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REGULAR NORMED BIMODULES

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ABSTRACT. In this article, we will give a characterization of Banach bimodules over C^*-algebras of compact operators that arise from operator spaces as well as a characterization of (F)-Banach bundles amongst all (H)-Banach bundles over a hyper-Stonian space. These two characterizations are concerned with whether a certain natural map from a Banach bimodule to its canonical bidual is isometric (we call such bimodule *regular*).

KEYWORDS: *Banach bimodules, operator spaces, Banach bundles.*

MSC (2000): Primary 46H25, 46L07.

INTRODUCTION

The aim of this paper is to study duality theory for essential normed bimodules. Given a pre-C^*-algebra A and an essential normed A-bimodule X, we would like to have a canonical definition of the dual object X^s of X which satisfies the following properties:

(i) X^s is also an essential normed A-bimodule (i.e. the dual object is in the same category);

(ii) X^s depends only on X and A;

(iii) when $A = \mathcal{K}(l^2)$ and X is defined by an operator space, X^s is defined by the corresponding dual operator space;

(iv) when A is commutative, X^s is the essential part of $\mathcal{L}_A(X, A)$ (i.e. the duality agrees with the usual one for commutative algebras).

Let's forget about the norm structure for the moment and consider a bimodule M over a unital algebra R. The natural "dual object" $L_R(M, R)$ fails to be a R-bimodule if R is not commutative. An easy way to rectify this situation is to "add another copy of R" and consider $L_R(M, \mathfrak{R})$ where $\mathfrak{R}$ is the algebraic tensor product $R \odot R$ together with the R-bimodule structure: $a \cdot (b \otimes c) \cdot d = abd \otimes c$. Therefore, $L_R(M, \mathfrak{R})$ becomes a R-bimodule (given by the bimodule structure on the second variable of $R \odot R$). However, when R is commutative, $L_R(M, \underline{R} \odot R) \neq$

$L_R(M,R)$ unless R is the scalar field. A natural way to correct this is to replace $R \odot R$ with $R \odot_Z R$ (where Z is the center of R).

We employ this simple idea in Section 1 to define the "regular dual object", X^s, of an essential normed A-bimodule X (for technical reasons, we will assume that A has a contractive approximate identity and $A^2 = A$). There is a canonical contraction $\kappa_X : X \to X^{ss}$ (the dual of X^s). In general, κ_X is not an isometry and X is called *regular* if κ_X happens to be an isometry. It is easy to see that X^s is always regular and so, $\kappa_X(X)$ is called the *regularization* of X. Regular bimodules are thought to be nice objects because of the results in Sections 2 and 3. It is natural to ask whether one can give a canonical characterisation of regularity without explicitly involving the duality. We will show that every regular bimodule will satisfy certain properties (known as *pseudo-regularity*) which do not involve the dual object explicitly.

In Section 2, we will consider the situation when $A = \bigoplus_{\lambda\in\Lambda} \mathcal{K}(H_\lambda)$ where Λ is any index set and H_λ is a Hilbert space. We will show that regularity and pseudo-regularity coincide in this case and will characterize regular Banach A-bimodules.

In Section 3, we will consider the situation when A is a commutative von Neumann algebra. In this case, pseudo-regular bimodules correspond to (H)-Banach bundles and we will show that regular bimodules correspond to (F)-Banach bundles. Hence, the regularization process gives us a canonical way to obtain an (F)-Banach bundle from any given Banach bundle (in particular, from an (H)-Banach bundle).

1. DUALITY AND REGULARITY OF BANACH BIMODULES

Throughout this section, A is a pre-C^*-algebra containing a contractive approximate identity $\{f_i\}_{i\in I}$ such that $A \cdot A = A$ (i.e. any element of A is a finite sum of elements of the form ab where $a,b \in A$). We denote by $\overline{A}$ the completion of A and recall that an A-bimodule X is *(algebraically) essential* if $X = A \cdot X \cdot A$ (where $A \cdot X \cdot A$ is the linear span of elements of the form $a \cdot x \cdot b$ with $a,b \in A$ and $x \in X$). For any A-module X, we denote by X_E the *essential part* $A \cdot X \cdot A$ of X.

Let B be a C^*-subalgebra of $M(\overline{A})$ (the multiplier algebra of $\overline{A}$) and $Z_A \cong C(\Omega)$ be the center of $M(\overline{A})$ (where Ω is a compact Hausdorff space). By [1], there exists a C^*-semi-norm $\|\cdot\|_m$ on the algebraic Z_A-tensor product $\overline{A} \odot_{Z_A} B$ which is minimum in some sense (see 2.8 of [1]). As in [1], we denote by $\overline{A} \overset{m}{\otimes}_{Z_A} B$ the Hausdorff completion of $\overline{A} \odot_{Z_A} B$ under $\|\cdot\|_m$.

From now on, we will denote by $a \circledast_A 1$ and $a \circledast_A b$ (or simply $a \circledast 1$ and $a \circledast b$) the canonical images of $a \in A$ and $a \otimes_{Z_A} b \in (A \odot_{Z_A} B)$ in $M(\overline{A} \overset{m}{\otimes}_{Z_A} B)$ and

$\overline{A}\overset{m}{\otimes}_{Z_A} B$ respectively. Note that the map that sends $a \in A$ to $a \circledast 1 \in M(\overline{A}\overset{m}{\otimes}_{Z_A} B)$ is a $*$-homomorphism.

LEMMA 1.1. *$\overline{A}\overset{m}{\otimes}_{Z_A} B$ is a normed A-bimodule under the multiplication $c \cdot (a \circledast b) \cdot d = cad \circledast b$ and $\{f_i\}_{i\in I}$ is an approximate identity in A for the A-bimodule $\overline{A}\overset{m}{\otimes}_{Z_A} B$. Similarly, if $\{g_j\}$ is a bounded approximate identity of B, then both $(1 \circledast g_j)\alpha$ and $\alpha(1 \circledast g_j)$ converge to α for any $\alpha \in \overline{A}\overset{m}{\otimes}_{Z_A} B$.*

The above lemma may be used implicitly throughout this article. In the following we denote by $\mathfrak{A}$ the normed A-bimodule $\overline{A}\overset{m}{\otimes}_{Z_A} \overline{A}$ (the A-bimodule structure is as given in Lemma 1.1).

