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THE GEOMETRIC MEANS IN BANACH $*$-ALGEBRAS

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Dedicated to Professor Joe Diestel for his 60th birthday

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ABSTRACT. The arithmetic-geometric-harmonic inequality has played a special role in elementary mathematics. During the past twenty five years (see [1], [2] and [8] etc.) a great many mathematicians have researched on various kinds of matrix versions of the arithmetic-geometric-harmonic inequality. It is interesting to see whether the arithmetic-geometric-harmonic inequality can be extended to the context of Banach $*$-algebras. In this article we will define the geometric means of positive elements in Banach $*$-algebras and prove that the arithmetic-geometric-harmonic inequality does hold in Banach $*$-algebras.

KEYWORDS: *Arithmetic mean, geometric mean, harmonic mean, Banach $*$-algebra.*

MSC (2000): 47A63, 47A64.

INTRODUCTION

Let A be a Banach $*$-algebra. An element $a \in A$ is called *self-adjoint* if $a^* = a$. A is *Hermitian* if every self-adjoint element a of A has real spectrum: $\sigma(a) \subset \mathbb{R}$, where $\sigma(a)$ denotes the spectrum of a. We assume in what follows that a Banach $*$-algebra A is Hermitian. Also we assume that A is unital with unit 1. Saying an element $a \geqslant 0$ means that $a = a^*$ and $\sigma(a) \subset [0, \infty)$; $a > 0$ means that $a \geqslant 0$ and $0 \notin \sigma(a)$. Thus, $a > 0$ implies its inverse a^{-1} exists. Denote the set of all invertible elements in A by $\mathrm{Inv}(A)$. If $a, b \in A$, then $a, b \in \mathrm{Inv}(A)$ imply $ab \in \mathrm{Inv}(A)$, and $(ab)^{-1} = b^{-1}a^{-1}$. Saying $a \geqslant b$ means $a - b \geqslant 0$, and $a > b$ means $a - b > 0$. The Shirali-Ford Theorem ([6] or [3], Theorem 41.5) asserted that $a^*a \geqslant 0$ for every $a \in A$. Based on the Shirali-Ford Theorem, Okayasu [5], Tanahashi and Uchiyama [7] proved the following inequalities:

(i) If $a, b \in A$, then $a \geqslant 0, b \geqslant 0$ imply $a + b \geqslant 0$, with $\alpha \geqslant 0$ implies $\alpha a \geqslant 0$.
(ii) If $a, b \in A$, then $a > 0, b \geqslant 0$ imply $a + b > 0$.
(iii) If $a, b \in A$, then either $a \geqslant b > 0$, or $a > b \geqslant 0$ imply $a > 0$.
(iv) If $a > 0$, then $a^{-1} > 0$.

(v) If $c > 0$, then $0 < b < a$ if and only if $cbc < cac$; also $0 < b \leqslant a$ if and only if $cbc \leqslant cac$.

(vi) If $0 < a < 1$, then $1 < a^{-1}$.

(vii) If $0 < b < a$, then $0 < a^{-1} < b^{-1}$; also if $0 < b \leqslant a$, then $0 < a^{-1} \leqslant b^{-1}$.

Also, Okayasu [5] showed that the following Löwner-Heinz inequality still holds in Banach $*$-algebras:

THEOREM 0.1. *Let A be a unital Hermitian Banach $*$-algebra with continuous involution. Let $a, b \in A$ and $p \in [0,1]$. Then $a^p > b^p$ if $a > b$, and $a^p \geqslant b^p$ if $a \geqslant b$.*

It is natural to ask if there is an arithmetic-geometric-harmonic means inequality in Banach $*$-algebras. In this paper, we will address this problem.

1. THE LAWS OF EXPONENTS

Let $a \in A$ and $a > 0$, then $0 \notin \sigma(a)$ and the fact of $\sigma(a)$ being nonempty compact subset of $\mathbb{C}$ implies that

$$\inf\{z : z \in \sigma(a)\} > 0 \quad \text{and} \quad \sup\{z : z \in \sigma(a)\} < \infty.$$

Choose γ to be a closed rectifiable curve in $\{\mathrm{Re} z > 0\}$, the right half open plane of the complex plane, such that $\sigma(a) \subset \mathrm{ins}\gamma$, the inside of γ. Let G be an open subset of $\mathbb{C}$ with $\sigma(a) \subset G$. If $f : G \to \mathbb{C}$ is analytic, we define an element $f(a)$ in A by

$$f(a) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} f(z)(z-a)^{-1} \mathrm{d}z.$$

It is known (see pp. 201–204 in [4]) that $f(a)$ does not depend on the choice of γ and the Spectral Mapping Theorem:

$$\sigma(f(a)) = f(\sigma(a))$$

holds.

