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THE COMPLETION OF A C^*-ALGEBRA WITH A LOCALLY CONVEX TOPOLOGY

FABIO BAGARELLO, MARIA FRAGOULOPOULOU,
ATSUSHI INOUE and CAMILLO TRAPANI

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ABSTRACT. There are examples of C^*-algebras $\mathcal{A}$ that accept a locally convex $*$-topology τ coarser than the given one, such that $\widetilde{\mathcal{A}}[\tau]$ (the completion of $\mathcal{A}$ with respect to τ) is a GB^*-algebra. The multiplication of $\mathcal{A}[\tau]$ may be or not be jointly continuous. In the second case, $\widetilde{\mathcal{A}}[\tau]$ may fail being a locally convex $*$-algebra, but it is a partial $*$-algebra. In both cases the structure and the representation theory of $\widetilde{\mathcal{A}}[\tau]$ are investigated. If $\overline{\mathcal{A}}_+^{\tau}$ denotes the τ-closure of the positive cone $\mathcal{A}_+$ of the given C^*-algebra $\mathcal{A}$, then the property $\overline{\mathcal{A}}_+^{\tau} \cap (-\overline{\mathcal{A}}_+^{\tau}) = \{0\}$ is decisive for the existence of certain faithful $*$-representations of the corresponding $*$-algebra $\widetilde{\mathcal{A}}[\tau]$.

KEYWORDS: *GB^*-algebra, unbounded C^*-seminorm, partial $*$-algebra.*

MSC (2000): 46K10, 47L60.

1. INTRODUCTION

A mapping p of a $*$-subalgebra $\mathcal{D}(p)$ of a $*$-algebra $\mathcal{A}$ into $\mathbb{R}_+ = [0, \infty)$ is said to be an *unbounded C^*-(semi)norm* if it is a C^*-(semi)norm on $\mathcal{D}(p)$. Unbounded C^*-seminorms on $*$-algebras have appeared in many mathematical and physical subjects (for example, locally convex $*$-algebras, the moment problem, the quantum field theory etc.; see, e.g., [1], [18], [31], [33]). But a systematical study seems far to be complete (cf. also Introduction of [19]). So we have tried to study methodically unbounded C^*-seminorms and to apply such studies to those locally convex $*$-algebras that accept such C^*-seminorms [8], [11], [12], [13]. A *locally convex $*$-algebra* is a $*$-algebra which is also a Hausdorff locally convex space such that the multiplication is separately continuous and the involution is continuous. The studies of locally convex $(*)$-algebras started with those of locally m-convex $(*)$-algebras by R. Arens [7] and E.A. Michael [25], in 1952. In fact, the notion of a locally m-convex algebra was introduced by R. Arens [6], in 1946. For a complete account on locally m-convex algebras, see [26]. A locally convex $*$-algebra $\mathcal{A}[\tau]$ is said to be *locally C^*-convex* if the topology τ is determined by a directed family $\{p_\lambda\}_{\lambda\in\Lambda}$ of C^*-seminorms. A complete locally C^*-convex algebra is said to be a *pro-C^*-algebra* [27] (or a *locally C^*-algebra* [22]). Every pro-C^*-algebra is a projective limit of C^*-algebras. But it is difficult to study general locally convex $*$-algebras which are not locally C^*-convex, even if the multiplication is jointly continuous. So the third author together with K.-D. Kürsten defined and studied recently in [24] the so-called C^*-like locally convex $*$-algebras, that read as follows: If $\mathcal{A}[\tau]$ is a locally convex $*$-algebra, a directed family $\Gamma = \{p_\lambda\}_{\lambda\in\Lambda}$ of seminorms determining the topology τ is said to be *C^*-like* if for any $\lambda \in \Lambda$ there exists $\lambda' \in \Lambda$ such that $p_\lambda(xy) \leqslant p_{\lambda'}(x)p_{\lambda'}(y)$, $p_\lambda(x^*) \leqslant p_{\lambda'}(x)$ and $p_\lambda(x)^2 \leqslant p_{\lambda'}(x^*x)$ for any $x, y \in \mathcal{A}$. Of course, $p_\lambda{}'s$ are not necessarily C^*-seminorms; nevertheless, an unbounded C^*-norm p_Γ of $\mathcal{A}$ is defined by them in the following way:

$$\mathcal{D}(p_\Gamma) = \left\{x \in \mathcal{A} : \sup_{\lambda\in\Lambda} p_\lambda(x) < \infty\right\} \quad \text{with} \;\; p_\Gamma(x) := \sup_{\lambda\in\Lambda} p_\lambda(x), x \in \mathcal{D}(p_\Gamma).$$

A locally convex $*$-algebra $\mathcal{A}[\tau]$ is said to be *C^*-like* if it is complete and there is a C^*-like family $\Gamma = \{p_\lambda\}_{\lambda\in\Lambda}$ of seminorms determining the topology τ such that $\mathcal{D}(p_\Gamma)$ is τ-dense in $\mathcal{A}[\tau]$. In 1967, G.R. Allan [3] introduced and studied a class of locally convex $*$-algebras called GB^*-algebras. In 1970, P.G. Dixon [16] modified Allan's definition in the class of topological $*$-algebras, so that this wider class of GB^*-algebras includes certain non-locally convex $*$-algebras. The notion of a GB^*-algebra is a generalization of a C^*-algebra. Given a locally convex $*$-algebra $\mathcal{A}[\tau]$ with identity 1, denote by $\mathcal{B}^*$ the collection of all closed, bounded, absolutely convex subsets $\mathbf{B}$ of $\mathcal{A}$ satisfying $1 \in \mathbf{B}$, $\mathbf{B}^* = \mathbf{B}$ and $\mathbf{B}^2 \subset \mathbf{B}$. For every $\mathbf{B} \in \mathcal{B}^*$, the linear span of $\mathbf{B}$ forms a normed $*$-algebra under the Minkowski functional $\|\cdot\|_{\mathbf{B}}$ of $\mathbf{B}$, and it is denoted by Alg$\mathbf{B}$ (simply, $A[\mathbf{B}]$). If $A[\mathbf{B}]$ is complete for every $\mathbf{B} \in \mathcal{B}^*$, then $\mathcal{A}[\tau]$ is said to be *pseudo-complete*. If $\mathcal{A}[\tau]$ is sequentially complete, then it is pseudo-complete. Let $\mathcal{A}[\tau]$ be a pseudo-complete locally convex $*$-algebra. If $\mathcal{B}^*$ has the greatest member $\mathbf{B}_0$ and $(1 + x^*x)^{-1} \in A[\mathbf{B}_0]$ for every $x \in \mathcal{A}$, then $\mathcal{A}[\tau]$ is said to be a *GB^*-algebra over* $\mathbf{B}_0$. If $\mathcal{A}[\tau]$ is a GB^*-algebra over $\mathbf{B}_0$, then $A[\mathbf{B}_0]$ is a C^*-algebra and $\|\cdot\|_{\mathbf{B}_0}$ is an unbounded C^*-norm of $\mathcal{A}[\tau]$. Thus, the study of unbounded C^*-seminorms may be useful for investigations on locally convex $*$-algebras of this type. Let $\mathcal{A}[\tau]$ be a locally convex $*$-algebra and p an unbounded C^*-norm of $\mathcal{A}[\tau]$. Then

$$\mathcal{D}(p) \subset \mathcal{A}[\tau] \subset \widetilde{\mathcal{A}}[\tau] \quad \text{and} \quad \mathcal{D}(p) \subset \mathcal{A}_p \equiv \widetilde{\mathcal{D}(p)}[p] \quad (C^*\text{-algebra}),$$

where $\widetilde{\mathcal{A}}[\tau]$ and $\mathcal{A}_p$ denote the completions of $\mathcal{A}[\tau]$ and $\mathcal{D}(p)[p]$, respectively. But we have no relation of $\widetilde{\mathcal{A}}[\tau]$ with the C^*-algebra $\mathcal{A}_p$, in general.

Suppose now that the following condition (N_1) holds:

(N_1) The topology defined by p is stronger than the topology τ on $\mathcal{D}(p)$ (simply, $\tau \prec p$).

Then the identity map $i : \mathcal{D}(p) \to \mathcal{A}[\tau]$ is continuous, therefore it can be extended to a continuous linear map $\widetilde{i}$ of $\mathcal{A}_p$ into $\widetilde{\mathcal{A}}[\tau]$, where $\widetilde{i}$ is not necessarily an injection. It is easily shown that $\widetilde{i}$ is an injection if and only if the following condition (N_2) is satisfied:

(N_2) τ and p are *compatible* in the sense that, for any Cauchy net $\{x_\alpha\}$ in $\mathcal{D}(p)[p]$ such that $x_\alpha \xrightarrow{\tau} 0$, then $x_\alpha \xrightarrow{p} 0$.

In this case we say that $\mathcal{A}_p$ is *imbedded* in $\widetilde{\mathcal{A}}[\tau]$ and we write $\widetilde{\mathcal{A}}[p] \hookrightarrow \widetilde{\mathcal{A}}[\tau]$. Moreover, we have

$$\mathcal{D}(p) \subset \mathcal{A}[\tau] \hookrightarrow \widetilde{\mathcal{A}}[\tau], \quad \text{respectively } \mathcal{D}(p) \subset \mathcal{A}_p \hookrightarrow \widetilde{\mathcal{A}}[\tau].$$

An unbounded C^*-norm p is said to be *normal*, if it satisfies the conditions (N_1) and (N_2).

The unbounded C^*-norms p_Γ and $\|\cdot\|_{\mathbf{B}_0}$ considered above are normal.

In this paper we shall investigate the structure and the representation theory of locally convex $*$-algebras with normal unbounded C^*-norms. As stated above, it is sufficient to investigate the completion $\widetilde{\mathcal{A}_0}[\tau]$ of the C^*-algebra $\mathcal{A}_0[\|\cdot\|]$ with respect to a locally convex topology τ on $\mathcal{A}_0$ such that $\tau \prec \|\cdot\|$. Then the following cases arise:

Case 1: If the multiplication in $\mathcal{A}_0$ is jointly continuous with respect to the topology τ, then $\widetilde{\mathcal{A}_0}[\tau]$ is a complete locally convex $*$-algebra containing the C^*-algebra $\mathcal{A}_0[\|\cdot\|]$ as a dense subalgebra.

Case 2: If the multiplication on $\mathcal{A}_0$ is not jointly continuous with respect to τ, then $\widetilde{\mathcal{A}_0}[\tau]$ is not necessarily a locally convex $*$-algebra, but it has the structure of a partial $*$-algebra [4].

Under this stimulus, we investigate in the sequel the structure and the representation theory of $\widetilde{\mathcal{A}_0}[\tau]$.

2. CASE 1

In this section we study the structure and the representation theory of $\widetilde{\mathcal{A}_0}[\tau]$ as described in Case 1 before.