We will now construct the dual bimodule of an essential normed A-bimodule. We have already stated in the introduction what is expected for dual bimodules. For any essential normed A-bimodule X, we denote by $\mathcal{L}_A(X, \mathfrak{A})$ the space of all continuous A-bimodule maps from X to $\mathfrak{A}$. Note that $\mathcal{L}_A(X, \mathfrak{A}) = \mathcal{L}_A(X, A \cdot \mathfrak{A} \cdot A)$ is a A-bimodule with multiplications given by $(a \cdot T \cdot b)(x) = (1 \circledast a)T(x)(1 \circledast b)$ ($T \in \mathcal{L}_A(X, \mathfrak{A})$; $a, b \in A$; $x \in X$). We set

$$X^s := \mathcal{L}_A(X, \mathfrak{A})_E$$

(the essential part of $\mathcal{L}_A(X, \mathfrak{A})$) and call it the *regular dual* of X.

It is clear that X^s is an essential normed A-bimodule (with the canonical norm on $\mathcal{L}_A(X, \mathfrak{A})$) and there is a contraction $\kappa_X : X \to X^{ss}$ given by $\kappa_X(x)(\varphi) = \varphi(x)^{(12)}$ (where (12) is the flip of the two variables in $\mathfrak{A}$ — it is not hard to see from the definition of $\overset{m}{\otimes}_{Z_A}$ that such flip map exists).

DEFINITION 1.2. Let X be an essential normed A-bimodule. An element $x \in X$ is said to be *regular in* X if for any $\varepsilon > 0$, there exists $T \in X^s$ such that $\|T\| \leqslant 1$ and $\|x\| \leqslant \|T(x)\| + \varepsilon$. We say that X is *regular* if κ_X is an isometry. Moreover, the closure of $\kappa_X(X)$ in X^{ss} together with the induced norm is called the *regularization* of X and is denoted by X_{reg}.

In [12], a type of dual bimodule, $X^\dagger$, was introduced. However, $X^\dagger$ is in general not a Banach A-bimodule. Furthermore, the regularity defined using $X^\dagger$ (i.e. $X \to (X^\dagger)^\dagger$ being isometric) is in general strictly weaker than ours. In particular, it will not give the relation between (H)-Banach bundles and (F)-Banach bundles as obtained using our notion of regularization (see Section 3 below).

In [8], yet another type of dual bimodule $X^*_{\mathcal{D}}$ was introduced. If $A \subseteq \mathcal{L}(H)$ such that A'' is standard in $\mathcal{L}(H)$, then $X^*_{\mathcal{L}(H)}$ is very similar to $X^\dagger$ (except that Hom in [12] are bounded maps while Hom in [8] are completely bounded). If $A = \mathcal{K}(H)$ or if A is commutative, then $X^*_{\mathfrak{A}} = X^s$ (in both cases, elements in X^s are automatically completely bounded) but it seems unlikely that one can use any result in [8] to shorten the proofs in this paper.

REMARK 1.3. Let X be an essential normed A-bimodule and $\overline{X}$ be its completion.

(i) If A is a C^*-algebra, then X^s is closed in $\mathcal{L}_A(X, \mathfrak{A})$ (because of the the Cohen factorization theorem).

(ii) $\overline{X}$ is an essential Banach $\overline{A}$-bimodule and X^s is dense in $\overline{X}^s$ (since $\mathcal{L}_A(X, \mathfrak{A}) = \mathcal{L}_{\overline{A}}(\overline{X}, \mathfrak{A})$). Thus, X is regular if and only if $\overline{X}$ is regular.

(iii) X is regular if and only if for any $x \in X$ and any $\varepsilon > 0$, there exists $T \in \mathcal{L}_A(X, \mathfrak{A}) = \mathcal{L}_A(X, A \cdot \mathfrak{A} \cdot A)$ such that $\|T\| \leqslant 1$ and $\|x\| < \|T(x)\| + \varepsilon$ (note that $\|f_i \cdot T(x) \cdot f_i - T(x)\| < \varepsilon/2$ for large enough i because of Lemma 1.1). Consequently, an essential submodule of a regular bimodule is again regular.

(iv) Since $(\kappa_X)^s \circ \kappa_{X^s} = \mathrm{id}_{X^s}$ and both κ_{X^s} and $(\kappa_X)^s : (X^{ss})^s \to X^s$ are contractions, κ_{X^s} is always an isometry. Therefore, X^s is regular and so is X_{reg} (by part (ii)). Moreover, if Y is a regular normed A-bimodule and $T \in \mathcal{L}_A(X, Y)$, there exists $T_{\mathrm{reg}} \in \mathcal{L}_{\overline{A}}(X_{\mathrm{reg}}, Y)$ such that $T = T_{\mathrm{reg}} \circ \kappa_X$.

It is natural to ask if one can characterise regularity without finding the regular dual. In some cases, this can be done using the notion of pseudo-regularity as defined in the following (although this is not the case in general; see e.g. Section 3).

DEFINITION 1.4. (i) A semi-norm p on a A-bimodule X is said to be *absolutely A-convex* if for any $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ and any $x_1, \ldots, x_n \in X$,

$$p\Big(\sum_{i=1}^{n} a_i x_i b_i\Big) \;\leqslant\; \sqrt{\Big\|\sum_{i=1}^{n} a_i a_i^*\Big\|}\max_{i=1,\ldots,n} p(x_i)\sqrt{\Big\|\sum_{i=1}^{n} b_i^* b_i\Big\|}.$$

Moreover, an essential normed A-bimodule is said to be *absolutely A-convex* if its norm is absolutely A-convex.

