\delta_\mu(f^k) = b_0 f^{k-1}$. Then $b_0\delta_\mu(f^{k-1}) = \delta_\mu(b_0 f^{k-1}) = \delta_\mu^2(f^k) = 0$ and since $b_0$ is injective $\delta_\mu(f^{k-1}) = 0$. Hence, $f^{k-1} = \delta_\mu(g^k)$ and so $\delta_\mu(f^k - b_0 g^k) = 0$. Thus $f^k - b_0 g^k = \delta_\mu(f^{k+1})$ which precisely means that the equivalence class containing $f^k$ is in the image of $\delta_\mu$. Now suppose that the complex $K_\bullet(\widetilde{\delta}, \Lambda^\bullet X)$ is exact. Assume that $\delta(f^k) = 0$. Since the quotient complex (4.1) with boundary operator $\delta$ is exact there are $g_0^k$ and $f_0^{k+1}$ such that $f^k = b_0 g_0^k + \delta(f_0^{k+1})$. Since $b_0$ is injective it follows that $\delta(g_0^k) = 0$ and so, again, there are $g_1^k$ and $f_1^{k+1}$ such that $g_0^k = b_0 g_1^k + \delta(f_1^{k+1})$ and $\delta(g_1^k) = 0$. Repeating this process we see that we can write

$$f^k = b_0^{k+1} g_k^k + \delta\Big(\sum_{j=0}^{k} b_0^j f_j^{k+1}\Big) \tag{4.2}$$

and $\delta(g_k^k) = 0$. Now, $0 = \Psi\delta(g_k^k) = \widetilde{\delta}\Psi(g_k^k)$ and so by assumption, $\Psi(g_k^k) = \widetilde{\delta}(h^{k+1})$ and thus $\Psi(b_0^{k+1} g_k^k) = \widetilde{\delta}(b_0^{k+1} h^{k+1})$. But the range of $b_0$ is contained in the domain of every $b_j^{-1}$ and so $b_0^{k+1}h^{k+1} \in \mathscr{R}(\Psi)$. Hence,

$$\Psi(b_0^{k+1} g_k^k) = \widetilde{\delta}(b_0^{k+1} h^{k+1}) = \widetilde{\delta}\Psi\Psi^{-1}(b_0^{k+1} h^{k+1}) = \Psi\delta\Psi^{-1}(b_0^{k+1} h^{k+1})$$

and since $\Psi$ is injective we get $b_0^{k+1} g_k^k = \delta\Psi^{-1}(b_0^{k+1} h^{k+1})$. According to (4.2) we obtain

$$f^k = \delta\Big(\Psi^{-1}(b_0^{k+1} h^{k+1}) + \sum_{j=0}^{k} b_0^j f_j^{k+1}\Big)$$

and then $K_\bullet(\delta, \Lambda^\bullet X)$ is exact. Note that in case some $\lambda_j = 0$ then automatically $K_\bullet(\widetilde{\delta}, \Lambda^\bullet X)$ is exact. Thus $\sigma(a) \subseteq \widetilde{\sigma}(a)$.

We now show that if $K_\bullet(\delta, \Lambda^\bullet X)$ is exact then $H_k(\widetilde{\delta}, \Lambda X) = 0$ for $k = 0$, $k = n$ and for $k = n-1$, thus implying that $\sigma(a) = \widetilde{\sigma}(a)$ for $a = (a_1, a_2)$. We may of course assume that $\lambda_j \neq 0$ for all $j$. If $\widetilde{\delta}(f^n) = 0$ then $0 = \Phi\widetilde{\delta}(f^n) = \delta\Phi(f^n)$. By assumption then $\Phi(f^n) = 0$ and since $\Phi$ is injective (at this level the identity) we have $f^n = 0$. If instead $\widetilde{\delta}(f^{n-1}) = 0$ we conclude that $0 = \Phi\widetilde{\delta}(f^{n-1}) = \delta\Phi(f^{n-1})$. Since $\Phi$ is the identity at the top level we get

$$\Phi(f^{n-1}) = \delta(f^n) = \delta\Phi(f^n) = \Phi\widetilde{\delta}(f^n),$$

which implies that $f^{n-1} = \widetilde{\delta}(f^n)$. Finally, given any $f^0 \in \Lambda^0 X$ we can write $f^0 = \delta(f^1)$. At the lowest level, $\Psi$ is the identity and we see that $f^0 = \Psi(f^0) = \Psi\delta(f^1) = \widetilde{\delta}\Psi(f^1)$ finishing the proof. ∎

