COMPACT TRIPOTENTS AND THE STONE-WIEERSTRASS THEOREM FOR C*-ALGEBRAS AND JB*-TRIPLES

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ABSTRACT. We establish some generalizations of Urysohn lemma for the hull-kernel structure in the setting of JB*-triples. These results are the natural extensions of those obtained by C.A. Akemann in the setting of C*-algebras. We also develop some connections with the classical Stone-Weierstrass problem for C*-algebras and JB*-triples.

KEYWORDS: Compact tripotents, compact projections, C*-algebras, JB*-triples, Urysohn’s lemma, Stone-Weierstrass Theorem.

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INTRODUCTION

Let $K$ be a topological compact Hausdorff space and let $C(K)$ denote the Banach space of all complex-valued continuous functions on $K$. The classical Urysohn lemma allows us to describe the open subsets of $K$ in the following way: a subset $A \subseteq K$ is open if and only if there is an increasing net $(x_\alpha)$ in $C(K)$ satisfying that $0 \leq x_\alpha(t) \nearrow 1$, for each $t \in A$, and $0 = x_\alpha(t)$ for each $t \in K \setminus A$. Clearly, a subset $C \subseteq K$ is closed (equivalently, compact) if and only if $K \setminus C$ is open. We can see the characteristic functions $\chi_A$ as projections in the bidual of $C(K)$.

In the more general setting of non-necessarily abelian C*-algebras the notions of open and compact projections in the bidual of a C*-algebra are mainly due to C.A. Akemann ([1], [3], see also [5], [33]). Let $A$ be a C*-algebra. A projection $p$ in $A^{**}$ is said to be open if $p$ is the weak*-limit of a increasing net of positive elements in $A$, equivalently, $pA^{**}p \cap A$ is weak*-dense in $pA^{**}p$ (compare Proposition 3.11.9 of [33]). We say that $p$ is closed whenever $1 - p$ is open. Finally, a projection $p$ is said to be compact if, and only if, $p$ is closed and there exists a positive element $a \in A$ such that $p \leq a \leq 1$, equivalently, there is a monotone decreasing net $(a_\lambda)$ in $A_+$ with $p \leq a_\lambda \leq 1$, converging strongly to $p$ (see for example [1] or Definition-Lemma 2.47 of [11]). If $A$ is unital then every closed
projection in $A^{**}$ is compact. Akemann called this collection of open projections in $A^{**}$ the hull-kernel structure (HKS) of $A$. In the HKS of a $C^*$-algebra, the following generalization of Urysohn lemma was obtained by Akemann in Theorem I.1 of [2]:

**Theorem 0.1.** Let $A$ be a unital $C^*$-algebra and let $p$ and $q$ be two closed projections in $A^{**}$ with $pq = 0$. Then there exists $a$ in $A$ with $0 \leq a \leq 1$, $ap = 0$ and $aq = q$.

The generalizations of Urysohn lemma to the setting of non-commutative $C^*$-algebras are closely related with the general Stone-Weierstrass problem for non-commutative $C^*$-algebras. This tool has been intensively developed since 1969 by C.A. Akemann [1], [2], [3], L.G. Brown [11], C.A. Akemann, J. Anderson and G. Pedersen [4] and C.A. Akemann and G. Pedersen [5], among others.

$C^*$-algebras belong to the more general class of complex Banach spaces known as JB*-triples (see definition below). In this setting the role of projections is played by those elements called tripotents. Moreover, in [20] and [22] the notions of open, compact and closed tripotents and the notion of maximality of these tripotents are introduced and developed. The aim of this paper is the study of the hull-kernel structure (HKS) of $A^{**}$. Theorem 1.4 assures that whenever $e$ is a Jordan triple in $A_{**}$ and $B$ is a JB*-triple, $e$ is a JB*-triple in the bidual of a JB*-triple $E$, with $e$ compact and $f$ minimal, then there exist two orthogonal norm-one elements $a_1$ and $a_2$ in $E$ such that $e \leq a_1$ and $f \leq a_2$. The second Urysohn lemma type result is Theorem 1.10, where we establish the following: Let $E$ be a JB*-triple, $x$ a norm-one element in $E$ and $u$ a compact tripotent in $E^{**}$ relative to $E$ satisfying that $u \leq r(x)$. Then there exists a norm-one element $y$ in the inner ideal of $E$ generated by $x$, such that $u \leq y \leq r(x)$.

In the last section we find some connections between the generalizations of Urysohn lemma to the HKS of a $C^*$-algebra or a JB*-triple with the Stone-Weierstrass problem. As main result (see Theorem 2.5) we prove that whenever $B$ is a JB*-subtriple of a JB*-triple $E$ such that for every couple of orthogonal tripotents $u, v$ in $E^{**}$ with $v$ minimal and $u$ minimal or zero, there exist orthogonal elements $x, y$ in $B$ such that $\|y\| = 1$, $\|x\| \in \{0, 1\}$ and $u \leq x$ and $v \leq y$ (when $u = 0$, then we mean $x = 0$), then $B$ separates the extreme points of the closed unit ball of $E^*$ and zero. This result combined with those obtained by C.A. Akemann [2] and B. Sheppard [39], on the Stone-Weierstrass theorem for $C^*$-algebras and JB*-triples, respectively, allow us to establish some new versions of the Stone-Weierstrass theorem in the setting of $C^*$-algebras and JB*-triples.

We recall (c.f. [31]) that a JB*-triple is a complex Banach space $E$ together with a continuous triple product $\{\cdot, \cdot, \cdot\} : E \times E \times E \to E$, which is conjugate linear in the middle variable and symmetric bilinear in the outer variables satisfying that:

(a) $L(a, b)L(x, y) = L(x, y)L(a, b) + L(L(a, b)x, y) - L(x, L(b, a)y)$, where $L(a, b)$ is the operator on $E$ given by $L(a, b)x = \{a, b, x\}$;
(b) $L(a,a)$ is an hermitian operator with non-negative spectrum;
(c) $\|L(a,a)\| = \|a\|^2$.