Suppose that $\mathcal{A}_0[\|\cdot\|_0]$ is a C^*-algebra with identity *1*, τ a locally convex topology on $\mathcal{A}_0$ such that $\tau \prec \|\cdot\|_0$ and $\mathcal{A}_0[\tau]$ a locally convex $*$-algebra with jointly continuous multiplication (take, for instance, the C^*-algebra $\mathcal{C}[0,1]$ of all continuous functions on $[0,1]$, with the topology τ of uniform convergence on the countable compact subsets of $[0,1]$). As we shall shown in Example 4.1, the C^*-algebra $\mathcal{A}_0[\|\cdot\|_0]$ that determines the locally convex $*$-algebra $\widetilde{\mathcal{A}_0}[\tau]$ is not unique. For this reason, we denote by $C^*(\mathcal{A}_0,\tau)$ the set of all C^*-algebras $\mathcal{A}[\|\cdot\|]$ such that $\mathcal{A}_0 \subset \mathcal{A} \subset \widetilde{\mathcal{A}_0}[\tau]$, $\tau \prec \|\cdot\|$ and $\|x\| = \|x\|_0, \forall x \in \mathcal{A}_0$. Then $C^*(\mathcal{A}_0,\tau)$ is an ordered set with the order:

$$\mathcal{A}_1[\|\cdot\|_1] \preceq \mathcal{A}_2[\|\cdot\|_2] \text{ if and only if } \mathcal{A}_1 \subset \mathcal{A}_2 \text{ and } \|x\|_1 = \|x\|_2, \forall x \in \mathcal{A}_1.$$

But we do not know whether there exists a maximal C^*-algebra in $C^*(\mathcal{A}_0, \tau)$.

LEMMA 2.1. *We denote by $\mathbf{B}_\tau$ the τ-closure of the unit ball $\mathcal{U}(\mathcal{A}_0) \equiv \{x \in \mathcal{A}_0 : \|x\|_0 \leqslant 1\}$ of the C^*-algebra $\mathcal{A}_0[\|\cdot\|_0]$. Then $\mathbf{B}_\tau \in \mathcal{B}^*$ and $A[\mathbf{B}_\tau]$ is a Banach $*$-algebra with the norm $\|\cdot\|_{\mathbf{B}_\tau}$, satisfying the following conditions:*

(i) *$(1+x^*x)^{-1}, x(1+x^*x)^{-1}$ and $(1+x^*x)^{-1}x$ exist in $\mathbf{B}_\tau$ for every $x \in \widetilde{\mathcal{A}}_0[\tau]$.*

(ii) *$\mathcal{A}_0 \subset A[\mathbf{B}_\tau]$ and $\|x\|_0 = \|x\|_{\mathbf{B}_\tau}$ for each $x \in \mathcal{A}_0$. Hence, $\mathcal{U}(\mathcal{A}_0) = \mathbf{B}_\tau \cap \mathcal{A}_0$ and $\mathcal{A}_0$ is a closed $*$-subalgebra of the Banach $*$-algebra $A[\mathbf{B}_\tau]$.*

(iii) *$A[\mathbf{B}_\tau]$ is $\|\cdot\|_{\mathbf{B}}$-dense in $A[\mathbf{B}]$ for each $\mathbf{B} \in \mathcal{B}^*$ containing $\mathcal{U}(\mathcal{A}_0)$.*

Proof. It is clear that $\mathbf{B}_\tau \in \mathcal{B}^*$ and $A[\mathbf{B}_\tau]$ is a Banach $*$-algebra since $\widetilde{\mathcal{A}}_0[\tau]$ is complete.

(i) Take an arbitrary $x \in \widetilde{\mathcal{A}}_0[\tau]$ and $\{x_\alpha\}$ a net in $\mathcal{A}_0$ such that $\tau\text{-}\lim_\alpha x_\alpha = x$. Then since $\mathcal{A}_0$ is a C^*-algebra, it follows first that $(1+x_\alpha^* x_\alpha)^{-1} \in \mathcal{U}(\mathcal{A}_0)$, for every α, and secondly that for any τ-continuous seminorm p

$$\begin{aligned}
p((1+x_\alpha^* x_\alpha)^{-1} &- (1+x_\beta^* x_\beta)^{-1}) \\
&= p((1+x_\alpha^* x_\alpha)^{-1}(x_\beta^* x_\beta - x_\alpha^* x_\alpha)(1+x_\beta^* x_\beta)^{-1}) \\
&\leqslant q((1+x_\alpha^* x_\alpha)^{-1}) q((1+x_\beta^* x_\beta)^{-1}) q(x_\beta^* x_\beta - x_\alpha^* x_\alpha) \\
&\leqslant \gamma \|(1+x_\alpha^* x_\alpha)^{-1}\|_0 \|(1+x_\beta^* x_\beta)^{-1}\|_0 q(x_\beta^* x_\beta - x_\alpha^* x_\alpha) \\
&\leqslant \gamma q(x_\beta^* x_\beta - x_\alpha^* x_\alpha)
\end{aligned}$$

for some $\gamma > 0$ and some τ-continuous seminorm q. Thus $\{(1+x_\alpha^* x_\alpha)^{-1}\}$ is a Cauchy net in $\widetilde{\mathcal{A}}_0[\tau]$ and $y \equiv \lim_\alpha (1+x_\alpha^* x_\alpha)^{-1}$ exists in $\widetilde{\mathcal{A}}_0[\tau]$. Since

$$1 = (1+x_\alpha^* x_\alpha)(1+x_\alpha^* x_\alpha)^{-1} = (1+x_\alpha^* x_\alpha)^{-1}(1+x_\alpha^* x_\alpha), \quad \forall \alpha,$$

it follows that $(1+x^*x)^{-1} \in \widetilde{\mathcal{A}}_0[\tau]$ and $y = (1+x^*x)^{-1}$. Also, $(1+x^*x)^{-1} \in \mathbf{B}_\tau$ and in a similar way we have that

$$x(1+x^*x)^{-1} \text{ and } (1+x^*x)^{-1}x \text{ belong to } \mathbf{B}_\tau.$$

(ii) Since $\mathcal{U}(\mathcal{A}_0) \subset \mathbf{B}_\tau$, it follows that $\mathcal{A}_0 \subset A[\mathbf{B}_\tau]$ and $\|x\|_{\mathbf{B}_\tau} \leqslant \|x\|_0$ for each $x \in \mathcal{A}_0$. From the theory of C^*-algebras (see, for example, Proposition I.5.3 of [32]), we have $\|x\|_0 \leqslant \|x\|_{\mathbf{B}_\tau}$ for each $x \in \mathcal{A}_0$. Hence, it follows that $\|x\|_0 = \|x\|_{\mathbf{B}_\tau}$, for each $x \in \mathcal{A}_0$, which implies that $\mathcal{U}(\mathcal{A}_0) = \mathbf{B}_\tau \cap \mathcal{A}_0$ and $\mathcal{A}_0$ is a closed $*$-subalgebra of $A[\mathbf{B}_\tau]$.

(iii) Take an arbitrary $\mathbf{B} \in \mathcal{B}^*$ containing $\mathcal{U}(\mathcal{A}_0)$. Since $\mathbf{B}$ is τ-closed, it follows that $\mathbf{B}_\tau \subset \mathbf{B}$, and so $A[\mathbf{B}_\tau] \subset A[\mathbf{B}]$ and $\|x\|_{\mathbf{B}} \leqslant \|x\|_{\mathbf{B}_\tau}$ for each $x \in A[\mathbf{B}_\tau]$. Let $x \in A[\mathbf{B}]$. By (i) we have

$$x\Big(1+\frac{1}{n}x^*x\Big)^{-1} \in A[\mathbf{B}_\tau], \quad \forall n \in \mathbb{N} \text{ and}$$

$$\begin{aligned}\lim_{n\to\infty}\left\|x\left(1+\frac{1}{n}x^*x\right)^{-1}-x\right\|_{\mathbf{B}} &= \lim_{n\to\infty}\frac{1}{n}\left\|xx^*x\left(1+\frac{1}{n}x^*x\right)^{-1}\right\|_{\mathbf{B}}\\ &\leqslant \lim_{n\to\infty}\frac{1}{n}\|xx^*x\|_{\mathbf{B}}\left\|\left(1+\frac{1}{n}x^*x\right)^{-1}\right\|_{\mathbf{B}}\\ &\leqslant \lim_{n\to\infty}\frac{1}{n}\|xx^*x\|_{\mathbf{B}}\left\|\left(1+\frac{1}{n}x^*x\right)^{-1}\right\|_{\mathbf{B}_\tau}\\ &\leqslant \lim_{n\to\infty}\frac{1}{n}\|xx^*x\|_{\mathbf{B}}=0.\end{aligned}$$

Hence, $A[\mathbf{B}_\tau]$ is $\|\cdot\|_{\mathbf{B}}$-dense in $A[\mathbf{B}]$. This completes the proof. ∎

By Lemma 2.1(i) $A[\mathbf{B}_\tau]$ is a symmetric Banach $*$-algebra, therefore by Pták's theory for hermitian algebras [28] (see, e.g., Corollary 3.4 and Theorem 3.2 of [20]) $A[\mathbf{B}_\tau]$ is hermitian and the Pták function defined as $p_{A[\mathbf{B}_\tau]}(x) := r_{A[\mathbf{B}_\tau]}(x^*x)^{1/2}$, $x \in A[\mathbf{B}_\tau]$, where $r_{A[\mathbf{B}_\tau]}$ is the spectral radius, is a C^*-seminorm satisfying $p_{A[\mathbf{B}_\tau]}(x) \leqslant \|x\|_{\mathbf{B}_\tau}$, for each $x \in A[\mathbf{B}_\tau]$ and $p_{A[\mathbf{B}_\tau]}(x) \leqslant \|x\|_0$, for each $x \in \mathcal{A}_0$. It is natural to consider the following question:

Question A. Is $\widetilde{\mathcal{A}}_0[\tau]$ a GB^*-algebra? When is $\widetilde{\mathcal{A}}_0[\tau]$ a GB^*-algebra?

An answer is provided by the following:

THEOREM 2.2. *The following statements are equivalent:*

(i) $\widetilde{\mathcal{A}}_0[\tau]$ *is a* GB^**-algebra.*

(ii) *There exists the greatest member* $\mathbf{B}_0$ *in* $\mathcal{B}^*$*.*

(iii) *There exists a member* $\mathbf{B}_0$ *in* $\mathcal{B}^*$ *containing* $\mathcal{U}(\mathcal{A}_0)$ *such that* $\|\cdot\|_{\mathbf{B}_0}$ *is a* C^**-norm.*

If (iii) *is true, then* $\mathbf{B}_0$ *in* (iii) *is the greatest member in* $\mathcal{B}^*$ *and* $\widetilde{\mathcal{A}}_0[\tau]$ *is a* GB^**-algebra over* $\mathbf{B}_0$*.*

Proof. (i) ⇒ (iii) Since $\widetilde{\mathcal{A}}_0[\tau]$ is a GB^*-algebra, there exists the greatest member $\mathbf{B}_0$ in $\mathcal{B}^*$. Then $\|\cdot\|_{\mathbf{B}_0}$ is a C^*-norm and $\mathcal{U}(\mathcal{A}_0) \subset \mathbf{B}_\tau \subset \mathbf{B}_0$, since $\mathbf{B}_\tau \in \mathcal{B}^*$.