EXAMPLE 4.7. Let $X = L^2(\mathbb{R})$ and let $b_0$ and $b_1$ be multiplication with $1/(\mathrm{i}+\xi)^2$ and $1/(\mathrm{i}+\xi)$ on $X$ and let $b_2$ be the identity. Then $[b_0, b_1, b_2]$ is a projective operator and $(a_1, a_2) = (b_0^{-1}b_1, b_0^{-1}b_2)$ is affine and has the property that each $a_j$ has a bounded inverse. It is straightforward to check explicitly that $\sigma(a_1, a_2) = \widetilde{\sigma}(a_1, a_2) = \{(\mathrm{i}+x, (\mathrm{i}+x)^2) \in \mathbb{C}^2 : x \in \mathbb{R}\}$, i.e., the (essential) range of the multiplication operator $(a_1, a_2)$.

## 5. CAYLEY TRANSFORMS

We summarize our results to see that the affine operators are precisely those operators which are Cayley transforms of bounded ones and that the Spectral mapping theorem holds.

Let $a = (a_1, \dots, a_n)$ be affine and let $[\lambda] \in \mathbb{CP}^n$ be such that condition (i) in Definition 3.12 is fulfilled. Then if $a_0 = \lambda_0 + \sum_1^n \lambda_j a_j$, the projective operator $[b] = [a_0^{-1}, a_1 a_0^{-1}, \dots, a_n a_0^{-1}]$ projects to $a$ and $[\lambda] \in \sigma[b]^*$ by Theorem 3.17 and its proof. Let $[\beta_1], \dots, [\beta_n]$ be points in $\mathbb{CP}^n$ such that $[\lambda], [\beta_1], \dots, [\beta_n]$ are in general position. Applying the projection $\rho_{\lambda,\beta}$ to $[b]$ we get the bounded commuting tuple

$$\rho_{\lambda,\beta}([b]) = \Big(\Big(\beta_{1,0} + \sum_1^n \beta_{1,j} a_j\Big) a_0^{-1}, \dots, \Big(\beta_{n,0} + \sum_1^n \beta_{n,j} a_j\Big) a_0^{-1}\Big)$$

and $\sigma(\rho_{\lambda,\beta}([b])) = \rho_{\lambda,\beta}(\sigma[b])$ by Theorem 2.8. Hence, if $\phi$ is the corresponding rational fractional transformation we see that $\phi(a) = \rho_{\lambda,\beta}([b])$ is a bounded commuting tuple and by Theorem 4.4 we have $\sigma(\phi(a)) = \phi(\widehat{\sigma}(a))$ naturally interpreted.

Conversely assume that a tuple of closed operators $a = (a_1, \dots, a_n)$ is the Cayley transform of a bounded commuting tuple $(b_1, \dots, b_n)$, that is

$$a_k = \Big(\lambda_{0,0} + \sum_1^n \lambda_{0,j} b_j\Big)^{-1} \Big(\lambda_{k,0} + \sum_1^n \lambda_{k,j} b_j\Big),$$

where $(\lambda_{j,k})$ is an invertible matrix and $\lambda_{0,0} + \sum_1^n \lambda_{0,j} b_j$ is injective, i.e., the affine hyperplane $\{z \in \mathbb{C}^n : \langle z, \lambda_0 \rangle = 0\}$ is admissible. Then clearly $[e, b_1, \dots, b_n]$ is a projective operator and $[1, 0, \dots, 0] \in \sigma[e, b_1, \dots, b_n]^*$. Moreover, the hyperplane $[\lambda_{0,0}, \dots, \lambda_{0,n}]$ has to be admissible and so $a$ is the projection of a projective operator from an admissible hyperplane. Since the spectrum of the projective operator also has a nonempty dual complement it follows from Theorem 3.17 that $a$ is affine.