Every $C^*$-algebra is a JB$^*$-triple via the triple product given by

$$2\{x,y,z\} = xy^*z + zy^*x,$$

and every JB$^*$-algebra is a JB$^*$-triple under the triple product

$$\{x,y,z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*.$$

A JBW$^*$-triple is a JB$^*$-triple which is also a dual Banach space (with a unique predual [9]). The second dual of a JB$^*$-triple is a JBW$^*$-triple [17]. Elements $a,b$ in a JB$^*$-triple, $E$, are orthogonal if $L(a,b) = 0$. With each tripotent $u$ (i.e. $u = \{u,u,u\}$) in $E$ is associated the Peirce decomposition

$$E = E_2(u) \oplus E_1(u) \oplus E_0(u),$$

where for $i = 0,1,2$, $E_i(u)$ is the $i/2$ eigenspace of $L(u,u)$. The Peirce rules are that $\{E_i(u),E_j(u),E_k(u)\}$ is contained in $E_{i-j+k}(u)$ if $i - j + k \in \{0,1,2\}$ and is zero otherwise. In addition,

$$\{E_2(u),E_0(u),E\} = \{E_0(u),E_2(u),E\} = 0.$$

The corresponding Peirce projections, $P_i(u) : E \to E_i(u)$, $(i = 0,1,2)$ are contractive and satisfy

$$P_2(u) = D(2D - I), \quad P_1(u) = 4D(I - D), \quad P_0(u) = (I - D)(I - 2D),$$

where $D$ is the operator $L(u,u)$ and $I$ is the identity map on $E$ (compare [23]). A non-zero tripotent $u \in E$ is called minimal if and only if $E_2(u) = \mathbb{C}u$.

Let $e$ and $x$ be two norm-one elements in a JB$^*$-triple, $E$, with $e$ tripotent. We shall say that $e \leqslant x$ (respectively, $x \leqslant e$) whenever $L(e,e)x = e$ (respectively, $x$ is a positive element in the JB$^*$-algebra $E_2(e)$).

The strong$^*$-topology in a JBW$^*$-triple was introduced by T.J. Barton and Y. Friedman in [8]. This strong$^*$-topology can be defined in the following way: Given a JBW$^*$-triple $W$, a norm-one element $\varphi$ in $W_*$ and a norm-one element $z$ in $W$ such that $\varphi(z) = 1$, it follows from Proposition 1.2 of [8] that the assignment

$$(x,y) \mapsto \varphi\{x,y,z\}$$

defines a positive sesquilinear form on $W$. Moreover, for every norm-one element $w$ in $W$ $\varphi(w) = 1$, we have $\varphi\{x,y,z\} = \varphi\{x,y,w\}$, for all $x,y \in W$. The law $x \mapsto \|x\|_{\varphi} := (\varphi\{x,x,z\})^{1/2}$, defines a prehilbertian seminorm on $W$. The strong$^*$-topology (noted by $S^*(W,W_*)$) is the topology on $W$ generated by the family

$$\{\| \varphi : \varphi \in W_{*+}, \|\varphi\| = 1 \}.$$
Let $W$ be a JBW$^*$-triple and let $a$ be a norm-one element in $W$. The sequence $(a^{2^n-1})$ defined by $a^1 = a$, $a^2 = a$, $a^{2n+1} = \{a, a^{2n-1}, a\}$ ($n \in \mathbb{N}$) converges in the strong*-topology (and hence in the weak*-topology) of $W$ to a tripotent $u(a)$ in $W$ (compare Lemma 3.3 of [20]). This tripotent will be called the support tripotent of $a$. There exists a smallest tripotent $r(a) \in W$ satisfying that $a$ is positive in the JBW$^*$-algebra $W_2(r(a))$, and $u(a) \leq a^{2n-1} \leq a \leq r(a)$. This tripotent $r(a)$ will be called the range tripotent of $a$. (Beware that in [20], $r(a)$ is called the support tripotent of $a$).

In [20], C.M. Edwards and G.T. Rüttimann introduced the concepts of open and compact tripotents in the bidual of a JB$^*$-triple. In [22], the authors of the present paper studied the notions of open and compact tripotents in a JBW$^*$-triple with respect to a weak*-dense subtriple. Concretely, given a JBW$^*$-triple $W$ and a weak*-dense JB$^*$-subtriple $E$ of $W$, a tripotent $u$ in $W$ is said to be compact-G$_\delta$ relative to $E$ if $u$ is the support tripotent of a norm one element in $E$. The tripotent $u$ is said to be compact relative to $E$ if $u = 0$ or there exist a decreasing net, $(u_\lambda) \subseteq W$, of compact-G$_\delta$ tripotents relative to $E$ converging, in the strong*-topology of $W$, to the element $u$ (compare Section 4 of [20]). A tripotent $u$ in $W$ is said to be open relative to $E$ if $E \cap W_2(u)$ is weak*-dense in $W_2(u)$. When $E$ is a JB$^*$-triple, the range (respectively, the support) tripotent of every norm-one element in $E$ is always an open (respectively, compact) tripotent in $E^{**}$ relative to $E$.

**Notation 0.2.** Given a Banach space $X$, we denote by $X_1$, $S_X$, and $X^*$ the closed unit ball, the unit sphere, and the dual space of $X$, respectively. If $K$ is any convex subset of $X$, then we write $\partial_e(K)$ for the set of extreme points of $K$.

1. THE NON-COMMUTATIVE URYSOHN LEMMA FOR JB$^*$-TRIPLES

This section is mainly devoted to obtain some Urysohn lemma type results for the HKS of a JB$^*$-triple. We begin by developing some new properties of compact tripotents in the bidual of a JB$^*$-triple.

**Proposition 1.1.** Let $W$ and $V$ be JBW$^*$-triples, $E$ a weak*-dense JB$^*$-subtriple of $W$ and $T : W \to V$ a surjective weak*-continuous triple homomorphism such that $\|T(x)\| = \|x\|$, for all $x$ in $E$. Suppose that $e$ is a tripotent in $W$, then $T(e)$ is compact relative to $T(E)$ in $V$ whenever $e$ is compact relative to $E$. Moreover, if $T$ is a triple isomorphism, then $e$ is compact relative to $E$ in $W$ if and only if $T(e)$ is compact relative to $T(E)$ in $V$.

