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COMPACT TRIPOTENTS AND THE STONE-WEIERSTRASS THEOREM FOR C^*-ALGEBRAS AND JB*-TRIPLES

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ABSTRACT. We establish some generalizations of Urysohn lemma for the *hull-kernel structure* in the setting of JB*-triples. These results are the natural extensions of those obtained by C.A. Akemann in the setting of C^*-algebras. We also develop some connections with the classical Stone-Weierstrass problem for C^*-algebras and JB*-triples.

KEYWORDS: *Compact tripotents, compact projections, C^*-algebras, JB*-triples, Urysohn's lemma, Stone-Weierstrass Theorem.*

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INTRODUCTION

Let K be a topological compact Hausdorff space and let $C(K)$ denote the Banach space of all complex-valued continuous functions on K. The classical Urysohn lemma allows us to describe the open subsets of K in the following way: a subset $A \subseteq K$ is open if and only if there is an increasing net (x_α) in $C(K)$ satisfying that $0 \leqslant x_\alpha(t) \nearrow 1$, for each $t \in A$, and $0 = x_\alpha(t)$ for each $t \in K\backslash A$. Clearly, a subset $C \subseteq K$ is closed (equivalently, compact) if and only if $K\backslash C$ is open. We can see the characteristic functions χ_A as projections in the bidual of $C(K)$.

In the more general setting of non-necessarily abelian C^*-algebras the notions of open and compact projections in the bidual of a C^*-algebra are mainly due to C.A. Akemann ([1], [3], see also [5], [33]). Let A be a C^*-algebra. A projection p in A^{**} is said to be *open* if p is the weak*-limit of a increasing net of positive elements in A, equivalently, $pA^{**}p \cap A$ is weak*-dense in $pA^{**}p$ (compare Proposition 3.11.9 of [33]). We say that p is *closed* whenever $1 - p$ is open. Finally, a projection p is said to be *compact* if, and only if, p is closed and there exists a positive element $a \in A$ such that $p \leqslant a \leqslant 1$, equivalently, there is a monotone decreasing net (a_λ) in A_+ with $p \leqslant a_\lambda \leqslant 1$, converging strongly to p (see for example [1] or Definition-Lemma 2.47 of [11]). If A is unital then every closed projection in A^{**} is compact. Akemann called this collection of open projections in A^{**} the *hull-kernel structure (HKS)* of A. In the HKS of a C^*-algebra, the following generalization of Urysohn lemma was obtained by Akemann in Theorem I.1 of [2]:

THEOREM 0.1. *Let A be a unital C^*-algebra and let p and q be two closed projections in A^{**} with $pq = 0$. Then there exists a in A with $0 \leqslant a \leqslant 1$, $ap = 0$ and $aq = q$.*

The generalizations of Urysohn lemma to the setting of non-commutative C^*-algebras are closely related with the general Stone-Weierstrass problem for non-commutative C^*-algebras. This tool has been intensively developed since 1969 by C.A. Akemann [1], [2], [3], L.G. Brown [11], C.A. Akemann, J. Anderson and G. Pedersen [4] and C.A. Akemann and G. Pedersen [5], among others.

C^*-algebras belong to the more general class of complex Banach spaces known as JB*-triples (see definition below). In this setting the role of projections is played by those elements called tripotents. Moreover, in [20] and [22] the notions of open, compact and closed tripotents in the bidual of a JB*-triple are introduced and developed. The aim of this paper is the study of the *hull-kernel structure* in a JB*-triple. In Section 2 we prove some generalizations of Urysohn lemma for this HKS. Theorem 1.4 assures that whenever e and f are two orthogonal tripotents in the bidual of a JB*-triple E, with e compact and f minimal, then there exist two orthogonal norm-one elements a_1 and a_2 in E such that $e \leqslant a_1$ and $f \leqslant a_2$. The second Urysohn lemma type result is Theorem 1.10, where we establish the following: Let E be a JB*-triple, x a norm-one element in E and u a compact tripotent in E^{**} relative to E satisfying that $u \leqslant r(x)$. Then there exists a norm-one element y in the inner ideal of E generated by x, such that $u \leqslant y \leqslant r(x)$.

In the last section we find some connections between the generalizations of Urysohn lemma to the HKS of a C^*-algebra or a JB*-triple with the Stone-Weierstrass problem. As main result (see Theorem 2.5) we prove that whenever B is a JB*-subtriple of a JB*-triple E such that for every couple of orthogonal tripotents u, v in E^{**} with v minimal and u minimal or zero, there exist orthogonal elements x, y in B such that $\|y\| = 1$, $\|x\| \in \{0, 1\}$ and $u \leqslant x$ and $v \leqslant y$ (when $u = 0$, then we mean $x = 0$), then B separates the extreme points of the closed unit ball of E^* and zero. This result combined with those obtained by C.A. Akemann [2] and B. Sheppard [39], on the Stone-Weierstrass theorem for C^*-algebras and JB*-triples, respectively, allow us to establish some new versions of the Stone-Weierstrass theorem in the setting of C^*-algebras and JB*-triples.

We recall (c.f. [31]) that a JB*-triple is a complex Banach space E together with a continuous triple product $\{\cdot,\cdot,\cdot\} : E \times E \times E \to E$, which is conjugate linear in the middle variable and symmetric bilinear in the outer variables satisfying that:

(a) $L(a, b)L(x, y) = L(x, y)L(a, b) + L(L(a, b)x, y) - L(x, L(b, a)y)$, where $L(a, b)$ is the operator on E given by $L(a, b)x = \{a, b, x\}$;

(b) $L(a,a)$ is an hermitian operator with non-negative spectrum;
(c) $\|L(a,a)\| = \|a\|^2$.