For any $\alpha \in \mathbb{R}$, we define

$$a^{\alpha} = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} z^{\alpha}(z-a)^{-1} \mathrm{d}z$$

where z^{α} is the principal α-power of z. Since A is a Banach $*$-algebra, $a^{\alpha} \in A$. Since z^{α} is analytic in $\{\mathrm{Re} z > 0\}$, by the Spectral Mapping Theorem

$$\sigma(a^{\alpha}) = (\sigma(a))^{\alpha} = \{z^{\alpha} : z \in \sigma(a)\} \subset (0, \infty).$$

Thus, we have

LEMMA 1.1. *If $0 < a \in A$ and $\alpha \in \mathbb{R}$, then $a^{\alpha} \in A$ with $a^{\alpha} > 0$.*

Moreover, one of the laws of exponents holds in Banach $*$-algebras.

LEMMA 1.2. *If $0 < a \in A$ and $\alpha, \beta \in \mathbb{R}$, then $a^{\alpha} a^{\beta} = a^{\alpha+\beta}$.*

Proof. Let γ be defined as in the discussion preceding Lemma 1.1. It is known that ([4], VII. 4.7, Riesz Functional Calculus) that the map

$$f \mapsto f(a) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} f(z)(z-a)^{-1}\mathrm{d}z$$

of Hol$(a) \to A$ is an algebra homomorphism, where Hol(a) = all of the functions that are analytic in a neighborhood of $\sigma(a)$. That is, $f(a)g(a) = (fg)(a)$. Moreover, $z^{\alpha}z^{\beta} = z^{\alpha+\beta}$ holds for principal powers of z implies that

$$a^{\alpha}a^{\beta} = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} z^{\alpha}z^{\beta}(z-a)^{-1}\mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} z^{\alpha+\beta}(z-a)^{-1}\mathrm{d}z = a^{\alpha+\beta}. \quad \blacksquare$$

LEMMA 1.3. *If* $0 < a \in A$ *and* $\alpha \in \mathbb{R}$, *then* $(a^{\alpha})^{-1} = (a^{-1})^{\alpha} = a^{-\alpha}$.

Proof. Note that $a^0 = 1$ ([3], Lemma 1, p. 31), and from Lemma 1.2 we have

$$a^{\alpha}a^{-\alpha} = a^{\alpha+(-\alpha)} = a^0 = 1.$$

By the uniqueness of the inverse of an element in A, $(a^{\alpha})^{-1} = a^{-\alpha}$.

Next we want to verify that $(a^{-1})^{\alpha} = a^{-\alpha}$. We know that $a > 0$ implies that

$$\inf\{z : z \in \sigma(a)\} > 0 \quad \text{and} \quad \sup\{z : z \in \sigma(a)\} < \infty.$$

Choose positive real numbers r_1 and r_2 such that:

$$0 < r_1 < \inf\{z : z \in \sigma(a)\}, \quad r_2 > \sup\{z : z \in \sigma(a)\}$$
$$\frac{1}{r_1} > \sup\{z : z \in \sigma(a)\}, \quad 0 < \frac{1}{r_2} < \inf\{z : z \in \sigma(a)\}.$$

Let γ be a closed rectifiable curve in $\{\mathrm{Re} z > 0\}$, which passes r_1 and r_2 and such that $\sigma(a) \subset \mathrm{ins}\gamma$. Then the curve $1/\gamma = \{1/z : z \in \gamma\}$ is also a closed rectifiable with $\sigma(a) \subset \mathrm{ins}(1/\gamma)$ and $1/\gamma \subset \{\mathrm{Re} z > 0\}$. Thus,

$$\begin{aligned}(a^{-1})^{\alpha} &= \frac{1}{2\pi \mathrm{i}} \int_{\gamma} z^{\alpha}(z-a^{-1})^{-1}\mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} z^{\alpha}\left(a - \frac{1}{z}\right)^{-1}\frac{a}{z}\mathrm{d}z \\ &= \frac{a}{2\pi \mathrm{i}} \int_{1/\gamma} \lambda^{-\alpha-1}(\lambda - a)^{-1}\mathrm{d}\lambda \quad (\text{substituting} : \lambda = 1/z) \\ &= aa^{-\alpha-1} = a^{-\alpha} \quad (\text{Lemma1.2}). \quad \blacksquare\end{aligned}$$

LEMMA 1.4. *If* $0 < a \in A, 0 < b \in A, \alpha, \beta \in \mathbb{R}$, *and* $ab = ba$, *then* $a^{\alpha}b^{\beta} = b^{\beta}a^{\alpha}$.

