CLASSIFYING THE TYPES OF PRINCIPAL GROUPOID C*-ALGEBRAS

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ABSTRACT. Suppose $G$ is a second countable, locally compact, Hausdorff groupoid with a fixed left Haar system. Let $G^0/G$ denote the orbit space of $G$ and $C^*(G)$ denote the groupoid $C^*$-algebra. Suppose that $G$ is a principal groupoid. We show that $C^*(G)$ is CCR if and only if $G^0/G$ is a $T_1$ topological space, and that $C^*(G)$ is GCR if and only if $G^0/G$ is a $T_0$ topological space. We also show that $C^*(G)$ is a Fell Algebra if and only if $G$ is a Cartan groupoid.

KEYWORDS: Locally compact groupoid, $C^*$-algebra.


1. INTRODUCTION

$C^*$-algebras can be classified as being continuous-trace, bounded trace, Fell Algebras, CCR (liminal), and GCR (postliminal). These are listed in order of containment. Recall that for separable $C^*$-algebras, an algebra is GCR if and only if it is Type I. Further, $C^*$-algebras that are not GCR are very poorly behaved. In the case of a transformation group $C^*$-algebra $C^*(H, X)$ (where $H$ is a group that acts continuously on the space $X$) each of these classifications correspond to a property of the transformation group itself. For example, Phil Green was able to prove in [7] that a freely acting transformation group $C^*$-algebra has continuous-trace if and only if the action of the transformation group is proper. In [12] the authors have generalized Green’s result to principal groupoids. In this paper we generalize three more such results.

In [6], Elliot Gootman showed the following:

**Theorem 1.1.** Suppose $H$ and $X$ are both second countable. Then $C^*(H, X)$ is GCR if and only if every stability group is GCR and the orbit space is $T_0$.

Dana Williams considered the case for CCR transformation group $C^*$-algebras in [20], and proved the theorem below.
Theorem 1.2. Suppose that $H$ and $X$ are both second countable. Suppose also that at every point of discontinuity $y$ of the map $x \mapsto S_x$, the stability group $S_y$ is amenable, then $C^*(H, X)$ is CCR if and only if the stability groups are CCR and the orbit space is $T_1$.

Remark 1.3. Gootman has shown that the hypothesis on $x \mapsto S_x$ in Theorem 1.2 is unnecessary; however, the details have not appeared.

We also note that Thierry Fack proved versions of Theorem 1.1 and Theorem 1.2 for foliation $C^*$-algebras in [3].

Finally, in [8], Astrid an Huef proved:

Theorem 1.4. $C^*(H, X)$ is a Fell algebra if and only if $(H, X)$ is a Cartan $G$-space.

We generalize each of the above three theorems to principal groupoids. The key comes in showing that there is a continuous injection between the orbit space of the groupoid and the spectrum of the associated groupoid $C^*$-algebra. In fact, when the orbit space is $T_0$, we show that these spaces are homeomorphic.

We have also been able to further generalize the CCR and GCR results to non-principal groupoids; however, these results will appear later.

2. Preliminaries

A groupoid $G$ is a small category in which every morphism is invertible. A principal groupoid is a groupoid in which there is at most one morphism between each pair of objects. We define maps $r$ and $s$ from $G$ to $G$ by $r(x) = xx^{-1}$ and $s(x) = x^{-1}x$. These are the maps Renault calls $r$ and $d$ in [17]. The common image of $r$ and $s$ is called the unit space which we denote $G^0$.

We will only consider second countable, locally compact, Hausdorff groupoids $G$. Our main results also requires $G$ to be principal; however, we will state this condition when it is needed. We will also assume that $G$ has a fixed left Haar system, $\{\lambda^u\}_{u \in G^0}$.

Now consider the vector space $C_c(G)$, the space of continuous functions with compact support from $G$ to the complex numbers, $\mathbb{C}$. We can view this space as an $*$-algebra by defining convolution and involution with the formulae:

$$f * g(x) = \int f(y)g(y^{-1}x) \, d\lambda^r(x)(y) = \int f(xy)g(y^{-1}) \, d\lambda^s(x)(y)$$

and

$$f^*(x) = \overline{f(x^{-1})}.$$

A representation of $C_c(G)$ is a $*$-homomorphism $\pi$ from $C_c(G)$ into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ that is continuous with respect to the inductive limit topology on $C_c(G)$ and the weak operator topology on $B(\mathcal{H})$, and that is non-degenerate in the sense that the linear span of $\{\pi(f)\eta : f \in C_c(G), \eta \in \mathcal{H}\}$ is dense in $\mathcal{H}$. We define the groupoid $C^*$-algebra with the following theorem.
Theorem 2.1. For \( f \in C_c(G) \), the quantity
\[
\|f\| := \sup\{\|\pi(f)\| : \pi \text{ is a representation of } C_c(G)\}
\] (2.1)
is finite and defines a C*-norm on \( C_c(G) \). The completion of \( C_c(G) \) with respect to this norm is a C*-algebra, denoted \( C^*(G) \).

The only real issue in proving Theorem 2.1 comes in showing that \( \|f\| < \infty \) for all \( f \in C_c(G) \). This is a consequence of Renault’s Disintegration Theorem ([18], Theorem 4.2 and [11], Theorem 3.23). The motivating example of a groupoid \( C^* \)-algebra is a transformation group \( C^*(H, X) \), defined in [20] and [8].

We define the map \( \pi : G \to G^0 \times G^0 \) by \( \pi(x) = (r(x), s(x)) \). Using \( \pi \), we define an equivalence relation on \( G^0 \) and endow the set of equivalence classes with the quotient topology. We call this topological space the orbit space of \( G \), denoted \( G^0/G \).