(iii) ⇒ (ii) Let $\mathbf{B}_0 \in \mathcal{B}^*$ such that $\|\cdot\|_{\mathbf{B}_0}$ is a C^*-norm and $\mathcal{U}(\mathcal{A}_0) \subset \mathbf{B}_0$. Take an arbitrary $\mathbf{B} \in \mathcal{B}^*$ and $h^* = h \in \mathbf{B}$. Let $\mathcal{C}$ be a maximal, commutative, locally convex $*$-algebra containing h. Then $\mathcal{C}$ is a complete commutative locally convex $*$-algebra. We denote by $\mathcal{B}^*_{\mathcal{C}}$ the collection of all closed, bounded, absolutely convex subsets $\mathbf{B}_1$ of $\mathcal{C}$ satisfying: $1 \in \mathbf{B}_1, \mathbf{B}_1^* = \mathbf{B}_1$ and $\mathbf{B}_1^2 \subset \mathbf{B}_1$. Then $\mathcal{B}^*_{\mathcal{C}} = \{\mathbf{B}_2 \cap \mathcal{C}; \mathbf{B}_2 \in \mathcal{B}^*\}$. We show that $\mathbf{B} \cap \mathcal{C} \subset \mathbf{B}_0 \cap \mathcal{C}$. Since $\mathcal{C}$ is commutative and complete, it follows from Theorem 2.10 of [3], that $\mathcal{B}^*_{\mathcal{C}}$ is directed, so that there exists $\mathbf{B}_1 \in \mathcal{B}^*_{\mathcal{C}}$ such that $(\mathbf{B} \cap \mathcal{C}) \cup (\mathbf{B}_0 \cap \mathcal{C}) \subset \mathbf{B}_1$. Then since the C^*-algebra $A[\mathbf{B}_0 \cap \mathcal{C}] = A[\mathbf{B}_0] \cap \mathcal{C}$ is contained in the Banach $*$-algebra $A[\mathbf{B}_1]$, it follows from Proposition I.5.3 of [32] that

$$\|x\|_{\mathbf{B}_0} = \|x\|_{\mathbf{B}_0\cap\mathcal{C}} \leqslant \|x\|_{\mathbf{B}_1}, \quad \forall x \in A[\mathbf{B}_0] \cap \mathcal{C}.$$

On the other hand, since $\mathbf{B}_0 \cap \mathcal{C} \subset \mathbf{B}_1$, it follows that

$$\|x\|_{\mathbf{B}_1} \leqslant \|x\|_{\mathbf{B}_0\cap\mathcal{C}} = \|x\|_{\mathbf{B}_0}, \quad \forall x \in A[\mathbf{B}_0] \cap \mathcal{C}.$$

Thus, we have

$$\|x\|_{\mathbf{B}_1} = \|x\|_{\mathbf{B}_0}, \quad \forall x \in A[\mathbf{B}_0] \cap \mathcal{C} \tag{2.1}$$

and the C^*-algebra $A[\mathbf{B}_0] \cap \mathcal{C}$ is $\|\cdot\|_{\mathbf{B}_1}$-dense in the Banach $*$-algebra $\mathcal{A}[\mathbf{B}_1]$. Indeed, from Lemma 2.1(i)

$$x\Big(1+\frac{1}{n}x^*x\Big)^{-1} \in A[\mathbf{B}_\tau], \quad \forall x \in A[\mathbf{B}_1] \text{ and } \forall n \in \mathbb{N}.$$

It is easily shown that $\{x, (1+y^*y)^{-1} : x, y \in \mathcal{C}\}$ is commutative, so that by the maximality of $\mathcal{C}$, $\{(1+y^*y)^{-1} : y \in \mathcal{C}\} \subset \mathcal{C}$. Furthermore, it follows from the assumption $\mathcal{U}(\mathcal{A}_0) \subset \mathbf{B}_0$, that $A[\mathbf{B}_\tau] \cap \mathcal{C} \subset A[\mathbf{B}_0] \cap \mathcal{C}$. Hence,

$$x\Big(1+\frac{1}{n}x^*x\Big)^{-1} \in A[\mathbf{B}_\tau] \cap \mathcal{C} \subset A[\mathbf{B}_0] \cap \mathcal{C}.$$

In a similar way as in the proof of Lemma 2.1(iii) we can show that

$$\Big\| x\Big(1+\frac{1}{n}x^*x\Big)^{-1} - x \Big\|_{\mathbf{B}_1} \leqslant \frac{1}{n}\|xx^*x\|_{\mathbf{B}_1}.$$

Hence, $A(\mathbf{B}_0] \cap \mathcal{C}$ is $\|\cdot\|_{\mathbf{B}_1}$-dense in $A[\mathbf{B}_1]$. By (2.1) $A[\mathbf{B}_0] \cap \mathcal{C} = A[\mathcal{C} \cap \mathbf{B}_0] = A[\mathbf{B}_1]$, and so $\mathbf{B}_0 \cap \mathcal{C} = \mathbf{B}_1$. Thus, $\mathbf{B} \cap \mathcal{C} \subset \mathbf{B}_0 \cap \mathcal{C}$. Therefore, $h \in \mathbf{B}_0$ and if $\mathbf{B}_h = \{x \in \mathbf{B} : x^* = x\}$, we have $\mathbf{B}_h \subset (\mathbf{B}_0)_h$, which implies that $\|x\|^2_{\mathbf{B}_0} = \|x^*x\|_{\mathbf{B}_0} \leqslant 1$ for each $x \in \mathbf{B}$. Hence, $\mathbf{B} \subset \mathbf{B}_0$ and $\mathbf{B}_0$ is the greatest member in $\mathcal{B}^*$.

(ii) ⇒ (i) This follows from Lemma 2.1(i) and so the proof is complete. ∎

By Theorem 2.2 we have the next:

COROLLARY 2.3. *Consider the following statements:*
(i) $\widetilde{\mathcal{A}}_0[\tau]$ *is a* GB^**-algebra over* $\mathcal{U}(\mathcal{A}_0)$.
(ii) $\mathcal{U}(\mathcal{A}_0)$ *is* τ*-closed.*
(iii) $\widetilde{\mathcal{A}}_0[\tau]$ *is a* GB^**-algebra over* $\mathbf{B}_\tau$.
(iv) $\mathbf{B}_\tau$ *is the greatest member in* $\mathcal{B}^*$.
(v) $\|\cdot\|_{\mathbf{B}_\tau}$ *is a* C^**-norm.*
Then the following implications hold: (i) ⇔ (ii) ⇒ (iii) ⇔ (iv) ⇔ (v).

We investigate now the representation theory of $\widetilde{\mathcal{A}}_0[\tau]$. We begin with some basic terminology. For more details see [23], [30]. Let $\mathcal{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. Denote by $\mathcal{L}(\mathcal{D})$ all linear operators from $\mathcal{D}$ into $\mathcal{D}$ and let

$$\mathcal{L}^\dagger(\mathcal{D}) := \{X \in \mathcal{L}(\mathcal{D}) : \mathcal{D}(X^*) \supset \mathcal{D} \text{ and } X^*\mathcal{D} \subset \mathcal{D}\}.$$

$\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra, under the usual algebraic operations and the involution $X \to X^\dagger := X^* \upharpoonright \mathcal{D}$. Furthermore, $\mathcal{L}^\dagger(\mathcal{D})$ is a locally convex $*$-algebra equipped with the topology τ_w (respectively τ_{s^*}) defined by the family $\{p_{\xi,\eta}(\cdot) : \xi, \eta \in \mathcal{D}\}$ of seminorms with $p_{\xi,\eta}(X) := |(X\xi|\eta)|, X \in \mathcal{L}^\dagger(\mathcal{D})$ (respectively the family $\{p^\dagger_\xi(\cdot) : \xi \in \mathcal{D}\}$ of seminorms with $p^\dagger_\xi(X) := \|X\xi\| + \|X^\dagger\xi\|, X \in \mathcal{L}^\dagger(\mathcal{D})$). A $*$-subalgebra of $\mathcal{L}^\dagger(\mathcal{D})$ is said to be an *O^*-algebra* on $\mathcal{D}$. Let $\mathcal{A}$ be a $*$-algebra. A $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{L}^\dagger(\mathcal{D})$ is called (unbounded) *$*$-representation* of $\mathcal{A}$

on the Hilbert space $\mathcal{H}$, with domain $\mathcal{D}$. If $\mathcal{A}$ has an identity, say 1, we suppose that $\pi(1) = I$, with I the identity operator in $\mathcal{L}^\dagger(\mathcal{D})$. From now on, we shall use the notation: $\mathcal{D}(\pi)$ for the domain of π and $\mathcal{H}_\pi$ for the corresponding Hilbert space. A $*$-representation π of $\mathcal{A}$ is said to be *faithful* if $\pi(a) = 0$, $a \in \mathcal{A}$, implies $a = 0$. A $*$-representation π of a locally convex $*$-algebra $\mathcal{A}[\tau]$ is said to be $(\tau - \tau_{\mathrm{w}})$*-continuous* (respectively $(\tau - \tau_{s^*})$*-continuous*) if it is continuous from $\mathcal{A}[\tau]$ to $\pi(\mathcal{A})[\tau_{\mathrm{w}}]$ (respectively to $\pi(\mathcal{A})[\tau_{s^*}]$).

We define now a wedge $\widetilde{\mathcal{A}}_0[\tau]_+$ of $\widetilde{\mathcal{A}}_0[\tau]$. Take an arbitrary C^*-algebra $\mathcal{A}[\|\cdot\|] \in C^*(\mathcal{A}_0, \tau)$. Then we have $\overline{\mathcal{A}}_+^\tau = \overline{(\mathcal{A}_0)}_+^\tau$, where $\mathcal{A}_+$ and $(\mathcal{A}_0)_+$ are positive cones in the C^*-algebras $\mathcal{A}$ and $\mathcal{A}_0$ respectively. Indeed, take an arbitrary $a \in \mathcal{A}_+$. Then there is a net $\{x_\alpha\}$ in $\mathcal{A}_0$ such that $\tau - \lim_\alpha x_\alpha = a^{1/2}$. Hence, $\{x_\alpha^* x_\alpha\} \subset (\mathcal{A}_0)_+$ and $\tau - \lim_\alpha x_\alpha^* x_\alpha = a$. This implies that $\overline{\mathcal{A}}_+^\tau \subset \overline{(\mathcal{A}_0)}_+^\tau$. The converse is clear. Thus, the τ-closure $\overline{\mathcal{A}_0}_+^\tau$ of $(\mathcal{A}_0)_+$ is independent of the method of taking C^*-algebras in $C^*(\mathcal{A}_0, \tau)$, therefore in the sequel we shall denote by $\widetilde{\mathcal{A}}_0[\tau]_+$ the τ-closure of $(\mathcal{A}_0)_+$. So $\widetilde{\mathcal{A}}_0[\tau]_+$ is a wedge (in the sense that if $x, y \in \widetilde{\mathcal{A}}_0[\tau]_+$ and $\lambda \geqslant 0$, then $x + y, \lambda x \in \widetilde{\mathcal{A}}_0[\tau]_+$), and $\widetilde{\mathcal{A}}_0[\tau]_+ = \overline{\mathcal{P}(\widetilde{\mathcal{A}}_0[\tau])}^\tau$ (the τ-closure of the algebraic wedge $\mathcal{P}(\widetilde{\mathcal{A}}_0[\tau]) \equiv \Big\{ \sum_{k=1}^n x_k^* x_k : x_k \in \widetilde{\mathcal{A}}_0[\tau]\ (k = 1, \ldots, n), n \in \mathbb{N} \Big\}$).

A linear functional f on $\widetilde{\mathcal{A}}_0[\tau]$ is said to be *strongly positive* (respectively *positive*) if $f(x) \geqslant 0$ for each $x \in \widetilde{\mathcal{A}}_0[\tau]_+$ (respectively $x \in \mathcal{P}(\widetilde{\mathcal{A}}_0[\tau])$).

THEOREM 2.4. *The following statements are equivalent:*

(i) $\widetilde{\mathcal{A}}_0[\tau]_+ \cap (-\widetilde{\mathcal{A}}_0[\tau]_+) = \{0\}$.

(ii) $A[\mathbf{B}_\tau]_+ \cap (-A[\mathbf{B}_\tau]_+) = \{0\}$.