(ii) An essential normed A-bimodule X is said to be *pseudo-regular* if it is absolutely A-convex and its completion $\overline{X}$ (which is a unital $M(\overline{A})$-bimodule in the canonical way) is a commutative Z_A-bimodule.

If A is a C^*-algebra and X is an essential Banach A-bimodule, then Proposition 2.2 of [12] tells us that one only needs to consider $n = 2$ in the definition of absolute A-convexity.

EXAMPLE 1.5. (i) If $A = \bigoplus_{i\in I}^{c_0} \mathcal{K}(H_i)$ (c_0-direct sum), then we have $\mathfrak{A} = \bigoplus_{i\in I}^{c_0} \mathcal{K}(H_i) \otimes \mathcal{K}(H_i)$.

(ii) If $A = M_\infty$ (the space of all infinite matrices with finite numbers of non-zero entries, considered as a subspace of $\mathcal{K}(l^2)$), then $\mathfrak{A} = \mathcal{K}(l^2) \otimes \mathcal{K}(l^2)$.

(iii) If $A = C_0(\Omega)$ for some locally compact space Ω, then $\mathfrak{A} = C_0(\Omega)$.

(iv) Let H be an infinite dimensional Hilbert space, $\mathbf{W}$ be an operator space, $X := \mathbf{W} \check{\otimes} \mathcal{K}(H)$ (spatial tensor product) and $X^{\#} := \mathcal{L}_{\mathcal{K}(H)}(X; \mathcal{K}(H) \check{\otimes} \mathcal{K}(H))$. It is clear that for any $T \in \mathrm{CB}(\mathbf{W};\mathcal{K}(H))$, we have $T \otimes \mathrm{id}_{\mathcal{K}(H)} \in X^{\#}$. Conversely, any $\varphi \in X^{\#}$ restricts to a map

$$\varphi_0 \ \in \ \mathcal{L}_{\mathcal{K}(l^2)}(\mathbf{W}\check{\otimes}\mathcal{K}(l^2);\mathcal{K}(H)\check{\otimes}\mathcal{K}(l^2)) \ \cong \ \mathrm{CB}(\mathbf{W};\mathcal{K}(H))$$

(see e.g. 1.2 of [9]). If $T \in \mathrm{CB}(\mathbf{W};\mathcal{K}(H))$ is the corresponding element of φ_0, then clearly $\varphi = T \otimes \mathrm{id}_{\mathcal{K}(H)}$. Consequently, $X^s \cong \mathrm{CB}(\mathbf{W};\mathcal{K}(H))_E$ (see the first paragraph).

(v) For any Hilbert space H, one can consider $\mathcal{K}(H)^*$ as an essential normed $\mathcal{K}(H)$-bimodule. In this case, $(\mathcal{K}(H)^*)_{\mathrm{reg}} = \mathcal{K}(H)$. In fact, $\mathcal{K}(H)^* \to (\mathcal{K}(H)^*)_{\mathrm{reg}}$ is the identification of $\mathcal{K}(H)^*$ as the set of trace-class operators.

(vi) Suppose that $A = \mathcal{K}(l^2)$ or M_∞ and X is an essential normed A-bimodule. Then a closed subset $D \subseteq X$ is absolutely A-convex if and only if for any disjoint projections $p, q \in A$ and any $a \in A$ with $\|a\| \leqslant 1$, we have $p \cdot D \cdot p + q \cdot D \cdot q \subseteq D$, $a \cdot D \subseteq D$ and $D \cdot a \subseteq D$. In fact, the case of $A = M_\infty$ is more or less the same as 3.2 of [4] and the case of $A = \mathcal{K}(l^2)$ follows from some completion arguments.

PROPOSITION 1.6. *If X is a regular normed A-bimodule, X is pseudo-regular.*

Proof. Let U and V be the closed unit balls of X and X^s respectively. The regularity of X means that $U = \{x \in X : \|\varphi(x)\| \leqslant 1 \text{ for any } \varphi \in V\}$. Therefore, X is absolutely A-convex (note that the norm on $\mathfrak{A}$ is absolutely A-convex). For any $\varphi \in \overline{X}^s$, $z \in Z_A$ and $x = \sum_{k=1}^{n} a_k x_k b_k \in X$ ($a_k, b_k \in A$ and $x_k \in X$), we have

$$\varphi(z \cdot x) \ = \ \sum_{k=1}^{n} (z a_k \circledast 1)\varphi(x_k)(b_k \circledast 1) \ = \ \sum_{k=1}^{n} (a_k \circledast 1)\varphi(x_k)(b_k z \circledast 1) \ = \ \varphi(x \cdot z)$$

and so $z \cdot x = x \cdot z$ (as $\overline{X}^s$ separates points of $\overline{X}$ by Remark 1.3(iii)). Thus, X is pseudo-regular because the multiplications are continuous. ∎

2. THE CASE WHEN $A = \bigoplus_{i \in I} \mathcal{K}(H_i)$

We will first consider the case when $A = \mathcal{K}(H)$ (where H is a Hilbert space). In the following, $\check{\otimes}$ is the spatial tensor product of two operator spaces.

THEOREM 2.1. *Let X be an essential Banach $\mathcal{K}(H)$-bimodule. The following statements are equivalent:*

(i) *X is regular.*

(ii) *X is pseudo-regular.*

(iii) *There exists a complete operator space $\mathbf{W}$ such that $X = \mathbf{W}\check{\otimes}\mathcal{K}(H)$.*

Proof. (i)⇒(ii). This follows from Proposition 1.6(ii).