Every C^*-algebra is a JB*-triple via the triple product given by

$$2\{x,y,z\} = xy^*z + zy^*x,$$

and every JB*-algebra is a JB*-triple under the triple product

$$\{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.$$

A JBW*-triple is a JB*-triple which is also a dual Banach space (with a unique predual [9]). The second dual of a JB*-triple is a JBW*-triple [17]. Elements a, b in a JB*-triple, E, are *orthogonal* if $L(a,b) = 0$. With each tripotent u (i.e. $u = \{u,u,u\}$) in E is associated the *Peirce decomposition*

$$E = E_2(u) \oplus E_1(u) \oplus E_0(u),$$

where for $i = 0,1,2$, $E_i(u)$ is the $i/2$ eigenspace of $L(u,u)$. The Peirce rules are that $\{E_i(u), E_j(u), E_k(u)\}$ is contained in $E_{i-j+k}(u)$ if $i - j + k \in \{0,1,2\}$ and is zero otherwise. In addition,

$$\{E_2(u), E_0(u), E\} = \{E_0(u), E_2(u), E\} = 0.$$

The corresponding *Peirce projections*, $P_i(u) : E \to E_i(u)$, $(i = 0,1,2)$ are contractive and satisfy

$$P_2(u) = D(2D - I), \quad P_1(u) = 4D(I - D), \quad \text{and} \quad P_0(u) = (I - D)(I - 2D),$$

where D is the operator $L(u,u)$ and I is the identity map on E (compare [23]). A non-zero tripotent $u \in E$ is called *minimal* if and only if $E_2(u) = \mathbb{C}u$.

Let e and x be two norm-one elements in a JB*-triple, E, with e tripotent. We shall say that $e \leqslant x$ (respectively, $x \leqslant e$) whenever $L(e,e)x = e$ (respectively, x is a positive element in the JB*-algebra $E_2(e)$).

The strong*-topology in a JBW*-triple was introduced by T.J. Barton and Y. Friedman in [8]. This strong*-topology can be defined in the following way: Given a JBW*-triple W, a norm-one element φ in W_* and a norm-one element z in W such that $\varphi(z) = 1$, it follows from Proposition 1.2 of [8] that the assignment

$$(x,y) \mapsto \varphi\{x,y,z\}$$

defines a positive sesquilinear form on W. Moreover, for every norm-one element w in W $\varphi(w) = 1$, we have $\varphi\{x,y,z\} = \varphi\{x,y,w\}$, for all $x,y \in W$. The law $x \mapsto \|x\|_\varphi := (\varphi\{x,x,z\})^{1/2}$, defines a prehilbertian seminorm on W. The strong*-topology (noted by $S^*(W,W_*)$) is the topology on W generated by the family $\{\|\cdot\|_\varphi : \varphi \in W_*, \|\varphi\| = 1\}$.

The strong*-topology is compatible with the duality (W,W_*) (see Theorem 3.2 of [8]). The strong*-topology was further developed in [36], [34]. In particular, the triple product is jointly strong*-continuous on bounded sets (see [36], [34]).

Let W be a JBW*-triple and let a be a norm-one element in W. The sequence (a^{2n-1}) defined by $a^1 = a$, $a^{2n+1} = \{a, a^{2n-1}, a\}$ $(n \in \mathbb{N})$ converges in the strong*-topology (and hence in the weak*-topology) of W to a tripotent $u(a)$ in W (compare Lemma 3.3 of [20]). This tripotent will be called the *support tripotent* of a. There exists a smallest tripotent $r(a) \in W$ satisfying that a is positive in the JBW*-algebra $W_2(r(a))$, and $u(a) \leqslant a^{2n-1} \leqslant a \leqslant r(a)$. This tripotent $r(a)$ will be called the *range tripotent* of a. (Beware that in [20], $r(a)$ is called the support tripotent of a).

In [20], C.M. Edwards and G.T. Rüttimann introduced the concepts of *open* and *compact* tripotents in the bidual of a JB*-triple. In [22], the authors of the present paper studied the notions of open and compact tripotents in a JBW*-triple with respect to a weak*-dense subtriple. Concretely, given a JBW*-triple W and a weak*-dense JB*-subtriple E of W, a tripotent u in W is said to be *compact-*G_δ *relative to* E if u is the support tripotent of a norm one element in E. The tripotent u is said to be *compact relative to* E if $u = 0$ or there exist a decreasing net, $(u_\lambda) \subseteq W$, of compact-G_δ tripotents relative to E converging, in the strong*-topology of W, to the element u (compare Section 4 of [20]). A tripotent u in W is said to be *open relative to* E if $E \cap W_2(u)$ is weak*-dense in $W_2(u)$. When E is a JB*-triple, the range (respectively, the support) tripotent of every norm-one element in E is always an open (respectively, compact) tripotent in E^{**} relative to E.

NOTATION 0.2. Given a Banach space X, we denote by X_1, S_X, and X^* the closed unit ball, the unit sphere, and the dual space of X, respectively. If K is any convex subset of X, then we write $\partial_e(K)$ for the set of extreme points of K.

1. THE NON-COMMUTATIVE URYSOHN LEMMA FOR JB*-TRIPLES

This section is mainly devoted to obtain some Urysohn lemma type results for the HKS of a JB*-triple. We begin by developing some new properties of compact tripotents in the bidual of a JB*-triple.

PROPOSITION 1.1. *Let W and V be* JBW**-triples, E a weak*-dense* JB**-subtriple of W and $T : W \to V$ a surjective weak*-continuous triple homomorphism such that $\|T(x)\| = \|x\|$, for all x in E. Suppose that e is a tripotent in W, then $T(e)$ is compact relative to $T(E)$ in V whenever e is compact relative to E. Moreover, if T is a triple isomorphism, then e is compact relative to E in W if and only if $T(e)$ is compact relative to $T(E)$ in V.*

Proof. Suppose that $e \in W$ is compact relative to E. If $T(e) = 0$, then there is nothing to prove. Suppose that $T(e)$ is a non-zero tripotent in V. By definition, there exists a decreasing net $(u_\lambda)_{\lambda \in \Lambda} \subset W$, of compact-$G_\delta$ tripotents relative to E (i.e., $\forall \lambda$ there exists $a_\lambda \in S_E$ such that $u_\lambda = u(a_\lambda)$), converging to e in the strong*-topology of W.

