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ACTION OF DISCRETE AMENABLE GROUPS ON REAL W^*-ALGEBRAS

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ABSTRACT. We consider a real analogue of the result of A. Ocneanu about the actions of discrete amenable groups on W^*-algebras. One gives the classification up to outer conjugacy of the actions of amenable groups on the hyperfinite real factor of type II_1. A main result is the uniqueness up to outer conjugacy of the free action of an amenable group on the hyperfinite real factor of type II_1.

KEYWORDS: *Real W^*-algebra, action of groups on real W^*-algebras, discrete amenable groups.*

MSC (2000): 46L10.

1. INTRODUCTION

The classical papers by Connes [2] and [4] showed that the structure of factors is closely connected with properties of their automorphisms. In [5] and [3] Connes gave the complete classification of periodic automorphisms of hyperfinite type II_1 factor and described the outer conjugation classes of automorphisms of injective type II_∞ factors. On the other hand the classification of periodic automorphisms of W^*-algebras is a classification of the actions of a finite cyclic group $\mathbb{Z}_n$ on W^*-algebra, where n is a period of automorphism. In [6] Jones generalized the Connes work for arbitrary finite groups. In [10] and [7] the classifications of the actions are given for amenable discrete and compact abelian groups.

The classification of periodic automorphisms of hyperfinite real types II_1, II_∞ factors were taken by Rakhimov and Usmanov in [12], [13]. In [11] those results generalized for arbitrary finite groups.

In the present paper the author will consider the actions of discrete amenable groups on real W^*-algebras. Similarly to the complex case (Ocneanu's work), one gives a complete classification (up to outer conjugacy) of the actions of a discrete amenable group on the hyperfinite real factor of type II_1.

2. PRELIMINARIES

Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. A weakly closed $*$-subalgebra $\mathfrak{A}$ with identity element $\mathbf{1}$ in $B(H)$ is called a W*-*algebra*. A real $*$-subalgebra $\mathfrak{R} \subset B(H)$ is called a *real* W*-*algebra* if it is closed in the weak operator topology and $\mathfrak{R} \cap i\mathfrak{R} = \{0\}$. A real W*-algebra $\mathfrak{R}$ is called a *real factor* if its center $Z(\mathfrak{R})$ contains only elements of the form $\{\lambda \mathbf{1}\}$, $\lambda \in \mathbb{R}$. We say that a real W*-algebra $\mathfrak{R}$ is of the type $\mathrm{I_{fin}}$, $\mathrm{I_\infty}$, $\mathrm{II_1}$, $\mathrm{II_\infty}$, or $\mathrm{III_\lambda}$, $(0 \leqslant \lambda \leqslant 1)$ if the enveloping W*-algebra $\mathfrak{A}(\mathfrak{R})$ has the corresponding type in the ordinary classification of W*-algebras. A linear mapping α of an algebra into itself with $\alpha(x^*) = \alpha(x)^*$ is called a $*$-*automorphism* if $\alpha(xy) = \alpha(x)\alpha(y)$; *involutive* $*$-*antiautomorphism* if $\alpha(xy) = \alpha(y)\alpha(x)$ and $\alpha^2(x) = x$. If α is an involutive $*$-antiautomorphism of W*-algebra M, we denote by (M, α) the real W*-algebra, generated by α, i.e. $(M, \alpha) = \{x \in M : \alpha(x) = x^*\}$ (see [1]).

