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THE LATTICE OF CLOSED IDEALS IN THE BANACH ALGEBRA OF OPERATORS ON A CERTAIN DUAL BANACH SPACE

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ABSTRACT. We determine the closed operator ideals of the Banach space

$$(\ell_2^1 \oplus \ell_2^2 \oplus \cdots \oplus \ell_2^n \oplus \cdots)_{\ell_1}.$$

KEYWORDS: *Ideal lattice, operator, Banach space, Banach algebra.*

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1. INTRODUCTION

The aim of this note is to classify the closed ideals in the Banach algebra $\mathscr{B}(F)$ of (bounded, linear) operators on the Banach space

$$F = (\ell_2^1 \oplus \ell_2^2 \oplus \cdots \oplus \ell_2^n \oplus \cdots)_{\ell_1}. \tag{1.1}$$

More precisely, we shall show that there are exactly four closed ideals in $\mathscr{B}(F)$, namely $\{0\}$, the compact operators $\mathscr{K}(F)$, the closure $\overline{\mathscr{G}}_{\ell_1}(F)$ of the set of operators factoring through ℓ_1, and $\mathscr{B}(F)$ itself.

The collection of Banach spaces E for which a classification of the closed ideals in $\mathscr{B}(E)$ exists is very sparse. Indeed, the following list appears to be the complete list of such spaces.

(i) For a finite-dimensional Banach space E, $\mathscr{B}(E)$ is isomorphic to the $(n \times n)$-matrices, where n is the dimension of E, and so it is ancient folklore that $\mathscr{B}(E)$ is simple in this case.

(ii) In 1941 Calkin [2] classified all the ideals in $\mathscr{B}(\ell_2)$. In particular he proved that there are only three closed ideals in $\mathscr{B}(\ell_2)$, namely $\{0\}$, $\mathscr{K}(\ell_2)$, and $\mathscr{B}(\ell_2)$.

(iii) In 1960 Gohberg, Markus, and Feldman [5] extended Calkin's theorem to the other classical sequence spaces. More precisely, they showed that $\{0\}$, $\mathscr{K}(E)$, and $\mathscr{B}(E)$ are the only closed ideals in $\mathscr{B}(E)$ for each of the spaces $E = c_0$ and $E = \ell_p$, where $1 \leqslant p < \infty$.

(iv) Later in the 1960's Gramsch [6] and Luft [10] independently extended Calkin's theorem in a different direction by classifying all the closed ideals in $\mathscr{B}(H)$ for each Hilbert space H (not necessarily separable). In particular, they showed that these ideals are well-ordered by inclusion.

(v) In 2003 Laustsen, Loy, and Read [8] proved that, for the Banach space

$$E = (\ell_2^1 \oplus \ell_2^2 \oplus \cdots \oplus \ell_2^n \oplus \cdots)_{c_0}, \tag{1.2}$$

there are exactly four closed ideals in $\mathscr{B}(E)$, namely $\{0\}$, the compact operators $\mathscr{K}(E)$, the closure $\overline{\mathscr{G}}_{c_0}(E)$ of the set of operators factoring through c_0, and $\mathscr{B}(E)$ itself.

Note that (1.1) is the dual Banach space of (1.2), and so the result of this note can be seen as a "dualization" of [8]. In fact, our strategy draws heavily on the methods introduced in [8]. However, the present case is more involved because in [8] it was possible to restrict attention to block-diagonal operators of a special kind. In the Banach space (1.1), however, one cannot even reduce to operators with a "locally finite matrix" (due to the fact that the unit vector basis of ℓ_1 is not shrinking), and so a new trick is required (see Remark 2.14 for details).

(vi) In 2004 Daws [4] extended Gramsch and Luft's result to the Gohberg-Markus-Feldman case by classifying the closed ideals in $\mathscr{B}(E)$ for $E = c_0(\mathbb{I})$ and $E = \ell_p(\mathbb{I})$, where $\mathbb{I}$ is an index set of arbitrary cardinality and $1 \leqslant p < \infty$. Again, these ideals are well-ordered by inclusion.

2. THE CLASSIFICATION THEOREM

All Banach spaces are assumed to be over the same scalar field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We denote by I_E the identity operator on the Banach space E.

We begin by recalling various definitions and results concerning ℓ_1-direct sums and operators between them.

DEFINITION 2.1. Let (E_n) be a sequence of Banach spaces. We denote by $\big(\bigoplus E_n\big)_{\ell_1}$ or $(E_1 \oplus E_2 \oplus \cdots)_{\ell_1}$ the ℓ_1-direct sum of $E_1, E_2, \ldots$, that is, the collection of sequences (x_n) such that $x_n \in E_n$ for each $n \in \mathbb{N}$ and

$$\|(x_n)\| \overset{\text{defn}}{=} \sum_{n=1}^{\infty} \|x_n\| < \infty. \tag{2.1}$$

This is a Banach space for coordinate-wise defined vector space operations and norm given by (2.1).