(ii)⇒(iii). By the theorem in p. 333 of [12], there exists a Hilbert space K, a non-degenerate $*$-representation π of $\mathcal{K}(H)$ on K as well as an isometry $J : X \to \mathcal{L}(K)$ such that $J(axb) = \pi(a)J(x)\pi(b)$ ($a, b \in \mathcal{K}(H); x \in X$). Note that there is a

Hilbert space E such that $K \cong E \otimes H$ as well as $\pi(a) = 1 \otimes a$ and so we can assume $\pi = 1 \otimes \mathrm{id}$ and $J = \mathrm{id}$. Let $\mathbf{W} := \{y \in \mathcal{L}(E) : y \otimes a \in X \text{ for any } a \in \mathcal{K}(H)\}$. Clearly, $\mathbf{W}\check{\otimes}\mathcal{K}(H) \subseteq X$. Suppose $\{\xi_i\}_{i\in I}$ is an orthonormal basis for H. For any $i, j \in I$, we set

$$\theta_{i,j}(\zeta) := \xi_i\langle\xi_j, \zeta\rangle \quad \text{and} \quad \omega_{i,j}(t) := \langle\xi_i, t\xi_j\rangle \quad (\zeta \in H; t \in \mathcal{L}(H)).$$

Then $(\mathrm{id} \otimes \omega_{j,k})(x) \otimes \theta_{i,l} = (1 \otimes \theta_{i,j})x(1 \otimes \theta_{k,l}) \in X$ ($x \in X; i, j, k, l \in I$) and so $(\mathrm{id} \otimes \omega_{j,k})(x) \in \mathbf{W}$. Furthermore, since X is essential, certain finite sums of elements of the form

$$(1 \otimes \theta_{i,i})x(1 \otimes \theta_{j,j}) = (\mathrm{id} \otimes \omega_{i,j})(x) \otimes \theta_{i,j} \in \mathbf{W}\check{\otimes}\mathcal{K}(H)$$

converge to x in norm and so $X = \mathbf{W}\check{\otimes}\mathcal{K}(H)$ as required.

(iii)⇒(i). If $\dim H = n$, then $X \cong M_n(W)$ as normed M_n-bimodules and this implication follows from 2.3.4 of [3] (note that $\mathrm{CB}(\mathbf{W}; M_n) \cong X^s$). On the other hand, if H is infinite dimensional, then $(\mathbf{V}\check{\otimes}\mathcal{K}(H))^s \cong \mathrm{CB}(\mathbf{V}; \mathcal{K}(H))_E$ for any operator space $\mathbf{V}$ (by Example 1.5(iv)). Therefore, we have:

$$\mathbf{W}\check{\otimes}\mathcal{K}(H) \subseteq \mathrm{CB}(\mathbf{W}^*; \mathcal{K}(H))_E = (\mathbf{W}^*\check{\otimes}\mathcal{K}(H))^s = (\mathrm{CB}(\mathbf{W}; \mathcal{K}(H))_E)^s = (\mathbf{W}\check{\otimes}\mathcal{K}(H))^{ss}$$

($\mathbf{W}^*\check{\otimes}\mathcal{K}(H) = \mathrm{CB}(\mathbf{W}; \mathcal{K}(H))_E$ because there is an approximate unit in $\mathcal{K}(H)$ consisting of finite rank projections). It is not hard to check that the above embedding is precisely κ_X and thus X is regular. ∎

REMARK 2.2. (i) The equivalence of Theorem 2.1(ii) and (iii) is probably known (e.g. one can use 2.1 of [6] and Proposition 3.3 of [14] to obtain this in the case of $H = l^2$). However, we decided to give a proof here for clarity and completeness.

(ii) Let $\{\xi_i\}_{i\in I}$ be an othonormal basis for H and A be the linear span of $\{\theta_{i,j} : i, j \in I\}$. One can use the completion consideration in Remark 1.3 to obtain a similar result as the above theorem for A. In fact, there is also an elementary proof for this fact (without using the theorem in [12]) but such a proof is much more lengthy.

Suppose that A is the c_0-direct sum $\bigoplus_{\lambda\in\Lambda}^{c_0} \mathcal{K}(H_\lambda)$ and $d_\lambda \in A$ corresponds to the identity in $\mathcal{L}(H_\lambda)$. Then $Z_A = c_0(\Lambda)$ and $\mathfrak{A} = \bigoplus_{\lambda\in\Lambda}^{c_0} \mathcal{K}(H_\lambda) \otimes \mathcal{K}(H_\lambda)$. Let X be a pseudo-regular Banach A-bimodule. Then it is not hard to see that $X_\lambda := d_\lambda \cdot X$ is a regular Banach $\mathcal{K}(H_\lambda)$-bimodule and X is the c_0-directed sum $\bigoplus_{\lambda\in\Lambda}^{c_0} X_\lambda$. Using this, one can check easily that X is also regular. Thus, we have the following theorem.

THEOREM 2.3. *Let Λ be an index set and H_λ be a Hilbert space for any $\lambda \in \Lambda$. Suppose that $A = \bigoplus_{\lambda\in\Lambda}^{c_0} \mathcal{K}(H_\lambda)$ and X is a pseudo-regular Banach A-bimodule. Then*

X is regular and there exists a family of operator spaces $\{\mathbf{W}_\lambda\}_{\lambda\in\Lambda}$ such that $X = \bigoplus_{\lambda\in\Lambda}^{c_0} \mathbf{W}_\lambda \check{\otimes} \mathcal{K}(H_\lambda)$.

The first two parts of the following corollary follow easily from Theorem 2.1 (or more precisely, Remark 2.2(ii)) and the final part follows from the above theorem.