**Proof.** Suppose that $e \in W$ is compact relative to $E$. If $T(e) = 0$, then there is nothing to prove. Suppose that $T(e)$ is a non-zero tripotent in $V$. By definition, there exists a decreasing net $(u_\lambda)_{\lambda \in \Lambda} \subseteq W$, of compact-G$_\delta$ tripotents relative to $E$ (i.e., $\forall \lambda$ there exists $u_{\lambda} \in S_E$ such that $u_\lambda = u(a_\lambda)$), converging to $e$ in the strong*-topology of $W$.
From the hypothesis we know that, for each \( \lambda \in \Lambda \), \( \|T(a_\lambda)\| = \|a_\lambda\| = 1 \). Since, for each \( \lambda \), \( u(T(a_\lambda)) \) coincides with the limit, in the weak*-topology of \( V \), of the sequence \( (T(a_\lambda)2n-1) = (T(a_\lambda2n-1)) \), and \( T \) is weak*-continuous, we have \( u(T(a_\lambda)) = T(u(a_\lambda)) \). The conditions \( (u_\lambda) \) decreasing and \( T \) triple homomorphism imply that \( u(T(a_\lambda)) = T(u(a_\lambda)) \) is also a decreasing net in \( V \). Since \( T \) is weak*-continuous, we deduce, from Corollary 3 in [36], that \( T \) is \( S^*(W,W_\ast) - S^*(V,V_\ast) \)-continuous. Therefore, \( u(T(a_\lambda)) = T(u(a_\lambda)) \) tends to \( T(e) \) in the \( S^*(V,V_\ast) \)-topology. This shows that \( T(e) \) is compact relative to \( T(E) \) in \( V \). \( \blacksquare \)

**Remark 1.2.** Note that under the assumptions of the previous proposition there is a relationship between compact-\( G_\delta \) tripotents in \( W \) (respectively, range tripotents in \( W \)) relative to \( E \) and compact-\( G_\delta \) tripotents in \( V \) (respectively, range tripotents in \( V \)) relative to \( T(E) \). Indeed, let \( x \in E \) be a norm-one element. The sequence \( x^{2n-1} \) (respectively, \( x^1/(2n-1) \)) tends to \( u(x) \) (respectively, \( r(x) \)) in the weak*-topology of \( W \). Since \( T \) is a weak*-continuous triple homomorphism isometric on \( E \), it follows that \( T(u(x)) = u(T(x)) \) (respectively, \( T(r(x)) = r(T(x)) \)). Moreover, since every compact-\( G_\delta \) (respectively, range) tripotent in \( V \) relative to \( T(E) \) is of the form \( u(T(x)) \) (respectively, \( r(T(x)) \)) for a suitable norm-one element \( x \in E \), it is clear that \( T \) maps the set of compact-\( G_\delta \) (respectively, range) tripotents in \( W \) relative to \( E \) onto the set of compact-\( G_\delta \) (respectively, range) tripotents in \( V \) relative to \( T(E) \).

In Theorem 3.4 of [16] it is proved that every minimal tripotent in the bidual of a JB*-triple, \( E \), is compact relative to \( E \). The next corollary shows that this result remains true for every minimal tripotent in a JBW*-triple \( W \) and for any weak*-dense JB*-subtriple of \( W \).

Let \( E \) be a JB*-triple. A subtriple \( I \) of \( E \) is said to be an *ideal* of \( E \) if \( \{E,E,I\} + \{E,E,I\} \subseteq I \). We shall say that \( I \) is an *inner ideal* of \( E \) whenever \( \{I,I,E\} \subseteq I \).

If \( E \) and \( F \) are two JB*-triples, a representation \( \pi : E \to F \) is any triple homomorphism from \( E \) to \( F \). Let \( j : E \to E^** \) be the canonical inclusion of \( E \) into its bidual. Each weak*-closed ideal \( I \) of \( E^** \) is an \( M \)-summand (see [27]). Therefore there exists a weak*-continuous contractive projection \( \pi : E^** \to I \). The representation \( E \to I \) given by \( x \mapsto \pi j(x) \) is called the *canonical representation* of \( E \) corresponding to \( I \). Suppose that \( E \) is a weak*-dense JB*-subtriple of a JBW*-triple \( W \) and let \( \lambda : E \to W \) be the natural inclusion. From Proposition 6 of [7], there exists a weak*-closed triple ideal \( M \) of \( E^** \) and a triple isomorphism \( \Psi : W \to M \) satisfying that \( \Psi \lambda \) is the canonical representation of \( E \) corresponding to \( M \).

**Corollary 1.3.** Let \( E \) be a weak*-dense JB*-subtriple of a JBW*-triple \( W \). Let \( M \) be the weak*-closed triple ideal of \( E^** \) and let \( \Psi : W \to M \) the triple isomorphism described in the above paragraph, satisfying that \( \Psi \lambda \) is the canonical representation of \( E \) corresponding to \( M \). Let \( e \) be a tripotent in \( W \). Then \( e \) is compact relative to \( E \) in \( W \).
whenever $\Psi(e)$ is compact relative to $E$ in $E^{**}$. In particular, every minimal tripotent in $W$ is compact relative to $E$.

**Proof.** Let $\pi : E^{**} \to M$ denote the canonical projection of $E^{**}$ onto $M$. Clearly, $\pi$ is a surjective weak*-continuous triple homomorphism and if $\lambda : E \to W$ and $j : E \to E^{**}$ denote the canonical inclusions of $E$ into $W$ and $E^{**}$, respectively, we have $\Psi \circ \lambda = \pi \circ j$.

Let $e \in W$ be a tripotent in $W$ such that $\Psi(e)$ is compact relative to $E$ in $E^{**}$. Proposition 1.1 applied to $\pi : E^{**} \to M$, $E^{**}$ and $E$, gives $\Psi(e)$ compact relative to $\pi(E)$ in $M$. Again, Proposition 1.1 assures that $e$ is compact relative to $E$ in $W$.

Finally, if $e$ is minimal in $W$, that is, $W_2(e) = Ce$, it is not hard to see that $M_2(\Psi(e)) = E_2^*(\Psi(e)) = C\Psi(e)$, and hence $\Psi(e)$ is a minimal tripotent in $E^{**}$. Therefore, from Theorem 3.4 of [16], it follows that $\Psi(e)$ is compact relative to $E$ in $E^{**}$, which implies that $e$ is compact relative to $E$ in $W$.  

Let $x$ be a norm-one element in a JB*-triple $E$. Throughout the paper, $E_x$ will denote the norm-closed JB*-subtriple of $E$ generated by $x$. It is known that $E_x$ is JB*-triple isomorphic (and hence isometric) to $C_0(\Omega)$ for some locally compact Hausdorff space $\Omega$ contained in $[0,1]$, such that $\Omega \cup \{0\}$ is compact and $C_0(\Omega)$ denotes the Banach space of all complex-valued continuous functions vanishing at 0. Moreover, if we denote by $\Psi$ the triple isomorphism from $E_x$ onto $C_0(\Omega)$, then $\Psi(x)(t) = t \ (t \in \Omega)$ (cf. 4.8 in [30], 1.15 in [31] and [23]).

The following result is a first generalization of Urysohn lemma to the setting of JB*-triples.