From the hypothesis we know that, for each $\lambda \in \Lambda$, $\|T(a_\lambda)\| = \|a_\lambda\| = 1$. Since, for each λ, $u(T(a_\lambda))$ coincides with the limit, in the weak*-topology of V, of the sequence $(T(a_\lambda)^{2n-1}) = (T(a_\lambda^{2n-1}))$, and T is weak*-continuous, we have $u(T(a_\lambda)) = T(u(a_\lambda))$. The conditions (u_λ) decreasing and T triple homomorphism imply that $u(T(a_\lambda)) = T(u(a_\lambda))$ is also a decreasing net in V. Since T is weak*-continuous, we deduce, from Corollary 3 in [36], that T is $S^*(W, W_*) - S^*(V, V_*)$-continuous. Therefore, $u(T(a_\lambda)) = T(u(a_\lambda))$ tends to $T(e)$ in the $S^*(V, V_*)$-topology. This shows that $T(e)$ is compact relative to $T(E)$ in V. ∎

REMARK 1.2. Note that under the assumptions of the previous proposition there is a relationship between compact-G$_\delta$ tripotents in W (respectively, range tripotents in W) relative to E and compact-G$_\delta$ tripotents in V (respectively, range tripotents in V) relative to $T(E)$. Indeed, let $x \in E$ be a norm-one element. The sequence x^{2n-1} (respectively, $x^{1/(2n-1)}$) tends to $u(x)$ (respectively, $r(x)$) in the weak*-topology of W. Since T is a weak*-continuous triple homomorphism isometric on E, it follows that $T(u(x)) = u(T(x))$ (respectively, $T(r(x)) = r(T(x))$). Moreover, since every compact-G$_\delta$ (respectively, range) tripotent in V relative to $T(E)$ is of the form $u(T(x))$ (respectively, $r(T(x))$) for a suitable norm-one element $x \in E$, it is clear that T maps the set of compact-G$_\delta$ (respectively, range) tripotents in W relative to E onto the set of compact-G$_\delta$ (respectively, range) tripotents in V relative to $T(E)$.

In Theorem 3.4 of [16] it is proved that every minimal tripotent in the bidual of a JB*-triple, E, is compact relative to E. The next corollary shows that this result remains true for every minimal tripotent in a JBW*-triple W and for any weak*-dense JB*-subtriple of W.

Let E be a JB*-triple. A subtriple I of E is said to be an *ideal* of E if $\{E, E, I\} + \{E, I, E\} \subseteq I$. We shall say that I is an *inner ideal* of E whenever $\{I, E, I\} \subseteq I$.

If E and F are two JB*-triples, a representation $\pi : E \to F$ is any triple homomorphism from E to F. Let $j : E \to E^{**}$ be the canonical inclusion of E into its bidual. Each weak*-closed ideal I of E^{**} is an M-summand (see [27]). Therefore there exists a weak*-continuous contractive projection $\pi : E^{**} \to I$. The representation $E \to I$ given by $x \mapsto \pi j(x)$ is called *the canonical representation* of E corresponding to I. Suppose that E is a weak*-dense JB*-subtriple of a JBW*-triple W and let $\lambda : E \to W$ be the natural inclusion. From Proposition 6 of [7], there exists a weak*-closed triple ideal M of E^{**} and a triple isomorphism $\Psi : W \to M$ satisfying that $\Psi\lambda$ is the canonical representation of E corresponding to M.

COROLLARY 1.3. *Let E be a weak*-dense* JB**-subtriple of a* JBW**-triple W. Let M be the weak*-closed triple ideal of E^{**} and let $\Psi : W \to M$ the triple isomorphism described in the above paragraph, satisfying that $\Psi\lambda$ is the canonical representation of E corresponding to M. Let e be a tripotent in W. Then e is compact relative to E in W*

whenever $\Psi(e)$ is compact relative to E in E^{**}. In particular, every minimal tripotent in W is compact relative to E.

Proof. Let $\pi : E^{**} \to M$ denote the canonical projection of E^{**} onto M. Clearly, π is a surjective weak*-continuous triple homomorphism and if $\lambda : E \to W$ and $j : E \to E^{**}$ denote the canonical inclusions of E into W and E^{**}, respectively, we have $\Psi \circ \lambda = \pi \circ j$.

Let $e \in W$ be a tripotent in W such that $\Psi(e)$ is compact relative to E in E^{**}. Proposition 1.1 applied to $\pi : E^{**} \to M$, E^{**} and E, gives $\Psi(e)$ compact relative to $\pi(E)$ in M. Again, Proposition 1.1 assures that e is compact relative to E in W.

Finally, if e is minimal in W, that is, $W_2(e) = \mathbb{C}e$, it is not hard to see that $M_2(\Psi(e)) = E_2^{**}(\Psi(e)) = \mathbb{C}\Psi(e)$, and hence $\Psi(e)$ is a minimal tripotent in E^{**}. Therefore, from Theorem 3.4 of [16], it follows that $\Psi(e)$ is compact relative to E in E^{**}, which implies that e is compact relative to E in W. ∎

Let x be a norm-one element in a JB*-triple E. Throughout the paper, E_x will denote the norm-closed JB*-subtriple of E generated by x. It is known that E_x is JB*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space Ω contained in $[0,1]$, such that $\Omega \cup \{0\}$ is compact and $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. Moreover, if we denote by Ψ the triple isomorphism from E_x onto $C_0(\Omega)$, then $\Psi(x)(t) = t$ $(t \in \Omega)$ (cf. 4.8 in [30], 1.15 in [31] and [23]).

The following result is a first generalization of Urysohn lemma to the setting of JB*-triples.

THEOREM 1.4. *Let E be a weak*-dense* JB**-subtriple of a* JBW**-triple W. Let u, v be two orthogonal tripotents in W with u compact relative to E and v minimal. Then there exist two orthogonal elements a_1 and a_2 in E such that $\|a_2\| = 1$, $\|a_1\| \in \{0,1\}$, $u \leqslant a_1$ and $v \leqslant a_2$.*

Proof. When $u = 0$, we take $a_1 = 0$ and the existence of a_2 follows from the last statement in Corollary 1.3 (see also [16]). We may therefore assume $u \neq 0$.

Since v is a minimal tripotent in W, from Proposition 4 of [23] it follows that there exists $\varphi \in \partial_e((W_*)_1)$ satisfying $\varphi(v) = 1$.

Corollary 1.3 implies v compact relative to E. Now, Proposition 2.3 of [22] assures that v and u are closed tripotents relative to E, that is, $W_0(u) \cap E$ and $W_0(v) \cap E$ are subtriples of W which are weak*-dense in $W_0(u)$ and $W_0(v)$, respectively. From the orthogonality of u and v we have $u \in W_0(v)$ and $v \in W_0(u)$.

Let us denote $F = W_0(u) \cap E$. Since Theorem 2.8 of [16] remains true when E^{**} is replaced with any JBW*-triple W such that E is weak*-dense in W, then applying this result to F and $W_0(u)$, it follows that for every $\varepsilon, \delta > 0$, there exist $y \in F$ and a tripotent $e \in W_0(u)$ such that $e \leqslant v$, $P_i(e)(v - y) = 0$, for $i = 1,2$, $\|y\| \leqslant (1+\delta)\|(P_2(e) + P_1(e))(v)\|$ and $|\varphi(v - e)| < \varepsilon$. Since ε can be chosen arbitrary small and v is a minimal tripotent in $W_0(u)$, we have $e = v$. The same arguments given in Lemma 3.1 of [16] assure the existence of a norm-one element $b_2 \in F$ such that $v \leqslant b_2$.