COROLLARY 2.4. *Let X be an essential normed A-bimodule.*

(i) *Suppose that $A = \bigcup_{n\in\mathbb{N}} M_n$. Then X is regular if and only if there exists an operator space $\mathbf{W}$ such that $X = \bigcup_{n\in\mathbb{N}} M_n(W)$.*

(ii) *Suppose that $A = M_n$. Then X is regular if and only if there exists an operator space $\mathbf{Y}$ such that $X \cong M_n(Y)$ under the norm induced by the operator space structure of $\mathbf{Y}$.*

(iii) *Suppose that $A = c_0$. Then X is regular if and only if there exists a sequence of Banach spaces $\{X_k\}$ such that X is a normed c_0-submodule of $\bigoplus_{\lambda\in\Lambda}^{c_0} X_k$.*

In the remainder of this section, we will give two remarks concerning the case when $A = \mathcal{K}(l^2)$. First of all, Theorem 2.1 allows us to detect some hidden operator space structures. For example, if $\mathbf{V}$ is a complete operator space, then any essential Banach $\mathcal{K}(l^2)$-submodule of $\mathcal{K}(l^2)\check{\otimes}\mathbf{V}$ is of the form $\mathcal{K}(l^2)\check{\otimes}\mathbf{U}$ for some operator subspace $\mathbf{U}$ of $\mathbf{V}$. As for another example, if Y is an essential operator A-bimodule of a C^*-algebra A, one can use the $\mathcal{K}(l^2)$-bimodule approach to show the existence of a canonical operator space structure on the space of double centralizers $M_A(Y)$ that turns it into a unital operator $M(A)$-bimodule in a canonical way (see e.g. p. 310 of [10]).

Secondly, "regularization" is a process that produces a canonical complete operator space from any essential Banach $\mathcal{K}(l^2)$-bimodule. The following corollary shows that it is actually a left adjoint of the forgetful functor from the category of complete operator spaces to the category of essential Banach $\mathcal{K}(l^2)$-bimodules (note that if $\mathbf{W}$ is the operator space such that $X_{\mathrm{reg}} = \mathbf{W}\check{\otimes}\mathcal{K}(l^2)$, then the following corollary shows that $\mathrm{CB}(\mathbf{W},\mathbf{V}) \cong \mathcal{L}_{\mathcal{K}(l^2)}(X, \mathcal{K}(l^2)\check{\otimes}\mathbf{V})$ canonically).

COROLLARY 2.5. *Let X and Y be essential Banach $\mathcal{K}(l^2)$-bimodules. Any $\varphi \in \mathcal{L}_{\mathcal{K}(l^2)}(X,Y)$ induces a map $\varphi_{\mathrm{reg}} \in \mathcal{L}_{\mathcal{K}(l^2)}(X_{\mathrm{reg}}, Y_{\mathrm{reg}})$ such that $\varphi_{\mathrm{reg}} \circ \kappa_X = \kappa_Y \circ \varphi$* and *$\|\varphi_{\mathrm{reg}}\| \leqslant \|\varphi\|$. If, in addition, Y is regular, then $\|\varphi_{\mathrm{reg}}\| = \|\varphi\|$. Consequently, the canonical map, $\widehat{\kappa}_X : \mathcal{L}_{\mathcal{K}(l^2)}(X_{\mathrm{reg}}, \mathcal{K}(l^2)\check{\otimes}\mathbf{V}) \to \mathcal{L}_{\mathcal{K}(l^2)}(X, \mathcal{K}(l^2)\check{\otimes}\mathbf{V})$ is an isometry for any complete operator space $\mathbf{V}$.*

Proof. Consider $\varphi^s : Y^s \to X^s$ given by $\varphi^s(f) = f \circ \varphi$. It is easy to see that $\|\varphi^s\| \leqslant \|\varphi\|$. Hence we have a bounded $\mathcal{K}(l^2)$-bimodule map $\varphi^{ss} : X^{ss} \to Y^{ss}$ such that

$$\varphi^{ss} \circ \kappa_X = \kappa_Y \circ \varphi$$

and $\|\varphi^{ss}\| \leqslant \|\varphi^{s}\| \leqslant \|\varphi\|$. Now, the restriction of φ^{ss} on X_{reg} is the required map φ_{reg}. Finally, if Y is regular, κ_Y is an isometry and so $\|\varphi\| \leqslant \|\varphi_{\mathrm{reg}}\|\|\kappa_X\| \leqslant \|\varphi_{\mathrm{reg}}\|$. ∎

3. THE CASE WHEN A IS A COMMUTATIVE VON NEUMANN ALGEBRAS

Throughout this section, Ω is a compact Hausdorff space and X is an essential Banach $C(\Omega)$-module (i.e. commutative Banach $C(\Omega)$-bimodule). For any $x \in X$, we denote by $X(x)$ the closed $C(\Omega)$-submodule $\overline{C(\Omega) \cdot x}$.

As noted in [12], X is pseudo-regular if and only if it is a $C(\Omega)$-convex module in the sense of [2], p. 40 . Therefore, this is the case if and only if X is the space of continuous sections of an (H)-Banach bundle (see p. 8 of [2] and 2.5 of [2]). Let us first give the following (probably well known) lemma.

LEMMA 3.1. *Let $f \in C(\Omega)_+$ and $h : \Omega \to \mathbb{R}_+$ be an upper semi-continuous function. Then $f \leqslant h$ if and only if $\|gf\| \leqslant \|gh\| := \sup_{\omega\in\Omega} g(\omega)h(\omega)$ for any $g \in C(\Omega)_+$.*

Proof. The necessity is clear. Suppose that there exist $\omega_0 \in \Omega$ and $r \in \mathbb{R}_+$ such that $f(\omega_0) > r > h(\omega_0)$. Then $W = \{\omega \in \Omega : h(\omega) < r < f(\omega)\}$ is an open set containing ω_0. If g is a continuous function from Ω to $[0,1]$ such that $0 \leqslant g \leqslant 1$, $g(\omega_0) = 1$ and g vanishes outside W, then $\|gf\| > r > \|gh\|$. ∎

REMARK 3.2. For any function $h : \Omega \to \mathbb{R}_+$, we define $\|h\| := \sup_{\omega\in\Omega} h(\omega)$ and

$$\|h\|_e := \inf\Big\{ \sup_{\omega\in\Delta} h(\omega) : \Delta \text{ is an open dense subset of } \Omega \Big\}.$$

If h is upper-semi-continuous, then

$$\|h\|_e = \inf\Big\{ \sup_{\omega\in\Xi} h(\omega) : \Xi \text{ is a dense subset of } \Omega \Big\}.$$

PROPOSITION 3.3. *Suppose that X is a $C(\Omega)$-convex Banach module and $x \in X$. Define $\|x\|_e := \||x|\|_e$ (where $|x|(\omega) = \|x(\omega)\|$). Then the following statements are equivalent:*

(i) $\|x\| = \|x\|_e$.