(iii) *The Pták function $p_{A[\mathbf{B}_\tau]}$ on the Banach $*$-algebra $A[\mathbf{B}_\tau]$ is a C^*-norm (see comments before Question A).*

(iv) *There exists a faithful $*$-representation of $\widetilde{\mathcal{A}}_0[\tau]$.*

(v) *There exists a faithful $(\tau - \tau_{s^*})$-continuous $*$-representation of $\widetilde{\mathcal{A}}_0[\tau]$.*

Proof. (i) $\Rightarrow$ (v) Let $\mathcal{F}$ be the set of all τ-continuous strongly positive linear functionals on $\widetilde{\mathcal{A}}_0[\tau]$. Let $(\pi_f, \lambda_f, \mathcal{H}_f)$ be the GNS-construction for $f \in \mathcal{F}$. We put

$$\mathcal{D}(\pi) := \Big\{ (\lambda_f(x_f)) \in \bigoplus_{f \in \mathcal{F}} \mathcal{H}_f : \lambda_f(x_f) = 0 \text{ except for a finite number of } f \in \mathcal{F} \Big\}$$

$$\pi(a)(\lambda_f(x_f)) := (\lambda_f(a x_f)), \quad a \in \widetilde{\mathcal{A}}_0[\tau],\ (\lambda_f(x_f)) \in \mathcal{D}(\pi).$$

Then it is easily shown that π is a $(\tau - \tau_{s^*})$-continuous $*$-representation of $\widetilde{\mathcal{A}}_0[\tau]$. We show that π is faithful. In fact, suppose $0 \neq a \in \widetilde{\mathcal{A}}_0[\tau]_h$ (the hermitian part of $\widetilde{\mathcal{A}}_0[\tau]$). Let $a \in \widetilde{\mathcal{A}}_0[\tau]_+$. Since $\widetilde{\mathcal{A}}_0[\tau]_+ \cap (-\widetilde{\mathcal{A}}_0[\tau]_+) = \{0\}$, we have $\widetilde{\mathcal{A}}_0[\tau]_+ \cap \{-a\} = \phi$. Then it follows from Chapter II, Section 5, Proposition 4 in [15], that there exists a τ-continuous strongly positive linear functional f on $\widetilde{\mathcal{A}}_0[\tau]$ such that $f(a) > 0$. Let $a \notin \widetilde{\mathcal{A}}_0[\tau]_+$. Since $\widetilde{\mathcal{A}}_0[\tau]_+ \cap \{a\} = \phi$, we can show in a similar way that there exists a τ-continuous strongly positive linear functional f on $\widetilde{\mathcal{A}}_0[\tau]$ such that $f(a) < 0$. Since $(\pi_f(a)\lambda_f(1)|\lambda_f(1)) = f(a) \neq 0$ this implies that $\pi_f(a) \neq 0$, and so $\pi(a) \neq 0$. Similarly, for any $0 \neq a \in \widetilde{\mathcal{A}}_0[\tau]$ we have $\pi(a) \neq 0$ by considering $a = a_1 + \mathrm{i}a_2$ $(a_1, a_2 \in \widetilde{\mathcal{A}}_0[\tau]_h)$.

(v) $\Rightarrow$ (iv) This is trivial.

(iv) $\Rightarrow$ (iii) Let π be a faithful $*$-representation of $\widetilde{\mathcal{A}}_0[\tau]$. Since $A[\mathbf{B}_\tau]$ is a symmetric Banach $*$-algebra by Lemma 2.1(i), it follows from Theorem 3.2 and Corollary 3.4 in [20], that the Pták function $p_{A[\mathbf{B}_\tau]}$ is a C^*-seminorm. In particular (Raikov criterion for symmetry),

$$p_{A[\mathbf{B}_\tau]}(x) = \sup_{\rho \in \mathrm{Rep}(A[\mathbf{B}_\tau])} \|\rho(x)\|, \quad x \in A[\mathbf{B}_\tau],$$

where $\mathrm{Rep}(A[\mathbf{B}_\tau])$ denotes the set of all $*$-representations of $A[\mathbf{B}_\tau]$. Suppose $p_{A[\mathbf{B}_\tau]}(x) = 0$. Since $\pi \restriction A[\mathbf{B}_\tau] \in \mathrm{Rep}(A[\mathbf{B}_\tau])$, we have $\pi(x) = 0$, and so $x = 0$. Thus $p_{A[\mathbf{B}_\tau]}$ is a C^*-norm.

(iii) $\Rightarrow$ (ii) We first show that

$$\mathrm{Sp}_{A[\mathbf{B}_\tau]}(x) \subset \mathbb{R}_+ \equiv \{\lambda \in \mathbb{R} : \lambda \geqslant 0\}, \quad \forall x \in A[\mathbf{B}_\tau]_+, \tag{2.2}$$

where $\mathrm{Sp}_{A[\mathbf{B}_\tau]}(x)$ stands for the spectrum of $x \in A[\mathbf{B}_\tau]$. In fact, take an arbitrary $x \in A[\mathbf{B}_\tau]_+$ and a net $\{x_\alpha\}$ in $(\mathcal{A}_0)_+$ that converges to x with respect to τ. Since $A[\mathbf{B}_\tau]$ is hermitian ([20], Corollary 3.4), it follows that $\mathrm{Sp}_{A[\mathbf{B}_\tau]}(x) \subset \mathbb{R}$. Let $\lambda < 0$. Notice that $\lambda(\lambda 1 - x_\alpha)^{-1} \in \mathcal{U}(\mathcal{A}_0)$, for every α. Then for any τ-continuous seminorm p on $\widetilde{\mathcal{A}}_0[\tau]$

$$\begin{aligned}
&p(\lambda(\lambda 1 - x_\alpha)^{-1} - \lambda(\lambda 1 - x_\beta)^{-1}) \\
&\quad = |\lambda| p((\lambda 1 - x_\alpha)^{-1}(x_\alpha - x_\beta)(\lambda 1 - x_\beta)^{-1}) \\
&\quad \leqslant |\lambda| q((\lambda 1 - x_\alpha)^{-1}) q(x_\alpha - x_\beta) q((\lambda 1 - x_\beta)^{-1}) \\
&\quad \leqslant \frac{1}{|\lambda|}\gamma \|\lambda(\lambda 1 - x_\alpha)^{-1}\|_0 \|\lambda(\lambda 1 - x_\beta)^{-1}\|_0 q(x_\alpha - x_\beta) \\
&\quad \leqslant \frac{\gamma}{|\lambda|} q(x_\alpha - x_\beta)
\end{aligned}$$

for some constant $\gamma > 0$ and a τ-continuous seminorm q on $\widetilde{\mathcal{A}}_0[\tau]$. It follows that $\lambda(\lambda 1 - x_\alpha)^{-1}$ converges to an element y of $\mathbf{B}_\tau$ with respect to τ, which implies that $\lambda(\lambda 1 - x)^{-1}$ exists and equals y. Hence, $\lambda \notin \mathrm{Sp}_{A[\mathbf{B}_\tau]}(x)$. Thus, we have $\mathrm{Sp}_{A[\mathbf{B}_\tau]}(x) \subset \mathbb{R}_+$. Take an arbitrary $x \in A[\mathbf{B}_\tau]_+ \cap (-A[\mathbf{B}_\tau]_+)$. Then from (2.2), it follows that $\mathrm{Sp}_{A[\mathbf{B}_\tau]}(x) = \{0\}$, therefore $p_{A[\mathbf{B}_\tau]}(x) = r_{A[\mathbf{B}_\tau]}(x) = 0$. Since $p_{A[\mathbf{B}_\tau]}$ is a norm, we have $x = 0$.

(ii) $\Rightarrow$ (i) Take an arbitrary $a \in \widetilde{\mathcal{A}}_0[\tau]_+ \cap (-\widetilde{\mathcal{A}}_0[\tau]_+)$. Then from Lemma 2.1(i) it follows that $a(1 + a^2)^{-1} \in A[\mathbf{B}_\tau]_+ \cap (-A[\mathbf{B}_\tau]_+) = \{0\}$, which implies $a = 0$. This completes the proof. ∎

In the case of C^*-algebras (respectively pro-C^*-algebras), condition (ii) of Theorem 2.4, is always true. Also see Example 4.4 in Section 4. In the case of symmetric Banach $*$-algebras (respectively symmetric topological $*$-algebras), which in fact can be viewed as a generalization of C^*-algebras [28] (respectively pro-C^*-algebras), it seems that such a property has not been investigated. Some information about the set $\mathcal{A}_+$, with $\mathcal{A}$ a certain involutive algebra can be found in [14] and [29].

Question B. (i) Is $\mathcal{P}(\widetilde{\mathcal{A}}_0[\tau])$ τ-closed? That is, does the equality $\widetilde{\mathcal{A}}_0[\tau]_+ = \mathcal{P}(\widetilde{\mathcal{A}}_0[\tau])$ hold? Equivalently, for each net $\{x_\alpha\}$ in $(\mathcal{A}_0)_+$ which converges to $x \in \widetilde{\mathcal{A}}_0[\tau]$, is $\{x_\alpha^{1/2}\}$ τ-Cauchy?

(ii) Does one of the conditions in Theorem 2.4 always hold?

If $\widetilde{\mathcal{A}}_0[\tau]$ is a GB^*-algebra, then the above questions (i) and (ii) have positive answers. Does the converse hold? That is, the following question arises.

Question C. If the answer to Question B is affirmative, is then $\widetilde{\mathcal{A}}_0[\tau]$ a GB^*-algebra?

To consider Question C, we define an unbounded C^*-seminorm r_π of $\widetilde{\mathcal{A}}_0[\tau]$ induced by a $*$-representation π of $\widetilde{\mathcal{A}}_0[\tau]$ as follows:

$$\mathcal{D}(r_\pi) = \widetilde{\mathcal{A}}_0[\tau]_b^\pi := \{x \in \widetilde{\mathcal{A}}_0[\tau] : \overline{\pi(x)} \in \mathcal{B}(\mathcal{H}_\pi)\},$$
$$r_\pi(x) = \|\overline{\pi(x)}\|, \quad x \in \mathcal{D}(r_\pi).$$

Then we have the next:

LEMMA 2.5. *Let π be a faithful $*$-representation of $\widetilde{\mathcal{A}}_0[\tau]$ and $\mathbf{B}$ any element of $\mathcal{B}^*$ containing $\mathcal{U}(\mathcal{A}_0)$. Then the following statements hold:*

(i) *$\mathcal{A}_0 \subset A[\mathbf{B}_\tau] \subset A[\mathbf{B}] \subset \mathcal{D}(r_\pi) = \widetilde{\mathcal{A}}_0[\tau]_b^\pi$ and $\|\pi(x)\| \leqslant \|x\|_{\mathbf{B}}, \forall x \in A[\mathbf{B}]$, as well as $\|\pi(x)\| = \|x\|_{\mathbf{B}_\tau} = \|x\|_0, \forall x \in \mathcal{A}_0$.*

(ii) *$\pi(A[\mathbf{B}])$ is τ_{s^*}-dense in $\pi(\widetilde{\mathcal{A}}_0[\tau])$, and it is also uniformly dense in $\pi(\widetilde{\mathcal{A}}_0[\tau]_b^\pi)$.*

(iii) *Suppose π is $(\tau - \tau_{\mathrm{w}})$-continuous. Then $\pi(\widetilde{\mathcal{A}}_0[\tau]_+) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))_+ \equiv \{X \in \mathcal{L}^\dagger(\mathcal{D}(\pi)) : X \geqslant 0\}$.*

Proof. (i) is easily shown.