Let F_{b_2} denote the JB*-subtriple of F generated by b_2. As we have commented above, there exists a locally compact Hausdorff space $L \subseteq [0,1]$ with $L \cup \{0\}$ compact such that F_{b_2} is isometrically isomorphic to $C_0(L)$ under some surjective isometry denoted by ψ and $\psi(b_2)(t) = t$, for any $t \in L$. Let a_2 and $\widetilde{a}_2 \in F_{b_2}$ the norm-one elements given by the expressions

$$\psi(a_2)(t) := \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant 3/4, \\ \text{affine} & \text{if } 3/4 \leqslant t \leqslant 1, \\ 1 & \text{if } t = 1; \end{cases} \quad \psi(\widetilde{a}_2)(t) := \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant 1/2, \\ \text{affine} & \text{if } 1/2 \leqslant t \leqslant 3/4, \\ 1 & \text{if } t \geqslant 3/4. \end{cases}$$

Clearly $v \leqslant u(b_2) \leqslant u(a_2) \leqslant a_2 \leqslant r(a_2) \leqslant \widetilde{a}_2$.

Now, Theorem 2.6 in [22] assures the existence of a norm-one element x in E such that $u \leqslant x$. We define

$$c_1 = P_0(\widetilde{a}_2)(x) := x - 2L(z,z)x + Q(z)^2(x) \in E,$$

where z is the element in $F_{\widetilde{a}_2} = E_{\widetilde{a}_2}$ satisfying $\{z, r(\widetilde{a}_2), z\} = \widetilde{a}_2$ (compare Section 2 of [22]). From Lemma 2.5 of [22], we have $c_1 \in E \cap W_0(r(a_2))$, which, in particular, implies that c_1 and a_2 are orthogonal. We claim that

$$L(u,u)\, c_1 = u.$$

Indeed, since $x \geqslant u$, then $x = u + P_0(u)(x)$. Moreover, since $z \in F_{\widetilde{a}_2} = E_{\widetilde{a}_2} \subseteq W_0(u)$, it follows, from Peirce rules, that

$$\begin{aligned} L(u,u)c_1 &= \{u,u,x - 2L(z,z)x + Q(z)^2(x)\} \\ &= \{u,u,u + P_0(u)(x) - 2L(z,z)(u + P_0(u)(x)) + Q(z)^2(u + P_0(u)(x))\} \\ &= \{u,u,u\} + \{u,u,P_0(u)(x) - 2L(z,z)(P_0(u)(x)) + Q(z)^2(P_0(u)(x))\} = u. \end{aligned}$$

Again, the same arguments given in Lemma 3.1 of [16] imply the existence of a norm-one element $a_1 \in E_{c_1}$ such that $u \leqslant a_1$. ∎

In the case of von Neumann algebras the above theorem generalizes Theorem II.19 in [1] from the setting of biduals of C^*-algebras to the more general setting of von Neumann algebras.

COROLLARY 1.5. *Let A be a weak*-dense C^*-subalgebra of a von Neumann algebra W. Let p, q be two orthogonal projections in W with p compact relative to A and q minimal. Then there exist two orthogonal positive elements a_1 and a_2 in A such that $\|a_2\| = 1$, $\|a_1\| \in \{0,1\}$, $p \leqslant a_1$ and $q \leqslant a_2$.*

In some particular triple representations the results stated in Proposition 1.1 and Remark 1.2 can be improved. This is the case of the canonical representation of a JB*-triple into the atomic part of its bidual. We recall that, given a JB*-triple E, then E^{**} decomposes into an orthogonal direct sum of two weak*-closed triple ideals A and N, where A (called *the atomic part of E^{**}*) coincides with the weak*-closure of the linear span of all minimal tripotents in E^{**}, $E^* = A_* \oplus^{\ell_1} N_*$ and the closed unit ball of N_* has no extreme points, which implies that $\partial_e(E_1^*) = \partial_e(A_{*,1})$ (compare Theorems 1 and 2 of [23]). If π denotes the natural weak*-continuous projection of E^{**} onto A and $j : E \to E^{**}$ is the canonical inclusion, then the mapping $\pi \circ j : E \to A$ is an isometric triple embedding called the canonical embedding of E into the atomic part of its bidual (see proof of Proposition 1 in [24]).

We recall some notation needed in what follows. Let X be a Banach space. For each pair of subsets G, F in the unit ball of X and X^*, respectively, let the subsets $G^{'}$ and $F_{,}$ be defined by

$$G^{'} = \{f \in B_{X^*} : f(x) = 1, \forall x \in G\} \quad \text{and} \quad F_{,} = \{x \in B_X : f(x) = 1, \forall f \in F\},$$

respectively.

PROPOSITION 1.6. *Let E be a* JB**-triple, let π denote the canonical projection of E^{**} onto its atomic part and let $i : E \to E^{**}$ be the canonical embedding of E into its bidual. The following assertions hold:*

(i) *Let u and v be two compact tripotents in E^{**} relative to E. Then $u \leqslant v$ if and only if $\pi(u) \leqslant \pi(v)$.*

(ii) *For each compact tripotent u in $\pi(E^{**})$ relative to $\pi(E)$ there exists a unique compact tripotent e in E^{**} relative to E such that $\pi(e) = u$.*

Proof. (i) Let us denote $A := \pi(E^{**})$. If $u \leqslant v$ in E^{**}, then $\pi(u) \leqslant \pi(v)$, since π is a triple homomorphism. Suppose now that $\pi(u) \leqslant \pi(v)$. From Theorem 4.4 of [18], we have

$$\{\pi(u)\}_{\underset{A_*}{,}} \subseteq \{\pi(v)\}_{\underset{A_*}{,}} \tag{1.1}$$

By Theorem 4.5 of [20] together with the comments preceding Corollary 3.5 in [16], every non-zero compact tripotent in E^{**} relative to E majorises a minimal tripotent of E^{**}. In particular, if e is a compact tripotent in E^{**} with $\pi(e) = 0$, then $e = 0$. We may therefore assume that $\pi(u)$ and hence $\pi(v)$ are not zero.