## 6. INTEGRAL FORMULAS FOR THE ANALYTIC FUNCTIONAL CALCULUS OF PROJECTIVE OPERATORS

We provide integral formulas realizing the functional calculus described in Section 2. Analogously to [1] we will construct a $\bar{\partial}$-closed $(n, n-1)$-form, $\omega_b^n x$, with values in $X \otimes L^n$, defined in $U \setminus \sigma[b]$, where $L^{-1}$ is the tautological line bundle and $U$ is $\mathbb{CP}^n$ minus some hyperplane, such that if $f \in \mathscr{O}(\sigma[b])$, then

$$f([b])x = \int_{\partial D} f \frac{\langle b, \lambda \rangle^n}{\langle z, \lambda \rangle^n} \omega_b^n x$$

where $\lambda \in \sigma[b]^*$ and $D$ is a suitable neighborhood of $\sigma[b]$.

We let $\delta_z$ denote interior multiplication with the vector field $\sum_0^n z_j \frac{\partial}{\partial z_j}$. Letting $f$ be a $k$-homogeneous $(p,0)$-form in some cone in $\mathbb{C}^{n+1}$ then $f$ is the pullback of an $L^k$-valued $(p,0)$-form in the projection of the cone in $\mathbb{CP}^n$ if and only if $\delta_z f = 0$. The statement is local and we may verify it when $z_0 \neq 0$. If $f$ is the pullback of an $L^k$-valued $(p,0)$-form then $f$ is $k$-homogeneous and can be written as $f = \sum_I f_I \mathrm{d}(z_{I_1}/z_0) \wedge \cdots \wedge \mathrm{d}(z_{I_p}/z_0)$. Since $\delta_z \mathrm{d}(z_i/z_0) = \delta_z(\mathrm{d}z_i/z_0 - z_i/z_0^2 \mathrm{d}z_0) = z_i/z_0 - z_0 z_i/z_0^2 = 0$ we have $\delta_z f = 0$. Conversely, a straight-forward calculation shows that if $f = \sum f_I dz_I$ is any $k$-homogeneous $(p,0)$-form then

$$f = z_0^p \sum_{0 \notin I} f_I \mathrm{d}(z_{I_1}/z_0) \wedge \cdots \wedge \mathrm{d}(z_{I_p}/z_0) + \frac{(-1)^{p-1}}{z_0} (\delta_z f) \wedge \mathrm{d}z_0.$$

So if $\delta_z f = 0$ then clearly $f$ is the pullback of a $(p,0)$-form which has to have values in $L^k$ since $f$ is $k$-homogeneous. In what follows we will identify the space of $X \otimes L^k$ valued $(p,0)$-forms on some subset of $\mathbb{CP}^n$ with the space of $k$-homogeneous $X$-valued $\delta_z$-closed $(p,0)$-forms on the cone over this subset in $\mathbb{C}^{n+1}$. Also if we are in e.g. $U = \mathbb{CP}^n \setminus \{z_0 = 0\}$ we will identify sections of $L^k$ with functions via the natural trivialization of $L^k$ over $U$ given by putting $z_0 = 1$ in the $k$-homogeneous polynomials representing $L^k$.

We let $\delta_b$ denote interior multiplication with $\sum_0^n b_j \frac{\partial}{\partial z_j}$. This operator commutes with $\delta_z$ so it maps $\delta_z$-closed $X$-valued forms to $\delta_z$-closed $X$-valued forms. However, $\delta_b$ reduces the homogeneity one step and therefore $\delta_b$ maps $k$-homogeneous $k$-forms to $k-1$-homogeneous $k-1$-forms. Moreover, $b$ is commuting so we have $\delta_b \circ \delta_b = 0$, and we get the complex

$$K_\bullet(\delta_b, X \otimes L^\bullet \otimes \Lambda^{\bullet,0} T^* \mathbb{CP}^n_{[z]}). \tag{6.1}$$

The operator $\delta_b$ depends on the choice of representative for $[b]$ but nevertheless we have the following proposition.

PROPOSITION 6.1. *Let* $[b]$ *be a projective operator and* $b$ *any representative. Then* $[z] \notin \sigma[b]$ *if and only if the complex* (6.1) *is exact.*

*Proof.* We may assume that $[z] = [1,0,\dots,0]$. We first claim that $[1,0,\dots,0] \notin \sigma[b]$ if and only if $0 \notin \sigma(b_1,\dots,b_n)$. Actually, if $0 \notin \sigma(b_1,\dots,b_n)$, that is $(b_1,\dots,b_n)$ is nonsingular, then $(z_0 - b_0, b_1,\dots,b_n)$ is nonsingular for all $z_0 \in \mathbb{C}$, see [8]. Hence $(z_0,0,\dots,0) \notin \sigma(b_0,\dots,b_n)$ for all $z_0 \in \mathbb{C}$, which means that $[1,0,\dots,0] \notin \sigma[b]$. On the other hand, if $[1,0,\dots,0] \notin \sigma[b]$ then $(z_0,0,\dots,0) \notin \sigma(b_0,\dots,b_n)$ for all $z_0 \in \mathbb{C}$. From the projection property for the Taylor spectrum, [8], we conclude that $0 \notin \sigma(b_1,\dots,b_n)$.