3. A MAP FROM \( G^0/G \) TO \( C^*(G)^\wedge \)

Following [12] and [17], pages 81–82, recall that for each \( u \in G^0 \) there is a representation \( L^u \) induced from the point mass measure \( \varepsilon_u \). When \( G \) is a principal groupoid, \( L^u \) acts on \( L^2(G, \lambda_u) \) so that for \( f \in C_c(G) \) and \( \xi \in L^2(G, \lambda_u) \),
\[
L^u(f)\xi(\gamma) = \int f(\gamma \alpha)\xi(\alpha^{-1})d\lambda^u(\alpha).
\]
The following lemma is Lemma 2.4 of [12].

Lemma 3.1. Suppose \( G \) is a principal groupoid. Then the representation \( L^u \) is irreducible for each \( u \in G^0 \). Further more, if \( [u] = [v] \) then \( L^u \) is unitarily equivalent to \( L^v \).

We can use this construction to define a map \( \psi : G^0/G \to C^*(G)^\wedge \) where \( \psi([u]) = L^u \). As usual, we view \( L^u \) as its unitary equivalence class in \( C^*(G)^\wedge \). Our notation is somewhat careless. We should denote the image of \( u \) under \( \psi \) by \([L^u]\) but the preceding lemma makes this carelessness less troubling.

Our goal is to show that for principal groupoids with \( T_0 \) orbit spaces, \( \psi \) is a homeomorphism. We will first show this for groupoids with \( T_1 \) orbit spaces and generalize this to \( T_0 \) orbit spaces later. Before we deal with \( \psi \), we must first determine what the representations of \( C^*(G) \) look like.

Fixing \( u \in G^0 \), recall from Lemma 2.13 in [17] that there is a representation \( M_u \) of \( C_0(G^0) \) on \( L^2(G, \lambda_u) \) defined by
\[
L^u(V(\phi)f) = M_u(\phi)L^u(f).
\] (3.1)

Proposition 3.2. Suppose that \( L \) is an irreducible representation of \( C^*(G) \), and that \( M \) is the representation of \( C_0(G^0) \) defined by \( M(\phi)L(f) = L(V(\phi)f) \). If \( \ker M = I_F := \{ \phi \in C_0(G^0) : \phi(x) = 0 \text{ for all } x \in F \} \), then there is a \( u \in G^0 \) such that \( F = [u] \).
Before we can prove this proposition, we need the following two lemmas.

**Lemma 3.3.** Let $U$ be an open subset of $G^0$. Then the ideal of $C^*(G)$ generated by $C_c(G|U)$ is $C_c(G|U) := \text{Ex}([U])$.

**Proof.** It suffices to see that

$$E_0 := C_c(G) * C_c(G|U) * C_c(G)$$

is dense in $C_c(G|U)$ in the inductive limit topology. In view of the Stone-Weierstrass Theorem ([19], Theorem 7.33) since $E_0$ is self-adjoint it suffices to show $E_0$ separates points of $G|U$ and vanishes at no point of $G|U$.

Because $G|U$ is Hausdorff, this is the same as showing that for each $\gamma \in G|U$ and each neighborhood $V$ of $\gamma$, there is an $F \in E_0$ with supp $F \subset V$ and $F(\gamma) \neq 0$. But if $\gamma \in G|U$, then $\gamma = \alpha \beta \delta$ with $\beta \in G|U$, $s(\alpha) = s(\gamma)$, and $r(\delta) = r(\gamma)$.

Now notice that

$$f * g * h(\gamma) = \int_G f * g(\gamma \eta) h(\eta^{-1}) d\lambda^s(\gamma)(\eta)$$

$$= \int_G \int_G f(\omega) g(\omega^{-1} \gamma \eta) h(\eta^{-1}) d\lambda^r(\gamma)(\omega) d\lambda^s(\gamma)(\eta).$$

$$= \int_G \int_G f(\omega) g(\omega^{-1} \gamma \eta^{-1}) h(\eta) d\lambda^r(\gamma)(\omega) d\lambda^s(\gamma)(\eta).$$

We can choose neighborhoods $V_1$, $V_2$ and $V_3$ of $\alpha$, $\beta$ and $\delta$, respectively, such that $V_1 V_2 V_3 \subset V$. Notice from the integral above that if $\gamma \in \text{supp}(f * g * h)$ then there exists $\omega \in \text{supp} f$, $\eta \in \text{supp} h$ so that $\omega^{-1} \gamma \eta^{-1} \in \text{supp} g$. Since $\gamma = \omega(\omega^{-1} \gamma) \eta^{-1} \eta$, we see that $\text{supp}(f * g * h) \subset (\text{supp} f)(\text{supp} g)(\text{supp} h)$, so we have $\text{supp}(f * g * h) \subset V$ provided $f \subset V_1$, $g \subset V_2$ and $h \subset V_3$. Thus it suffices to take non-negative functions $f, h \in C_c(G)$ and $g \in C_c(G|U)$ with the appropriate supports and $f(\alpha) = g(\beta) = h(\delta) = 1$ and $F = f * g * h$.]

**Lemma 3.4.** Suppose that $L$ is a non-degenerate representation of $C^*(G, \lambda)$, and that $M$ is the representation of $C_0(G^0)$ defined by $M(\phi)L(f) = L(V(\phi)f)$. Then ker $M = I_F$ for a closed, G-invariant set $F \subset G^0$.

**Proof.** We know ker $M = I_F$ for closed subset $F$ of $G^0$. Let $U := G^0 \setminus F$. It will suffice to see that $U$ is G-invariant; that is, $U = [U]$.

If $f \in C_c(G|U)$, then $K = \text{supp} f$ is a compact subset of $G|U$. Thus $C = r(K)$ is a compact subset of $U$. Therefore we can choose $\phi \in C_c(U)$ such that $\phi(u) = 1$ for all $u \in C$. Then $V(\phi)f = f$. Since $\phi$ vanishes on $F$, $M(\phi)L(f) = L(V(\phi)f) = 0$. So $f \in \ker L$, and we have shown that

$$C_c(G|U) \subset \ker L.$$
Lemma 3.3 implies that $C_c(G|U) \subset \ker L$. If $[U] \neq U$, then there is a $\phi \in C_c(G^0)$ such that supp $\phi \subset |U|$ and $\phi$ is not identically zero on $F$. Since $V(\phi)f \in C_c(G|U)$ for all $f \in C_c(G)$, it follows that $V(\phi)f \in \ker L$. Therefore $M(\phi) = 0$, which contradicts $\ker M = J_F$. 