(ii) *For any $\varepsilon > 0$, there exists $f \in C(\Omega)_+$ such that $f \leqslant |x|$ and $\|x\| \leqslant \|f\| + \varepsilon$.*

(iii) *x is regular in $X(x) = \overline{C(\Omega) \cdot x}$ (see Definition 1.2).*

Consequently, if X is regular, then $\|x\| = \|x\|_e$ for any $x \in X$.

Proof. (i)⇒(ii) Since $G := \{\omega \in \Omega : |x|(\omega) \geqslant \|x\|_e - \varepsilon\}$ is a closed set in Ω (as $|x|$ is upper semi-continuous), G contains an open set V (otherwise, the open set $\{\omega \in \Omega : |x|(\omega) < \|x\|_e - \varepsilon\}$ is dense which contradicts the definition of $\|x\|_e$). Take any $\omega_0 \in V$. Let $f \in C(\Omega)$ be such that $0 \leqslant f(\omega) \leqslant \|x\|_e - \varepsilon$

$(\omega \in \Omega)$, $f(\omega_0) = \|x\|_e - \varepsilon$ and f vanishes outside V. Then clearly $f \leqslant |x|$ and $\|x\|_e = \|f\| + \varepsilon$.

(ii)⇒(iii) For any $\varepsilon > 0$, let f be the function as given in statement (ii). We first show that $\varphi : X(x) \to C(\Omega)$ given by $\varphi(g \cdot x) = gf$ is well defined. Suppose that $g \in C(\Omega)$ such that $g(\omega)x(\omega) = 0$ for any $\omega \in \Omega$. If $g(\omega) \neq 0$, then $x(\omega) = 0$ and so $f(\omega) = 0$ which implies that $gf = 0$. Thus, $\varphi \in X(x)^s$ is a well defined contraction such that $\|x\| \leqslant \|\varphi(x)\| + \varepsilon$.

(iii)⇒(i) It is clear that $\|x\|_e \leqslant \|x\|$. For any $\varepsilon > 0$, let $\varphi \in X(x)^s$ such that $\|\varphi\| \leqslant 1$ and $\|x\| \leqslant \|\varphi(x)\| + \varepsilon$. Put $f = |\varphi(x)| \in C(\Omega)_+$. Then for any $g \in C(\Omega)_+$, we have $(g|\varphi(x)|)(\omega) = |\varphi(g \cdot x)|(\omega)$ for any $\omega \in \Omega$ and so,

$$\|gf\| = \sup_{\omega\in\Omega} |\varphi(g \cdot x)(\omega)| = \|\varphi(g \cdot x)\| \leqslant \|g \cdot x\| = \sup_{\omega\in\Omega} g(\omega)\|x(\omega)\| = \|g|x|\|.$$

Hence $f \leqslant |x|$ (by Lemma 3.1) and $\|f\| = \|f\|_e \leqslant \|x\|_e$. Thus,

$$\|x\| \ \leqslant \ \|\varphi(x)\| + \varepsilon \ = \ \|f\| + \varepsilon \leqslant \|x\|_e + \varepsilon. \quad \blacksquare$$

LEMMA 3.4. *Let X be a $C(\Omega)$-convex Banach module. The map $x \mapsto \|x\|_e$ is an absolutely $C(\Omega)$-convex seminorm on X.*

Proof. Let $f_1, f_2 \in C(\Omega)_+$ with $\|f_1 + f_2\| \leqslant 1$ and $x_1, x_2 \in X$ with $\|x_1\|_e, \|x_2\|_e \leqslant 1$. For any $\varepsilon > 0$, there exist open dense subsets Δ_1 and Δ_2 such that $\sup_{\omega\in\Delta_i} \|x_i(\omega)\| < 1 + \varepsilon$ $(i = 1, 2)$. If $\Delta = \Delta_1 \cap \Delta_2$,

$$\begin{aligned}\|f_1 \cdot x_1 + f_2 \cdot x_2\|_e &\leqslant \sup_{\omega\in\Delta} \|f_1(\omega)x_1(\omega) + f_2(\omega)x_2(\omega)\| \\ &\leqslant \sup_{\omega\in\Delta} f_1(\omega)\|x_1(\omega)\| + f_2(\omega)\|x_2(\omega)\| \ = \ 1 + \varepsilon. \quad \blacksquare\end{aligned}$$

REMARK 3.5. $\|\cdot\|_e$ is a norm if the underlying topology of the (H)-Banach bundle (p, E, Ω) associated with X is Hausdorff. In fact, suppose that $y \in X$ such that $\|y\|_e = 0$. For any $n \in \mathbb{N}$, the open set $\{\omega : \|y(\omega)\| < 1/n\}$ is dense in Ω. Therefore, by the Baire's Category theorem,

$$K_y \ := \ \{\omega \in \Omega : y(\omega) = 0_\omega\} \ = \ \bigcap_{n\in\mathbb{N}} \{\omega : \|y(\omega)\| < 1/n\}$$

is dense in Ω (where 0_ω is the zero of the fibre at ω). Consider the map $j : \Omega \to E$ defined by $j(\omega) = 0_\omega$. By condition (4) of 1.1 in [2], we see that j is a continuous map and so $j(\Omega)$ is compact in the Hausdorff space E. Thus, $K_y = y^{-1}(j(\Omega))$ is also closed in Ω and hence $K_y = \Omega$. This shows that $y \equiv 0$. Thus, $\|\cdot\|_e$ is a norm on X.