To finish the proof we show that $0 \notin \sigma(b_1,\dots,b_n)$ if and only if the complex (6.1) is exact at $[z] = [1,0,\dots,0]$. Note that for any $f \in X \otimes L^k \otimes \Lambda^{k,0}T^*\mathbb{CP}^n_{[1,0,\dots,0]}$ we have $\delta_{[1,0,\dots,0]}f = z_0\frac{\partial}{\partial z_0}f = 0$ so $f$ does not contain any $\mathrm{d}z_0$. Hence $\delta_b$ acts just as interior multiplication with $\sum_1^n b_j\frac{\partial}{\partial z_j}$, which we denote by $\delta_{b'}$, and we can identify the complex (6.1) with the complex

$$0 \longleftarrow \Lambda^0 X \xleftarrow{\delta_{b'}} \Lambda^1 X \xleftarrow{\delta_{b'}} \cdots \xleftarrow{\delta_{b'}} \Lambda^n X \longleftarrow 0.$$

However, by definition, this complex is exact precisely when $0 \notin \sigma(b_1,\dots,b_n)$, and we are done. ∎

Assume $[1,0,\dots,0] \in \sigma[b]^*$ and let $(\zeta_1,\dots,\zeta_n) = (z_1/z_0,\dots,z_n/z_0)$ be local coordinates around $[1,0,\dots,0]$. In these local coordinates $\delta_b$ is interior multiplication with

$$b_0\sum_1^n(b_0^{-1}b_j - \zeta_j)\frac{\partial}{\partial\zeta_j}$$

if we work in the natural trivialization of $L^k$ around $[1,0,\dots,0]$. We abbreviate this operator $b_0\delta_{b_0^{-1}b-\zeta}$.

PROPOSITION 6.2. *Let $[b]$ be a projective operator with $\sigma[b]^*$ nonempty and let $U$ be a neighborhood of $\sigma[b]$ which does not intersect a hyperplane. Then for any $q$ the following complex is exact:*

$$K_\bullet(\delta_b, \mathcal{E}_{\bullet,q}(U\setminus\sigma[b], X\otimes L^\bullet)).$$

*Proof.* We may assume that $U$ does not intersect the hyperplane $[1,0,\dots,0]$. We know that pointwise for $[z] \in U\setminus\sigma[b]$ the complex (6.1) is exact. In the local coordinates $(\zeta_1,\dots,\zeta_n) = (z_1/z_0,\dots,z_n/z_0)$ this means that the complex $K_\bullet(b_0\delta_{b_0^{-1}b-\zeta}, X\otimes\Lambda^{\bullet,0}T^*\mathbb{C}^n)$ is exact for $\zeta \in U\setminus\sigma[b]$. From the theory of parameterized complexes it follows that

$$K_\bullet(b_0\delta_{b_0^{-1}b-\zeta}, \mathcal{E}_{\bullet,0}(U\setminus\sigma[b], X))$$

is exact, see e.g. [11]. But this is the statement in the proposition (in local coordinates) for $q = 0$. Taking exterior products with barred differentials does not affect exactness since $\delta_b$ commutes with this operation. Hence the statement is true for any $q$. ∎