Set $E = \big(\bigoplus E_n\big)_{\ell_1}$. For each $m \in \mathbb{N}$, we write J_m^E (or just J_m) for the canonical embedding of E_m into E and Q_m^E (or Q_m) for the canonical projection of E onto E_m. Both J_m^E and Q_m^E are operators of norm one; in fact, the former is an isometry, and the latter is a quotient map.

We use similar notation for finite direct sums.

DEFINITION 2.2. For each $n \in \mathbb{N}$, let $T_n\colon E_n \to F_n$ be an operator, where E_n and F_n are Banach spaces. Suppose that $\sup \|T_n\| < \infty$. Then we can define the *diagonal operator*

$$\operatorname{diag}(T_n)\colon \ \big(\bigoplus E_n\big)_{\ell_1} \to \big(\bigoplus F_n\big)_{\ell_1}, \quad (x_n) \mapsto (T_n x_n).$$

Clearly, we have $\|\operatorname{diag}(T_n)\| = \sup \|T_n\|$. In the finite case, we also use the notation $T_1 \oplus \cdots \oplus T_n$ for the diagonal operator mapping from $(E_1 \oplus \cdots \oplus E_n)_{\ell_1}$ to $(F_1 \oplus \cdots \oplus F_n)_{\ell_1}$.

DEFINITION 2.3. Let $T\colon \big(\bigoplus E_n\big)_{\ell_1} \to \big(\bigoplus F_n\big)_{\ell_1}$ be an operator, where (E_n) and (F_n) are sequences of Banach spaces. We associate with T the infinite matrix $(T_{m,n})$, where

$$T_{m,n} = Q_m^F T J_n^E\colon \ E_n \to F_m \quad (m, n \in \mathbb{N}).$$

The *support of the n^{th} column* of T is

$$\operatorname{colsupp}_n(T) = \{m \in \mathbb{N} : T_{m,n} \neq 0\} \quad (n \in \mathbb{N}).$$

We say that T has *finite columns* if each column has finite support.

The significance of operators with finite columns lies in the fact that, in the case where each of the spaces E_n $(n \in \mathbb{N})$ is finite-dimensional, given an operator $T\colon \big(\bigoplus E_n\big)_{\ell_1} \to \big(\bigoplus F_n\big)_{\ell_1}$, there is a compact operator $K\colon \big(\bigoplus E_n\big)_{\ell_1} \to \big(\bigoplus F_n\big)_{\ell_1}$ such that $T + K$ has finite columns; in fact K can be picked with arbitrarily small norm (see Lemma 2.7(i) in [8]).

We next introduce a parameter n_ε that is at the heart of our main result (Theorem 2.13). It is the dual version of the parameter m_ε that was introduced in [8].

DEFINITION 2.4. Let G be a closed subspace of a Hilbert space H. We denote by $G^\perp$ the orthogonal complement of G in H, and write proj_G for the orthogonal projection of H onto G (so that proj_G is the idempotent operator on H with image G and kernel $G^\perp$).

Let k be a positive integer, let E be a Banach space, let $H_1, \dots, H_k$ be Hilbert spaces, and denote by $\mathbb{N}_0$ the set of non-negative integers. For each $\varepsilon > 0$ and each operator $T\colon E \to (H_1 \oplus \cdots \oplus H_k)_{\ell_1}$, set

$$n_\varepsilon(T) = \sup \left\{ n \in \mathbb{N}_0 : \begin{array}{l} \|(\operatorname{proj}_{G_1^\perp} \oplus \cdots \oplus \operatorname{proj}_{G_k^\perp})T\| > \varepsilon \\ \text{whenever } G_j \subset H_j \text{ are subspaces} \\ \text{with } \dim G_j \leqslant n \text{ for } j = 1, \dots, k \end{array} \right\} \in \mathbb{N}_0 \cup \{\pm\infty\}.$$

The parameter n_ε gives quantitative information about certain factorizations. This is the content of parts (i) and (ii) of Lemma 2.5, which are dual to the corresponding statements about the parameter m_ε in Lemma 5.3 in [8]. We shall indeed prove parts (i) and (ii) via Lemma 5.3 in [8], but would like to emphasize that their proofs are fairly elementary (and indeed we could easily have translated them into direct proofs here). The important point in [8] is the definition of m_ε itself. Part (iii) of Lemma 2.5 has no counterpart in [8]; it will be used to deal with the extra difficulty that on ℓ_1-direct sums one has to consider operators whose matrices may have infinite rows.