In the following, we denote by X_{ess} the completion of $(X/N, \|\cdot\|_e)$ (where $N = \{x \in X : \|x\|_e = 0\}$). A natural question is whether $X = X_{\text{ess}}$ for any absolutely $C(\Omega)$-convex Banach module X. This is, of course, true if Ω is a finite set. The following example shows that it is not the case in general.

EXAMPLE 3.6. Let Ω be a compact Hausdorff space with a non-isolated point $\omega \in \Omega$. One can turn $\mathbb{C}$ into a Banach $C(\Omega)$-module, denoted by X, through the multiplication $f \cdot r = f(\omega)r$. It is not hard to check that X is $C(\Omega)$-convex and so $X = \Gamma(E)$ for an (H)-Banach bundle E over Ω. By the construction in [2], pp. 35–36 we see that the fibre E_ω equals $\mathbb{C}$ while $E_\nu = (0)$ for any $\nu \in \Omega \setminus \{\omega\}$. There exists $y \in \Gamma(E)$ such that $y(\omega) = 1$. Thus,

$$|y|(\nu) = \begin{cases} 0 & \text{if } \nu \neq \omega, \\ 1 & \text{if } \nu = \omega, \end{cases}$$

(which is clearly not continuous as ω is not an isolated point) and so $\|y\|_e = 0$ (in fact, $\|x\|_e = 0$ for any $x \in X$).

THEOREM 3.7. *Let Ω be a Stonian space and X be a $C(\Omega)$-convex Banach module. Then $X_{\mathrm{reg}} = X_{\mathrm{ess}}$. Consequently, if X comes from an* (F)*-Banach bundle, then X is regular.*

Proof. By the argument of "(i)⇒(ii)" in Proposition 3.3, we see that for any $x \in X$ and $\varepsilon > 0$, there exists $f \in C(\Omega)_+$ with $f \leqslant |x|$ and $\|f\| = \|x\|_e - \varepsilon$. Moreover, by the argument of "(ii)⇒(iii)" in Proposition 3.3, the map $\varphi : X(x) \to C(\Omega)$ given by $\varphi(g \cdot x) = gf$ is well defined. As $f \leqslant |x|$ and f is continuous,

$$\|gf\| = \|gf\|_e \leqslant \|g \cdot x\|_e$$

for any $g \in C(\Omega)$. Therefore, φ is a contraction from the semi-normed space $(X(x), \|\cdot\|_e)$ to $C(\Omega)$ and so it defines a contraction in $(X(x)_{\mathrm{ess}})^s$, also denoted by φ, such that $\|x_0\|_e \leqslant \|\varphi(x_0)\| + \varepsilon$ (where x_0 is the image of x in $X(x)_{\mathrm{ess}}$). It is not hard to check that $X(x)_{\mathrm{ess}} = X_{\mathrm{ess}}(x_0)$. Thus, as Ω is Stonian, φ extends to $\psi \in (X_{\mathrm{ess}})^s$ such that $\|\psi\| = \|\varphi\| \leqslant 1$ (by 3.10 of [5]). Hence X_{ess} is regular.

On the other hand, suppose that Y is a regular Banach $C(\Omega)$-bimodule and $\Phi \in \mathcal{L}_{C(\Omega)}(X, Y)$. Let E and F be (H)-Banach bundles over Ω such that $X = \Gamma(E)$ and $Y = \Gamma(F)$ (note that as Y is regular, it is $C(\Omega)$-convex and such F exists). Then Φ induces a Banach bundle map $\Psi : E \to F$. By Proposition 3.3, $\|y\| = \|y\|_e$ for any $y \in Y$. Thus for any $z \in X$,

$$\begin{aligned} \|\Phi(z)\| = \|\Psi \circ z\| = \|\Psi \circ z\|_e &= \inf\Big\{ \sup_{\omega \in \Delta} \|\Psi(z(\omega))\| : \Delta \text{ is dense in } \Omega \Big\} \\ &\leqslant \|\Phi\| \cdot \inf\Big\{ \sup_{\omega \in \Delta} \|z(\omega)\| : \Delta \text{ is dense in } \Omega \Big\} = \|\Phi\| \cdot \|z\|_e. \end{aligned}$$

Consequently, Φ factors through an element in $\mathcal{L}_{C(\Omega)}(X_{\mathrm{ess}}, Y)$ uniquely. Since X_{ess} is regular, it is the regularization of X. The second statement comes from the fact that $\|x\| = \|x\|_e$ if X comes from an (F)-Banach bundle. ∎

This theorem and Remark 3.5 show that if Ω is a Stonian space and E is a Hausdorff (H)-Banach bundle over Ω with $X = \Gamma(E)$, then κ_X is injective.

REMARK 3.8. Suppose that Ω is a compact Hausdorff space, Y is a normed $C(\Omega)$-module and $n : Y \to C(\Omega)_+$ satisfies the three conditions in p. 47 of [2]. Then n extends to the completion $\widetilde{Y}$ of Y which also satisfies the same three conditions. In fact, $-n(y-z) \leqslant n(y) - n(z) \leqslant n(y-z)$ implies that $\|n(y) - n(z)\| \leqslant \|n(y-z)\| = \|y-z\|$ $(y, z \in Y)$. It is not hard to see that this gives a well-defined map from $\widetilde{Y}$ to $C(\Omega)_+$ (as $C(\Omega)_+$ is complete) which satisfies the three conditions in p. 47 of [2].