We now construct the integral representation of the functional calculus. Let $f \in \mathscr{O}(U)$ where $U$ is a neighborhood of $\sigma[b]$ that avoids a hyperplane. Let $x$ be the function which is identically $x$ in $U \setminus \sigma[b]$. From Proposition 6.2 we see that there is a form $\omega_b^1 x \in \mathcal{E}_{1,0}(U \setminus \sigma[b], X \otimes L^1)$ such that $x = \delta_b \omega_b^1 x$. Now $\delta_b$ and $\overline{\partial}$ anti-commute and so $\delta_b \overline{\partial} \omega_b^1 x = -\overline{\partial} \delta_b\, \omega_b^1 x = -\overline{\partial} x = 0$. Hence, by Proposition 6.2 there is a form $\omega_b^2 \in \mathcal{E}_{2,1}(U \setminus \sigma[b], X \otimes L^2)$ such that $\overline{\partial} \omega_b^1 x = \delta_b \omega_b^2 x$. Continuing in this way and successively solving the equations $\overline{\partial} \omega_b^j x = \delta_b \omega_b^{j+1} x$ we finally arrive at a form $\omega_b^n x \in \mathcal{E}_{n,n-1}(U \setminus \sigma[b], X \otimes L^n)$. This form is $\overline{\partial}$-closed because, as above $\delta_b \overline{\partial} \omega_b^n x = 0$ and since $\delta_b$ is injective on this level we must have $\overline{\partial} \omega_b^n x = 0$. If we start with another solution $x = \delta_b \widetilde{\omega}_b^1 x$ and solve the equations $\overline{\partial} \widetilde{\omega}_b^j x = \delta_b \widetilde{\omega}_b^{j+1} x$ then $\omega_b^n x$ and $\widetilde{\omega}_b^n x$ define the same $\overline{\partial}$-cohomology class. In fact, since $\delta_b(\omega_b^2 x - \widetilde{\omega}_b^2 x) = \overline{\partial}(\omega_b^1 x - \widetilde{\omega}_b^1 x)$ and $\delta_b(\omega_b^1 x - \widetilde{\omega}_b^1 x) = 0$ we get from Proposition 6.2 that $\delta_b(\omega_b^2 x - \widetilde{\omega}_b^2 x) = \overline{\partial} \delta_b w^1 = -\delta_b \overline{\partial}\, w^1$, that is $\delta_b(\omega_b^2 x - \widetilde{\omega}_b^2 x + \overline{\partial} w^1) = 0$, for some $w^1$. Inductively we obtain $\delta_b(\omega_b^n x - \widetilde{\omega}_b^n x + \overline{\partial} w^{n-1}) = 0$ and since $\delta_b$ is injective on that level we get $\omega_b^n x - \widetilde{\omega}_b^n x + \overline{\partial} w^{n-1} = 0$. Hence we get a well defined mapping (depending on the representative $b$)

$$x \mapsto [\omega_b^n x]_{\overline{\partial}}.$$

From the construction it is clear that this map is linear in $x$.

PROPOSITION 6.3. *Let $b$ be a projective operator and assume that $[\lambda] \in \sigma[b]^*$. Then the $\overline{\partial}$-cohomology class of $\langle b, \lambda\rangle^n \langle z, \lambda\rangle^{-n} \omega_b^n x$ does not depend on the representative for $[b]$.*

*Proof.* Clearly $\langle z, \lambda\rangle \langle b, \lambda\rangle^{-1} \delta_b$ does not depend on the representative. Let $\widetilde{\omega}^j$, $j = 1, \ldots, n$ be solutions to the equations $x = \langle z, \lambda\rangle \langle b, \lambda\rangle^{-1} \delta_b\, \widetilde{\omega}^1$, $\overline{\partial} \widetilde{\omega}^j = \langle z, \lambda\rangle \langle b, \lambda\rangle^{-1} \delta_b\, \widetilde{\omega}^{j+1}$ in $U \setminus \sigma[b]$. Then $\widetilde{\omega}^j$ can not depend on the representative. Moreover $\omega^j := \langle z, \lambda\rangle^j \langle b, \lambda\rangle^{-j} \widetilde{\omega}^j$, $j = 1, \ldots, n$ must satisfy the equations $x = \delta_b \omega^1$, $\overline{\partial} \omega^j = \delta_b \omega^{j+1}$ in $U \setminus \sigma[b]$. Hence we get that $\langle b, \lambda\rangle^n \langle z, \lambda\rangle^{-n}\, \omega_b^n x$ defines the same $\overline{\partial}$-cohomology class as $\widetilde{\omega}^n$ and we are done. ∎

THEOREM 6.4. *Let $[b]$ be a projective operator with $[\lambda] \in \sigma[b]^*$. Assume that $f \in \mathscr{O}(\sigma[b])$ and let $D$ be a neighborhood of $\sigma[b]$ such that its closure is contained in an open set, which avoids some hyperplane and in which $f$ is holomorphic. Then*

$$f([b])x = \int_{\partial D} f \frac{\langle b, \lambda\rangle^n}{\langle z, \lambda\rangle^n} \omega_b^n x.$$