(ii) Take an arbitrary $a \in \widetilde{\mathcal{A}}_0[\tau]$. Then it follows that

$$(1 + \varepsilon a^* a)^{-1} a = \frac{1}{\sqrt{\varepsilon}}(1 + (\sqrt{\varepsilon} a)^*(\sqrt{\varepsilon} a))^{-1}(\sqrt{\varepsilon} a) \in A[\mathbf{B}_\tau], \quad \forall \varepsilon > 0$$

and for each $\xi \in \mathcal{D}(\pi)$

$$\begin{aligned}\|\pi((1 + \varepsilon a^* a)^{-1} a)\xi - \pi(a)\xi\| &= \varepsilon \|\pi((1 + \varepsilon a^* a)^{-1})\pi(a^* a^2)\xi\| \\ &\leqslant \varepsilon \|\pi((1 + \varepsilon a^* a)^{-1})\| \|\pi(a^* a^2)\xi\| \\ &\leqslant \varepsilon \|(1 + \varepsilon a^* a)^{-1}\|_{B_\tau} \|\pi(a^* a^2)\xi\| \\ &\leqslant \varepsilon \|\pi(a^* a^2)\xi\| \underset{\varepsilon \downarrow 0}{\longrightarrow} 0,\end{aligned}$$

so that $\pi(A[\mathbf{B}_\tau])$ is τ_{s^*}-dense in $\pi(\widetilde{\mathcal{A}}_0[\tau])$. Take an arbitrary $a \in \widetilde{\mathcal{A}}_0[\tau]_b^\pi$. Then since

$$\|\pi((1+\varepsilon a^*a)^{-1}a)\xi - \pi(a)\xi\| \leqslant \varepsilon\|\pi(a^*a^2)\|\|\xi\|$$

for each $\xi \in \mathcal{D}(\pi)$, it follows that $\lim_{\varepsilon\downarrow 0} \pi((1+\varepsilon a^*a)^{-1}a) = \pi(a)$ uniformly, which implies that $\pi(A[\mathbf{B}_\tau])$ is uniformly dense in $\pi(\widetilde{\mathcal{A}}_0[\tau]_b^\pi)$. Since $A[\mathbf{B}_\tau] \subset A[\mathbf{B}]$, (ii) follows.

(iii) This follows from $(\tau - \tau_{\mathrm{w}})$-continuity of π and $\pi((\mathcal{A}_0)_+) \subset \mathcal{L}^\dagger(\mathcal{D}(\pi))_+$. This completes the proof. ∎

We simply sketch how Lemma 2.5 looks:

$$\begin{array}{llcl}
\pi: & \widetilde{\mathcal{A}}_0[\tau] & \longrightarrow & \pi(\widetilde{\mathcal{A}}_0[\tau]) \\
 & \cup & & \cup \quad \tau_{s^*}\text{-dense} \\
 & \widetilde{\mathcal{A}}_0[\tau]_b^\pi & \longrightarrow & \pi(\widetilde{\mathcal{A}}_0[\tau]_b^\pi) \\
 & \cup & & \cup \quad \text{uniformly dense} \\
 & A[\mathbf{B}_\tau] & \longrightarrow & \pi(A[\mathbf{B}_\tau]) \\
 & \text{symmetric} & & \\
 & \text{Banach } *\text{-algebra} & & \\
 & \cup & & \cup \\
 & \mathcal{A}_0[\|\cdot\|] & \longrightarrow & \pi(\mathcal{A}_0) \\
 & C^*\text{-algebra} & & C^*\text{-algebra on } \mathcal{H}_\pi\ .
\end{array}$$

The following theorem gives an answer to Question C.

THEOREM 2.6. *The following statements are equivalent:*

(i) $\widetilde{\mathcal{A}}_0[\tau]$ *is a* GB^**-algebra.*

(ii) *There exists a faithful* $(\tau - \tau_{s^*})$*-continuous* $*$*-representation* π *of* $\widetilde{\mathcal{A}}_0[\tau]$*, such that* $\tau \prec r_\pi$.

Proof. (i) ⇒ (ii) Suppose $\widetilde{\mathcal{A}}_0[\tau]$ is a GB^*-algebra over $\mathbf{B}_0$. Since $A[\mathbf{B}_\tau]_+ \cap (-A[\mathbf{B}_\tau]_+) \subset A[\mathbf{B}_0]_+ \cap (-A[\mathbf{B}_0]_+) = \{0\}$, Theorm 2.4 implies the existence of a faithful $(\tau - \tau_{s^*})$-continuous $*$-representation of $\widetilde{\mathcal{A}}_0[\tau]$. Furthermore, since $\pi(A[\mathbf{B}_0])$ is a C^*-algebra, Lemma 2.5(ii) yields that

$$\pi(A[\mathbf{B}_0]) = \pi(\widetilde{\mathcal{A}}_0[\tau]_b^\pi) \quad \text{and} \quad r_\pi(x) = \|\pi(x)\| = \|x\|_{\mathbf{B}_0}, \quad \forall x \in \mathcal{D}(r_\pi),$$

which implies $\tau \prec r_\pi$.

(ii) ⇒ (i) Since $\tau \prec r_\pi$ and π is $(\tau - \tau_{s^*})$-continuous, it follows that τ and r_π are compatible, whence one gets that the completion $\mathcal{A}_{r_\pi}$ of $\mathcal{D}(r_\pi)[r_\pi]$ is embedded in $\widetilde{\mathcal{A}}_0[\tau]$. We denote by $\mathbf{B}_0$ the τ-closure of the unit ball $\mathcal{U}(\mathcal{A}_{r_\pi})$ of the

C^*-algebra A_{r_π}. Then $\mathbf{B}_0 \in \mathcal{B}^*$ and from Lemma 2.5(i) we get

$$\mathbf{B} \subset \mathcal{U}(\widetilde{\mathcal{A}}_0[\tau]_b^\pi) \subset \mathbf{B}_0, \quad \forall \mathbf{B} \in \mathcal{B}^*,$$

which implies that $\mathbf{B}_0 = \mathcal{U}(\widetilde{\mathcal{A}}_0[\tau]_b^\pi)$, with $\mathbf{B}_0$ the greatest member in $\mathcal{B}^*$. Thus, from Theorem 2.2, we conclude that $\widetilde{\mathcal{A}}_0[\tau]$ is a GB^*-algebra over $\mathcal{U}(\widetilde{\mathcal{A}}_0[\tau]_b^\pi)$ and this completes the proof. ∎

It is known that every $*$-representation π of a Fréchet $*$-algebra $\mathcal{A}[\tau]$ is $(\tau - \tau_{s^*})$-continuous. Indeed, take an arbitrary $\xi \in \mathcal{D}(\pi)$ and put $f_\xi(x) := (\pi(x)\xi|\xi)$, $x \in \mathcal{A}$. Then f_ξ is a positive linear functional on the Fréchet $*$-algebra $\mathcal{A}[\tau]$, which is continuous by Theorem 4.3 of [17]. Furthermore, since the multiplication of a Fréchet $*$-algebra is jointly continuous, it follows that π is $(\tau - \tau_{s^*})$-continuous. From this fact, as well as Theorem 2.6, we conclude the following:

COROLLARY 2.7. *Let $\widetilde{\mathcal{A}}_0[\tau]$ be a Fréchet $*$-algebra. Then the following are equivalent:*

(i) *$\widetilde{\mathcal{A}}_0[\tau]$ is a GB^*-algebra.*

(ii) *There exists a faithful $*$-representation π of $\widetilde{\mathcal{A}}_0[\tau]$ such that $\tau \prec r_\pi$.*

3. CASE 2

In this section we shall investigate the structure and the representation theory of $\widetilde{\mathcal{A}}_0[\tau]$ as it appears in Case 2 in the Introduction. First we recall some basic definitions and properties of partial $*$-algebras and quasi $*$-algebras (for more details, refer to [4]). A *partial $*$-algebra* is a vector space $\mathcal{A}$, endowed with a vector space involution $x \to x^*$ and a partial multiplication defined by a set $\Gamma \subset \mathcal{A} \times \mathcal{A}$ (a binary relation) with the properties:

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$.

(ii) $(x, y_1), (x, y_2) \in \Gamma$ implies $(x, \lambda y_1 + \mu y_2) \in \Gamma$ for all $\lambda, \mu \in \mathbb{C}$.

(iii) For any $(x, y) \in \Gamma$, a multiplication $xy \in \mathcal{A}$, is defined on $\mathcal{A}$, which is distributive with respect to addition and satisfies the relation $(xy)^* = y^*x^*$. Whenever $(x, y) \in \Gamma$, we say that x is a *left multiplier* of y and y is a *right multiplier* of x, and write $x \in L(y)$ respectively $y \in R(x)$.

Let $\mathcal{A}$ be a vector space and let $\mathcal{A}_0$ be a subspace of $\mathcal{A}$, which is also a $*$-algebra. $\mathcal{A}$ is said to be a *quasi $*$-algebra* with distinguished $*$-algebra $\mathcal{A}_0$ (or, simply, over $\mathcal{A}_0$) if

(i_1) the left multiplication ax and the right multiplication xa of an element a of $\mathcal{A}$ with an element x of $\mathcal{A}_0$, that extend the multiplication of $\mathcal{A}_0$, are always defined and are bilinear;

(i_2) $x_1(x_2a) = (x_1x_2)a$, $(ax_1)x_2 = a(x_1x_2)$ and $x_1(ax_2) = (x_1a)x_2$, for any $x_1, x_2 \in \mathcal{A}_0$ and $a \in \mathcal{A}$;

(i_3) an involution $*$ that extends the involution of $\mathcal{A}_0$ is defined in $\mathcal{A}$ with the property $(ax)^* = x^*a^*$ and $(xa)^* = a^*x^*$ for each $x \in \mathcal{A}_0$ and $a \in \mathcal{A}$.

Let $\mathcal{A}_0[\tau]$ be a locally convex $*$-algebra. Then the completion $\widetilde{\mathcal{A}}_0[\tau]$ of $\mathcal{A}_0[\tau]$ is a quasi $*$-algebra over $\mathcal{A}_0$ equipped with the following left and right multiplications:

$$ax := \lim_\alpha x_\alpha x \quad \text{and} \quad xa := \lim_\alpha x x_\alpha, \quad \forall\, x \in \mathcal{A}_0 \ \text{and}\ a \in \mathcal{A},$$

where $\{x_\alpha\}$ is a net in $\mathcal{A}_0$ converging to a with respect to the topology τ. Furthermore, the left and right multiplications are separately continuous. A $*$-invariant subspace $\mathcal{A}$ of $\widetilde{\mathcal{A}}_0[\tau]$ containing $\mathcal{A}_0$ is said to be a (*quasi-*) *$*$-subalgebra* of $\widetilde{\mathcal{A}}_0[\tau]$ if ax and xa belong to $\mathcal{A}$ for any $x \in \mathcal{A}_0$ and $a \in \mathcal{A}$. Then it is readily shown that $\mathcal{A}$ is a quasi $*$-algebra over $\mathcal{A}_0$. Moreover, $\mathcal{A}[\tau]$ is a locally convex space containing $\mathcal{A}_0$ as a dense subspace and the right and left multiplications are separately continuous. Such an algebra $\mathcal{A}$ is said to be a *locally convex quasi $*$-algebra* over $\mathcal{A}_0$.