**Theorem 1.4.** Let $E$ be a weak*-dense JB*-subtriple of a JBW*-triple $W$. Let $u, v$ be two orthogonal tripotents in $W$ with $u$ compact relative to $E$ and $v$ minimal. Then there exist two orthogonal elements $a_1$ and $a_2$ in $E$ such that $\|a_2\| = 1, \|a_1\| \in \{0,1\}$, $u \leq a_1$ and $v \leq a_2$.

**Proof.** When $u = 0$, we take $a_1 = 0$ and the existence of $a_2$ follows from the last statement in Corollary 1.3 (see also [16]). We may therefore assume $u \neq 0$.

Since $v$ is a minimal tripotent in $W$, from Proposition 4 of [23] it follows that there exists $\varphi \in \partial_v((W_x)_1)$ satisfying $\varphi(v) = 1$.

Corollary 1.3 implies $v$ compact relative to $E$. Now, Proposition 2.3 of [22] assures that $v$ and $u$ are closed tripotents relative to $E$, that is, $W_0(u) \cap E$ and $W_0(v) \cap E$ are subtriples of $W$ which are weak*-dense in $W_0(u)$ and $W_0(v)$, respectively. From the orthogonality of $u$ and $v$ we have $u \in W_0(v)$ and $v \in W_0(u)$.

Let us denote $F = W_0(u) \cap E$. Since Theorem 2.8 of [16] remains true when $E^{**}$ is replaced with any JBW*-triple $W$ such that $E$ is weak*-dense in $W$, then applying this result to $F$ and $W_0(u)$, it follows that for every $\varepsilon, \delta > 0$, there exist $y \in F$ and a tripotent $e \in W_0(u)$ such that $e \leq v$, $P_i(e)(v - y) = 0$, for $i = 1, 2$, $\|y\| \leq (1 + \delta)\|(P_2(e) + P_1(e))(v)\|$ and $|\varphi(v - e)| < \varepsilon$. Since $\varepsilon$ can be chosen arbitrary small and $v$ is a minimal tripotent in $W_0(u)$, we have $e = v$. The same
arguments given in Lemma 3.1 of [16] assure the existence of a norm-one element $b_2 \in F$ such that $v \leq b_2$.

Let $F_{b_2}$ denote the JB$^*$-subtriple of $F$ generated by $b_2$. As we have commented above, there exists a locally compact Hausdorff space $L \subseteq [0,1]$ with $L \cup \{0\}$ compact such that $F_{b_2}$ is isometrically isomorphic to $C_0(L)$ under some surjective isometry denoted by $\psi$ and $\psi(b_2)(t) = t$, for any $t \in L$. Let $a_2$ and $\tilde{a}_2 \in F_{b_2}$ the norm-one elements given by the expressions

$$\psi(a_2)(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq 3/4, \\ \text{affine} & \text{if } 3/4 < t < 1, \\ 1 & \text{if } t = 1; \end{cases}$$

$$\psi(\tilde{a}_2)(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq 1/2, \\ \text{affine} & \text{if } 1/2 < t \leq 3/4, \\ 1 & \text{if } t \geq 3/4. \end{cases}$$

Clearly $v \leq u(b_2) \leq u(a_2) \leq a_2 \leq r(a_2) \leq \tilde{a}_2$.

Now, Theorem 2.6 in [22] assures the existence of a norm-one element $x$ in $E$ such that $u \leq x$. We define

$$c_1 = P_0(\tilde{a}_2)(x) := x - 2L(z,z)x + Q(z)^2(x) \in E,$$

where $z$ is the element in $F_{\tilde{a}_2} = E_{\tilde{a}_2}$ satisfying $\{z, r(\tilde{a}_2), z\} = \tilde{a}_2$ (compare Section 2 of [22]). From Lemma 2.5 of [22], we have $c_1 \in E \cap W_0(r(a_2))$, which, in particular, implies that $c_1$ and $a_2$ are orthogonal. We claim that

$$L(u,u)c_1 = u.$$

Indeed, since $x \geq u$, then $x = u + P_0(u)(x)$. Moreover, since $z \in F_{\tilde{a}_2} = E_{\tilde{a}_2} \subseteq W_0(u)$, it follows, from Peirce rules, that

$$L(u,u)c_1 = \{u,u,x-2L(z,z)x+Q(z)^2(x)\}$$

$$= \{u,u,u + P_0(u)(x) - 2L(z,z)(u + P_0(u)(x)) + Q(z)^2(u + P_0(u)(x))\}$$

$$= \{u,u,u\} + \{u,u,P_0(u)(x)-2L(z,z)(P_0(u)(x))+Q(z)^2(P_0(u)(x))\} = u.$$

Again, the same arguments given in Lemma 3.1 of [16] imply the existence of a norm-one element $a_1 \in E_{c_1}$ such that $u \leq a_1$. \hfill $\blacksquare$

In the case of von Neumann algebras the above theorem generalizes Theorem II.19 in [1] from the setting of biduals of $C^*$-algebras to the more general setting of von Neumann algebras.

**Corollary 1.5.** Let $A$ be a weak$^*$-dense $C^*$-subalgebra of a von Neumann algebra $W$. Let $p, q$ be two orthogonal projections in $W$ with $p$ compact relative to $A$ and $q$ minimal. Then there exist two orthogonal positive elements $a_1$ and $a_2$ in $A$ such that $\|a_2\| = 1, \|a_1\| \in \{0, 1\}$, $p \leq a_1$ and $q \leq a_2$.

In some particular triple representations the results stated in Proposition 1.1 and Remark 1.2 can be improved. This is the case of the canonical representation of a JB$^*$-triple into the atomic part of its bidual. We recall that, given a JB$^*$-triple $E$, then $E^{**}$ decomposes into an orthogonal direct sum of two weak$^*$-closed triple ideals $A$ and $N$, where $A$ (called the atomic part of $E^{**}$) coincides with the weak$^*$-closure of the linear span of all minimal tripotents in $E^{**}$, $E^* = A_\omega \oplus \ell^1 N_\omega$ and the
closed unit ball of \( N \) has no extreme points, which implies that \( \partial_e(E^*_1) = \partial_e(A_{*,1}) \) (compare Theorems 1 and 2 of [23]). If \( \pi \) denotes the natural weak*-continuous projection of \( E^{**} \) onto \( A \) and \( j : E \to E^{**} \) is the canonical inclusion, then the mapping \( \pi \circ j : E \to A \) is an isometric triple embedding called the canonical embedding of \( E \) into the atomic part of its bidual (see proof of Proposition 1 in [24]).