LEMMA 2.5. *Let k be a positive integer, let $H, K_1, \dots, K_k$ be Hilbert spaces, let $T\colon H \to (K_1 \oplus \cdots \oplus K_k)_{\ell_1}$ be an operator, and let $0 < \varepsilon < \|T\|$.*

(i) *Suppose that $n_\varepsilon(T)$ is finite. Then there exist a positive integer d and operators $R\colon H \to \ell_1^d$ and $S\colon \ell_1^d \to (K_1 \oplus \cdots \oplus K_k)_{\ell_1}$ such that $\|R\| \leqslant \|T\| \sqrt{n_\varepsilon(T)+1}$, $\|S\| \leqslant 1$, and $\|T - SR\| \leqslant \varepsilon$.*

(ii) *For each positive integer $n \leqslant (1/2)\, n_\varepsilon(T) + 1$, there exist operators $U\colon \ell_2^n \to H$ and $V\colon (K_1 \oplus \cdots \oplus K_k)_{\ell_1} \to \ell_2^n$ such that $\|U\| \leqslant 1/\varepsilon$, $\|V\| \leqslant 1$, and $I_{\ell_2^n} = VTU$.*

(iii) *Let g be a positive integer, let H_0 be a closed subspace of finite codimension in H, and suppose that $n_\varepsilon(T) \geqslant \dim H_0^\perp + g$. Then $n_\varepsilon(T|_{H_0}) \geqslant g$.*

Proof. In Definition 5.2(ii) in [8] the quantity

$$m_\varepsilon(W) = \sup \left\{ m \in \mathbb{N}_0 : \begin{array}{l} \|W(\mathrm{proj}_{G_1^\perp} \oplus \cdots \oplus \mathrm{proj}_{G_k^\perp})\| > \varepsilon \\ \text{whenever } G_j \subset K_j \text{ are subspaces} \\ \text{with } \dim G_j \leqslant m \text{ for } j = 1, \dots, k \end{array} \right\} \tag{2.2}$$

in $\mathbb{N}_0 \cup \{\pm\infty\}$ is introduced for each operator $W\colon (K_1 \oplus \cdots \oplus K_k)_{\ell_\infty} \to H$. Making standard identifications of dual spaces, we may regard the adjoint of the operator $T\colon H \to (K_1 \oplus \cdots \oplus K_k)_{\ell_1}$ as an operator $T^*\colon (K_1 \oplus \cdots \oplus K_k)_{\ell_\infty} \to H$, where the subscript ℓ_∞ indicates that we equip the direct sum with the norm

$$\|(x_1, \dots, x_k)\| = \max\{\|x_1\|, \dots, \|x_k\|\} \quad (x_1 \in K_1, \dots, x_k \in K_k).$$

It follows that we may insert $W = T^*$ in (2.2). Standard properties of adjoint operators show that

$$m_\varepsilon(T^*) = n_\varepsilon(T). \tag{2.3}$$

We use this identity and Lemma 5.3 in [8] to prove (i) and (ii).

(i) Suppose that $n_\varepsilon(T) < \infty$. By (2.3) and Lemma 5.3(i) in [8], we can find a positive integer d and operators $A\colon (K_1 \oplus \cdots \oplus K_k)_{\ell_\infty} \to \ell_\infty^d$ and $B\colon \ell_\infty^d \to H$ such that $\|A\| \leqslant 1$, $\|B\| \leqslant \|T\| \sqrt{n_\varepsilon(T)+1}$, and $\|T^* - BA\| \leqslant \varepsilon$. Dualizing this gives us operators $R = B^*\colon H \to \ell_1^d$ and $S = A^*\colon \ell_1^d \to (K_1 \oplus \cdots \oplus K_k)_{\ell_1}$ such that (i) holds because the adjoint operation is antimultiplicative and an operator has the same norm as its adjoint.

(ii) Suppose that $n \leqslant (1/2)\, n_\varepsilon(T) + 1$. Then (2.3) and Lemma 5.3(ii) in [8] imply that there are operators $C\colon \ell_2^n \to (K_1 \oplus \cdots \oplus K_k)_{\ell_\infty}$ and $D\colon H \to \ell_2^n$ such that $\|C\| \leqslant 1$, $\|D\| \leqslant 1/\varepsilon$, and $I_{\ell_2^n} = DT^*C$. As before, we dualize this to obtain operators $U = D^*\colon \ell_2^n \to H$ and $V = C^*\colon (K_1 \oplus \cdots \oplus K_k)_{\ell_1} \to \ell_2^n$ such that (2.5) is satisfied.