*Proof.* After a projective transformation we can assume that $[\lambda] = [1, 0, \ldots, 0]$ and since the $\overline{\partial}$-cohomology class of $\langle b, \lambda\rangle^n \langle z, \lambda\rangle^{-n} \omega_b^n x$ does not depend on the representative we may assume that $b$ is the representative such that $e = \langle b, \lambda\rangle = b_0$ given by Proposition 3.3. We recapitulate the definition of $f([b])$. Let $\widetilde{f}$ be the canonical lift of $f$ to $\mathbb{C}^{n+1}$. Then $f([b])x = \widetilde{f}(b)x$. Let $p$ denote the mapping

$V = \{z \in \mathbb{C}^{n+1} : z_0 \neq 0\} \to \mathbb{C}^n$ given by $(z_0, \ldots, z_n) \mapsto (z_1/z_0, \ldots, z_n/z_0)$ and let $\phi$ be the local chart $(\zeta_1, \ldots, \zeta_n) \mapsto [1, \zeta_1, \ldots, \zeta_n]$. Then

$$\begin{tikzcd} V \arrow[d, "p"] \arrow[dr, "\widetilde{f}"] & \\ \mathbb{C}^n \arrow[r, "\phi^* f"'] & \mathbb{C} \end{tikzcd}$$

must commute. From the composition rule in Taylor's functional calculus we get that $\widetilde{f}(b) = \phi^* f(b_1, \ldots, b_n)$. We will show that

$$\int_{\partial D} f \omega_b^n x = \phi^* f(b_1, \ldots, b_n) x.$$

In the local chart $\phi$ and in the natural trivialization over it, $\delta_b$ is the operator $\delta_{b'-\zeta}$ where $b' = (b_1, \ldots, b_n)$ because of our choice of $b$. So our solutions $\omega_b^j x$ to the $\delta_b$-equations must satisfy

$$\begin{aligned} x &= \delta_{b'-\zeta} \phi^*(\omega_b^1 x) \\ \bar{\partial} \phi^*(\omega_b^1 x) &= \delta_{b'-\zeta} \phi^*(\omega_b^2 x) \\ &\vdots \\ \bar{\partial} \phi^*(\omega_b^{n-1} x) &= \delta_{b'-\zeta} \phi^*(\omega_b^n x) \end{aligned}$$

in $\phi^{-1}(U \setminus \sigma[b])$. But from the Spectral Mapping Theorem $\phi^{-1}(U \setminus \sigma[b]) = \phi^{-1}(U) \setminus \sigma(b')$. Hence $[\phi^*(\omega_b^n x)]_{\bar{\partial}}$ must be the same $\bar{\partial}$-cohomology class as the resolvent class Andersson defines in [1] corresponding to $b'$. Moreover, it is shown in [1] that integrating against this resolvent realizes the functional calculus. Thus we obtain

$$\phi^* f(b_1, \ldots, b_n) x = \int \phi^* f \phi^*(\omega_b^n x) = \int \phi^*(f \omega_b^n x) = \int f \omega_b^n x. \quad \blacksquare$$

We have seen that the resolvent, that is the $\bar{\partial}$-cohomology class determined by $\langle b, \lambda \rangle^n \langle z, \lambda \rangle^{-n} \omega_b^n x$, does not depend on the representative for $[b]$ and that the functional calculus is realized by integrating against it. Actually, the resolvent is even independent of the choice of $[\lambda] \in \sigma[b]^*$ in the following sense.