We recall some notation needed in what follows. Let \( X \) be a Banach space. For each pair of subsets \( G, F \) in the unit ball of \( X \) and \( X^* \), respectively, let the subsets \( G' \) and \( F' \) be defined by

\[
G' = \{ f \in B_{X^*} : f(x) = 1, \forall x \in G \} \quad \text{and} \quad F' = \{ x \in B_X : f(x) = 1, \forall f \in F \},
\]

respectively.

**Proposition 1.6.** Let \( E \) be a JB*-triple, let \( \pi \) denote the canonical projection of \( E^{**} \) onto its atomic part and let \( i : E \to E^{**} \) be the canonical embedding of \( E \) into its bidual. The following assertions hold:

(i) Let \( u \) and \( v \) be two compact tripotents in \( E^{**} \) relative to \( E \). Then \( u \leq v \) if and only if \( \pi(u) \leq \pi(v) \).

(ii) For each compact tripotent \( u \) in \( \pi(E^{**}) \) relative to \( \pi(E) \) there exists a unique compact tripotent \( v \) in \( E^{**} \) relative to \( E \) such that \( \pi(v) = u \).

**Proof.** (i) Let us denote \( A := \pi(E^{**}) \). If \( u \leq v \) in \( E^{**} \), then \( \pi(u) \leq \pi(v) \), since \( \pi \) is a triple homomorphism. Suppose now that \( \pi(u) \leq \pi(v) \). From Theorem 4.4 of [18], we have

\[
\{ \pi(u) \}_{A_*} \subseteq \{ \pi(v) \}_{A_*}.
\]

By Theorem 4.5 of [20] together with the comments preceding Corollary 3.5 in [16], every non-zero compact tripotent in \( E^{**} \) relative to \( E \) majorises a minimal tripotent of \( E^{**} \). In particular, if \( e \) is a compact tripotent in \( E^{**} \) with \( \pi(e) = 0 \), then \( e = 0 \). We may therefore assume that \( \pi(u) \) and hence \( \pi(v) \) are not zero.

From Theorem 4.2 of [20], it follows that the sets \( \{ u \}_{E_*} \) and \( \{ v \}_{E_*} \) are non-empty \( \sigma(E^*,E) \)-compact and convex subsets of \( E^*_1 \). By the Krein-Milman theorem we have

\[
\{ u \}_{E_*} = \overline{\partial_e(E^*_1)}(\partial_e(\{ u \}_{E_*}))
\]

and

\[
\{ v \}_{E_*} = \overline{\partial_e(E^*_1)}(\partial_e(\{ v \}_{E_*})).
\]

Since \( \partial_e(E^*_1) = \partial_e(A_{*,1}) \), we have

\[
\{ \pi(u) \}_{A_*} \cap \partial_e(A_{*,1}) = \{ \pi(u) \}_{E_*} \cap \partial_e(E^*_1) = \{ u \}_{E_*} \cap \partial_e(E^*_1) = \partial_e(\{ u \}_{E_*}).
\]

Similarly,

\[
\{ \pi(v) \}_{A_*} \cap \partial_e(A_{*,1}) = \partial_e(\{ v \}_{E_*}).
\]
Finally, we deduce, from (1.1), (1.2), (1.3) and the last two expressions, that
\[ \{u\} \subseteq \{v\} \]
which shows that \( u \leq v \) (compare Theorem 4.4 of [18]).

(ii) Let \( u \) be a non-zero compact tripotent in \( A = \pi(E^{**}) \) relative to \( \pi(E) \). Then there exists a decreasing net \( (u_\lambda) \) of compact-G\( \delta \) tripotents in \( A \) relative to \( \pi(E) \) converging in the strong*-topology of \( A \) to \( u \). By Remark 1.2, for each \( \lambda \), there is a norm-one element \( x_\lambda \in E \) such that
\[ u_\lambda = u(\pi(x_\lambda)) = \pi(u(x_\lambda)). \]
Since \( \pi(u(x_\lambda)) \) is a decreasing net of compact-G\( \delta \) tripotents, then (i) implies that \( (u(x_\lambda)) \) is a decreasing net in \( E^{**} \). By Theorem 4.5 of [20] there exists a non-zero compact tripotent \( e \in E^{**} \) relative to \( E \) such that \( e \) coincides with the infimum of the family \( (u(x_\lambda)) \). Since \( \pi \) is weak*-continuous and \( (u(x_\lambda)) \) tends to \( e \) in the weak*-topology of \( E^{**} \), we have that \( \pi((u(x_\lambda)) \to \pi(e) \) in the \( \sigma(E^{**},E^{**}) \)-topology, and hence \( \pi(e) = u \). Finally, the uniqueness of \( e \) follows from (i).

The above result is a partial generalization of Theorem II.17 in [1]. In the more particular setting of JB*-algebras we have:

**Corollary 1.7.** Let \( A \) be a JB*-algebra, let \( \pi \) denote the canonical projection of \( A^{**} \) onto its atomic part and let \( j : A \to A^{**} \) be the canonical embedding of \( A \) into its bidual. The following assertions hold:

(i) Let \( p \) and \( q \) be two compact projections in \( A^{**} \) relative to \( A \). Then \( p \leq q \) if and only if \( \pi(p) \leq \pi(q) \).

(ii) For each compact projection \( p \) in \( \pi(A^{**}) \) relative to \( \pi(A) \) there exists a unique compact projection \( q \) in \( A^{**} \) relative to \( A \) such that \( \pi(q) = p \).

Given a JB*-algebra \( A \), the cone of all positive elements in \( A \) will be denoted by \( A_+ \), while \( A^*_+ \) will denote the set of positive elements in \( A^* \). Let \( W \) be a JBW*-algebra. The symbol \( Q_+(W) \) will denote the set of all positive elements in \( W \), with norm less or equal to one. \( Q_+(W) \) will be called the normal quasi-state space of \( W \). The normal state space, \( S_+(W) \), is the set of all elements in \( Q_+(W) \) with norm equal to one. Given a projection \( p \) in \( W \) we shall denote \( F(p) = F_W(p) := \{ \varphi \in Q_+(W) : \varphi(p) = \|\varphi\| \} \). If \( A \) is a JB*-algebra, then the set \( Q(A) \) (respectively, \( S(A) \)) of quasi-states (respectively, states) of \( A \) is defined as \( Q_+(A^{**}) \) (respectively, \( S_+(A^{**}) \)).

The following result was proved by M. Neal in Lemma 3.2 and Theorem 5.2 of [32].