(iii) For each $j = 1, \ldots, k$, let G_j be a subspace of K_j with $\dim G_j \leqslant g$, and define

$$F_j = G_j + Q_j T(H_0^\perp) \subset K_j.$$

Then F_j is finite-dimensional with $\dim F_j \leqslant n_\varepsilon(T)$, and so we can find a unit vector $x \in H$ such that $\|(\mathrm{proj}_{F_1^\perp} \oplus \cdots \oplus \mathrm{proj}_{F_k^\perp})Tx\| > \varepsilon$. It follows that

$$\begin{aligned}\|(\mathrm{proj}_{G_1^\perp} \oplus \cdots \oplus \mathrm{proj}_{G_k^\perp})T|_{H_0}\| &\geqslant \|(\mathrm{proj}_{G_1^\perp} \oplus \cdots \oplus \mathrm{proj}_{G_k^\perp})T(\mathrm{proj}_{H_0} x)\| \\ &\geqslant \|(\mathrm{proj}_{F_1^\perp} \oplus \cdots \oplus \mathrm{proj}_{F_k^\perp})T(\mathrm{proj}_{H_0} x)\| \\ &= \|(\mathrm{proj}_{F_1^\perp} \oplus \cdots \oplus \mathrm{proj}_{F_k^\perp})Tx\| > \varepsilon,\end{aligned}$$

and so $n_\varepsilon(T|_{H_0}) \geqslant g$. ∎

REMARK 2.6. Let T be an operator on $\left(\bigoplus K_n\right)_{\ell_1}$ with finite columns, where (K_n) is an (infinite) sequence of Hilbert spaces. As in Remark 5.4 in [8], there is a natural way to define $n_\varepsilon(TJ_m)$ for each $\varepsilon > 0$ and each $m \in \mathbb{N}$, namely by ignoring the cofinite number of Hilbert spaces K_k such that $Q_k T J_m = 0$.

The proof of our classification result (Theorem 2.13) has two non-trivial parts. The first part is done in Proposition 2.9 relying on older results. The second part is dealt with in Proposition 2.11 using the parameter n_ε and a small trick to take care of matrices with infinite rows. Before proceeding we prove a little lemma which will be useful in a number of places.

LEMMA 2.7. *Let $\mathscr{J}$ be an ideal in a Banach algebra $\mathscr{A}$. If $P \in \overline{\mathscr{J}}$ is idempotent, then in fact $P \in \mathscr{J}$.*

Proof. Let (T_n) be a sequence in $\mathscr{J}$ converging to P. Replacing T_n with PT_nP, we may assume that $T_n \in P\mathscr{A}P$ for each $n \in \mathbb{N}$. Note that $P\mathscr{A}P$ is a Banach algebra with identity P, and so there exists a positive integer n such that T_n is invertible in $P\mathscr{A}P$. Thus there is $S \in \mathscr{A}$ with $P = (PSP)T_n$, which implies that $P \in \mathscr{J}$. ∎

DEFINITION 2.8. For each pair (E, F) of Banach spaces, set

$$\mathscr{G}_{\ell_1}(E, F) = \{TS : S \in \mathscr{B}(E, \ell_1),\ T \in \mathscr{B}(\ell_1, F)\}.$$

The fact that ℓ_1 is isomorphic to $\ell_1 \oplus \ell_1$ implies that $\mathscr{G}_{\ell_1}$ is an operator ideal, and so its closure $\overline{\mathscr{G}}_{\ell_1}$ is a closed operator ideal. As usual, we write $\overline{\mathscr{G}}_{\ell_1}(E)$ instead of $\overline{\mathscr{G}}_{\ell_1}(E, E)$.

PROPOSITION 2.9. *Set $F = \left(\bigoplus \ell_2^n\right)_{\ell_1}$. Then $\overline{\mathscr{G}}_{\ell_1}(F)$ is a proper ideal in $\mathscr{B}(F)$.*

Proof. Assume towards a contradiction that $I_F \in \overline{\mathscr{G}}_{\ell_1}(F)$. Then $I_F \in \mathscr{G}_{\ell_1}(F)$ by Lemma 2.7, and so F is isomorphic to ℓ_1, which is false. (It is well-known that F is not isomorphic to ℓ_1, but this is by no means obvious. One may for example use the fact that ℓ_1 has a unique unconditional basis up to equivalence (see Section 2.b in [9], or Section 5 in [7] for a simpler proof relying only on Khintchine's inequality), whereas it is easy to see that F does not have this property.) ∎

The following construction is a dual version of Construction 4.2 in [8].