Let N be a real or complex W*-algebra and G be a group, the identity of G will be written as $\mathbf{1}$. An *action* of G on N is a homomorphism $\theta : G \to \mathrm{Aut}(N)$; θ is called *free* if $\theta_g \in \mathrm{Int}(N)$ $(g \neq \mathbf{1})$; *crossed* if $\theta_{\mathbf{1}} = \mathrm{Id}$ and $\theta_g\theta_h\theta_{gh}^{-1} \in \mathrm{Int}(N)$, for any $g, h \in G$, where $\mathrm{Aut}(N)$ (respectively $\mathrm{Int}(N)$) is the group of all $*$-automorphisms (respectively inner $*$-automorphisms) of N. Two actions θ and θ' of G on N are called *conjugate* if there is a $*$-automorphism σ of N such that $\sigma\theta_g\sigma^{-1} = \theta'_g$, for all $g \in G$; *outer conjugate* if there are a unitary cocycle u for θ, i.e. unitaries $u_g \in N$, $g \in G$ with $u_{gh} = u_g\theta_g(u_h)$ and a $*$-automorphism σ of N such that $\sigma \mathrm{Ad} u_g \theta_g \sigma^{-1} = \theta'_g$, for all $g \in G$.

3. MODEL ACTION

Let G be an amenable group and $\mathcal{K}$ be a paving structure of G, S_i^n, K_i^n and M^n the sets of G, constructed in the Chapter 3 of [10]. We use $\mathcal{K}$ and those sets to index the matrix units of an UHF-algebra. Let $\mathcal{E}^0$ be a finite dimensional factor of dimension $|\overline{S}^0|$; $\mathcal{F}^n$ be a factor of dimension $|M^n|$ $(n \geqslant 0)$ and $\mathcal{E}^{n+1} = \mathcal{E}^n \otimes \mathcal{F}^n$. Let $\mathcal{E}$ be the finite factor obtained as weak closure of the UHF-algebra $\overline{\bigcup_n \mathcal{E}^n}$ on the GNS representation associated to its canonical trace. Let $(e^n_{s_1,s_2})$ $(s_i \in S^n)$ be a system of matrix units in $\mathcal{E}^n$ and u^n_g be a unitary of $\mathcal{E}^n$ given by

$$u^n_g = \sum_i \sum_{(k,s)} e^n_{(k_1,s),(k,s)}$$

where $g \in G$, $i \in I_n$, $(k,s) \in K^n_i \times S^n_i$, $k_1 = \ell^n_g(k)$ and $\ell^n_g : K^n \to K^n$ is the approximate left g-translation defined in 3.4 of [10].

We define the canonical involutive $*$-antiautomorphism α_n of $\mathcal{E}^n$ as

$$\alpha_n(e^n_{s_1,s_2}) = e^n_{s_2,s_1}.$$

It is easy see that $\alpha_n(u^n_g) = (u^n_g)^*$, $g \in G$.

Since $|\overline{S}^n| \to \infty$, $\mathcal{E}$ is a hyperfinite factor of type II_1; for each $g \in G$, $u_g = \lim_{n\to\infty} u_g^n$ $*$-strongly was shown in 4.4 of [10] to exist and yield a faithful representation of G in $\mathcal{E}$. Let α be the canonical involutive $*$-antiautomorphism of $\mathcal{E}$, generated by $(\alpha_n)_{n\in\mathbb{N}}$. For each n, $\mathcal{E} = \mathcal{E}^n \otimes ((\mathcal{E}^n)' \cap \mathcal{E})$ and $(\mathcal{E}^n)' \cap \mathcal{E}$ is a hyperfinite subfactor of $\mathcal{E}$ type II_1, on which $\mathrm{Ad}u_g$ acts almost trivially. The three $(\mathcal{E}, (u_g), \alpha)$ is called the *submodel*; $\mathrm{Ad}u_g$ the *submodel action* and α the *submodel involution*.

Let R be a countably infinite tensor product of copies of the submodel factor $\mathcal{E}$, taken with respect to the normalized trace, and for each $g \in G$, we let θ_g^o and α^o be the corresponding tensor product of copies of the submodel action $\mathrm{Ad}u_g$ and submodel involution α respectively. Then R is the hyperfinite factor of type II_1, θ^o is a free action of G on R (for each $g \in G$ we have $\theta_g^o \in \mathrm{Aut}(R)$), and α^o is an involutive $*$-antiautomorphism of R with $\theta^o \cdot \alpha^o = \alpha^o \cdot \theta^o$. We call $\mathfrak{R} = (R, \alpha^o)$ the *real model* and $\theta^o : G \to \mathrm{Aut}(R)$ the *model action*. The restriction of θ^o to $\mathfrak{R}$ we denote again by θ^o and we call it the *real model action*.