THEOREM 6.5. *Let $[b]$ be a projective operator and assume that $[\lambda]$, $[\widetilde{\lambda}] \in \sigma[b]^*$. Let $U$ be a pseudoconvex neighborhood of $\sigma[b]$ such that none of the hyperplanes $[\lambda]$ and $[\widetilde{\lambda}]$ intersect $U$. Then $\langle b, \lambda \rangle^n \langle z, \lambda \rangle^{-n}\, \omega_b^n x$ and $\langle b, \widetilde{\lambda} \rangle^n \langle z, \widetilde{\lambda} \rangle^{-n}\, \omega_b^n x$ are $\bar{\partial}$-cohomologous in $U \setminus \sigma[b]$.*

In order to prove Theorem 6.5 we have to look more closely at the relation between the homological construction of the functional calculus and the integral construction. We recapitulate the homological construction. Let $c = (c_1, \ldots, c_n)$

be a commuting tuple of bounded operators on $X$. We let $\mathcal{E}_{p,q}(U,X)$ denote the set of smooth $X$-valued $(p,q)$-forms in $U \subseteq \mathbb{C}^n$ and we put

$$\mathscr{L}^k(U,X) = \bigoplus_{q-p=k} \mathcal{E}_{p,q}(U,X).$$

The operator $\nabla_{z-c} = \delta_{z-c} - \overline{\partial}$ is an anti-derivative on $\bigoplus_k \mathscr{L}^k(U,X)$ and maps $\mathscr{L}^k(U,X)$ to $\mathscr{L}^{k+1}(U,X)$. Moreover $\nabla_{z-c} \circ \nabla_{z-c} = 0$ and we get the complex $\mathrm{Tot}\mathscr{L}(U,X)$:

$$\cdots \xrightarrow{\nabla_{z-c}} \mathscr{L}^{k-1}(U,X) \xrightarrow{\nabla_{z-c}} \mathscr{L}^k(U,X) \xrightarrow{\nabla_{z-c}} \mathscr{L}^{k+1}(U,X) \xrightarrow{\nabla_{z-c}} \cdots.$$

This complex is exact if $U$ is disjoint with $\sigma(c)$ since the Koszul complex is exact outside of $\sigma(c)$. The crucial part of the homological construction of the functional calculus for $c$ is to show that for any neighborhood $U$ of $\sigma(c)$ we have that $X$ and $H^0(\mathrm{Tot}\mathscr{L}(U,X))$ are isomorphic as $\mathscr{O}(\mathbb{C}^n)$-modules. Since $H^0(\mathrm{Tot}\mathscr{L}(U,X))$ has a natural $\mathscr{O}(U)$-module structure, which extends the $\mathscr{O}(\mathbb{C}^n)$-module structure, the isomorphism yields an $\mathscr{O}(U)$-module structure on $X$ extending the $\mathscr{O}(\mathbb{C}^n)$-module structure. Furthermore one shows that if $U' \subseteq U$ are neighborhoods of $\sigma(c)$ then the $\mathscr{O}(U')$-module structure on $X$ extends the $\mathscr{O}(U)$-module structure. Hence we get a $\mathscr{O}(\sigma(c))$-module structure on $X$ and this is our functional calculus. Given a function $f \in \mathscr{O}(U)$ ($U$ a neighborhood of $\sigma(c)$) the $X$-valued function $z \mapsto xf(z)$ determines an element in $H^0(\mathrm{Tot}\mathscr{L}(U,X))$ and the isomorphism maps this element to $f(c)x$ by definition. This construction is due to Taylor; see [8] and [9].

The integral construction of $f(c)x$ is first to solve the equation $\nabla_{z-c}\omega_{z-c}x = x$ in $U \setminus \sigma(c)$, then identifying the component, $\omega^n_{z-c}x$, of $\omega_{z-c}x$ of bidegree $(n, n-1)$, and put

$$f(c)x = \int_{\partial D} f(z)\, \omega^n_{z-c}x.$$

Note that for bidegree reasons, solving $\nabla_{z-c}\omega_{z-c}x = x$ is exactly the same as solving the equations $x = \delta_{z-c}\,\omega^1_{z-c}x$, $\overline{\partial}\,\omega^k_{z-c}x = \delta_{z-c}\,\omega^{k+1}_{z-c}x$, $k = 1, \dots, n-1$. In [1] Andersson shows that the two definitions of $f(c)x$ coincide. The crucial step in proving Theorem 6.5 is the following lemma.