Proof. Suppose that $z \notin \sigma(a)$, then $ab = ba \Longrightarrow (z-a)b = b(z-a) \Longrightarrow b(z-a)^{-1} = (z-a)^{-1}b$. Let γ be defined as in the discussion preceding Lemma 1.1.

Then

$$a^\alpha b = \Big(\frac{1}{2\pi \mathrm{i}} \int\limits_\gamma z^\alpha (z-a)^{-1} \mathrm{d}z\Big) b = \frac{1}{2\pi \mathrm{i}} \int\limits_\gamma z^\alpha (z-a)^{-1} b \mathrm{d}z$$

$$= \frac{1}{2\pi \mathrm{i}} \int\limits_\gamma z^\alpha b (z-a)^{-1} \mathrm{d}z = b\Big(\frac{1}{2\pi \mathrm{i}} \int\limits_\gamma z^\alpha (z-a)^{-1} \mathrm{d}z\Big) = b a^\alpha.$$

Thus,

$$ab = ba \Longrightarrow a^\alpha b = b a^\alpha \Longrightarrow a^\alpha b^\beta = b^\beta a^\alpha. \quad \blacksquare$$

2. THE ARITHMETIC MEAN, GEOMETRIC MEAN AND HARMONIC MEAN

Naturally, for $a, b \in A$, and w_1, w_2 are positive numbers summing to 1, their *weighted arithmetic mean* can be defined as

$$A_w(a,b) := w_1 a + w_2 b.$$

If $a > 0, b > 0$, their *weighted harmonic mean* can be defined as

$$H_w(a,b) := (w_1 a^{-1} + w_2 b^{-1})^{-1}.$$

From the point view of matrix analysis (see [1]), if $a > 0, b > 0$, and w_1, w_2 are positive numbers summing to 1, their *weighted geometric mean* can be defined as

$$G_w(a,b) := b^{1/2}(b^{-1/2} a b^{-1/2})^{w_1} b^{1/2}.$$

Denote $A_w(a,b), G_w(a,b)$ and $H_w(a,b)$ by $A(a,b), G(a,b)$ and $H(a,b)$ respectively if $w_1 = w_2 = 1/2$. It is clear that $A_w(a,b), G_w(a,b), H_w(a,b) \in A$ and $H_w(a,b) > 0$ and $G_w(a,b) > 0$ by inequalities (ii), (iv), (v) and Lemma 1.1 above. Does the following arithmetic-geometric-harmonic inequalities hold

$$H_w(a,b) \leqslant G_w(a,b) \leqslant A_w(a,b)$$

in Banach $*$-algebras?

Based on the lemmas above we can prove some properties of arithmetic mean, geometric mean and harmonic mean mentioned by Ando [1].

THEOREM 2.1. *Suppose that $a, b \in A$ with $a > 0, b > 0$, then*

$$H(a,b) = H(b,a) \quad \text{and} \quad G(a,b) = G(b,a).$$

Proof. $H(a,b) = H(b,a)$ follows the definition of the harmonic mean and the fact that A is an Abelian group.

Observe that $G(a,b) = G(b,a)$ is equivalent to

$$a^{-1/2} b^{1/2} (b^{-1/2} a b^{-1/2})^{1/2} b^{1/2} a^{-1/2} = (a^{-1/2} b a^{-1/2})^{1/2}.$$

Since positive elements are equal if and only if their squares are equal (see Lemma 6 of [7]), using Lemma 1.2 this is in turn equivalent to

$$a^{-1/2}b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}[b^{1/2}a^{-1}b^{1/2}](b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}a^{-1/2} = a^{-1/2}ba^{-1/2}.$$

Since the term in square brackets is just $(b^{-1/2}ab^{-1/2})^{-1}$ by Lemma 1.3, the left hand side of the expression above does indeed reduce to the right hand side when we use Lemma 1.2 again. ∎