Concerning $*$-representations of partial $*$-algebras and quasi $*$-algebras, start with a dense subspace $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ and denote by $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the set of all linear operators X from $\mathcal{D}$ to $\mathcal{H}$ such that $\mathcal{D}(X^*) \supset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a partial $*$-algebra with respect to the usual sum, scalar multiplication and involution $X^\dagger = X^*\lceil_{\mathcal{D}}$ and the (weak) partial multiplication $X \square Y = X^{\dagger *}Y$, defined whenever X is a left multiplier of Y ($X \in L(Y)$), that is, if and only if $Y\mathcal{D} \subset \mathcal{D}(X^{\dagger *})$ and $X^\dagger \mathcal{D} \subset \mathcal{D}(Y^*)$. A (partial) $*$-subalgebra of the partial $*$-algebra $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is said to be a *partial O*-algebra* on $\mathcal{D}$. A *$*$-representation* of a partial $*$-algebra $\mathcal{A}$ is a $*$-homomorphism π of $\mathcal{A}$ into a partial O*-algebra $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, in the sense of Definition 2.1.6 in [4], satisfying $\pi(1) = I$, whenever $1 \in \mathcal{A}$.

In this case too, the spaces $\mathcal{D}$ and $\mathcal{H}$ will be denoted by $\mathcal{D}(\pi)$ and $\mathcal{H}_\pi$ respectively. The algebraic conjugate dual $\mathcal{D}^\dagger$ of $\mathcal{D}$ (i.e., the set of all conjugate linear functionals on $\mathcal{D}$) becomes a vector space in a natural way. Denote by $\mathcal{L}(\mathcal{D},\mathcal{D}^\dagger)$ the set of all linear maps from $\mathcal{D}$ to $\mathcal{D}^\dagger$. Then $\mathcal{L}(\mathcal{D},\mathcal{D}^\dagger)$ is a $*$-invariant vector space under the usual operations and the involution $T \to T^\dagger$ with $\langle T^\dagger\xi,\eta\rangle := \overline{\langle T\eta,\xi\rangle}, \xi,\eta \in \mathcal{D}$, where $\langle T^\dagger\xi,\eta\rangle \equiv T^\dagger\xi(\eta)$. Any linear operator X defined on $\mathcal{D}$ is regarded as an element of $\mathcal{L}(\mathcal{D},\mathcal{D}^\dagger)$ such that $\langle X\xi,\eta\rangle = (X\xi|\eta)$, $\xi,\eta \in \mathcal{D}$. For $\mathcal{L}(\mathcal{D},\mathcal{D}^\dagger)$ we have the following:

LEMMA 3.1. (i) *$\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is regarded as a $*$-subalgebra of $\mathcal{L}(\mathcal{D},\mathcal{D}^\dagger)$.*

(ii) *For any $X \in \mathcal{L}^\dagger(\mathcal{D})$ and $T \in \mathcal{L}(\mathcal{D},\mathcal{D}^\dagger)$ we may define the multiplications $X \circ T$ and $T \circ X$ by*

$$\langle X \circ T\xi,\eta\rangle := \langle T\xi, X^\dagger\eta\rangle \quad \text{and} \quad \langle T \circ X\xi,\eta\rangle := \langle TX\xi,\eta\rangle;$$

under these multiplications, $\mathcal{L}(\mathcal{D},\mathcal{D}^\dagger)$ is a quasi $*$-algebra over $\mathcal{L}^\dagger(\mathcal{D})$.

(iii) *The locally convex topology τ_w on $\mathcal{L}(\mathcal{D},\mathcal{D}^\dagger)$ is defined by the family $\{p_{\xi,\eta}(\cdot) : \xi,\eta \in \mathcal{D}\}$ of seminorms with $p_{\xi,\eta}(T) := |\langle T\xi,\eta\rangle|$, $T \in \mathcal{L}(\mathcal{D},\mathcal{D}^\dagger)$, and it is called* weak topology. *It particular,*

$$\mathcal{L}(\mathcal{D},\mathcal{D}^\dagger) = \text{ the set of all sesquilinear forms on } \mathcal{D}\times\mathcal{D} = \widetilde{\mathcal{L}^\dagger(\mathcal{D})}[\tau_{\mathrm{w}}]$$

and $\mathcal{L}(\mathcal{D},\mathcal{D}^\dagger)[\tau_{\mathrm{w}}]$ *is a locally convex quasi* $*$*-algebra over* $\mathcal{L}^\dagger(\mathcal{D})$*. More generally, for any* O^**-algebra* $\mathcal{M}$ *on* $\mathcal{D}$*,* $\widetilde{\mathcal{M}}[\tau_{\mathrm{w}}]$ *is a locally convex quasi* $*$*-algebra over* $\mathcal{M}$*.*

A quasi $*$*-representation of a quasi* $*$*-algebra* $\mathcal{A}$ *over* $\mathcal{A}_0$ is naturally defined as a linear map π of $\mathcal{A}$ into a quasi $*$-algebra $\mathcal{L}(\mathcal{D},\mathcal{D}^\dagger)$ over $\mathcal{L}^\dagger(\mathcal{D})$ such that:

(i) π is a $*$-representation of the $*$-algebra $\mathcal{A}_0$;
(ii) $\pi(a)^\dagger = \pi(a^*), \forall a \in \mathcal{A}$;
(iii) $\pi(ax) = \pi(a)\circ\pi(x)$ and $\pi(xa) = \pi(x)\circ\pi(a), \forall a\in\mathcal{A}, \forall x\in\mathcal{A}_0$.

It is easily shown that if π is a quasi $*$-representation of $\mathcal{A}$, then $\pi(\mathcal{A})$ is a quasi $*$-algebra over $\pi(\mathcal{A}_0)$.

LEMMA 3.2. *Let* $\mathcal{A}[\tau]$ *be a locally convex quasi* $*$*-algebra over* $\mathcal{A}_0$ *and* π *a quasi* $*$*-representation of* $\mathcal{A}$*. Suppose* π *is* $(\tau-\tau_{\mathrm{w}})$*-continuous. Then* $\pi(\mathcal{A})$ *is a locally convex quasi* $*$*-algebra over* $\pi(\mathcal{A}_0)$*.*

Proof. From Lemma 3.1(iii) and the $(\tau-\tau_{\mathrm{w}})$-continuity of π we have

$$\pi(\mathcal{A}_0)\subset\pi(\mathcal{A})\subset\widetilde{\pi(\mathcal{A}_0)}[\tau_{\mathrm{w}}] \text{ and}$$
$$\pi(x)\circ\pi(a) = \pi(xa), \quad \pi(a)\circ\pi(x) = \pi(ax)$$

for each $a\in\mathcal{A}$ and $x\in\mathcal{A}_0$, which implies that $\pi(\mathcal{A})$ is a quasi $*$-subalgebra of $\widetilde{\pi(\mathcal{A}_0)}[\tau_{\mathrm{w}}]$. Hence, $\pi(\mathcal{A})$ is a locally convex quasi $*$-algebra over $\pi(\mathcal{A}_0)$. So the proof is complete. ∎

Let $\mathcal{A}_0[\|\cdot\|_0]$ be a C^*-algebra with *1* and τ a locally convex topology on $\mathcal{A}_0$ such that $\tau \prec \|\cdot\|_0$ and $\mathcal{A}_0[\tau]$ a locally convex $*$-algebra whose multiplication is not jointly continuous.

In general, $\widetilde{\mathcal{A}}_0[\tau]$ is a quasi $*$-algebra over $\mathcal{A}_0$ (but not a $*$-algebra!). For this reason, the theory of quasi $*$-algebras must be used. We remark that for any $\mathcal{A}\in C^*(\mathcal{A}_0,\tau)$, $\widetilde{\mathcal{A}}[\tau] = \widetilde{\mathcal{A}}_0[\tau]$ as locally convex spaces, but $\widetilde{\mathcal{A}}[\tau]$ is different from $\widetilde{\mathcal{A}}_0[\tau]$ as a quasi $*$-algebra. Moreover, the wedge $\widetilde{\mathcal{A}}_0[\tau]_+$ of the quasi $*$-algebra $\widetilde{\mathcal{A}}_0[\tau]$ over $\mathcal{A}_0$, defined as the τ-closure of the positive cone $(\mathcal{A}_0)_+$, does not necessarily coincide with the wedge $\widetilde{\mathcal{A}}[\tau]_+$ of the quasi $*$-algebra $\widetilde{\mathcal{A}}[\tau]$ over $\mathcal{A}$, in contrast with Case 1 (see the discussion before Theorem 2.4).

A linear functional f on $\widetilde{\mathcal{A}}_0[\tau]$, such that $f(x)\geqslant 0$, for each $x\in\overline{\mathcal{A}}_0[\tau]_+$, is said to be *a strongly positive* linear functional on the *quasi* $*$*-algebra* $\widetilde{\mathcal{A}}_0[\tau]$ *over* $\mathcal{A}_0$. Regarding the representation theory of $\widetilde{\mathcal{A}}_0[\tau]$ we have the next:

THEOREM 3.3. *The following statements are equivalent:*

(i) $\widetilde{\mathcal{A}}_0[\tau]_+\cap(-\widetilde{\mathcal{A}}_0[\tau]_+) = \{0\}$.
(ii) *There exists a faithful* $(\tau-\tau_{\mathrm{w}})$*-continuous quasi* $*$*-representation of the quasi* $*$*-algebra* $\widetilde{\mathcal{A}}_0[\tau]$ *over* $\mathcal{A}_0$*.*

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{F}$ be the set of all τ-continuous strongly positive linear functionals on the quasi $*$-algebra $\widetilde{\mathcal{A}}_0[\tau]$ over $\mathcal{A}_0$. For any $f \in \mathcal{F}$ we denote by $(\pi_f, \lambda_f, \mathcal{H}_f)$ the GNS-construction for $f \upharpoonright \mathcal{A}_0$. Let $f \in \mathcal{F}$. For any $a \in \widetilde{\mathcal{A}}_0[\tau]$ we put

$$\langle \widetilde{\lambda}_f(a), \lambda_f(x) \rangle = f(x^* a), \quad x \in \mathcal{A}_0.$$

Then since f is τ-continuous, it follows that

$$|f(x^* a)|^2 = \lim_\alpha |f(x^* x_\alpha)|^2 \leqslant \lim_\alpha f(x^* x) f(x_\alpha^* x_\alpha),$$

for each $a \in \widetilde{\mathcal{A}}_0[\tau]$ and $x \in \mathcal{A}_0$, where $\{x_\alpha\}$ is a net in $\mathcal{A}_0$ converging to a with respect to τ; it follows that $\widetilde{\lambda}_f(a)$ is well-defined and belongs to the algebraic conjugate dual $\lambda_f(\mathcal{A}_0)^\dagger$ of the vector space $\lambda_f(\mathcal{A}_0)$. It is clear that $\widetilde{\lambda}_f$ is a linear map of $\widetilde{\mathcal{A}}_0[\tau]$ into the vector space $\lambda_f(\mathcal{A}_0)^\dagger$, which is an extension of λ_f. Put

$$\mathcal{D}(\pi) := \Big\{ (\lambda_f(x_f))_{f \in \mathcal{F}} \in \bigoplus_{f \in \mathcal{F}} \mathcal{H}_f : x_f \in \mathcal{A}_0 \text{ and } \lambda_f(x_f) = 0 \text{ except for a finite number of } f \in \mathcal{F} \Big\},$$

and for $(\lambda_f(x_f)) \in \mathcal{D}(\pi)$

$$\langle (\widetilde{\lambda}_f(a_f)), (\lambda_f(x_f)) \rangle = \sum_{f \in \mathcal{F}} \langle \widetilde{\lambda}_f(a_f), \lambda_f(x_f) \rangle = \sum_{f \in \mathcal{F}} f(x_f^* a_f), \quad a_f \in \widetilde{\mathcal{A}}_0[\tau].$$

Then $(\widetilde{\lambda}_f(a_f)) \in \mathcal{D}(\pi)^\dagger$. Furthermore, for any $a \in \mathcal{A}$, put

$$\pi(a)(\lambda_f(x_f)) = (\widetilde{\lambda}_f(a x_f)), \quad (\lambda_f(x_f)) \in \mathcal{D}(\pi).$$

It is easily shown that π is a quasi $*$-representation of the quasi $*$-algebra $\widetilde{\mathcal{A}}_0[\tau]$ over $\mathcal{A}_0$. Moreover, the $(\tau - \tau_{\mathrm{w}})$-continuity of π follows from

$$\langle \pi(a)(\lambda_f(x_f)), (\lambda_f(y_f)) \rangle = \sum_{f \in \mathcal{F}} f(y_f^* a x_f),$$

for any $a \in \mathcal{A}$, $(\lambda_f(x_f))$ and $(\lambda_f(y_f))$ in $\mathcal{D}(\pi)$ and from the τ-continuity of $f \in \mathcal{F}$. The faithfulness of π is shown in a similar way as in the proof of Theorem 2.4(i) $\Rightarrow$ (v).