Proof of Proposition 3.2. Since $G^0/G$ is a second countable Baire space, we know from lemma on page 222 preceding Corollary 19 in [7] that every irreducible closed set must be a point closure. Lemma 3.4 tells us that $\ker M = J_F$ where $F$ is a closed $G$-invariant subset of $G^0$. Thus the image of $F$ in $G^0/G$ is closed. Suppose $F$ is not an orbit closure. Then $F$ is not irreducible. That is $F$ can be written as the union $C_1 \cup C_2$ where each $C_i$ is a closed $G$-invariant set such that $F \not\subset C_i$. In particular, $C_i \cap F \neq \emptyset$ for $i = 1$ or $i = 2$.

Let $U_i$ be the $G$-invariant open set $G^0 \setminus C_i$. Since $\text{Ex}(U_1) \cap \text{Ex}(U_2) = \text{Ex}(U_1) \setminus \text{Ex}(U_2)$, it follows from Lemma 2.10 in [13] that

$$C_c(G|U_i)C_c(G|U_2)$$

is dense in $C^*(G|U_1) \cap C^*(G|U_2)$. On the other hand

$$C_c(G|U_1)C_c(G|U_2) \subset C_c(G|U_1 \cap U_2) = C_c(G|G^0 \setminus (C_1 \cup C_2)) = C_c(G|G^0 \setminus F) = C_c(G|U).$$

Thus, (3.3) implies that

$$\text{Ex}(U_1) \cap \text{Ex}(U_2) \subset \ker L.$$ 

Since $L$ is irreducible, $\ker L$ is prime. Thus

$$\text{Ex}(U_i) \subset \ker L \quad \text{for some } i = 1, 2.$$ 

We may as well assume that $i = 1$. Since $U_1 \cap F \neq \emptyset$ (otherwise, we would have $F$ in $C_1$), we can choose $\phi \in C_c(G^0)$ such that supp $\phi \subset U_1$ and $\phi|_F \neq 0$. If $f \in C_c(G)$, we know

$$V(\phi)f(\gamma) = \phi(r(\gamma))f(\gamma),$$

thus $r(\gamma) \in U_1$ and because $U_1$ is invariant, $s(\gamma) \in U_1$ also. This means that $V(\phi)f$ is in $C_c(G|U_1)$. Thus $V(\phi)f \in \ker L$ for all $f \in C_c(G)$. It follows that $M(\phi) = 0$. But this contradicts $\phi|_F \neq 0$. Thus $F$ must be an orbit closure as claimed.

Corollary 3.5. Every irreducible representation of $C^*(G)$ factors through $C^*(G|u)$ for some $u \in G^0$.

Proof. Suppose $L$ is an irreducible representation and $M$ is the associated representation satisfying (3.1). We know $\ker M = J_F$ and that $F = \overline{[u]}$ by Proposition 3.2. Let $U := G^0 \setminus F$. We must show that $\text{Ex}(U) \subset \ker L$ by Lemma 2.10 of [13]. It suffices to show $C_c(G|U) \subset \ker L$. We will do this as we did in the proof of Lemma 3.4. If $f \in C_c(G|U)$, then $K = \text{supp } f$ is a compact subset of $G|U$. Thus $C = r(K)$ is a compact subset of $U$. Therefore we can choose $\phi \in C_c(U)$ such that $\phi(u) = 1$ for all $u \in C$. Then $V(\phi)f = f$. Since $\phi$ vanishes on $F$, $M(\phi)L(f) = L(V(\phi)f) = 0$. So $f \in \ker L$, and we have shown that $C_c(G|U) \subset \ker L$. □
We now have all the pieces needed to show that for principal groupoids, the map \( \psi \) is a continuous open injection. Further, if the orbit space is \( T_1 \), then \( \psi \) is a homeomorphism.

**Proposition 3.6.** Suppose \( G \) is a principal groupoid. Then the map \( \psi \) defined above is a continuous, open, injection.

**Proof.** We know that \( \psi \) is a continuous injection by Proposition 2.5 of [12]. We will show \( \psi \) is an open map using the criteria from Proposition II.13.2 in [4]. Let \( L^{u_n} \to L^u \) be a convergent net in \( C^*(G)^\wedge \). Thus \( M_{u_n} \to M_u \) in \( C_0(G^0)^\wedge \). Each \( M_{u_n} \) corresponds to a closed subset, namely \([u_n]\). By Lemma 2.4 of [20], we may pass to a subnet and relabel if necessary and find \( v_n \in [u_n] \) so \( v_n \to u \). Therefore \( \psi \) is open.

**Remark 3.7.** We will eventually weaken the hypothesis of Proposition 3.8 and require only that \( G \) be a principal groupoid and \( G^0/G \) be \( T_0 \).

**Proposition 3.8.** Suppose \( G \) is a principal groupoid in which orbits are closed. Then the map \( \psi \) defined above is a homeomorphism.

**Proof.** All that is left to show is that \( \psi \) is surjective. Let \( L \) be any irreducible representation of \( C^*(G) \). Since orbits are closed, we know that \( L \) is lifted from a representation on \( C^*(G_{|[u]}) \) from Corollary 3.5. The representation \( L^u \) is also a representation on \( C^*(G_{|[u]}) \). Since \( C^*(G_{|[u]}) \) is a transitive groupoid, and \( G \) is principal, Lemma 2.4 of [10] tells us that \( C^*(G_{|[u]}) \cong K(H) \). However, the compact operators have only one irreducible representation. Therefore \( L^u \cong L \).