From Theorem 4.2 of [20], it follows that the sets $\{u\}_{\underset{E^*}{,}}$ and $\{v\}_{\underset{E^*}{,}}$ are non-empty $\sigma(E^*, E)$-compact and convex subsets of E_1^*. By the Krein-Milman theorem we have

$$\{u\}_{\underset{E^*}{,}} = \overline{\text{co}}^{\sigma(E^*,E)}(\partial_e(\{u\}_{\underset{E^*}{,}})), \tag{1.2}$$

$$\{v\}_{\underset{E^*}{,}} = \overline{\text{co}}^{\sigma(E^*,E)}(\partial_e(\{v\}_{\underset{E^*}{,}})). \tag{1.3}$$

Since $\partial_e(E_1^*) = \partial_e(A_{*,1})$, we have

$$\{\pi(u)\}_{\underset{A_*}{,}} \cap \partial_e(A_{*,1}) = \{\pi(u)\}_{\underset{E^*}{,}} \cap \partial_e(E_1^*) = \{u\}_{\underset{E^*}{,}} \cap \partial_e(E_1^*) = \partial_e(\{u\}_{\underset{E^*}{,}}).$$

Similarly,

$$\{\pi(v)\}_{\underset{A_*}{,}} \cap \partial_e(A_{*,1}) = \partial_e(\{v\}_{\underset{E^*}{,}}).$$

Finally, we deduce, from (1.1), (1.2), (1.3) and the last two expressions, that

$$\{u\}'_{E^*} \subseteq \{v\}'_{E^*}$$

which shows that $u \leqslant v$ (compare Theorem 4.4 of [18]).

(ii) Let u be a non-zero compact tripotent in $A = \pi(E^{**})$ relative to $\pi(E)$. Then there exists a decreasing net (u_λ) of compact-G$_\delta$ tripotents in A relative to $\pi(E)$ converging in the strong*-topology of A to u. By Remark 1.2, for each λ, there is a norm-one element $x_\lambda \in E$ such that

$$u_\lambda = u(\pi(x_\lambda)) = \pi(u(x_\lambda)).$$

Since $\pi(u(x_\lambda))$ is a decreasing net of compact-G$_\delta$ tripotents, then (i) implies that $(u(x_\lambda))$ is a decreasing net in E^{**}. By Theorem 4.5 of [20] there exists a non-zero compact tripotent $e \in E^{**}$ relative to E such that e coincides with the infimum of the family $(u(x_\lambda))$. Since π is weak*-continuous and $(u(x_\lambda))$ tends to e in the weak*-topology of E^{**}, we have that $\pi((u(x_\lambda)) \to \pi(e)$ in the $\sigma(E^{**}, E^*)$-topology, and hence $\pi(e) = u$. Finally, the uniqueness of e follows from (i). ∎

The above result is a partial generalization of Theorem II.17 in [1]. In the more particular setting of JB*-algebras we have:

COROLLARY 1.7. *Let A be a* JB*-*algebra, let π denote the canonical projection of A^{**} onto its atomic part and let $j : A \to A^{**}$ be the canonical embedding of A into its bidual. The following assertions hold:*

(i) *Let p and q be two compact projections in A^{**} relative to A. Then $p \leqslant q$ if and only if $\pi(p) \leqslant \pi(q)$.*

(ii) *For each compact projection p in $\pi(A^{**})$ relative to $\pi(A)$ there exists a unique compact projection q in A^{**} relative to A such that $\pi(q) = p$.*

Given a JB*-algebra A, the cone of all positive elements in A will be denoted by A_+, while A^*_+ will denote the set of positive elements in A^*. Let W be a JBW*-algebra. The symbol $Q_*(W)$ will denote the set of all positive elements in W_* with norm less or equal to one. $Q_*(W)$ will be called the *normal quasi-state space* of W. The *normal state space*, $S_*(W)$, is the set of all elements in $Q_*(W)$ with norm equal to one. Given a projection p in W we shall denote $F(p) = F_W(p) := \{\varphi \in Q_*(W) : \varphi(p) = \|\varphi\|\}$. If A is a JB*-algebra, then the set $Q(A)$ (respectively, $S(A)$) of quasi-states (respectively, states) of A is defined as $Q_*(A^{**})$ (respectively, $S_*(A^{**})$).

The following result was proved by M. Neal in Lemma 3.2 and Theorem 5.2 of [32].

PROPOSITION 1.8. *Let A be a* JB*-*algebra and let p be a projection in A^{**}. Then we have:*

(i) *p is open relative to A if and only if there exists an increasing net (a_λ) in $A_{1,+}$ with least upper bound p.*

(ii) *p is closed relative to A if and only if $F(p)$ is $\sigma(A^*, A)$-closed in $Q(A)$.*

The next result gives a characterization of compact projections in JB*-algebra biduals. A similar result was obtained by C.A. Akemann, J. Anderson and G.K. Pedersen in the setting of C^*-algebra biduals (see Lemma 2.4 of [4]).

Given a JB*-algebra A, $\widetilde{A} = A \oplus \mathbb{C}1$ will stand for the result of adjoining a unit to A (compare Section 3.3 of [26]). $\widetilde{A}$ is also called the *unitization* of A.

PROPOSITION 1.9. *Let A be a* JB**-algebra and let p be a projection in A^{**}. Then p is compact relative to A if and only if $F(p) \cap S(A)$ is $\sigma(A^*, A)$-closed in $Q(A)$.*

Proof. The proof given in Lemma 2.4 of [4] can be literally adapted to the present setting. We include here a sketch of the proof for completeness reasons. Suppose first that p is a non-zero compact projection in A^{**}. From Theorem 4.2 of [20] we have $F(p) \cap S(A) = \{p\}$, is $\sigma(A^*, A)$-closed in $Q(A)$.