(ii) $\Rightarrow$ (i) Let π be a faithful $(\tau - \tau_{\mathrm{w}})$-continuous quasi $*$-representation of $\widetilde{\mathcal{A}}_0[\tau]$ and $a \in \widetilde{\mathcal{A}}_0[\tau]_+ \cap (-\widetilde{\mathcal{A}}_0[\tau]_+)$. Then there is a net $\{x_\alpha\}$ in $(\mathcal{A}_0)_+$ such that $x_\alpha \xrightarrow{\tau} a$. By the $(\tau - \tau_{\mathrm{w}})$-continuity of π we now have

$$\langle \pi(a)\xi, \xi \rangle = \lim_\alpha (\pi(x_\alpha)\xi | \xi) \geqslant 0 \quad \text{and similarly} \quad \langle \pi(-a)\xi, \xi \rangle \geqslant 0,$$

for each $\xi \in \mathcal{D}(\pi)$. Hence, $\langle \pi(a)\xi, \xi \rangle = 0$ for each $\xi \in \mathcal{D}(\pi)$, which implies $\langle \pi(a)\xi, \eta \rangle = 0$ for any $\xi, \eta \in \mathcal{D}(\pi)$, that is $\pi(a) = 0$. By the faithfulness of π we have $a = 0$. This completes the proof. ∎

It is natural to consider the question: When there exists a faithful $*$-representation π of the quasi $*$-algebra $\widetilde{\mathcal{A}}_0[\tau]$ over $\mathcal{A}_0$ (into $\mathcal{L}^\dagger(\mathcal{D}(\pi),\mathcal{H}_\pi)$)? For that, we define the following notion: A subset $\mathcal{G}$ of $\mathcal{F}$ is said to be *separating* if $a \in \widetilde{\mathcal{A}}_0[\tau]$ with $f(a)=0$, for each $f \in \mathcal{G}$, implies $a=0$. For example, if $\mathcal{F}$ is separating and $\mathcal{G}$ is dense in $\mathcal{F}$ with respect to the weak*-topology, then $\mathcal{G}$ is separating.

PROPOSITION 3.4. *The following statements are equivalent:*

(i) *There exists a faithful $(\tau-\tau_{\mathrm{w}})$-continuous $*$-representation π of the quasi $*$-algebra $\widetilde{\mathcal{A}}_0[\tau]$ over $\mathcal{A}_0$ (into $\mathcal{L}^\dagger(\mathcal{D}(\pi),\mathcal{H}_\pi)$).*

(ii) $\widetilde{\mathcal{A}}_0[\tau]_+ \cap (-\widetilde{\mathcal{A}}_0[\tau]_+) = \{0\}$ *and $\mathcal{F}_{\mathrm{b}}$ is separating, where*

$$\mathcal{F}_{\mathrm{b}} = \{f \in \mathcal{F} : \forall a \in \widetilde{\mathcal{A}}_0[\tau]\ {}^\exists\gamma_a > 0 \textit{ with } |f(a^*x)|^2 \leqslant \gamma_a f(x^*x), \forall x \in \mathcal{A}_0\}.$$

Proof. (i) $\Rightarrow$ (ii) By Theorem 3.3 we have $\widetilde{\mathcal{A}}_0[\tau]_+ \cap (-\widetilde{\mathcal{A}}_0[\tau]_+) = \{0\}$. For each $\xi \in \mathcal{D}(\pi)$ we put $f_\xi(a) = (\pi(a)\xi|\xi)$, $a \in \widetilde{\mathcal{A}}_0[\tau]$. Then it is easily shown that $\{f_\xi : \xi \in \mathcal{D}\}$ is contained in $\mathcal{F}_{\mathrm{b}}$ and it is separating by the faithfulness of π. Hence, $\mathcal{F}_{\mathrm{b}}$ is separating.

(ii) $\Rightarrow$ (i) As shown in the proof of (i) $\Rightarrow$ (ii) in Theorem 3.3, $\widetilde{\lambda}_f(a) \in \lambda_f(\mathcal{A}_0)^\dagger$ for each $f \in \mathcal{F}$ and $a \in \widetilde{\mathcal{A}}_0[\tau]$. Take arbitrary $f \in \mathcal{F}_{\mathrm{b}}$ and $a \in \widetilde{\mathcal{A}}_0[\tau]$. Then since

$$|\langle \widetilde{\lambda}_f(a), \lambda_f(x)\rangle|^2 = |f(x^*a)|^2 \leqslant \gamma_a f(x^*x),$$

for each $x \in \mathcal{A}_0$, it follows from the Riesz theorem that $\widetilde{\lambda}_f(a)$ is regarded as an element of $\mathcal{H}_f$. Now we put

$$\mathcal{D}(\pi) = \{(\lambda_f(x_f))_{f\in\mathcal{F}_{\mathrm{b}}} : x_f \in \mathcal{A}_0 \text{ and } \lambda_f(x_f) = 0 \\ \text{except for a finite number of } f \in \mathcal{F}_{\mathrm{b}}\}$$

and for $a \in \widetilde{\mathcal{A}}_0[\tau]$,

$$\pi(a)((\lambda_f(x_f))) = ((\widetilde{\lambda}_f(ax_f))), \quad (\lambda_f(x_f)) \in \mathcal{D}(\pi).$$

Then π is a $*$-representation of $\widetilde{\mathcal{A}}_0[\tau]$ into $\mathcal{L}^\dagger(\mathcal{D}(\pi),\mathcal{H}_\pi)$. Furthermore, by the τ-continuity of the elements of $\mathcal{F}_{\mathrm{b}}$ it is easily shown that π is $(\tau-\tau_{\mathrm{w}})$-continuous, while π is faithful since $\mathcal{F}_{\mathrm{b}}$ is separating. This completes the proof. ∎

4. EXAMPLES

In this section we give some examples, illustrating the results presented in Sections 2 and 3.

EXAMPLE 4.1. Let $\mathcal{A}[\tau]$ be a pro-C^*-algebra, or more generally a C^*-like locally convex $*$-algebra with a C^*-like family $\Gamma = \{p_\lambda\}_{\lambda\in\Lambda}$ of seminorms determining the topology τ. Then $p_\Gamma \equiv \sup\limits_\lambda p_\lambda$ is a C^*-norm on the C^*-algebra $\mathcal{A}_0 \equiv \mathcal{D}(p_\Gamma) := \{x \in \mathcal{A} : p_\Gamma(x) < \infty\}$ and $\mathcal{A} = \widetilde{\mathcal{A}}_0[\tau]$. In this case, $B_\tau \equiv \overline{\mathcal{U}(p_\Gamma)}^\tau = \mathcal{U}(p_\Gamma)$. Here we give a concrete example.

Let Ω be a locally compact space. We consider the following locally convex $*$-algebras of functions on Ω with the usual operations $f+g, \lambda f, fg$ and the complex conjugate as involution:

$C_0(\Omega)$: the C^*-algebra of all continuous functions on Ω which converge to 0 at the infinite point;

$C_{\rm b}(\Omega)$: the C^*-algebra of all continuous and bounded functions on Ω;

$B(\Omega)$: the C^*-algebra of all bounded functions on Ω;

$C(\Omega)$: the pro-C^*-algebra of all continuous functions on Ω equipped with the locally uniform topology $\tau_{\rm lu}$ defined by the family $\{\|\cdot\|_K : K$ a compact subset of $\Omega\}$ of C^*-seminorms with $\|f\|_K := \sup_{t\in K} |f(t)|$;

$F(\Omega)$: the pro-C^*-algebra of all functions on Ω with the simple convergence topology $\tau_{\rm s}$ defined by the family of C^*-seminorms $\{p_t : t \in \Omega\}$ with $p_t(f) := |f(t)|$.

Then

$$\begin{array}{ccccccccc} C_0(\Omega) & \subset & C_{\rm b}(\Omega) & \subset & C(\Omega) & = & \widetilde{C_0(\Omega)}[\tau_{\rm lu}] & = & \widetilde{C_{\rm b}(\Omega)}[\tau_{\rm lu}] \\ & & \cap & & & & \cap & & \\ & & B(\Omega) & \subset & \widetilde{B(\Omega)}[\tau_{\rm s}] & = & \widetilde{C_0(\Omega)}[\tau_{\rm s}] & = & \widetilde{C_{\rm b}(\Omega)}[\tau_{\rm s}] = \mathcal{F}(\Omega). \end{array}$$

EXAMPLE 4.2. Let $\mathcal{A}[\tau]$ be a GB^*-algebra over $\mathbf{B}_0$. Then $A[\mathbf{B}_0][\|\cdot\|_{\mathbf{B}_0}]$ is a C^*-algebra and $\widetilde{A[\mathbf{B}_0]}[\tau] = \widetilde{\mathcal{A}}[\tau]$. In this case, $\mathbf{B}_\tau = \overline{\mathcal{U}(A[\mathbf{B}_0])}^{\tau} = \mathcal{U}(A[\mathbf{B}_0])$. The Arens algebra (see [5]) $\mathcal{A} = L^\omega[0,1] := \bigcap_{1\leqslant p<\infty} L^p[0,1]$ is a GB^*-algebra with the usual operations $f+g, \lambda f, fg$, usual involution f^* and the topology τ_ω defined by the family $\{\|\cdot\|_p : 1 \leqslant p < \infty\}$ of the L^p-norms; moreover,

$$A[\mathbf{B}_0] = L^\infty[0,1] \subset L^\omega[0,1] = \widetilde{L^\infty[0,1]}[\tau_\omega]$$

and

$$\widetilde{L^\infty[0,1]}[\|\cdot\|_p] = L^p[0,1], \quad 1 \leqslant p \leqslant \infty,$$

where $L^p[0,1]$ is a Banach quasi $*$-algebra over $L^\infty[0,1]$.