4. CCR GROUPOID \( C^* \)-ALGEBRAS

In order to prove the theorem below, a generalization of Williams’ Theorem 1.2, we only use the property of Proposition 3.6 that \( \psi \) is a continuous injection.

**Theorem 4.1.** Let \( G \) be a principal groupoid. Then \( G \) is CCR if and only if \( G^0/G \) is \( T_1 \).

**Proof.** Suppose \( C^*(G) \) is CCR. This implies that points of the spectrum, \( C^*(G)^\wedge \), are closed. We know the map

\[
\psi : G^0/G \to C^*(G)^\wedge,
\]

where \( \psi([u]) = L^u \), is a continuous injection by Proposition 3.6. Thus the inverse image of a point of the spectrum is one orbit which must also be closed.

Now suppose that the orbit space is \( T_1 \). Suppose \( L \) is a representation of \( C^*(G) \). We know from Corollary 3.5 that \( L \) factors through \( C^*(G_{|[u]}) = C^*(G_{|[u]}) \).
for some \( u \in \mathcal{G}^0 \). But \( \mathcal{C}^* (\mathcal{G}_{[\mathcal{U}]}) \) is a transitive groupoid thus
\[
\mathcal{C}^* (\mathcal{G}_{[\mathcal{U}]}) \cong \mathcal{C}^* (\mathcal{G}^0_{[\mathcal{U}^0]}) \otimes \mathcal{K}
\]
by Theorem 3.1 of [10]. This is CCR because we are assuming \( \mathcal{G} \) is a principal groupoid. This means that \( \mathcal{L} \) is lifted from a representation of a CCR \( \mathcal{C}^* \)-algebra making \( \mathcal{L} \) a representation onto the compact operators. That is, \( \mathcal{C}^* (\mathcal{G}) \) is CCR.

**Corollary 4.2.** If \( \mathcal{G} \) is a principal groupoid and \( \mathcal{C}^* (\mathcal{G}) \) is CCR then \( \psi \) is a homeomorphism.

**Proof.** This is immediate from Theorem 4.1 and Proposition 3.8.

5. GCR \( \mathcal{C}^* \)-Algebras

We can weaken the conditions in Proposition 3.8 and show that, for principal groupoids, \( \psi \) is a homeomorphism when \( \mathcal{G}^0 / \mathcal{G} \) is a \( T_0 \) space. In doing this, we actually describe the ideal structure of the associated groupoid \( \mathcal{C}^* \)-algebra. We will also prove a generalization of Gootman’s Theorem 1.1 for principal groupoids that says \( \mathcal{C}^* (\mathcal{G}) \) is GCR if and only if \( \mathcal{G}^0 / \mathcal{G} \) is \( T_0 \).

We know that for principal groupoids \( \psi \) is a continuous, injective, open map from Proposition 3.6. Therefore to show \( \psi \) is a homeomorphism, we must show that \( \psi \) is onto. What we will do is show that when we require the orbit space to be \( T_0 \) rather than \( T_1 \), we can show that every irreducible representation of \( \mathcal{C}^* (\mathcal{G}) \) is lifted from a representation of \( \mathcal{C}^* (\mathcal{G}_{[C]}) \) where \( C \) is a Hausdorff subset of \( \mathcal{G}^0 / \mathcal{G} \). This will suffice.

We will begin Proposition 5.1 below by assuming that \( \mathcal{G}^0 / \mathcal{G} \) is \( T_0 \). We will also show that the orbit equivalence relation \( \mathcal{R} \) on \( \mathcal{G}^0 \) is an \( F_\sigma \) subset of \( \mathcal{G}^0 \times \mathcal{G}^0 \). When this is the case, Arlan Ramsay proved in Theorem 2.1 of [16] that there is a list of 14 different properties that are each equivalent to saying that \( \mathcal{G}^0 / \mathcal{G} \) is \( T_0 \). Some of these equivalent properties include:

1. each orbit is locally closed,
2. \( \mathcal{G}^0 / \mathcal{G} \) is almost Hausdorff, and
3. \( \mathcal{G}^0 / \mathcal{G} \) is a standard Borel space.

We will use property (2) in our proof. The idea for this proof comes from Lemma 2.3 in [21].

**Proposition 5.1.** Suppose \( \mathcal{G} \) is a groupoid. If \( \mathcal{G}^0 / \mathcal{G} \) is \( T_0 \) then there is an ordinal \( \gamma \) and ideals \( \{ I_\alpha : \alpha \leq \gamma \} \) such that:

1. \( \alpha < \beta \) implies that \( I_\alpha \subset I_\beta \);
2. \( I_0 = 0 \) and \( I_\gamma = \mathcal{C}^* (\mathcal{G}) \);
3. if \( \delta \) is a limit ordinal, then \( I_\delta \) is the ideal generated by \( \{ I_\alpha \} _{\alpha < \delta} \);
4. if \( \alpha \) is not a limit ordinal, then \( I_\alpha / I_{\alpha-1} \cong \mathcal{C}^* (\mathcal{G}_{[U_\alpha \setminus U_{\alpha-1}]}) \) where \( U_\alpha \) is a saturated subset of \( \mathcal{G} \) and each space \( U_{\alpha+1} \setminus U_\alpha \) is Hausdorff;
(v) if \( L \) is an irreducible representation of \( C^*(G) \), then \( L \) is the canonical extension of an irreducible representation of \( C^*(G|_{U_\alpha \setminus U_{\alpha-1}}) \).

Also, if \( G \) is a principal groupoid, then the map \( \psi \) defined above is a homeomorphism from \( G^0/G \) into \( C^*(G)^\wedge \).

**Remark 5.2.** The \( C^* \)-algebra \( C^*(G|_{U_\alpha \setminus U_{\alpha-1}}) \) is actually the quotient of \( C^*(G|_{U_\alpha}) \) by \( C^*(G|_{U_{\alpha-1}}) \).