THEOREM 2.2. *Suppose that $a, b, c \in A$ with $a > 0, b > 0$ and $c \in \mathrm{Inv}(A)$, then*

$$c^*H(a,b)c = H(c^*ac, c^*bc) \quad \text{and} \quad c^*G(a,b)c = G(c^*ac, c^*bc).$$

Proof. Since $c \in \mathrm{Inv}(\mathrm{A})$, c^{-1} exists. Hence

$$\begin{aligned} c^*H(a,b)c &= c^*\Big(\frac{1}{2}a^{-1}+\frac{1}{2}b^{-1}\Big)^{-1}c = \Big(c^{-1}\Big(\frac{1}{2}a^{-1}+\frac{1}{2}b^{-1}\Big)(c^*)^{-1}\Big)^{-1} \\ &= \Big(\frac{1}{2}c^{-1}a^{-1}(c^*)^{-1}+\frac{1}{2}c^{-1}b^{-1}(c^*)^{-1}\Big)^{-1} = \Big(\frac{1}{2}(c^*ac)^{-1}+\frac{1}{2}(c^*bc)^{-1}\Big)^{-1} \\ &= H(c^*ac, c^*bc). \end{aligned}$$

It is analogous with the proof of Theorem 2.1, we now verify the second equality:

$$\begin{aligned} & c^*G(a,b)c = G(c^*ac, c^*bc) \\ &\iff c^*b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c \\ &\qquad = (c^*bc)^{1/2}((c^*bc)^{-1/2}(c^*ac)(c^*bc)^{-1/2})^{1/2}(c^*bc)^{1/2} \\ &\iff (c^*bc)^{-1/2}c^*b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2} \\ &\qquad = ((c^*bc)^{-1/2}(c^*ac)(c^*bc)^{-1/2})^{1/2} \\ &\iff ((c^*bc)^{-1/2}c^*b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2})^2 \\ &\qquad = (c^*bc)^{-1/2}(c^*ac)(c^*bc)^{-1/2}. \end{aligned}$$

The last equality is true, since by Lemma 1.2

$$\begin{aligned} & ((c^*bc)^{-1/2}c^*b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2})^2 \\ &= ((c^*bc)^{-1/2}c^*b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2}) \\ &\qquad ((c^*bc)^{-1/2}c^*b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2}) \\ &= (c^*bc)^{-1/2}c^*b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1}c^*b^{1/2} \\ &\qquad (b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2} \end{aligned}$$

$$= (c^*bc)^{-1/2}c^*b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2}$$
$$= (c^*bc)^{-1/2}c^*b^{1/2}(b^{-1/2}ab^{-1/2})b^{1/2}c(c^*bc)^{-1/2}$$
$$= (c^*bc)^{-1/2}c^*ac(c^*bc)^{-1/2}. \quad \blacksquare$$

THEOREM 2.3. *Suppose that $a, b \in A$ with $a > 0, b > 0$. Then*

$$H_w(a,b)^{-1} = A_w(a^{-1}, b^{-1}) \quad \text{and} \quad G_w(a^{-1}, b^{-1}) = G_w(a,b)^{-1}.$$

Proof. The first equality is obvious from its definitions. Using Lemma 1.2 and Lemma 1.3, we have

$$\begin{aligned} G_w(a^{-1}, b^{-1}) &= (b^{-1})^{1/2}((b^{-1})^{-1/2}a^{-1}(b^{-1})^{-1/2})^{w_1}(b^{-1})^{1/2} \\ &= (b^{1/2})^{-1}((b^{-1/2}ab^{-1/2})^{-1})^{w_1}(b^{1/2})^{-1} \\ &= (b^{1/2}(b^{-1/2}ab^{-1/2})^{w_1}b^{1/2})^{-1} = G_w(a,b)^{-1}. \quad \blacksquare \end{aligned}$$