4. PROPERLY AND STRONGLY OUTER AUTOMORPHISM. NONABELIAN ROHLIN THEOREM

Let M be a W*-algebra and ω be a free ultrafilter on $\mathbb{N}$ (i.e. a maximal filter which doesn't contain finite sets). A sequence $(x_n)_{n\in\mathbb{N}}$ of elements in M is called *central* (respectively *ω-central*), if it is the element of the C^*-algebra $L^\infty(\mathbb{N}, M)$, and for each $\psi \in M_*$ we have $\|[\psi, x]\| \to 0$, when $n \to \infty$ (respectively $n \to \omega$). Let $\bigoplus_\infty M$ be the direct sum of a countable number of copies of M and let $J_\omega = \{(x_n) \in \bigoplus_\infty M\colon x_n \to 0$ $*$-strongly, when $n \to \omega\}$, $\widetilde{M} = \{(x_n) \in \bigoplus_\infty M\colon x_n = x, \forall n\}$. Let ρ be the canonical homomorphism of $\bigoplus_\infty M$ onto $\bigoplus_\infty M/J_\omega$. Put $M_\omega = (\bigoplus_\infty M/J_\omega) \cap \rho(\widetilde{M})'$. It is known that M_ω is the algebra of all equivalence classes of ω-centralizing sequences in M (see [9]). Moreover, the quotient C^*-algebra $(\mathcal{M}^\omega/J_\omega) \cap \rho(\widetilde{M})'$ we denote by M^ω, where $\mathcal{M}^\omega$ is the normalizing algebra of J_ω.

Similarly we define $\mathfrak{R}_\omega$, where $\mathfrak{R}$ is a real W*-algebra, moreover, in [14] it is proved that the $*$-algebra of central sequences $\mathfrak{R}_\omega$ is a real W*-algebra and $\mathfrak{A}(\mathfrak{R})_\omega = \mathfrak{R}_\omega + \mathrm{i}\mathfrak{R}_\omega$.

For a $*$-automorphism (or $*$-antiautomorphism) β of M (respectively of $\mathfrak{R}$) the mapping $(x_n)_{n\in\mathbb{N}} \to (\beta(x_n))_{n\in\mathbb{N}}$ defines a $*$-automorphism (or $*$-antiautomorphism) β_ω of M_ω (respectively of $\mathfrak{R}_\omega$). If α is an involutive $*$-antiautomorphism of M it is easy to see that the real W*-algebra $(M_\omega, \alpha_\omega)$ coincides with $(M, \alpha)_\omega$. a $*$-automorphism β is called *properly outer* if none of its restrictions under a nonzero invariant central projection is inner. By Lemma 2.4.1 of [1] a $*$-automorphism β of σ-finite real W*-algebra $\mathfrak{R}$ is properly outer if and only if its linear extension on $\mathfrak{A}(\Re)$ is so. In other words, a $*$-automorphism β of W*-algebra M with $\beta\alpha = \alpha\beta$ is properly outer if and only if β is properly outer on (M, α). We call β *strongly outer* if the restriction of β to the relative commutant of any countable β-invariant subset of M_ω is properly outer. An action θ of G on M_ω is *strongly free* if all θ_g $(g \neq 1)$ are strongly outer. It is easy to show that a $*$-automorphism β of M with $\beta\alpha = \alpha\beta$ is strongly outer if and only if β is strongly outer on (M, α), and an action θ of G on M_ω with $\theta_g\alpha = \alpha\theta_g$ $(\forall g)$ is strongly free if and only if the action $\theta|_{\Re_\omega}$ of G on $(M, \alpha)_\omega$ is strongly free.

Now we shall give a real analogue of Rohlin Theorem, the proof of which is carried out easily, similarly to the proof of Theorem 6.1 in [10], if we also follow the scheme of Subsections 2.3 and 2.4 of [1].