**Proposition 1.8.** Let \( A \) be a JB*-algebra and let \( p \) be a projection in \( A^{**} \). Then we have:

(i) \( p \) is open relative to \( A \) if and only if there exists an increasing net \( (a_\lambda) \) in \( A_{1,+} \) with least upper bound \( p \).

(ii) \( p \) is closed relative to \( A \) if and only if \( F(p) \) is \( \sigma(A^*,A) \)-closed in \( Q(A) \).
The next result gives a characterization of compact projections in JB*-algebra biduals. A similar result was obtained by C.A. Akemann, J. Anderson and G.K. Pedersen in the setting of C*-algebra biduals (see Lemma 2.4 of [4]).

Given a JB*-algebra \( A \), \( \tilde{A} = A \oplus \mathbb{C}1 \) will stand for the result of adjoining a unit to \( A \) (compare Section 3.3 of [26]). \( \tilde{A} \) is also called the unitization of \( A \).

**Proposition 1.9.** Let \( A \) be a JB*-algebra and let \( p \) be a projection in \( A^{**} \). Then \( p \) is compact relative to \( A \) if and only if \( F(p) \cap S(A) \) is \( \sigma(A^*, A) \)-closed in \( Q(A) \).

**Proof.** The proof given in Lemma 2.4 of [4] can be literally adapted to the present setting. We include here a sketch of the proof for completeness reasons. Suppose first that \( p \) is a non-zero compact projection in \( A^{**} \). From Theorem 4.2 of [20] we have \( F(p) \cap S(A) = \{ p \} \), is \( \sigma(A^*, A) \)-closed in \( Q(A) \).

Let \( \tilde{A} \) be the unitization of \( A \). Each element \( \phi \in Q(\tilde{A}) \) can be written in the form \( \phi = \psi + \alpha \phi_0 \), with \( \psi \in Q(A) \), \( \| \phi \| = \| \psi \| + |\alpha| \), where \( \phi_0 \) is the unique state of \( \tilde{A} \) satisfying \( \phi_0(A) = 0 \) (compare Lemma 3.6.6 of [26]). Since \( p \in A^{**} \) and hence \( \phi_0(p) = 0 \), we easily check that

\[
F_{A^*}(p) \cap S(A) = F_{\tilde{A}^*}(p) \cap S(\tilde{A}).
\]

Therefore, \( F_{A^*}(p) \cap S(A) \) is \( \sigma(A^*, A) \)-closed in \( Q(A) \) if and only if \( F_{\tilde{A}^*}(p) \cap S(\tilde{A}) \) is \( \sigma(\tilde{A}^*, \tilde{A}) \)-closed in \( Q(\tilde{A}) \). By Proposition 1.8, it follows that \( p \) is closed in \( (\tilde{A})^{**} \) and in \( A^{**} \). Since clearly \( p \leq 1_{\tilde{A}} \), we deduce from Theorem 2.6 of [22] that \( p \) is compact in \( (\tilde{A})^{**} \) relative to \( \tilde{A} \). Let \( p_0 \) be the minimal projection in \( (\tilde{A})^{**} \) satisfying \( \phi_0(p_0) = 1 \). Theorem 1.4 implies the existence of a norm-one element \( x \in \tilde{A} \) such that \( p_0 \) and \( x \) are orthogonal and \( L(p, p)x = x \circ p = p \). In particular \( x \in A \), which gives \( p \) compact in \( A^{**} \) relative to \( A \) (compare Theorem 2.6 of [22]).

Let \( B \) be a JB*-subtriple of a JB*-triple \( E \). Throughout the paper, we shall identify the weak*-closure of \( B \) in \( E^{**} \) with \( B^{**} \). Let \( x \) be a norm-one element and let \( E(x) \) denote the norm closure of \( \{ x, E, x \} \) in \( E \). It was proved by L.J. Bunce, Ch.-H. Chu and B. Zalar in [14], [15], that \( E(x) \) coincides with the norm-closed inner ideal of \( E \) generated by \( x \), \( E(x) \) is a JB*-subalgebra of the JBW*-algebra \( E(x)^{**} = E_2^{**}(r(x)) \), where \( r(x) \) is the range tripotent of \( x \) in \( E^{**} \). Moreover, \( x \in E(x)_+ \).

We can now state the following version of Urysohn lemma which is a partial generalization of the result obtained by C.A. Akemann, J. Anderson and G.K. Pedersen in Lemma 2.5 of [4] (see also Lemma III.1 of [3], Corollary 2.48 of [11], Lemma 2.7 of [5]).

**Theorem 1.10.** Let \( E \) be a JB*-triple, \( x \) a norm-one element in \( E \) and \( u \) a compact tripotent in \( E^{**} \) relative to \( E \) satisfying that \( u \leq r(x) \). Then there exists a norm-one element \( y \) in \( E(x) \) such that \( u \leq y \leq r(x) \). Moreover, \( u \) is a compact tripotent in \( E_2^{**}(r(x)) = (E(x))^{**} \) relative to \( E(x) \).
Proof. We may assume that $0 \neq u \leq r(x)$. From Theorem 4.2 of [20], there exists a set of norm-one elements $\{a_\lambda\} \subset E$ satisfying that

$$\{u\}_{E^*} \cap S(E(x)) = \{u\}_{E^*} = \Psi(\{u\}_{E^*}).$$

Since $u \leq r(x)$, then $u$ is a projection in $E(x)''' = E_2'''(r(x))$.

Since $E(x)$ is a norm-closed inner ideal of $E$, it follows from Theorem 2.6 of [19] that every element $\phi \in E(x)^*$ has a unique norm-preserving linear extension to $E$. The restriction mapping $\Psi : E_1^* \to E(x)^*$, $\phi \mapsto \phi|_{E(x)}$, is $\sigma(E^*, E) - \sigma(E(x)^*, E(x))$-continuous. Let $\phi \in \{u\}_{E^*}$. Since $u$ is a projection in $E_2'''(r(x))$ and $\phi(u) = 1 = \|\phi|_{E_2'''(r(x))}\|$, we deduce that $\phi|_{E_2'''(r(x))}$ belongs to $S_*(E_2'''(r(x))) = S(E(x))$, and hence $\|\phi|_{E(x)}\| = 1$. Again, the unique extension property (see Theorem 2.6 of [19]) assures that