CONSTRUCTION 2.10. Let $E_1, E_2, E_3, \ldots$ and F be Banach spaces. Set $E = \big(\bigoplus E_n\big)_{\ell_1}$ and $\widetilde{F} = (F \oplus F \oplus \cdots)_{\ell_1}$, and let $T\colon E \to F$ be an operator. Since $\|TJ_n^E\| \leqslant \|T\|$ for each $n \in \mathbb{N}$, we have a diagonal operator $\operatorname{diag}(TJ_n^E)\colon E \to \widetilde{F}$. For each $y \in \widetilde{F}$ the series $\sum_{n=1}^{\infty} Q_n^{\widetilde{F}} y$ converges absolutely in F, and it is easy to check that

$$W\colon\ \widetilde{F} \to F, \quad y \mapsto \sum_{n=1}^{\infty} Q_n^{\widetilde{F}} y,$$

defines an operator of norm 1 satisfying

$$T = W \operatorname{diag}(TJ_n^E). \tag{2.4}$$

PROPOSITION 2.11. *Set $F = \big(\bigoplus \ell_2^n\big)_{\ell_1}$. For each operator T on F with finite columns, the following three conditions are equivalent:*

(i) $T \notin \overline{\mathscr{G}}_{\ell_1}(F)$;
(ii) $\sup\{n_\varepsilon(TJ_k^F) : k \in \mathbb{N}\} = \infty$ *for some* $\varepsilon > 0$;
(iii) *there are operators U and V on F such that $VTU = I_F$.*

Proof. We begin by proving the implication "not (ii) $\Rightarrow$ not (i)". We may suppose that $T \neq 0$. Let $0 < \varepsilon < \|T\|$, and suppose that $n' = \sup\{n_\varepsilon(TJ_k^F) : k \in \mathbb{N}\}$ is finite. Lemma 2.5(i) implies that, for each $k \in \mathbb{N}$, we can find a positive integer d_k and operators $R_k\colon \ell_2^k \to \ell_1^{d_k}$ and $S_k\colon \ell_1^{d_k} \to F$ such that $\|R_k\| \leqslant \|T\|\sqrt{n'+1}$, $\|S_k\| \leqslant 1$, and $\|TJ_k^F - S_kR_k\| \leqslant \varepsilon$. Put $\widetilde{F} = (F \oplus F \oplus \cdots)_{\ell_1}$ as in Construction 2.10. Then the diagonal operators $\operatorname{diag}(R_k)\colon F \to \big(\bigoplus \ell_1^{d_k}\big)_{\ell_1} = \ell_1$ and $\operatorname{diag}(S_k)\colon \ell_1 = \big(\bigoplus \ell_1^{d_k}\big)_{\ell_1} \to \widetilde{F}$ exist and satisfy

$$\|\operatorname{diag}(TJ_k^F) - \operatorname{diag}(S_k)\operatorname{diag}(R_k)\| = \sup \|TJ_k^F - S_kR_k\| \leqslant \varepsilon.$$

It follows that $\operatorname{diag}(TJ_k^F) \in \overline{\mathscr{G}}_{\ell_1}(F, \widetilde{F})$, and so $T \in \overline{\mathscr{G}}_{\ell_1}(F)$ by (2.4), as required.

To show "(ii)$\Rightarrow$(iii)", suppose that $\sup\{n_\varepsilon(TJ_k^F) : k \in \mathbb{N}\} = \infty$ for some $\varepsilon > 0$. We construct inductively a strictly increasing sequence (k_j) in $\mathbb{N}$ such that the following three conditions are satisfied:

(a) $\operatorname{colsupp}_{k_j}(T) \neq \emptyset$ for each $j \in \mathbb{N}$.

(b) Set $m_j = \max\Big(\operatorname{colsupp}_{k_j}(T)\Big) \in \mathbb{N}$. Then $m_{j+1} > m_j$ for each $j \in \mathbb{N}$.

(c) Set $E_j = \Big(\bigoplus_{i=m_{j-1}+1}^{m_j} \ell_2^i \Big)_{\ell_1}$, where $m_0 = 0$ and m_j is defined as in (b) for $j \in \mathbb{N}$, and let

$$P_j = \sum_{i=m_{j-1}+1}^{m_j} J_i^{E_j} Q_i^F \colon F \to E_j$$

be the canonical projection. Then there are operators $U_j \colon \ell_2^j \to \ell_2^{k_j}$ and $V_j \colon E_j \to \ell_2^j$ with $\|U_j\| \leqslant 1/\varepsilon$ and $\|V_j\| \leqslant 1$ such that the diagram

$$\begin{array}{ccccccc} \ell_2^j & & & \xrightarrow{\quad I_{\ell_2^j} \quad} & & & \ell_2^j \\ {\scriptstyle U_j}\downarrow & & & & & & \uparrow{\scriptstyle V_j} \\ \ell_2^{k_j} & \xrightarrow{J_{k_j}^F} & F & \xrightarrow{T} & F & \xrightarrow{P_j} & E_j \end{array} \tag{2.5}$$

is commutative, and $U_j(\ell_2^j) \subset \bigcap_{i=1}^{m_{j-1}} \ker T_{i,k_j}$ for each $j \in \mathbb{N}$.