**Proof.** First we will show that the orbit equivalence relation \( R \) on \( G^0 \) is an \( F_\sigma \) subset of \( G^0 \times G^0 \). To show that \( R \) is an \( F_\sigma \) set, we must show it is a countable union of closed sets of \( G^0 \times G^0 \). Notice that \( G \) is \( \sigma \)-compact and that \( R = \pi(G) \) where \( \pi(\gamma) = (r(\gamma), s(\gamma)) \). Therefore \( R \) is an \( F_\sigma \) subset because \( \pi \) is continuous.

Now from Theorem 2.1 in [16], we know that \( G^0/G \) is almost Hausdorff. Therefore, the discussion on page 125 of [5] gives us an ordinal \( \gamma \) and open subsets \( \{ U_\alpha : \alpha \leq \gamma \} \) of \( G^0/G \) such that:

(a) \( \alpha < \beta \) implies that \( U_\alpha \subset U_\beta \);
(b) \( \alpha < \gamma \) implies that \( U_\alpha \setminus U_{\alpha-1} \) is a dense Hausdorff subspace in the relative topology;
(c) if \( \delta \) is a limit ordinal, then \( U_\delta = \bigcup_{\alpha < \delta} U_\alpha \);
(d) \( U_0 = \emptyset \) and \( U_\gamma = G^0/G \).

In the sequel, we will abuse notation and consider each \( U_\alpha \) as an open invariant subset of \( G^0 \). Thus from Proposition 6.1 each \( U_\alpha \) corresponds to an ideal \( C^*(G|_{U_\alpha}) \) of \( C^*(G) \), which we will call \( I_\alpha \). Now properties (i), (ii), and (iii) follow immediately. Property (iv) follows immediately from the short exact sequence

\[
0 \longrightarrow C^*(U|_{\alpha-1}) \longrightarrow C^*(G|_{U_\alpha}) \longrightarrow C^*(G|_{U_\alpha \setminus U_{\alpha-1}}) \longrightarrow 0
\]

of Lemma 2.10 in [13].

Now we must show (v). Suppose \( L \) is an irreducible representation of \( C^*(G) \). Since \( L \) is an non-degenerate irreducible representation, the restriction of \( L \) to an ideal gives us an irreducible representation of the ideal. Define the set

\[
S = \{ \lambda : L(I_\lambda) \neq 0 \}.
\]

Since \( S \) is a set of ordinals, it has a smallest element. Let \( \alpha \) be the smallest element of \( S \). We know that \( \alpha \) is not a limit ordinal because of property (iii). Therefore \( \alpha - 1 \) exists and we have

\[
L(I_\alpha) \neq 0 \quad \text{and} \quad L(I_{\alpha-1}) = 0.
\]

Therefore, \( L \) is the canonical extension of a representation of \( I_\alpha / I_{\alpha-1} \) as needed.

Suppose \( G \) is a principal groupoid. We know that \( \psi \) is continuous, open, and injective from Proposition 3.6. Thus, to show \( \psi \) is a homeomorphism, we need only show that \( \psi \) is onto. In this proof, we need to be careful and define the following representations. Let \( \text{Ind}(G, u) \) be the representation \( L^u \) on \( C^*(G) \) and
let \( \text{Ind}(G_{U_{\alpha}}, u) \) be the representation \( L^u \) as a representation of \( C^* (G|_{U_{\alpha}}) \) for some \( u \in U_{\alpha} \).

Now let \( L \) be any representation of \( C^* (G) \). Our goal is to show that \( L \) is equivalent to \( L^u = \text{Ind}(G, u) \) for some \( u \in G^0 \). We know from part (v) that \( L \) is the canonical extension of a representation \( L' \) of \( I_{\alpha}/I_{\alpha-1} = C^*(G|_{U_{\alpha}\setminus U_{\alpha-1}}) \). We also know that \( U_{\alpha} \setminus U_{\alpha-1} \) is Hausdorff which means that \( L' \) is equivalent to \( \text{Ind}(G|_{U_{\alpha}}, u) \) for some \( u \in U_{\alpha} \). It suffices to show that the canonical extension of \( \text{Ind}(G|_{U_{\alpha}}, u) \) to \( C^* (G) \) must be equal to \( \text{Ind}(G, u) \). Notice that the spaces each of these representations act upon are the same. The representation \( \text{Ind}(G|_{U_{\alpha}}, u) \) extends to a representation \( \text{Ind}(G|_{U_{\alpha}}, u) \) on all of \( C^* (G) \). Notice that for \( f \in C_c(G), g \in L^2(G, \lambda_u), x \in G_u \) we have

\[
\text{Ind}(G|_{U_{\alpha}}, u)(f)(\text{Ind}(G|_{U_{\alpha}}, u)(g))\xi = \text{Ind}(G|_{U_{\alpha}}, u)(f * g)\xi = \text{Ind}(G, u)(f * g)\xi.
\]

Thus, \( \text{Ind}(G, u) \) is the canonical extension of \( \text{Ind}(G|_{U_{\alpha}}, u) \) as needed.  

We now have more than enough to prove the following theorem.

**Theorem 5.3.** Suppose \( G \) is a principal groupoid. Then \( C^* (G) \) is GCR if and only if \( G^0 / G \) is \( T_0 \).

**Proof.** Suppose \( C^* (G) \) is GCR. Then the spectrum of \( C^* (G) \) is \( T_0 \). From Lemma 3.8, we know there is a continuous injection from the orbit space into the spectrum. Therefore, the orbit space must also be \( T_0 \).