EXAMPLE 4.3. (i) The $*$-algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$ is a locally convex $*$-algebra equipped with the weak topology $\tau_{\rm w}$. We investigate the structure of $\widetilde{\mathcal{B}(\mathcal{H})}[\tau_{\rm w}]$. Let $S(\mathcal{H})$ be the set of all sesquilinear forms on $\mathcal{H}\times\mathcal{H}$. Then $S(\mathcal{H})$ is a complete locally convex space under the weak topology $\tau_{\rm w}$ defined by the family $\{p_{\xi,\eta}(\cdot) : \xi,\eta \in \mathcal{H}\}$ of seminorms with $p_{\xi,\eta}(\varphi) = |\varphi(\xi,\eta)|, \varphi \in S(\mathcal{H})$. An element φ of $S(\mathcal{H})$ is said to be *bounded* if there exists a constant $\gamma > 0$ such that $|\varphi(\xi,\eta)| \leqslant \gamma\|\xi\|\|\eta\|$ for each $\xi,\eta \in \mathcal{H}$. Denote by $S_{\rm b}(\mathcal{H})$ the set of all bounded sesquilinear forms on $\mathcal{H}\times\mathcal{H}$, and put $S(\mathcal{H})_+ \equiv \{\varphi \in S(\mathcal{H}) : \varphi \geqslant 0$ if and only if $\varphi(\xi,\xi) \geqslant 0, \forall\xi \in \mathcal{H}\}$ and $S_{\rm b}(\mathcal{H})_+ \equiv \{\varphi \in S_{\rm b}(\mathcal{H}) : \varphi \geqslant 0\}$. It is easily shown that $\varphi \in S_{\rm b}(\mathcal{H})$ *if and only if* there exists an element A of $\mathcal{B}(\mathcal{H})$ such that $\varphi(\xi,\eta) = \varphi_A(\xi,\eta) := (A\xi|\eta)$ for any

$\xi, \eta \in \mathcal{H}$, and $\varphi \in S_{\rm b}(\mathcal{H})_+$ *if and only if* $A \geqslant 0$. Hence, $S_{\rm b}(\mathcal{H})[\tau_{\rm w}]$ is a locally convex $*$-algebra equipped with the multiplication $\varphi_A\varphi_B := \varphi_{AB}$ and the involution $\varphi_A^* := \varphi_{A^*}$; it is also isomorphic to the locally convex $*$-algebra $\mathcal{B}(\mathcal{H})[\tau_{\rm w}]$ with respect to the map $\mathcal{B}(\mathcal{H})[\tau_{\rm w}] \ni A \mapsto \varphi_A \in S_{\rm b}(\mathcal{H})[\tau_{\rm w}]$. So $\widetilde{\mathcal{B}(\mathcal{H})}[\tau_{\rm w}]$ is isomorphic to $\widetilde{S_{\rm b}(\mathcal{H})}[\tau_{\rm w}] = S(\mathcal{H})$ and it is a locally convex quasi $*$-algebra over $\mathcal{B}(\mathcal{H})$ under the multiplications

$$(\varphi \circ \varphi_A)(\xi,\eta) := \varphi(A\xi,\eta), \quad (\varphi_A \circ \varphi)(\xi,\eta) := \varphi(\xi, A^*\eta), \quad \xi,\eta \in \mathcal{H},$$

for $A \in \mathcal{B}(\mathcal{H})$ and $\varphi \in \widetilde{S_{\rm b}(\mathcal{H})}[\tau_{\rm w}]$. Furthermore, it is easily shown that

$$\widetilde{\mathcal{B}(\mathcal{H})}[\tau_{\rm w}]_+ \cap (-\widetilde{\mathcal{B}(\mathcal{H})}[\tau_{\rm w}]_+) = \{0\}.$$

(ii) Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$. We introduce on $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ the strong $*$-topology $\tau_{\rm s^*}^{\mathcal{D}}$ defined by the family $\{p_\xi, p_\xi^\dagger : \xi \in \mathcal{D}\}$ of seminorms with $p_\xi(X) := \|X\xi\|$, $p_\xi^\dagger(X) := \|X^\dagger\xi\|$, $X \in \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$. Then $(\widetilde{\mathcal{B}(\mathcal{H}) \restriction \mathcal{D}})\,[\tau_{\rm s^*}^{\mathcal{D}}] = \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, but $(\widetilde{\mathcal{B}(\mathcal{H}) \restriction \mathcal{D}})\,[\tau_{\rm s^*}^{\mathcal{D}}]$ is not a locally convex $*$-algebra, and so $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is not a locally convex $*$-algebra over $\mathcal{B}(\mathcal{H}) \restriction \mathcal{D}$. We put

$$\mathcal{B}(\mathcal{D}) := \{A \restriction \mathcal{D} : A \in \mathcal{B}(\mathcal{H}), A\mathcal{D} \subset \mathcal{D} \text{ and } A^*\mathcal{D} \subset \mathcal{D}\}.$$

Then $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$ is a quasi $*$-algebra over $\mathcal{B}(\mathcal{D})$, but as $\widetilde{\mathcal{B}(\mathcal{D})}[\tau_{\rm s^*}^{\mathcal{D}}] \subsetneq \mathcal{L}^\dagger(\mathcal{D},\mathcal{H})$, in general, $\mathcal{L}^\dagger(\mathcal{D},\mathcal{H})[\tau_{\rm s^*}^{\mathcal{D}}]$ is not necessarily a locally convex quasi $*$-algebra over $\mathcal{B}(\mathcal{D})$. Let H be an unbounded positive self-adjoint operator on $\mathcal{H}$ with $H \geqslant I$, $H = \int_1^\infty \lambda\, {\rm d}E_H(\lambda)$ the spectral resolution of H and $\mathcal{D}^\infty(H) = \bigcap_{n=1}^\infty \mathcal{D}(H^n)$. Then for any $A \in \mathcal{B}(\mathcal{H})$, $E_H(n)AE_H(n) \in \mathcal{B}(\mathcal{D}^\infty(H))$, for each $n \in \mathbb{N}$ and for $n \to \infty$ it converges to A with respect to $\tau_{\rm s^*}^{\mathcal{D}^\infty(H)}$; so $\mathcal{L}^\dagger(\mathcal{D}^\infty(H),\mathcal{H})[\tau_{\rm s^*}^{\mathcal{D}^\infty(H)}]$ is a locally convex quasi $*$-algebra over $\mathcal{B}(\mathcal{D}^\infty(H))$.

EXAMPLE 4.4. Let $\mathcal{A}_\flat$ be a unital C^*-algebra, with norm $\|\cdot\|_\flat$ and involution $\flat$. Let $\mathcal{A}[\|\cdot\|]$ be a right Banach module over the C^*-algebra $\mathcal{A}_\flat$, with isometric involution $*$ and such that $\mathcal{A}_\flat \subset \mathcal{A}$. Set $\mathcal{A}_\# = (\mathcal{A}_\flat)^*$. We say that $\{\mathcal{A}, *, \mathcal{A}_\flat, b\}$ is a *CQ*-algebra* if

(i) $\mathcal{A}_\flat$ is dense in $\mathcal{A}$ with respect to its norm $\|\cdot\|$;
(ii) $\mathcal{A}_0 \equiv \mathcal{A}_\flat \cap \mathcal{A}_\#$ is dense in $\mathcal{A}_\flat$ with respect to its norm $\|\cdot\|_\flat$;
(iii) $(xy)^* = y^*x^*, \forall x,y \in \mathcal{A}_0$;
(iv) $\|x\|_\flat = \sup\limits_{a\in\mathcal{A},\|a\|\leqslant 1} \|ax\|, x \in \mathcal{A}_\flat$.

Since $*$ is isometric, it is easy to see that the space $\mathcal{A}_\#$ itself is a C^*-algebra with respect to the involution $x^\# \equiv (x^*)^{\flat *}$ and the norm $\|x\|_\# \equiv \|x^*\|_\flat$. A CQ*-algebra is called *proper* if $\mathcal{A}_\# = \mathcal{A}_\flat$. For CQ*-algebras we refer to [9], [10].

Let $\{\mathcal{A}, *, \mathcal{A}_\flat, \flat\}$ be a proper CQ*-algebra. Then we have

$$\|xy\| \leqslant \|x\|\|y\|_\flat, \quad \|xy\| \leqslant \|y\|\|x\|_\#, \quad \|x^*\| = \|x\|, \quad \text{and} \quad (xy)^* = y^*x^*,$$

for any $x, y \in \mathcal{A}_\flat$, and so $\mathcal{A}_\flat[\|\cdot\|]$ is a locally convex $*$-algebra with the involution $*$. Furthermore, since $\mathcal{A} = \widetilde{\mathcal{A}_\flat}[\|\cdot\|]$, it follows that $\mathcal{A}[\|\cdot\|]$ is a locally convex quasi $*$-algebra over $\mathcal{A}_\flat$. Consider the set $S_\flat(\mathcal{A})_+$ of all sesquilinear forms φ on $\mathcal{A} \times \mathcal{A}$ such that:

(i_1) $\varphi(a,a) \geqslant 0, \forall a \in \mathcal{A}$;
(i_2) $\varphi(ax,y) = \varphi(x,a^*y), \forall a \in \mathcal{A}, \forall x,y \in \mathcal{A}_\flat$;
(i_3) $|\varphi(a,b)| \leqslant \|a\|\|b\|, \forall a,b \in \mathcal{A}$.

Then $(\mathcal{A}, *, \mathcal{A}_\flat, \flat)$ is called *$*$-semisimple* if $a \in \mathcal{A}$ and $\varphi(a,a) = 0$, for every $\varphi \in S_\flat(\mathcal{A})_+$, implies $a = 0$. Suppose $(\mathcal{A}, *, \mathcal{A}_\flat, \flat)$ is a $*$-semisimple proper CQ*-algebra. Then $\mathcal{A}_+ \cap (-\mathcal{A}_+) = \{0\}$. Indeed, for any $\varphi \in S_\flat(\mathcal{A})_+$ we define a strongly positive linear functional on the quasi $*$-algebra $\mathcal{A}$ over $\mathcal{A}_\flat$ by $f_\varphi(a) = \varphi(a, 1)$, $a \in \mathcal{A}$. Take an arbitrary $h \in \mathcal{A}_+ \cap (-\mathcal{A}_+)$. Then

$$f_\varphi(h) = \lim_{n\to\infty} f_\varphi(x_n) \geqslant 0,$$

where $\{x_n\} \subset (\mathcal{A}_\flat)_+$ converges to h with respect to $\|\cdot\|$. Thus, $f_\varphi(h) = 0$, for each $\varphi \in S_\flat(\mathcal{A})_+$. We want to prove that $\varphi(h,h) = 0$ for each $\varphi \in S_\flat(\mathcal{A})_+$. Let $x \in A_\flat$ with $\|x\| \leqslant 1$. Then we may define an element φ_x of $S_\flat(\mathcal{A})_+$ by $\varphi_x(a,b) = \varphi(ax,bx)$ with $a,b \in \mathcal{A}$. Hence, $\varphi(hx,x) = 0$ for each $x \in \mathcal{A}_\flat$, which implies that $\varphi(hx,y) = 0$ for all $x,y \in \mathcal{A}_\flat$. Thus,

$$\varphi(h,h) = \lim_{n\to\infty} \varphi(h,x_n) = 0, \quad \forall \varphi \in S_\flat(\mathcal{A})_+ \quad \text{and therefore} \quad h = 0,$$

from the $*$-semisimplity of $(\mathcal{A}, *, \mathcal{A}_\flat, \flat)$.

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FABIO BAGARELLO, DIPARTIMENTO DI METODI E MODELLI MATEMATICI, FACOLTÀ DI INGEGNERIA, UNIVERSITA DI PALERMO, PALERMO, I-90128, ITALY
E-mail address: bagarell@unipa.it

MARIA FRAGOULOPOULOU, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ATHENS, ATHENS, 15784, GREECE
E-mail address: fragoulop@math.uoa.gr

ATSUSHI INOUE, DEPARTMENT OF APPLIED MATHEMATICS, FUKUOKA UNIVERSITY, FUKUOKA, 814-0180, JAPAN
E-mail address: a-inoue@fukuoka-u.ac.jp

CAMILLO TRAPANI, DIPARTIMENTO DI MATEMATICA ED APPLICAZIONI, UNIVERSITA DI PALERMO, PALERMO, I-90123, ITALY
E-mail address: trapani@unipa.it

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