$$F_{E(x)^*}(u) \cap S(E(x)) = \{u\}_{E(x)^*} = \Psi(\{u\}_{E^*}).$$

If we show that $F_{E(x)^*}(u) \cap S(E(x))$ is $\sigma(E(x)^*, E(x))$-closed in $Q(E(x))$, the thesis of the theorem will follow from Proposition 1.9 and Theorem 2.6 of [22]. To see this, let $(\phi_\mu)$ be a net in $F_{E(x)^*}(u) \cap S(E(x))$ converging to some $\phi$ in $F_{E(x)^*}(u) \cap S(E(x))$ in the $\sigma(E(x)^*, E(x))$-topology. Since $\Psi$ is surjective, there exist a net $(\phi_\mu)$ in $\{u\}_{E^*}$ and $\phi \in E_1^*$ such that $\Psi(\phi_\mu) = \phi_\mu$ and $\Psi(\phi) = \phi$. Since $E_1^*$ is $\sigma(E^*, E)$-compact, there exists a subnet $(\phi_\delta)$ converging to some $\phi'$ in the $\sigma(E^*, E)$-topology. For each $\lambda \in \Lambda$ we have $\phi_\delta(a_\lambda) \to \phi'(a_\lambda)$. In particular, since $(\phi_\delta) \subset \{u\}_{E^*}$, we have, by (1.4), $\phi_\delta(a_\lambda) = 1$ for all $\delta, \lambda$, which implies $\phi' \in \{u\}_{E^*}$. Finally, $\Psi(\phi_\delta) = \phi_\delta$ tends to $\Psi(\phi')$ in the $\sigma(E(x)^*, E(x))$-topology, thus

$$\varphi = \Psi(\phi) = \Psi(\phi') \in \Psi(\{u\}_{E^*}) = F_{E(x)^*}(u) \cap S(E(x)),$$

which finishes the proof. \hfill \blacksquare

Theorem 1.10 allows us to get the following generalization of Theorem II.17 of [1] and [3].

**Proposition 1.11.** Let $E$ be a JB$^*$-triple, let $\pi$ denote the canonical projection of $E''''$ onto its atomic part and let $j : E \to E''''$ be the canonical embedding of $E$ into its bidual. Then, for each range tripotent $e$ in $\pi(E'')$ relative to $\pi(E)$ there exists a unique range tripotent $r$ in $E''$ relative to $E$ such that $\pi(r) = e$.

**Proof.** Remark 1.2 assures the existence of such a tripotent, so the proof ends by proving the uniqueness. Suppose that there exist norm-one elements $x, y \in E$ such that $\pi(r(x)) = \pi(r(y)) = e$. By [31], there exists a locally compact Hausdorff space $L \subseteq [0, 1]$ with $L \cup \{0\}$ compact such that $E_x$ is isometrically isomorphic to $C_0(L)$. Let us define $u_n = \chi_{L \cap [1/n, 1]}$, $n \in \mathbb{N}$. Clearly, $u_n$ is a compact tripotent in $E''$ relative to $E$ and $u_n$ is an increasing sequence converging to $r(x)$ in the weak* topology of $E''$. $\pi(u_n) \leq \pi(r(x)) = e = \pi(r(y))$ and by Proposition 1.6
and Theorem 1.10 there is a sequence of norm-one positive elements \((z_n) \subset E(y)\) satisfying that \(\pi(u_n) \leq u(\pi(z_n)) \leq \pi(z_n) \leq \pi(r(y))\). Again, Proposition 1.6 gives \(u_n \leq u(z_n) \leq r(y)\). Finally, since \(E_2^*(r(y))\) is weak*-closed and \((u_n)\) tends to \(r(x)\) in the weak*-topology we have \(r(x) \leq r(y)\). Symmetrically, we get \(r(y) \leq r(x)\).

In the setting of C*-algebras, C.A. Akemann, J. Anderson and G.K. Pedersen proved, in Proposition 2.6 of [4], the following stronger version of the Urysohn lemma. Let \(A\) be a C*-algebra and let \(p\) and \(q\) be two closed orthogonal projections in \(A^{**}\) with \(p\) compact and \(\|ap\| < \varepsilon\) for some \(a\) in \(A\). Then there are orthogonal open projections \(r, s \in A^{**}\) such that \(p \leq r, q \leq s\) and \(\|ar\| < \varepsilon\). We do not know if we can obtain a similar result in the setting of JB*-triples.

**Problem 1.12.** Let \(E\) be a JB*-triple and let \(e, f\) be two non-zero orthogonal compact tripotents in \(E^{**}\) relative to \(E\). Do there exist orthogonal norm-one elements \(x, y\) in \(E\) such that \(e \leq x\) and \(f \leq y\)?

**Problem 1.13.** Can one replace in Theorem 1.10 the range tripotent, \(r(x)\), with any open tripotent in \(E^{**}\) relative to \(E\)?

2. CONNECTIONS WITH THE STONE-WEIERSTRASS THEOREM FOR C*-ALGEBRAS AND JB*-TRIPLES

As we have commented in the introduction, the generalizations of Urysohn lemma to the setting of non-commutative C*-algebras are closely related with the general Stone-Weierstrass problem for non-commutative C*-algebras. This tool has been intensively developed and applied to the Stone-Weierstrass problem in papers like [1], [2], [3], [4], [5] and [11].

The Stone-Weierstrass problem for C*-algebras can be concretely stated as follows:

Let \(B\) be a C*-subalgebra of a C*-algebra \(A\). Suppose that \(B\) separates the pure states of \(A\) and zero. Is \(B\) equal to \(A\)?

I. Kaplansky gave a positive answer to the above problem for the special class of type I C*-algebras in [29]. For general C*-algebras, many authors gave partial answer to the Stone-Weierstrass problem by including various additional conditions (see for example [29], [28], [25], [1], [2], [37], [21], [12], [6] and [10] among others).

We are particularly interested in the following Stone-Weierstrass type Theorem proved by C.A. Akemann in Theorem II.7 of [2].

**Theorem 2.1.** Let \(B\) be a C*-subalgebra of a unital C*-algebra \(A\) such that \(B\) separates the pure states of \(A\) and zero. Suppose that for every pair of orthogonal projections \(p, q\) in \(A^{**}\) with \(q\) minimal and \(p\) compact relative to \(A\), there exist orthogonal (positive) elements \(x, y\) in \(B\) such that \(\|y\| = 1, \|x\| \in \{0, 1\}\), \(p \leq x\) and \(q \leq y\). Then \(B = E\).
In the statement of Theorem II.7 in [2] it is not explicitly included in the hypothesis that $B$ separates the pure states of $A$ and zero. However, the proof uses the results in Section 3 of [1], where this condition is assumed (see page 285 of [1] and page 305 of [2]).