Now suppose we know \( G^0 / G \) is \( T_0 \). From Proposition 5.1, we know that every irreducible representation \( L \) of \( C^* (G) \) is the canonical extension of a representation of \( C^* (G|_{U_{\alpha}\setminus U_{\alpha-1}}) \) where \( U_{\alpha} \setminus U_{\alpha-1} \) is Hausdorff. Thus \( C^* (G|_{U_{\alpha}\setminus U_{\alpha-1}}) \) is CCR by Theorem 4.1. Therefore, the image of \( L \) contains the compact operators and \( C^* (G) \) is GCR.  

6. IDEALS

We know that for an open saturated subset \( U \) of \( G^0, C^* (G|_U) \) is an ideal in \( C^* (G) \). When \( G \) is principal and \( C^* (G) \) is GCR, all the ideals of \( C^* (G) \) are of this form.

**Proposition 6.1.** Suppose \( G \) is a principal groupoid and \( C^* (G) \) is GCR. Then the map \( U \mapsto \text{Ex}(U) \cong C^* (G|_U) \) from the collection of open saturated subsets of \( G^0 \) to the ideals of \( C^* (G) \) is a bijection.

**Proof.** Recall that if \( C^* (G) \) is GCR, \( C^* (G)^\wedge \cong \text{Prim}(C^* (G)) \). We also know that there is a natural correspondence between open subsets of \( \text{Prim}(C^* (G)) \) and ideals of \( C^* (G) \). Thus in order to show that \( \text{Ex} \) is a bijection, it suffices to show

\[
C^* (G|_U) \cong \bigcap_{v \not\in U} \ker L^v.
\]
Notice that \( C^*(G|_U) = \bigcap \{ \ker L^v : L^v(C^*(G|_U)) = 0 \} \).

It follows from the definition of \( L^v \) that if \( v \in U \), \( L^v(C_c(G|_U)) \neq 0 \) and if \( v \notin U \), \( L^v(C^*(G|_U)) = 0 \). Therefore

\[
C^*(G|_U) = \bigcap_{v \notin U} \ker L^v
\]
as needed. 

7. FELL ALGEBRAS

Finally, we generalize an Huef’s Theorem 1.4. Many of the results involving Cartan \( G \)-spaces that an Huef used to prove (1.4) came from [14]. Thus we first must generalize some of Palais’ work for Cartan \( G \)-spaces. This process leads us to some interesting results in their own right.

**DEFINITION 7.1.** A subset, \( N \) of \( G^0 \) is wandering if and only if the set

\[
G|_N = \pi^{-1}(N, N) = \{ \gamma \in G : s(g) \in N \text{ and } r(g) \in N \}
\]
is relatively compact.

**LEMMA 7.2.** A groupoid \( G \) is proper if and only if every compact subset of \( G^0 \) is wandering.

**Proof.** Suppose \( G \) is proper so that by definition \( \pi \) is a proper map. That is, the inverse image of a compact set is compact. Let \( K \) be a compact subset of \( G^0 \). By assumption \( \pi^{-1}(K, K) \) is compact; thus \( K \) is wandering.

Now suppose that every compact subset of \( G^0 \) is wandering. Let \( L \) be a compact subset of \( G^0 \times G^0 \). We must show \( \pi^{-1}(L) \) is compact. Note that \( L \subset W \times W \) where \( W \) is a compact subset of \( G^0 \).

Thus,

\[
\pi^{-1}(L) \subset \pi^{-1}(W, W)
\]
which is compact. Thus \( \pi^{-1}(L) \) is a closed subset of a compact set. Therefore \( \pi^{-1}(L) \) is compact. 

**DEFINITION 7.3.** We call a groupoid \( G \) a Cartan groupoid if and only if for every \( x \in G^0 \), \( x \) has a wandering neighborhood.

It is not difficult to show that a transformation group is a Cartan \( G \)-space if and only if the associated transformation group groupoid is a Cartan groupoid.

**LEMMA 7.4.** If \( G \) is a Cartan groupoid, then for each \( u \in G^0 \), \( [u] \) is closed in \( G^0 \).

**Proof.** Let \( u \in G^0 \). Let \( v \) be a limit point of \( [u] \) in \( G^0 \). Because \( G \) is a Cartan groupoid, \( v \) has a wandering neighborhood, \( U \). We will assume that \( U \) is closed. Thus, we can find a sequence of elements \( \{v_n\} \) in \( U \) that converge to \( v \) where each \( v_n \in [u] \). There also exists a sequence of elements \( \{\gamma_n\} \subset G \) such that for each \( n \),
s(γ_n) = v_n and r(γ_n) = u. Now choose one of the \{γ_n\}, call it \γ_{n_0}. Notice that \( r(γ_{n_0}^{-1}) = v_{n_0} \) and \( s(γ_{n_0}^{-1}) = u \). Thus \( γ_{n_0}^{-1}γ_n \in G |_U \) which is compact because it is relatively compact and closed. Thus we can pass to a subsequence, relabel, and assume \{γ_n\} converges to \γ. Since \( r \) and \( s \) are continuous, \( r(γ) = u \) and \( s(γ) = v \). Thus \( v \in \{u\} \).

Clearly, if \( G \) is proper, by Lemma 7.2 we see that \( G \) is a Cartan groupoid. We will prove a partial converse of this but first we need the following lemma.

**Lemma 7.5.** A groupoid \( G \) is proper if and only if every sequence \{γ_n\} ∈ \( G \) such that \{π(γ_n)\} converges has a convergent subsequence.

**Proof.** Suppose that \( G \) is proper. Let \{γ_n\} be a sequence where \{π(γ_n)\} converges to \( (u, v) \). Now, let \( K \) be a compact neighborhood of \( (u, v) \). Thus \{π(γ_n)\} is eventually inside of \( K \). Since \( π^{-1}(K) \) is compact, there is a subsequence \{γ_{n_k}\} that converges to \γ as needed.