THEOREM 4.1 (Nonabelian Real Rohlin Theorem). *Let G be a discrete countable amenable group, M be a W*-algebra with separable predual and α be an involutive $*$-antiautomorphism of M. Let $\theta : G \to \mathrm{Aut}(M_\omega)$ be a crossed action which is semiliftable (see 5.2 of [10]), strongly free and α_ω-invariant. Let ϕ be a faithful normal α-invariant state on M such that $\theta|_{Z(M)}$ leaves $\phi|_{Z(M)}$ invariant. Let $\varepsilon > 0$ and $K_1, \ldots, K_N$ be an ε-paving family of subsets of G. Then there is a partition of unity $(e_{i,k})_{i=1,\ldots,N_j; k\in K_i}$ in $(M, \alpha)_\omega$ such that:*

(i) $\displaystyle\sum_{i=1}^{N} |K_i|^{-1} \sum_{k,\ell\in K_i} |\theta_{k\ell^{-1}}(e_{i,\ell}) - e_{i,k}|_\phi \leqslant 5\sqrt{\varepsilon}\,;$

(ii) $[e_{i,k}, \theta_g(e_{j,\ell})] = 0$, *for all* g, i, j, k, ℓ;

(iii) $\theta_g\theta_h(e_{i,k}) = \theta_{gh}(e_{i,k})$, *for all* g, h, i, k.

Moreover, $(e_{i,k})_{i,k}$ can be chosen in the relative commutant in $(M, \alpha)_\omega$ of any given countable subset of $(M, \alpha)_\omega$.

5. MAIN RESULTS

A (real) cocycle crossed action of countable discrete group G on real W*-algebra (M, α) is a pair (θ, u), where $\theta : G \to \mathrm{Aut}(M)$ and $u : G \times G \to U(M)$ satisfy for $g, h, k \in G$

$$\theta_g\theta_h = \mathrm{Ad}u_{g,h}\theta_{gh}, \quad u_{g,h}u_{gh,k} = \theta_g(u_{h,k})u_{g,hk},$$
$$\theta_g\alpha = \alpha\theta_g, \quad \alpha(u_{g,h}) = u^*_{g,h}, \quad u_{1,g} = u_{g,1} = \mathbf{1}.$$

(θ, u) is called *centrally free* if θ is free with the obvious adaptation of the definition. The real cocycle u is the real coboundary of v (denote as $u = \partial\mathrm{v}$), if $\mathrm{v} : G \to U(M)$ satisfies $u_{g,h} = \theta_g(\mathrm{v}^*_h)\mathrm{v}^*_g\mathrm{v}_{gh}$ and $\alpha(\mathrm{v}_g) = \mathrm{v}^*_g$, $\forall g, h \in G$.

Throughout in future, G will be an amenable group.

THEOREM 5.1. *Let M be a W*-algebra with separable predual and α be an involutive $*$-antiautomorphism of M. Let $\phi \in M^+_*$ be faithful and α-invariant. If (θ, u) is* *a centrally free (real) cocycle crossed action of G on* (M, α)*, such that* $\theta|_{Z(M)}$ *preserves* $\phi|_{Z(M)}$*, then u is a real coboundary.*

Moreover, given any $\varepsilon > 0$ *and any finite* $F \subset G$*, there exists* $\delta > 0$ *and a finite* $K \subset G$ *such that if* $\|u_{g,h} - \mathbf{1}\|^{\#}_{\phi} < \delta$ *(*$g, h \in K$*), then* $u = \partial \mathrm{v}$ *with* $\|\mathrm{v}_g - \mathbf{1}\|^{\#}_{\phi} < \delta$*,* $g \in F$*.*

The proof of theorem follows from Theorem 7.5 of [10] with regard to $\alpha(\mathrm{v}_g) = \mathrm{v}_g^*$ ($g \in G$), since it is given for the (real) cocycle u.