Let $\widetilde{A}$ be the unitization of A. Each element $\phi \in Q(\widetilde{A})$ can be written in the form $\phi = \psi + \alpha\phi_0$, with $\psi \in Q(A)$, $\|\phi\| = \|\psi\| + |\alpha|$, where ϕ_0 is the unique state of $\widetilde{A}$ satisfying $\phi_0(A) = 0$ (compare Lemma 3.6.6 of [26]). Since $p \in A^{**}$ and hence $\phi_0(p) = 0$, we easily check that

$$F_{A^*}(p) \cap S(A) = F_{\widetilde{A}^*}(p) \cap S(\widetilde{A}).$$

Therefore, $F_{A^*}(p) \cap S(A)$ is $\sigma(A^*, A)$-closed in $Q(A)$ if and only if $F_{\widetilde{A}^*}(p) \cap S(\widetilde{A})$ is $\sigma(\widetilde{A}^*, \widetilde{A})$-closed in $Q(\widetilde{A})$. By Proposition 1.8, it follows that p is closed in $(\widetilde{A})^{**}$ and in A^{**}. Since clearly $p \leqslant 1_{\widetilde{A}}$, we deduce from Theorem 2.6 of [22] that p is compact in $(\widetilde{A})^{**}$ relative to $\widetilde{A}$. Let p_0 be the minimal projection in $(\widetilde{A})^{**}$ satisfying $\phi_0(p_0) = 1$. Theorem 1.4 implies the existence of a norm-one element $x \in \widetilde{A}$ such that p_0 and x are orthogonal and $L(p,p)x = x \circ p = p$. In particular $x \in A$, which gives p compact in A^{**} relative to A (compare Theorem 2.6 of [22]). ∎

Let B be a JB*-subtriple of a JB*-triple E. Throughout the paper, we shall identify the weak*-closure of B in E^{**} with B^{**}. Let x be a norm-one element and let $E(x)$ denote the norm closure of $\{x, E, x\}$ in E. It was proved by L.J. Bunce, Ch.-H. Chu and B. Zalar in [14], [15], that $E(x)$ coincides with the norm-closed inner ideal of E generated by x, $E(x)$ is a JB*-subalgebra of the JBW*-algebra $E(x)^{**} = E_2^{**}(r(x))$, where $r(x)$ is the range tripotent of x in E^{**}. Moreover, $x \in E(x)_+$.

We can now state the following version of Urysohn lemma which is a partial generalization of the result obtained by C.A. Akemann, J. Anderson and G.K. Pedersen in Lemma 2.5 of [4] (see also Lemma III.1 of [3], Corollary 2.48 of [11], Lemma 2.7 of [5]).

THEOREM 1.10. *Let E be a* JB**-triple, x a norm-one element in E and u a compact tripotent in E^{**} relative to E satisfying that $u \leqslant r(x)$. Then there exists a norm-one element y in $E(x)$ such that $u \leqslant y \leqslant r(x)$. Moreover, u is a compact tripotent in $E_2^{**}(r(x)) = (E(x))^{**}$ relative to $E(x)$.*

Proof. We may assume that $0 \neq u \leqslant r(x)$. From Theorem 4.2 of [20], there exists a set of norm-one elements $\{a_\lambda\} \subset E$ satisfying that

$$\{u\}_{\underset{E^*}{\prime}} = \bigcap_{\lambda \in \Lambda} \{u(a_\lambda)\}_\prime = \bigcap_{\lambda \in \Lambda} \{a_\lambda\}^\prime. \tag{1.4}$$

Since $u \leqslant r(x)$, then u is a projection in $E(x)^{**} = E_2^{**}(r(x))$.

Since $E(x)$ is a norm-closed inner ideal of E, it follows from Theorem 2.6 of [19] that every element $\varphi \in E(x)^*$ has a unique norm-preserving linear extension to E. The restriction mapping $\Psi : E_1^* \to E(x)_1^*$, $\phi \mapsto \phi|_{E(x)}$, is $\sigma(E^*, E) - \sigma(E(x)^*, E(x))$-continuous. Let $\phi \in \{u\}_{\underset{E^*}{\prime}}$. Since u is a projection in $E_2^{**}(r(x))$ and $\phi(u) = 1 = \|\phi|_{E_2^{**}(r(x))}\|$, we deduce that $\phi|_{E_2^{**}(r(x))}$ belongs to $S_*(E_2^{**}(r(x))) = S(E(x))$, and hence $\|\phi|_{E(x)}\| = 1$. Again, the unique extension property (see Theorem 2.6 of [19]) assures that

$$F_{E(x)^*}(u) \cap S(E(x)) = \{u\}_{\underset{E(x)^*}{\prime}} = \Psi(\{u\}_{\underset{E^*}{\prime}}).$$

If we show that $F_{E(x)^*}(u) \cap S(E(x))$ is $\sigma(E(x)^*, E(x))$-closed in $Q(E(x))$, the thesis of the theorem will follow from Proposition 1.9 and Theorem 2.6 of [22]. To see this, let (φ_μ) be a net in $F_{E(x)^*}(u) \cap S(E(x))$ converging to some φ in $F_{E(x)^*}(u) \cap S(E(x))$ in the $\sigma(E(x)^*, E(x))$-topology. Since Ψ is surjective, there exist a net (ϕ_μ) in $\{u\}_{\underset{E^*}{\prime}}$ and $\phi \in E_1^*$ such that $\Psi(\phi_\mu) = \varphi_\mu$ and $\Psi(\phi) = \varphi$. Since E_1^* is $\sigma(E^*, E)$-compact, there exists a subnet (ϕ_δ) converging to some $\phi^\prime$ in the $\sigma(E^*, E)$-topology. For each $\lambda \in \Lambda$ we have $\phi_\delta(a_\lambda) \to \phi^\prime(a_\lambda)$. In particular, since $(\phi_\delta) \subset \{u\}_{\underset{E^*}{\prime}}$, we have, by (1.4), $\phi_\delta(a_\lambda) = 1$ for all δ, λ, which implies $\phi^\prime \in \{u\}_{\underset{E^*}{\prime}}$. Finally, $\Psi(\phi_\delta) = \varphi_\delta$ tends to $\Psi(\phi^\prime)$ in the $\sigma(E(x)^*, E(x))$-topology, thus

$$\varphi = \Psi(\phi) = \Psi(\phi^\prime) \in \Psi(\{u\}_{\underset{E^*}{\prime}}) = F_{E(x)^*}(u) \cap S(E(x)),$$

which finishes the proof. ∎

Theorem 1.10 allows us to get the following generalization of Theorem II.17 of [1] and [3].