Now suppose for every \{γ_n\} ∈ \( G \) such that \( π(γ_n) \) converges to \( (u, v) \), \{γ_n\} has a convergent subsequence \{γ_{n_k}\} where \{γ_{n_k}\} converges to \γ. Let \( K \) be a compact subset of \( G^0 × G^0 \). We must show \( π^{-1}(K) \) is compact. Let \{γ_n\} ⊂ \( π^{-1}(K) \). It suffices to show \{γ_n\} has a convergent subsequence. Since \{π(γ_n)\} ⊂ \( K \), \{π(γ_n)\} has a convergent subsequence in \( K \), call it \{π(γ_{n_k})\} where \{π(γ_{n_k})\} → \( (u, v) \). So, by assumption, we can find a subsequence and relabel so that \{γ_{n_k}\} converges to \γ ∈ \( π^{-1}(K) \).

**Lemma 7.6.** A groupoid \( G \) is proper if and only if \( G \) is Cartan and \( G^0/G \) is Hausdorff.

**Proof.** Suppose \( G \) is Cartan and \( G^0/G \) is Hausdorff. Let \{γ_n\} be a sequence in \( G \) such that \{π(γ_n)\} converges to \( (u, v) \). By Lemma 7.5, we must show that there exists a convergent subsequence of \{γ_n\} that converges to \γ.

Because the quotient map is continuous,

\[
[r(γ_n)] → [u] \quad \text{and} \quad [s(γ_n)] → [v]
\]

in \( G^0/G \). Since the orbit space is Hausdorff, and for each \( n \)

\[
[r(γ_n)] = [s(γ_n)],
\]

we must have \([u] = [v]\). Thus there exist \γ ∈ \( G \) so that \( r(γ) = u \) and \( s(γ) = v \). Which also means that

\[
r(γ_n) → r(γ) \quad \text{and} \quad s(γ_n) → s(γ).
\]

That is,

\[
π(γ_n) → π(γ) = (u, v).
\]

Since \( r \) is open, we can pass to a subsequence, relabel, and find \( η_n → γ \) with \( r(η_n) = r(γ_n) \). Then \( η_n^{-1}γ_n \) makes sense and \( π(η_n^{-1}γ_n) → (v, v) \). By taking a wandering neighborhood \( U \) of \( v \), we can pass to a subsequence, relabel, and assume that \( η_n^{-1}γ_n → β \) with \( β ∈ G |_{[v]} \). But then \( γ_n → γβ \) as needed.
Now suppose $G$ is proper. Since $G$ is locally compact, Lemma 7.2 tells us that $G$ is Cartan. We must show that $G^0/G$ is Hausdorff. It suffices to show that limits of convergent nets are unique.

Suppose $\{x_n\} \in G^0$ and

$$[x_n] \to [u] \quad \text{and} \quad [x_n] \to [v].$$

Notice that the quotient map

$$q : G^0 \to G^0/G$$

is open. This is true because $q(U) = s(r^{-1}(U))$ for any open set $U \in G^0$ and $r$ and $s$ are continuous and open. Thus using Proposition 2.13.2 of [4], we can pass to a subnet, relabel, and assume that $x_n$ converges to $x$ in $G^0$ and that there are $\{v_n\} \subset G^0$ such that $[v_n] = [x_n]$ with $v_n$ converging to some $v$. Similarly, we can find $\{u_n\} \subset G^0$ such that $[u_n] = [x_n] = [v_n]$.

Let $\gamma_n \in G$ be such that $r(\gamma_n) = u_n$ and $s(\gamma_n) = v_n$. If $K$ is a compact neighborhood of $u$ and $v$, then $\{\gamma_n\}$ is eventually in the compact set $\pi^{-1}(K,K)$. Thus we can pass to a subnet, relabel, and assume that $\gamma_n$ converges to $\gamma$ in $G$. But then $(\gamma) = u$ and $s(\gamma) = v$. That is $[u] = [v]$. \hfill \blacksquare

Because of the correspondence between open saturated subsets and ideals, saturated sets give us a key to the structure of $C^*(G)$. For Cartan groupoids, we can take the saturation of wandering neighborhoods and see that in addition to getting a saturated set, some of the useful properties of wandering neighborhoods are preserved.

**Lemma 7.7.** Suppose $G$ is a principal Cartan groupoid and $U$ is an open wandering neighborhood. Let $V := [U]$ be the saturation of $U$. Then $V/G|_V$ and $U/G|_U$ are homeomorphic.

**Proof.** Suppose that

$$q_U : U \to U/G|_U \quad \text{and} \quad q_V : V \to V/G|_V$$

are the corresponding quotient maps for the orbit spaces for $G|_U$ and $G|_V$. Now consider the map

$$f : U/G|_U \to V/G|_V \quad \text{so that} \quad f(q_U(x)) = q_V(x)$$

for $x \in U$. We will show $f$ is a homeomorphism. Clearly, $f$ is well defined.

Suppose

$$q_V(x_1) = q_V(x_2) \quad \text{where} \quad x_1, x_2 \in U.$$

This means there exist $\gamma \in G|_V$ so that $r(\gamma) = x_1$ and $s(\gamma) = x_2$. Since we know $x_1$ and $x_2$ are in $U$, $\gamma \in G|_U$. Therefore

$$q_U(x_1) = q_U(x_2)$$

and $f$ is injective.
Now let \( q_V(y) \in V/G|_V \). Since \( y \in V \) and \( V = [U] \), \( y \) is in the orbit of \( x \) for some \( x \in U \). This means that \( q_V(y) = q_V(x) = f(q_U(x)) \) and \( f \) is surjective.

Suppose that \( \{q_U(x_n)\} \) converges to \( q_U(x) \). We must show that \( \{q_V(x_n)\} \) converges to \( q_V(x) \). Suppose the contrary. Thus we can find a neighborhood, \( W \), of \( q_V(x) \) for which there is a subsequence which we relabel and assume \( \{q_V(x_n)\} \notin W \) for all \( n \). Because \( \{q_U(x_n)\} \) converges to \( q_U(x) \), and \( q_U \) is an open map, it follows from Proposition 2.13.2 in [4] that we can find a sequence \( \{y_n\} \) and a subsequence of \( \{x_n\} \) and relabel so that \( y_n \to x \) and \( [y_n] = [x_n] \) in \( U \). Therefore \( q_V(y_n) = q_V(x_n) \) for all \( n \) and, since \( q_V \) is continuous, \( \{q_V(x_n)\} \) converges to \( q_V(x) \). This is a contradiction; thus \( f \) is continuous.