THEOREM 3.9. *Let Ω be a hyper-Stonian space and X be a $C(\Omega)$-convex Banach module. Then X_{ess} is the space of continuous sections of an* (F)*-Banach bundle.*

Proof. Let $\{\mu_i\}_{i\in I}$ be a maximal family of positive normal measures on Ω with disjoint supports and let Ξ_i be the support of μ_i. Then $\Xi := \bigcup_{i\in I} \Xi_i$ is an open dense subset of Ω and $\{\mu_i\}_{i\in I}$ defines a Radon measure μ on Ξ such that $C(\Omega) \cong L^\infty(\Xi, \mu)$ (see the argument of "(i)$\Rightarrow$(ii)" in III.1.18 of [13]). Denote by $\mathrm{USC}(\Omega)_+$ the set of all upper semi-continuous functions from Ω to $\mathbb{R}_+$. For any $h \in \mathrm{USC}(\Omega)_+$, we let $\psi(h)$ be the equivalence class of $h\,|_{\,\Xi}$ in $L^\infty(\Xi, \mu)$. We first show that

$$\|h\|_e = \|\psi(h)\|_\infty.$$

Let $\Lambda \subseteq \Xi$ be a measurable set such that $\mu(\Lambda) = 0$. Then $\mu_i(\Lambda) = 0$ for all $i \in I$ and so Λ is nowhere dense in Ω (see III.1.15 of [13]). The set $\Delta = \overline{\Lambda} \cup (\Omega \setminus \Xi)$ is closed and nowhere dense in Ω. Since $\Omega \setminus \Delta \subseteq \Xi \setminus \Lambda$, we see that

$$\sup_{\omega\in\Omega\setminus\Delta} h(\omega) \leqslant \sup_{\omega\in\Xi\setminus\Lambda} h(\omega).$$

As Λ is an arbitrary measure zero set, $\|h\|_e \leqslant \|\psi(h)\|_\infty$. Conversely, suppose that Δ is a closed nowhere dense subset of Ω and let $\Lambda = \Delta \cap \Xi$. Let C be a compact subset of Λ. Then C is also nowhere dense in Ω and $\mu_i(C) = 0$ for any $i \in I$ (because of III.1.15 in [13]). Therefore, $\mu(C) = \sum_{i\in I} \mu_i(C) = 0$ and so by the regularity of μ, we have $\mu(\Lambda) = 0$. Since

$$\sup_{\omega\in\Xi\setminus\Lambda} h(\omega) \leqslant \sup_{\omega\in\Omega\setminus\Delta} h(\omega)$$

and Δ is an arbitrary closed nowhere dense subset of Ω, we see that $\|\psi(h)\|_\infty \leqslant \|h\|_e$. Now, as $g \mapsto \psi(g)$ is the canonical isomorphism from $C(\Omega)$ to $L^\infty(\Xi, \mu)$ (see the argument of III.1.18 in [13]), the above shows that for any $h \in \mathrm{USC}(\Omega)_+$, there exists a unique $g \in C(\Omega)_+$ such that $g = h$ on an open and dense subset of Ξ (and hence of Ω). This induces a map

$$\phi : \mathrm{USC}(\Omega)_+ \to C(\Omega)_+$$

such that $\|\phi(h)\| = \|h\|_e$. For any $g \in C(\Omega)_+$ and $h \in \mathrm{USC}(\Omega)_+$, it is not hard to see that $\phi(gh) = g\phi(h)$. If we set $n(x) := \phi(|x|) \in C(\Omega)_+$ $(x \in X)$, then $\|x\|_e = \|n(x)\|$ (because $|x|$ is upper semi-continuous). Hence, if $x, y \in X$ such that $\|x - y\|_e = 0$, then $\|(|x-y|)\,|_{\,\Xi}\|_\infty = 0$ and so $|x - y| = 0$ a.e. on Ξ which implies that $|x| = |y|$ on an open dense subset of Ξ. This shows that if X^0_{ess} is the image of X in X_{ess}, then n induces

$$\widetilde{n} : X^0_{\mathrm{ess}} \to C(\Omega)_+$$

which satisfies conditions (1)–(3) in [2], p. 47. In fact, for any $g \in C(\Omega)$ and any $y \in X$,

$$\widetilde{n}(g \cdot y_0) \;=\; \widetilde{n}((g \cdot y)_0) \;=\; \phi(|g \cdot y|) \;=\; |g|\phi(|y|) \;=\; |g|\widetilde{n}(y_0)$$

(where $(g \cdot y)_0$ and y_0 are the images of $g \cdot y$ and y respectively in X_{ess}). By Remark 3.8, we see that $\widetilde{n}$ can be extended to X_{ess}. Thus, X_{ess} is the space of continuous sections of an (F)-Banach bundle (see e.g. pp. 47–48 of [2]). ∎

REMARK 3.10. We would like to thank the referee for informing us about [7] and for telling us that one can use the results in Section 6 of [7] to obtain the above theorem in an easier way. We left it to the readers to check the details. We decided to keep the proof as above because it is more elementary and our approach is completely different from the results in [7].

DEFINITION 3.11. Let E and E_c be respectively an (H)-Banach bundle and an (F)-Banach bundle over Ω and let Ψ be a Banach bundle map from E to E_c. Then (E_c, Ψ) is called the *continuous envelope of E* if any Banach bundle map from E to any (F)-Banach bundle over Ω factors through Ψ uniquely.

It is not known if (E_c, Ψ) always exists but it is the case when Ω is a hyper-Stonian space because of Theorems 3.7 and 3.9.

COROLLARY 3.12. *Suppose that Ω is a hyper-Stonian space, E is an* (H)*-Banach bundle over Ω and $X = \Gamma(E)$. Then E_c exists and $X_{\mathrm{ess}} = \Gamma(E_c)$.*

COROLLARY 3.13. *Let Ω be a hyper-Stonian space and X be a $C(\Omega)$-convex Banach module. The following are equivalent:*

(i) $X = X_{\mathrm{ess}}$.
(ii) *X is regular.*
(iii) $X = \Gamma(E)$ *for an* (F)*-Banach bundle over Ω.*

REMARK 3.14. As seen in Example 3.6, not every (H)-Banach bundle over a hyper-Stonian space is an (F)-Banach bundle. Therefore, regularity and pseudo-regularity do not coincide in this case.

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