A real factor $\Re$ is called a real McDuff factor if it is isomorphic to $R \otimes \Re$, where R is the hyperfinite real factor of type II_1. It is easy to see that the enveloping W*-algebra of a real McDuff factor is also a McDuff factor, since $\mathfrak{A}(R \otimes \Re) = \mathfrak{A}(R) \otimes \mathfrak{A}(\Re)$ ([8]) and $\mathfrak{A}(R)$ is the hyperfinite factor of type II_1 ([1]).

LEMMA 5.2. *Let* $\Re$ *be a real McDuff factor. If* $\theta : G \to \mathrm{Aut}(\Re_\omega)$ *is a semiliftable strongly free action, then* $(\Re_\omega)^\theta$ *is of the type* II_1*.*

Proof. Since $\mathfrak{A}(\Re)$ is a McDuff factor and the linear extension $\overline{\theta} : G \to \mathrm{Aut}\, \mathfrak{A}(\Re)_\omega$ of θ is also a semiliftable strongly free action by Lemma 8.3 of [10] the fixed point algebra $(\mathfrak{A}(\Re)_\omega)^{\overline{\theta}}$ is of the type II_1. Hence $(\Re_\omega)^\theta$ is also of the type II_1. ∎

By means of the lemma that follows we can lift constructions from $\Re_\omega$ to $\Re$.

LEMMA 5.3. *Let M be a factor,* α *be an involutive* $*$*-antiautomorphism of M and* $\theta : G \to \mathrm{Aut}(M)$ *be a centrally free* α*-invariant action. Let* $(\mathrm{v}_g) \subset (M, \alpha)^\omega$ *(i.e.* $(\mathrm{v}_g) \subset M^\omega$ *with* $\alpha^\omega(\mathrm{v}_g) = \mathrm{v}_g^*$ *) be a (real) cocycle for* $(\theta_g)^\omega$ *and* $(e_{i,j})_{i,j \in I}$ $(|I| < \infty)$ *be matrix units in* $(M, \alpha)^\omega$ *such that*

$$(\mathrm{Adv}_g\, \theta_g^\omega)(e_{i,j}) = e_{i,j}, \quad i, j \in I,\ g \in G.$$

Then there exist representing sequences $(E^\nu_{i,j})_\nu$ *for* $e_{i,j}$*, which for* $\nu \in \mathbb{N}$ *are matrix units in* (M, α)*, and* $(\mathrm{v}^\nu_g)_\nu$ *for* v_g*, which for each* ν *form a* (θ_g)*-cocycle in* (M, α)*, such that*

$$(\mathrm{Adv}^\nu_g\, \theta_g)(E^\nu_{i,j}) = E^\nu_{i,j}, \quad i, j \in I,\ g \in G,\ \nu \in \mathbb{N}.$$

The proof of lemma follows from Lemma 8.4 of [10] with regard to $\alpha\theta_g = \theta_g\alpha$, $\alpha^\omega(\mathrm{v}_g) = \mathrm{v}_g^*$ and $\alpha^\omega(e_{i,j}) = e_{j,i}$ ($i, j \in I$, $g \in G$).

In future let (M, α) be a real McDuff factor with separable predual and $\theta : G \to \mathrm{Aut}(M)$ be a centrally free α-invariant action, let $\varepsilon > 0$, Ψ be a finite α-invariant subset of M_*^+ and F be a finite subset of G. If we use lemmas and the scheme of proof of Theorem 8.5 in [10], we obtain

THEOREM 5.4. *There exists a cocycle* (v_g) *for* (θ_g) *with* $\alpha(\mathrm{v}_g) = \mathrm{v}_g^*$ *and a* II_1 *hyperfinite real subfactor* $R \subset (M, \alpha)$ *such that*

$$(M, \alpha) = R \otimes (R' \cap (M, \alpha)), \quad (\mathrm{Adv}_g\, \theta_g)|_R = \mathrm{id}_R \quad \text{and}$$
$$\|\mathrm{v}_g - \mathbf{1}\|^{\#}_{\psi} < \varepsilon, \quad \|\psi \circ P_{R' \cap (M,\alpha)} - \psi\| < \varepsilon, \quad \psi \in \Psi,\ g \in F.$$

This implies

COROLLARY 5.5. *θ is outer conjugate to* $\mathrm{id}_R \otimes \theta$.