Suppose \( q_V(u_n) \to q_V(u) \) where we can suppose that each \( u_n \) as well as each \( u \) belong to \( U \). Since \( q_V \) is open, we can pass to a subsequence, relabel, and assume that there are \( v_n \) in \( V \) such that \( q_V(v_n) = q_V(u_n) \) and \( v_n \to u \). Since \( U \) is open, we eventually have each \( v_n \in U \). Since \( q_U \) is continuous, for large \( n \), \( q_U(v_n) \to q_U(u) \). It follows from Proposition II.13.2 in [4] that \( f \) is open. 

**Lemma 7.8.** Suppose \( V \) is the saturation of an open wandering set, then \( G|_V \) is proper.

**Proof.** Because \( G \) is a Cartan groupoid, \( G|_V \) is also a Cartan groupoid. Thus, to show that \( G|_V \) is proper, it suffices to show that the orbit space, \( V/G|_V \), is Hausdorff. From Lemma 7.2, we know that \( G|_U \) is proper, thus by Lemma 7.6, \( U/G|_U \) is Hausdorff. But Lemma 7.7 tells us that \( U/G|_U \cong V/G|_V \). Therefore \( V/G|_V \) is also Hausdorff. 

With this newly defined structure of a Cartan groupoid, we have the machinery to generalize Theorem 1.4.

**Theorem 7.9.** Suppose \( G \) is a principal groupoid. Then \( G \) is a Cartan Groupoid if and only if \( A = C^*(G) \) is a Fell algebra.

**Proof.** Suppose \( G \) is a Cartan groupoid. We must show that for every irreducible representation, \( \pi \) of \( A \), \( \pi \) is a Fell point of \( \hat{A} \). Let \( x \in C^0 \) and \( U \) be an open wandering neighborhood of \( x \). Let \( V \) be the saturation of \( U \) which is also open.

Since \( G \) is a Cartan groupoid, the orbits of \( G \) are closed by Lemma 7.4. Therefore \( C^0/G \cong \hat{A} \) by Proposition 3.8. Let \( \pi \) be the representation of \( A \) that corresponds to \( [x] \).

Since \( V \) is a saturated open subset of \( G \), Lemma 2.10 of [13] tells us \( C^*(G|_V) \) is an ideal in \( A \). Thus \( \pi \) is an irreducible representation of \( C^*(G|_V) \). Also, from Lemma 7.8, we know that \( G|_V \) is a principal proper groupoid; thus Theorem 2.3 of [12] tells us that the ideal \( C^*(G|_V) \) has continuous-trace. We know continuous-trace \( C^*-\)algebras are Fell algebras, thus \( \pi \) is a Fell point of the open subset \( C^*(G|_V)^\wedge \) of \( \hat{A} \) which means \( \pi \) is a Fell point of \( \hat{A} \) also.
Now suppose $A$ is a Fell algebra. Let $x \in G^0$. We must show $x$ has a wandering neighborhood.

Since $A$ is CCR, $G^0 / G \cong \hat{A}$ by Corollary 4.2.

Let $\pi_x$ be the representation corresponding to $[x]$. Since $\pi_x$ is a Fell point, from Corollary 3.4 in [1] we know it has an open Hausdorff neighborhood in $\hat{A}$. This neighborhood is of the form $\hat{J}$ where $J$ is an ideal of $A$. We also know from Lemma 6.1 that

$$J \cong C^*(G|_V)$$

for some open, saturated subset $V$ of $G^0$. Notice that $x \in V$.

Since $J$ has Hausdorff spectrum and is a Fell algebra, $J$ has continuous-trace. Therefore by Theorem 2.3 of [12], $G|_V$ is proper. Thus, we know from Lemma 7.2 that every compact subset of $V$ is wandering.

Let $N$ be a compact neighborhood of $x$ in $V$. Therefore $N$ is a wandering neighborhood of $x$ in $G^0$. 

The proof of the following corollary is trivial in the transformation group case; however it requires much of the machinery established thus far to prove it in the groupoid case.

**Corollary 7.10.** Suppose $G$ is a principal groupoid. If $x \in G^0$ has a wandering neighborhood and $y \in [x]$, then $y$ has a wandering neighborhood.

**Proof.** Let $U$ be an open wandering neighborhood of $x$. We know that $G|_{[U]}$ is proper. Therefore $C^*(G|_{[U]})$ has continuous-trace which means it is a Fell algebra. Thus by Theorem 7.9, $G|_U$ is a Cartan groupoid. So we know every element of $[U]$ has a wandering neighborhood in $[U]$, therefore, every element has a wandering neighborhood in $G^0$.

**Corollary 7.11.** Let $G$ be a principal groupoid so that $C^*(G)$ is GCR. The largest Fell ideal of $C^*(G)$ is $C^*(G|_Y)$ where

$$Y = \{ x \in G^0 : \text{there exists a wandering neighborhood of } x \}.$$ 

**Proof.** Since $G$ is principal and $C^*(G)$ is GCR, by Lemma 6.1 we know every closed ideal is of the form $C^*(G|_Y)$ for some open $G$-invariant subset $Y \subseteq G^0$. From Corollary 7.10 we see that the $Y$ defined above is $G$-invariant. Also notice that $Y$ is open. Now apply Theorem 7.9 and we see that $C^*(G|_Y)$ is a Fell algebra and that any ideal that is also a Fell algebra, must be contained in $C^*(G|_Y)$.

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