THEOREM 2.4. *Suppose that $a, b \in A$ with $a > 0, b > 0$, and w_1, w_2 are positive numbers summing to 1, then*

$$H_w(a,b) \leqslant G_w(a,b) \leqslant A_w(a,b).$$

Proof. Firstly we verify the arithmetic-geometric means inequality: $G_w(a,b) \leqslant A_w(a,b)$. With the help of inequality (v),

$$\begin{aligned} &G_w(a,b) \leqslant A_w(a,b) \\ &\qquad \iff b^{1/2}(b^{-1/2}ab^{-1/2})^{w_1}b^{1/2} \leqslant w_1a + w_2b \\ &\qquad \iff b^{1/2}(b^{-1/2}ab^{-1/2})^{w_1}b^{1/2} \leqslant b^{1/2}(w_1b^{-1/2}ab^{-1/2} + w_2)b^{1/2} \\ &\qquad \iff (b^{-1/2}ab^{-1/2})^{w_1} \leqslant w_1b^{-1/2}ab^{-1/2} + w_2 \\ &\qquad \iff w_1n + w_2 - n^{w_1} \geqslant 0, \end{aligned}$$

where $n := b^{-1/2}ab^{-1/2}$. Lemma 1.1 and inequality (v) imply $n > 0$, and hence $\sigma(n) \subset (0, \infty)$.

Let $f(z) = w_1z + w_2 - z^{w_1}$, where z^{w_1} is the principal of the power function. Then $f(z)$ is analytic in the right half open plane $\{\mathrm{Re} z > 0\}$ of the complex plane. Next we claim that $f(z) \geqslant 0$ on the positive real line. In fact, let $x = z - 1$ in the Bernoulli inequality:

$$(1+x)^{w_1} \leqslant 1 + w_1x, \quad \text{if } 0 < w_1 < 1 \quad \text{and} \quad -1 < x.$$

We have

$$z^{w_1} \leqslant w_1z + (1 - w_1), \quad \text{if } 0 < w_1 < 1 \quad \text{and} \quad 0 < z,$$

that is,

$$f(z) \geqslant 0, \quad \text{if } 0 < w_1 < 1 \quad \text{and} \quad 0 < z.$$

The Spectral Mapping Theorem implies

$$\sigma(f(n)) = f(\sigma(n)) \subset [0, \infty).$$

So

$$f(n) = w_1 n + w_2 - n^{w_1} \geqslant 0.$$

Hence

$$G_w(a,b) \leqslant A_w(a,b).$$

Replacing a and b by a^{-1} and b^{-1} respectively in the arithmetic-geometric means inequality, Theorem 2.3 and inequality (vii) guarantee that

$$H_w(a,b) \leqslant G_w(a,b). \quad \blacksquare$$

In general, for $a_1, a_2, \ldots, a_n \in A$, and an n-tuple of positive numbers $w_1, w_2, \ldots, w_n$ summing to 1, their weighted arithmetic mean in A can be defined as

$$A_w(a_1, a_2, \ldots, a_n) := w_1 a_1 + w_2 a_2 + \cdots + w_n a_n.$$

If $a_i > 0, 1 \leqslant i \leqslant n$, their weighted harmonic mean in A can be defined as

$$H_w(a_1, a_2, \ldots, a_n) := (w_1 a_1^{-1} + w_2 a_2^{-1} + \cdots + w_n a_n^{-1})^{-1}.$$

From the point of view of matrix analysis (see [8]), if $a_i > 0, 1 \leqslant i \leqslant n$, and $w_1, \ldots, w_n$ are positive numbers summing to 1, their weighted geometric mean in A can be defined as

$$G_w(a_1, a_2, \ldots, a_n) := a_n^{1/2}(a_n^{-1/2} a_{n-1}^{1/2} \cdots (a_3^{-1/2} a_2^{1/2} (a_2^{-1/2} a_1 a_2^{-1/2})^{\alpha_1} \\ a_2^{1/2} a_3^{-1/2})^{\alpha_2} \cdots a_{n-1}^{1/2} a_n^{-1/2})^{\alpha_{n-1}} a_n^{1/2},$$

where $\alpha_i = 1 - \Big(w_{i+1} / \sum\limits_{j=1}^{i+1} w_j\Big)$ for $i = 1, \ldots, n-1$. Note that this geometric mean is just the inductive generalization of $n = 2$ case, which was discussed in Theorem 2.3 and 2.4.

Based on Theorem 2.4 with the same inductive proof as in [8], we have

THEOREM 2.5. *Suppose that $a_i \in A$, $1 \leqslant i \leqslant n$, with $a_i > 0$, $1 \leqslant i \leqslant n$, and $w_1, \ldots, w_n$ are positive numbers summing to* 1, *then*

$$H_w(a_1, \ldots, a_n) \leqslant G_w(a_1, \ldots, a_n) \leqslant A_w(a_1, \ldots, a_n).$$

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