PROPOSITION 1.11. *Let E be a* JB**-triple, let π denote the canonical projection of E^{**} onto its atomic part and let $j : E \to E^{**}$ be the canonical embedding of E into its bidual. Then, for each range tripotent e in $\pi(E^{**})$ relative to $\pi(E)$ there exists a unique range tripotent r in E^{**} relative to E such that $\pi(r) = e$.*

Proof. Remark 1.2 assures the existence of such a tripotent, so the proof ends by proving the uniqueness. Suppose that there exist norm-one elements $x, y \in E$ such that $\pi(r(x)) = \pi(r(y)) = e$. By [31], there exists a locally compact Hausdorff space $L \subseteq [0, 1]$ with $L \cup \{0\}$ compact such that E_x is isometrically isomorphic to $C_0(L)$. Let us define $u_n = \chi_{L \cap [1/n, 1]}$, $n \in \mathbb{N}$. Clearly, u_n is a compact tripotent in E^{**} relative to E and u_n is an increasing sequence converging to $r(x)$ in the weak*-topology of E^{**}. $\pi(u_n) \leqslant \pi(r(x)) = e = \pi(r(y))$ and by Proposition 1.6

and Theorem 1.10 there is a sequence of norm-one positive elements $(z_n) \subset E(y)$ satisfying that $\pi(u_n) \leqslant u(\pi(z_n)) \leqslant \pi(z_n) \leqslant \pi(r(y))$. Again, Proposition 1.6 gives $u_n \leqslant u(z_n) \leqslant r(y)$. Finally, since $E_2^{**}(r(y))$ is weak*-closed and (u_n) tends to $r(x)$ in the weak*-topology we have $r(x) \leqslant r(y)$. Symmetrically, we get $r(y) \leqslant r(x)$. ∎

In the setting of C^*-algebras, C.A. Akemann, J. Anderson and G.K. Pedersen proved, in Proposition 2.6 of [4], the following stronger version of the Urysohn lemma. Let A be a C^*-algebra and let p and q be two closed orthogonal projections in A^{**} with p compact and $\|ap\| < \varepsilon$ for some a in A. Then there are orthogonal open projections $r, s \in A^{**}$ such that $p \leqslant r$, $q \leqslant s$ and $\|ar\| < \varepsilon$. We do not know if we can obtain a similar result in the setting of JB*-triples.

PROBLEM 1.12. Let E be a JB*-triple and let e, f be two non-zero orthogonal compact tripotents in E^{**} relative to E. Do there exist orthogonal norm-one elements x, y in E such that $e \leqslant x$ and $f \leqslant y$?

PROBLEM 1.13. Can one replace in Theorem 1.10 the range tripotent, $r(x)$, with any open tripotent in E^{**} relative to E ?

2. CONNECTIONS WITH THE STONE-WEIERSTRASS THEOREM FOR C^*-ALGEBRAS AND JB*-TRIPLES

As we have commented in the introduction, the generalizations of Urysohn lemma to the setting of non-commutative C^*-algebras are closely related with the general Stone-Weierstrass problem for non-commutative C^*-algebras. This tool has been intensively developed and applied to the Stone-Weierstrass problem in papers like [1], [2], [3], [4], [5] and [11].

The Stone-Weierstrass problem for C^*-algebras can be concretely stated as follows:

> Let B be a C^*-subalgebra of a C^*-algebra A. Suppose that B separates the pure states of A and zero. Is B equal to A?

I. Kaplansky gave a positive answer to the above problem for the special class of type I C^*-algebras in [29]. For general C^*-algebras, many authors gave partial answer to the Stone-Weierstrass problem by including various additional conditions (see for example [29], [28], [25], [1], [2], [37], [21], [12], [6] and [10] among others).

We are particularly interested in the following Stone-Weierstrass type Theorem proved by C.A. Akemann in Theorem II.7 of [2].

THEOREM 2.1. *Let B be a C^*-subalgebra of a unital C^*-algebra A such that B separates the pure states of A and zero. Suppose that for every pair of orthogonal projections p, q in A^{**} with q minimal and p compact relative to A, there exist orthogonal (positive) elements x, y in B such that $\|y\| = 1$, $\|x\| \in \{0, 1\}$, $p \leqslant x$ and $q \leqslant y$. Then $B = E$.*

In the statement of Theorem II.7 in [2] it is not explicitly included in the hypothesis that B separates the pure states of A and zero. However, the proof uses the results in Section 3 of [1], where this condition is assumed (see page 285 of [1] and page 305 of [2]).

In the setting of JB-algebras and JB*-triples an intensive study of the Stone-Weierstrass problem was developed by B. Sheppard [38], [39]. Among other results, B. Sheppard generalizes the result obtained by Kaplansky for postliminal JB*-algebras and JB*-triples in the following result.

THEOREM 2.2 ([39], Theorem 5.7). *Let B be a* JB**-subtriple of a* JB**-triple E such that B separates the extreme points of the closed unit ball of E^*. Then, if E or B is postliminal, $E = B$.*

The aim of this section is an analysis of the connections between the Stone-Weierstrass theorem and the Urysohn lemma type results for JB*-triples, analogous to that made by C.A. Akemann in the setting of C^*-algebras.

The following definition is inspired by Urysohn lemma for JB*-triples proved in Theorem 1.4. We introduce this property just to simplify the notation in this paper.

DEFINITION 2.3. Let B be a JB*-subtriple of a JB*-triple E. We say that B satisfies the *SW-property* with respect to E if and only if for every couple of orthogonal tripotents u, v in E^{**} with v minimal and u compact relative to E, there exist orthogonal elements $x, y \in B$ such that $\|y\| = 1$, $\|x\| \in \{0, 1\}$, $u \leqslant x$ and $v \leqslant y$. When $u = 0$, we mean $x = 0$ in $u \leqslant x$.

Theorem 1.4 shows that every JB*-triple has the SW-property with respect to itself.

LEMMA 2.4. *Let A be a* JBW**-algebra and let p, q be minimal projections in A. Suppose that $q = q_2 + q_1 + q_0$ is the Peirce decomposition of q with respect to p and φ_q in $\partial_e(A_{*,1})$ such that $\varphi_q(q) = 1$. Then, either $p = q$ or $\varphi_q(q_0) \neq 0$.*

Proof. By 2.4.16 and 2.4.21 of [26] we have

$$P_2(p) = U_p^2 \circ * = U_{p^2} \circ *, \quad P_0(p) = U_{1-p} \circ *,$$

where $U_p(x) := \{p, x^*, p\}$ and $*$ denotes the canonical involution of A. Suppose that $\varphi_q(q_0) = 0$. We claim that $q = p$. Indeed, by Proposition 1 of [23] and the hypothesis we have

$$0 = \varphi_q(q_0) = \varphi_q(U_{1-p}(q)) = \varphi_q(U_q U_{1-p}(q)).$$

Since q is minimal and φ_q is faithful in $A_2(q) = \mathbb{C}q$, we have

$$U_q U_{1-p}(q) = 0.$$

Now by 2.4.18 of [26] it follows that

$$U_q U_{1-p}(q) = U_q U_{1-p} U_q(q) = U_{\{q,1-p,q\}}(q) = 0.$$

However, since $1 - p \geqslant 0$, by 3.3.6 of [26], we have $\{q, 1-p, q\}$ is a positive element in $A_2(q)$. Moreover, since q is the unit element in $A_2(q)$ and $U_{\{q,1-p,q\}}(q) = 0$, it follows that $\{q, 1-p, q\} = q - P_2(q)p = 0$. Finally, the equality $p = q$ can be derived from the minimality of p, since $q - P_2(q)p = 0$ and Lemma 1.6 of [23] imply that $p = q + P_0(q)p$. ∎

Let E be a JB*-triple. Throughout the paper MinTri(E) will stand for the set of all minimal tripotents in E.