LEMMA 6.6. *Let $c = (c_1, \dots, c_n)$ be bounded commuting operators on $X$ and let $U$ be a pseudoconvex neighborhood of $\sigma(c)$. If $f \in \mathscr{O}(U)$ and $f(c) = 0$ then $[f(z)\,\omega^n_{z-c}x]_{\overline{\partial}} = 0$, where $\omega^n_{z-c}x$ is the component of bidegree $(n, n-1)$ of a solution $\omega_{z-c}x$ to $\nabla_{z-c}\,\omega_{z-c}x = x$ in $U \setminus \sigma(c)$.*

*Proof.* Clearly we have $\nabla_{z-c}\, f(z)\omega_{z-c}x = f(z)x$ in $U \setminus \sigma(c)$. From the homological construction we see that $xf(z)$ must be $\nabla_{z-c}$-exact in $U$ since $f(c)x = 0$. Hence, $xf(z) = \nabla_{z-c}\, u(z)$ for some $u \in \mathscr{L}^{-1}(U,X)$. Thus, $u - f(z)\omega_{z-c}x$ is $\nabla_{z-c}$-closed in $U \setminus \sigma(c)$. Since $\mathrm{Tot}\mathscr{L}(U \setminus \sigma(c), X)$ is exact there is a $v \in \mathscr{L}^{-2}(U \setminus \sigma(c), X)$

such that $u(z) - f(z)\omega_{z-c}x = \nabla_{z-c}\, v(z)$ in $U \setminus \sigma(c)$. Identifying terms of bidegree $(n, n-1)$ we see that

$$u_{n,n-1} - f(z)\omega^n_{z-c}x = \bar{\partial} v_{n,n-2} \tag{6.2}$$

in $U \setminus \sigma(c)$. Moreover, $\nabla_{z-c}\, u = xf(z)$ so for bidegree reasons $\bar{\partial}\, u_{n,n-1} = 0$. Since $U$ is pseudoconvex $u_{n,n-1}$ is actually $\bar{\partial}$-exact and letting $u_{n,n-1} = \bar{\partial}\, \widetilde{v}_{n,n-2}$ we get from (6.2) that

$$f(z)\omega^n_{z-c}x = \bar{\partial}(\widetilde{v}_{n,n-2} - v_{n,n-2})$$

in $U \setminus \sigma(c)$ which is what we wanted to show. ∎

We proceed and prove Theorem 6.5.

*Proof of Theorem* 6.5. By Theorem 6.4 we know that both of the forms $\langle b, \lambda\rangle^n \langle z, \lambda\rangle^{-n}\omega^n_b x$ and $\langle b, \widetilde{\lambda}\rangle^n \langle z, \widetilde{\lambda}\rangle^{-n}\omega^n_b x$ represent the functional calculus. We have to show that they are $\bar{\partial}$-cohomologous in $U \setminus \sigma[b]$. We let $\rho$ be a projection from $[\lambda]$. From the proof of Theorem 6.4 we see that $\rho_*(\langle b, \lambda\rangle^n \langle z, \lambda\rangle^{-n}\omega^n_b x)$ defines the resolvent class $\omega_{\zeta-\rho([b])}$ corresponding to $\rho([b])$ if we choose $b \in [b]$ such that $\langle b, \lambda\rangle = e$. Hence, in the local coordinates $\zeta = \rho([z])$ the difference between the two forms has to be on the form

$$(1 - f(\zeta))\omega_{\zeta-\rho([b])}$$

where $f$ is holomorphic in $\rho(U)$. Now since both of the forms realize the functional calculus we must have $1(\rho([b])) - f(\rho([b])) = 0$. Hence from Lemma 6.6 we see that in the local coordinates, the two forms has to be $\bar{\partial}$-cohomologuos in $\rho(U) \setminus \sigma(\rho([b]))$. ∎

The function $f(\zeta)$ is the function $\langle b, \widetilde{\lambda}\rangle^n \langle z, \widetilde{\lambda}\rangle^{-n}$ in the local coordinates $\zeta$. Hence, we see that making a change of variables by a rational fractional transform of $\mathbb{C}^n$, computing the resolvent in the new coordinates and pulling it back, we get $\langle b, \widetilde{\lambda}\rangle^n \langle z(\zeta), \widetilde{\lambda}\rangle^{-n}$ times the resolvent we get if we compute it directly. Theorem 6.5 implies that the two forms are $\bar{\partial}$-cohomologous in suitable domains.

*Acknowledgements.* I am grateful to Mats Andersson who came up with the original idea of this paper and with whom I have had many rewarding discussions. I am also grateful to Lyudmila Turowska for many insightful comments and careful reading, and to Florian-Horia Vasilescu for valuable discussions on topics in this paper and on general questions about tuples of unbounded operators.

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Received December 12, 2004.