We start the induction by choosing $k_1 \in \mathbb{N}$ such that $n_\varepsilon(TJ_{k_1}^F) \geqslant 1$. Then $\operatorname{colsupp}_{k_1}(T)$ is non-empty and $\|TJ_{k_1}^F\| > \varepsilon$. Take a unit vector $x \in \ell_2^{k_1}$ such that $\|TJ_{k_1}^F x\| > \varepsilon$, and define

$$U_1 \colon\ \ell_2^1 = \mathbb{K} \to \ell_2^{k_1}, \quad \alpha \mapsto \frac{\alpha}{\|TJ_{k_1}^F x\|}\, x.$$

Further, take a functional $V_1 \colon E_1 \to \mathbb{K} = \ell_2^1$ of norm 1 such that

$$V_1(P_1 TJ_{k_1}^F x) = \|P_1 TJ_{k_1}^F (x)\|.$$

Then the diagram (2.5) is commutative because $\|P_1 TJ_{k_1}^F(x)\| = \|TJ_{k_1}^F(x)\|$, and the inclusion $U_1(\ell_2^1) \subset \bigcap_{i=1}^{m_0} \ker T_{i,k_1}$ is trivially satisfied because $\bigcap_{i \in \emptyset} \ker T_{i,k_1} = \ell_2^{k_1}$ by convention.

Now let $j \geqslant 2$, and suppose that $k_1 < k_2 < \cdots < k_{j-1}$ have been chosen. Set $h = \sum_{i=1}^{m_{j-1}} i$, take $k_j > k_{j-1}$ such that $n_\varepsilon(TJ_{k_j}^F) \geqslant h + 2(j-1)$, and set

$$H_0 = \bigcap_{i=1}^{m_{j-1}} \ker T_{i,k_j} = \ker((Q_1^F \oplus \cdots \oplus Q_{m_{j-1}}^F) TJ_{k_j}^F) \subset \ell_2^{k_j}.$$

Since $\dim H_0 \geqslant k_j - h$, it follows that $\dim H_0^\perp \leqslant h$. Hence Lemma 2.5(iii) implies that $n_\varepsilon(TJ_{k_j}^F|_{H_0}) \geqslant 2(j-1)$. In particular $TJ_{k_j}^F|_{H_0} \neq 0$, so that $\operatorname{colsupp}_{k_j}(T) \neq \emptyset$, and $m_j > m_{j-1}$ by the choice of H_0. Further, we note that $n_\varepsilon(P_j TJ_{k_j}^F|_{H_0}) =$

$n_\varepsilon(TJ^F_{k_j}|_{H_0})$ because $Q^F_i TJ^F_{k_j}|_{H_0} = 0$ whenever $i \leqslant m_{j-1}$ or $i > m_j$. Lemma 2.5(ii) then shows that there are operators $U_j\colon \ell_2^j \to H_0 \subset \ell_2^{k_j}$ and $V_j\colon E_j \to \ell_2^j$ with $\|U_j\| \leqslant 1/\varepsilon$ and $\|V_j\| \leqslant 1$ making the diagram (2.5) commutative, and the induction continues.

Next we "glue" the sequences of operators (U_j) and (V_j) together to obtain operators U and V on F. Specifically, given $x \in F$, we define $y_i \in \ell_2^i$ by

$$y_i = \begin{cases} U_j Q^F_j x & \text{if } i = k_j \text{ for some } j \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases} \qquad (i \in \mathbb{N}).$$

Then

$$\sum_{i=1}^\infty \|y_i\| = \sum_{j=1}^\infty \|U_j Q^F_j x\| \leqslant \frac{\|x\|}{\varepsilon},$$

and so $Ux = (y_i)_{i=1}^\infty$ defines an operator U on F. Further, since

$$\sum_{j=1}^\infty \|V_j P_j x\| \leqslant \sum_{j=1}^\infty \|P_j x\| = \|x\|,$$

we can define an operator V on F by $Vx = (V_j P_j x)_{j=1}^\infty$.