THEOREM 2.5. *Let B be a* JB**-subtriple of a* JB**-triple E. Suppose that for every $u \neq v$ in* MinTri$(E) \cup \{0\}$, *with u and v orthogonal, there exist orthogonal elements $x, y \in B$ such that $\|y\|, \|x\| \in \{0,1\}$ and $u \leqslant x$ and $v \leqslant y$ (if $u = 0$ or $v = 0$, we mean $x = 0$ or $y = 0$, respectively). Then B separates $\partial_e(E_1^*) \cup \{0\}$.*

Proof. Let $\varphi_1 \neq \varphi_2$ in $\partial_e(E_1^*) \cup \{0\}$. If $\varphi_1 = 0$, then there is a minimal tripotent u_2 in E^{**} such that $\varphi_2(u_2) = 1$ (compare Proposition 4 of [23]). Now, the hypothesis on B applied to 0 and u_2, assure the existence of orthogonal elements $x, y \in B$ such that $\|y\|, \|x\| \in \{0,1\}$ and $0 \leqslant x$ and $u_2 \leqslant y$. In particular $0 = \varphi_1(y) \neq \varphi_2(y) = 1$. We may therefore assume $\varphi_1, \varphi_2 \neq 0$.

Take $u_1 \neq u_2$ minimal tripotents in E^{**}, such that $\varphi_i(u_i) = 1$, for $i = 1,2$. As we have commented in the previous paragraph, the hypothesis implies the existence of a norm-one element $a \in B$, such that $u_1 \leqslant a$ and hence $\varphi_1(a) = 1$. If $\varphi_2(a) \neq 1$, then B separates φ_1, φ_2 and we finish. We may therefore assume that $\varphi_2(a) = 1$. In this case, by Propositions 1, 2 and Lemma 1.6 of [23] $u_2 \leqslant a$. Therefore, $u_1, u_2 \leqslant a \leqslant r(a)$, which implies that u_1 and u_2 are minimal projections in the JBW*-algebra $E_2^{**}(r(a))$. From Lemma 2.4 and the hypothesis, we have $\varphi_2(P_0(u_1)(u_2)) \neq 0$. Moreover, from page 258 of [8], it follows that $0 < |(P_0(u_1)(u_2))| \leqslant \|\varphi_2(P_0(u_1)(u_2))\|_{\varphi_2}$.

Let A denote the atomic part of E^{**}. Clearly, $P_0(u_1)(A) \subset A$ and hence $P_0(u_1)(A)$ coincides with the weak*-closure of the linear span of MinTri$(E^{**}) \cap E_0^{**}(u_1)$ (compare [23]). Since

$$0 < |\varphi_2(P_0(u_1)(u_2))|$$

we have $\varphi_2|_{P_0(u_1)(A)} \neq 0$, and hence there exists a minimal tripotent $w \in$ MinTri$(E^{**}) \cap E_0^{**}(u_1)$, such that $0 < \varphi_2(w) \leqslant \|w\|_{\varphi_2}$.

Finally, by hypothesis, there are two orthogonal norm-one elements x, y in B such that $u_1 \leqslant x$ and $w \leqslant y$. In particular $0 < \|w\|_{\varphi_2} \leqslant \|y\|_{\varphi_2}$ and $\varphi_1(x) = 1$. Therefore,

$$|\varphi_2(x)|^2 \leqslant \|x\|_{\varphi_2}^2 < \|x\|_{\varphi_2}^2 + \|y\|_{\varphi_2}^2 = \|x+y\|_{\varphi_2}^2 \leqslant \|x+y\|^2 = 1,$$

which proves the desired statements. ∎

Since every minimal tripotent in the bidual of a JB*-triple is compact (see Theorem 3.4 of [16]) we have:

COROLLARY 2.6. *Let B be a* JB**-subtriple of a* JB**-triple E. Suppose that B has the* SW*-property with respect to E. Then B separates $\partial_e(E_1^*) \cup \{0\}$.*

The significant results obtained by B. Sheppard on the Stone-Weierstrass theorem for JB*-triples in [39] allow us to get the following result connecting the SW-property and the Stone-Weierstrass Theorem for postliminal JB*-triples.

COROLLARY 2.7. *Let B a* JB**-subtriple of a* JB**-triple E. Suppose that B has the* SW*-property with respect to E, and E or B is postliminal. Then $B = E$.*

Proof. This follows from Theorems 2.5 and 2.2 (see Theorem 5.7 of [39]). ∎

REMARK 2.8. Let A be a C^*-algebra regarded as a JB*-triple and let p be a projection in A^{**}. Let $\circ$ denote the Jordan product on A. Suppose that x is a norm-one element in A such that $L(p,p)x = p$ (that is, $p \leqslant x$ in A^{**} regarded the latter as a JB*-triple), and hence $x = p + P_0(p)(x)$. In this case $L(p,p)(x \circ x^*) = p$. This shows that $p \leqslant x \circ x^*$.

Now, the proof given in Theorem 2.5 can be literally adapted, via Remark 2.8, to show that the assumption of B separating the pure states of A and zero can be dropped in Theorem 2.1 (see also Theorem II.7 of [2]).

COROLLARY 2.9. *Let B be a C^*-subalgebra of a C^*-algebra A. Suppose that for every pair of orthogonal projections p, q in A^{**} with q minimal and p compact relative to A, there exists orthogonal (positive) elements x, y in B such that $\|y\| = 1$, $\|x\| \in \{0,1\}$, $p \leqslant x$ and $q \leqslant y$. Then $B = A$.*

Proof. The proof of Theorem 2.5 can be literally followed up to its last part. To finish, in this case, we note that the element w can be chosen as a minimal projection, for example ww^* or w^*w. ∎

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