Moreover, given any $\varepsilon > 0$, finite $F \subset G$, and $\psi \in M_*^+$ with $\psi \cdot \alpha = \psi$, there exists an (θ_g)-cocycle (v_g) such that $\alpha(\mathrm{v}_g) = \mathrm{v}_g^*$, $(\mathrm{Adv}_g\theta_g)$ is conjugate to $\mathrm{id}_R \otimes \theta$ and $\|\mathrm{v}_g - \mathbf{1}\|_\psi^\# < \varepsilon$ $(g \in F)$.

Similarly to Theorem 5.4 we may obtain the following theorem.

THEOREM 5.6. *There exists a cocycle (v_g) for (θ_g) with $\alpha(\mathrm{v}_g) = \mathrm{v}_g^*$ and a II_1 hyperfinite real subfactor $R \subset (M, \alpha)$ such that*

$$(M, \alpha) = R \otimes (R' \cap (M, \alpha)), \quad (\mathrm{Adv}_g\, \theta_g)(R) = R,$$

$(\mathrm{Adv}_g\, \theta_g|_R)$ is conjugate to the model action θ^o and

$$\|\mathrm{v}_g - \mathbf{1}\|_\psi^\# < \varepsilon, \quad \|\psi \circ P_{R' \cap (M,\alpha)} - \psi\| < \varepsilon, \quad \psi \in \Psi,\ g \in F.$$

This implies

COROLLARY 5.7. *θ is outer conjugate to $\theta^o \otimes \theta$.*

Applying the scheme of proof of [11] and 9.1–9.4 of [10] we obtain

THEOREM 5.8. *If θ is an approximately inner and $\psi_o \in M_*^+$ with $\psi_o\alpha = \psi_o$, then there exists a cocycle (v_g) for (θ_g) with $\alpha(\mathrm{v}_g) = \mathrm{v}_g^*$ and a II_1 hyperfinite real subfactor $R \subset (M, \alpha)$ such that*

$$(M, \alpha) = R \otimes (R' \cap (M, \alpha)), \quad (\mathrm{Adv}_g\, \theta_g)(R) = R,$$

$(\mathrm{Adv}_g\, \theta_g|_R)$ is conjugate to the model action θ^o and

$$(\mathrm{Adv}_g\, \theta_g|_{R' \cap (M,\alpha)}) = \mathrm{id}_{R' \cap (M,\alpha)}, \quad \|\mathrm{v}_g - \mathbf{1}\|_{\psi_o}^\# < \varepsilon, \quad g \in F.$$

From the above results we can easily obtain

THEOREM 5.9. *If θ is an approximately inner, θ is outer conjugate to $\theta^o \otimes \mathrm{id}_{(M,\alpha)}$.*

Proof. By Corollary 5.7 θ^o is outer conjugate to $\theta^o \otimes \mathrm{id}_R$. From Theorem 5.8 we infer that θ is outer conjugate to $\theta^o \otimes \mathrm{id}_{R' \cap (M,\alpha)}$ and hence to $\theta^o \otimes \mathrm{id}_R \otimes \mathrm{id}_{R' \cap (M,\alpha)} = \theta^o \otimes \mathrm{id}_{(M,\alpha)}$. ∎

From the uniqueness, up to conjugacy, of an involutive $*$-antiautomorphism of the hyperfinite type II_1 factor M ([14]) and from $Ct(M, \alpha) = \mathrm{Int}(M, \alpha)$, $\overline{\mathrm{Int}}(M, \alpha) = \mathrm{Aut}(M, \alpha)$ ([1]) we obtain the main result of the paper

COROLLARY 5.10. *Any two free actions of the amenable group G on the hyperfinite real factor of type II_1 are outer conjugate.*

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