It remains to prove that $VTU = I_F$. For this, it suffices to check that

$$Q^F_i VTUJ^F_j(x) = \begin{cases} x & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \qquad (i, j \in \mathbb{N},\ x \in \ell_2^j).$$

By definition, we have $Q^F_i VTUJ^F_j(x) = V_i P_i TJ^F_{k_j} U_j(x)$. For $i = j$, the latter equals x by (2.5). For $i < j$, we have $P_i TJ^F_{k_j} U_j(x) = \sum_{h=m_{i-1}+1}^{m_i} J^{E_i}_h T_{h,k_j} U_j(x) = 0$ because $U_j x \in \ker T_{h,k_j}$ for each $h \leqslant m_{j-1}$. For $i > j$,

$$P_i TJ^F_{k_j} = \sum_{h=m_{i-1}+1}^{m_i} J^{E_i}_h T_{h,k_j} = 0$$

because $T_{h,k_j} = 0$ for each $h > m_j$. This completes the proof of the implication "(ii)⇒(iii)".

Finally, the implication "(iii)⇒(i)" follows from Proposition 2.9. ∎

In fact conditions (i) and (iii), above, are equivalent also for operators that do not have finite columns.

COROLLARY 2.12. *Let T be an operator on the Banach space $F = \left(\bigoplus \ell_2^n\right)_{\ell_1}$. Then $T \notin \overline{\mathscr{G}}_{\ell_1}(F)$ if and only if there exist operators R and S on F such that $I_F = STR$.*

Proof. As before, the implication "⇐" follows from Proposition 2.9.

Conversely, suppose that $T \notin \overline{\mathscr{G}}_{\ell_1}(F)$, and let K be a compact operator on F such that $T - K$ has finite columns (cf. Lemma 2.7(i) in [8]). By the ideal property we have $T - K \notin \overline{\mathscr{G}}_{\ell_1}(F)$. Proposition 2.11 implies that there are operators U and

V on F such that $I_F = V(T-K)U$. Thus VTU is a compact perturbation of the identity, and hence it is a Fredholm operator. It follows that, for some $W \in \mathscr{B}(F)$, the operator $WVTU$ is a cofinite-rank projection. This completes the proof because F is isomorphic to its closed subspaces of finite codimension. (This latter fact is a consequence of the existence of a left and a right shift operator on the basis of F obtained by stringing together the natural bases of $\ell_2^1, \ell_2^2, \ldots, \ell_2^n, \ldots$). ∎

Our main result classifying the closed ideals in $\mathscr{B}(F)$ is now easy to deduce.

THEOREM 2.13. *The lattice of closed ideals in* $\mathscr{B}(F)$, *where* $F = \left(\bigoplus \ell_2^n\right)_{\ell_1}$, *is given by*

$$\{0\} \subsetneq \mathscr{K}(F) \subsetneq \overline{\mathscr{G}}_{\ell_1}(F) \subsetneq \mathscr{B}(F). \tag{2.6}$$

Proof. It is clear that $\mathscr{B}(F)$ contains the chain of closed ideals (2.6). The right-hand inclusion is proper by Proposition 2.9. The middle inclusion is proper because F contains ℓ_1 as a complemented subspace, the projection onto which is an example of a non-compact operator in $\overline{\mathscr{G}}_{\ell_1}(F)$.

It remains to show that the ideals in (2.6) are the *only* closed ideals in $\mathscr{B}(F)$. Standard basis arguments show that the identity on ℓ_1 factors through any non-compact operator in $\mathscr{B}(F)$ (see Section 3 of [8] for details). It follows that, for each non-zero, closed ideal $\mathscr{J}$ in $\mathscr{B}(F)$, either $\mathscr{J} = \mathscr{K}(F)$ or $\overline{\mathscr{G}}_{\ell_1}(F) \subset \mathscr{J}$. However, Corollary 2.12 implies that $\overline{\mathscr{G}}_{\ell_1}(F)$ is a maximal ideal in $\mathscr{B}(F)$, and so there are no other closed ideals in $\mathscr{B}(F)$ than the four listed in (2.6). ∎

REMARK 2.14. We can now explain where the present proof differs in an essential way from the proof for the Banach space $E = \left(\bigoplus \ell_2^n\right)_{c_0}$ given in [8]. Indeed, each operator on E has a compact perturbation which has a "locally finite matrix" in the sense that its associated matrix (cf. Definition 2.3) has only finitely many non-zero entries in each row and in each column. This is not true for all operators on $F = \left(\bigoplus \ell_2^n\right)_{\ell_1}$ (an example of this is given below). We circumvent this difficulty by arranging that the operators U_j map into $\bigcap_{i=1}^{m_{j-1}} \ker T_{i,k_j}$ in the proof of Proposition 2.11.

An operator T on F such that no compact perturbation of T has a locally finite matrix can be constructed as follows. Let $(N_m)_{m=1}^\infty$ be a partition of $\mathbb{N}$ such that N_m is infinite for each $m \in \mathbb{N}$, and define an operator of norm 1 by

$$T\colon\ F \to F, \quad (y_n) \mapsto \Big(\sum_{n\in N_m} \langle y_n, x_n\rangle\, x_m\Big)_{m=1}^\infty,$$

where $\langle\cdot\,,\cdot\rangle$ denotes the inner product in ℓ_2^n and $x_n = (1,0,\ldots,0) \in \ell_2^n$ for each $n \in \mathbb{N}$.

Suppose that $S \in \mathscr{B}(F)$ has a locally finite matrix. Inductively we choose a strictly increasing sequence (n_m) in $\mathbb{N}$ such that $n_m \in N_m$ and $S_{m,j} = 0$ for each $j \geqslant n_m$ and $m \in \mathbb{N}$. We note that no subsequence of $((T-S)J^F_{n_m} x_{n_m})$ is convergent because

$$\begin{aligned}\|(T-S)(J^F_{n_k}x_{n_k}-J^F_{n_m}x_{n_m})\| &\geqslant \|Q^F_m(T-S)(J^F_{n_k}x_{n_k}-J^F_{n_m}x_{n_m})\|\\ &= \|T_{m,n_k}x_{n_k}-S_{m,n_k}x_{n_k}-T_{m,n_m}x_{n_m}+S_{m,n_m}x_{n_m}\|\\ &= \|0-0-x_m+0\| = 1\end{aligned}$$

whenever $k > m$. Since the sequence $(J^F_{n_m}x_{n_m})$ is bounded, we conclude that the operator $T-S$ is not compact. In other words, no compact perturbation of T has a locally finite matrix, as claimed.

3. AN APPLICATION

In Section 8 of [1] Bourgain, Casazza, Lindenstrauss, and Tzafriri prove that every infinite-dimensional, complemented subspace of the Banach space $F = \big(\bigoplus \ell_2^n\big)_{\ell_1}$ is isomorphic to either F or ℓ_1. Here we present a new proof of this fact using only the ideal structure of $\mathscr{B}(F)$. More precisely, we shall deduce it from Corollary 2.12.

THEOREM 3.1 ([1]). *Each infinite-dimensional, complemented subspace of $F = \big(\bigoplus \ell_2^n\big)_{\ell_1}$ is isomorphic to either F or ℓ_1.*

Proof. Let G be an infinite-dimensional, complemented subspace of F, and let $P \in \mathscr{B}(F)$ be an idempotent operator with image G. If $P \in \overline{\mathscr{G}}_{\ell_1}(F)$, then by Lemma 2.7 we have $P \in \mathscr{G}_{\ell_1}(F)$, and hence G is isomorphic to ℓ_1. If $P \notin \overline{\mathscr{G}}_{\ell_1}(F)$, then by Corollary 2.12 the identity on F factors through P, i.e. F is isomorphic to a complemented subspace of G. We can thus write $F \sim G \oplus X$ and $G \sim F \oplus Y$ for suitable Banach spaces X and Y. We now use Pełczyński's decomposition method and the fact that F is isomorphic to $(F \oplus F \oplus \cdots)_{\ell_1}$ to show that G is isomorphic to F:

$$\begin{aligned}F \sim G \oplus X \sim F \oplus Y \oplus X &\sim (F \oplus F \oplus \cdots)_{\ell_1} \oplus Y \oplus X\\ &\sim (G \oplus X \oplus G \oplus X \oplus \cdots)_{\ell_1} \oplus Y \oplus X\\ &\sim (G \oplus X \oplus G \oplus X \oplus \cdots)_{\ell_1} \oplus Y \sim F \oplus Y \sim G. \quad ∎\end{aligned}$$

REMARK 3.2. In Section 6 of [8] a new proof is presented for the corresponding result of Bourgain, Casazza, Lindenstrauss, and Tzafriri for the Banach space $E = \big(\bigoplus \ell_2^n\big)_{c_0}$, which says that every infinite-dimensional, complemented subspace of E is isomorphic to either E or c_0. The proof in [8] relies on a theorem of Casazza, Kottman, and Lin [3] that implies that E is primary. The results of [3], however, do not show that our space $F = \big(\bigoplus \ell_2^n\big)_{\ell_1}$ is primary, and so the argument in [8] cannot be used here. We note in passing that F is in fact primary — this follows easily from Theorem 3.1. Further, we note that the proof presented above works also for the space $E = \big(\bigoplus \ell_2^n\big)_{c_0}$.

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