In the setting of JB-algebras and JB*-triples an intensive study of the Stone-Weierstrass problem was developed by B. Sheppard [38], [39]. Among other results, B. Sheppard generalizes the result obtained by Kaplansky for postliminal JB*-algebras and JB*-triples in the following result.

**Theorem 2.2 ([39], Theorem 5.7).** Let $B$ be a JB*-subtriple of a JB*-triple $E$ such that $B$ separates the extreme points of the closed unit ball of $E^*$. Then, if $E$ or $B$ is postliminal, $E = B$.

The aim of this section is an analysis of the connections between the Stone-Weierstrass theorem and the Urysohn lemma type results for JB*-triples, analogous to that made by C.A. Akemann in the setting of $C^*$-algebras.

**Definition 2.3.** Let $B$ be a JB*-subtriple of a JB*-triple $E$. We say that $B$ satisfies the SW-property with respect to $E$ if and only if for every couple of orthogonal tripotents $u, v$ in $E^{**}$ with $v$ minimal and $u$ compact relative to $E$, there exist orthogonal elements $x, y \in B$ such that $\|y\| = 1$, $\|x\| \in \{0, 1\}$, $u \leq x$ and $v \leq y$. When $u = 0$, we mean $x = 0$ in $u \leq x$.

Theorem 1.4 shows that every JB*-triple has the SW-property with respect to itself.

**Lemma 2.4.** Let $A$ be a JBW*-algebra and let $p, q$ be minimal projections in $A$. Suppose that $q = q_2 + q_1 + q_0$ is the Peirce decomposition of $q$ with respect to $p$ and $\varphi_q$ in $\partial e(A_{*,1})$ such that $\varphi_q(q) = 1$. Then, either $p = q$ or $\varphi_q(q_0) \neq 0$.

**Proof.** By 2.4.16 and 2.4.21 of [26] we have

$$P_2(p) = U_p^2 \circ * = U_p^2 \circ *, \quad P_0(p) = U_{1-p} \circ *,$$

where $U_p(x) := \{p, x^*, p\}$ and * denotes the canonical involution of $A$. Suppose that $\varphi_q(q_0) = 0$. We claim that $q = p$. Indeed, by Proposition 1 of [23] and the hypothesis we have

$$0 = \varphi_q(q_0) = \varphi_q(U_{1-p}(q)) = \varphi_q(U_qU_{1-p}(q)).$$

Since $q$ is minimal and $\varphi_q$ is faithful in $A_2(q) = \mathbb{C}q$, we have

$$U_qU_{1-p}(q) = 0.$$

Now by 2.4.18 of [26] it follows that

$$U_qU_{1-p}(q) = U_qU_{1-p}U_q(q) = U_{\{q, 1-p, q\}}(q) = 0.$$
However, since $1 - p \geq 0$, by 3.3.6 of [26], we have $\{q, 1 - p, q\}$ is a positive element in $A_2(q)$. Moreover, since $q$ is the unit element in $A_2(q)$ and $U_{(q, 1 - p, q)}(q) = 0$, it follows that $\{q, 1 - p, q\} = q - P_2(q)p = 0$. Finally, the equality $p = q$ can be derived from the minimality of $p$, since $q - P_2(q)p = 0$ and Lemma 1.6 of [23] imply that $p = q + P_0(q)p$. 

Let $E$ be a $\text{JB}^*$-triple. Throughout the paper $\text{MinTri}(E)$ will stand for the set of all minimal tripotents in $E$.

**Theorem 2.5.** Let $B$ be a $\text{JB}^*$-subtriple of a $\text{JB}^*$-triple $E$. Suppose that for every $u \neq v$ in $\text{MinTri}(E) \cup \{0\}$, with $u$ and $v$ orthogonal, there exist orthogonal elements $x, y \in B$ such that $\|y\|, \|x\| \in \{0, 1\}$ and $u \leq x$ and $v \leq y$ (if $u = 0$ or $v = 0$, we mean $x = 0$ or $y = 0$, respectively). Then $B$ separates $\partial_{\text{e}}(E^*_1) \cup \{0\}$.

**Proof.** Let $\varphi_1 \neq \varphi_2$ in $\partial_{\text{e}}(E^*_1) \cup \{0\}$. If $\varphi_1 = 0$, then there is a minimal tripotent $u_2$ in $E^{**}$ such that $\varphi_2(u_2) = 1$ (compare Proposition 4 of [23]). Now, the hypothesis on $B$ applied to 0 and $u_2$, assure the existence of orthogonal elements $x, y \in B$ such that $\|y\|, \|x\| \in \{0, 1\}$ and $0 \leq x$ and $u_2 \leq y$. In particular $0 = \varphi_1(y) \neq \varphi_2(y) = 1$. We may therefore assume $\varphi_1, \varphi_2 \neq 0$.

Take $u_1 \neq u_2$ minimal tripotents in $E^{**}$, such that $\varphi_i(u_i) = 1$, for $i = 1, 2$. As we have commented in the previous paragraph, the hypothesis implies the existence of a norm-one element $a \in B$, such that $u_1 \leq a$ and hence $\varphi_1(a) = 1$. If $\varphi_2(a) \neq 1$, then $B$ separates $\varphi_1, \varphi_2$ and we finish. We may therefore assume that $\varphi_2(a) = 1$. In this case, by Propositions 1, 2 and Lemma 1.6 of [23] $u_2 \leq a$. Therefore, $u_1, u_2 \leq a \leq r(a)$, which implies that $u_1$ and $u_2$ are minimal projections in the $\text{JBW}^*$-algebra $E^*_2(r(a))$. From Lemma 2.4 and the hypothesis, we have $\varphi_2(P_0(u_1)(u_2)) \neq 0$. Moreover, from page 258 of [8], it follows that $0 < |(P_0(u_1)(u_2))| \leq \|\varphi_2(P_0(u_1)(u_2))\|_{\varphi_2}$.

Let $A$ denote the atomic part of $E^{**}$. Clearly, $P_0(u_1)(A) \subset A$ and hence $P_0(u_1)(A)$ coincides with the weak*-closure of the linear span of $\text{MinTri}(E^{**}) \cap E_0^{**}(u_1)$ (compare [23]). Since

$$0 < |\varphi_2(P_0(u_1)(u_2))|$$

we have $\varphi_2|_{P_0(u_1)(A)} \neq 0$, and hence there exists a minimal tripotent $w \in \text{MinTri}(E^{**}) \cap E_0^{**}(u_1)$, such that $0 < \varphi_2(w) \leq \|w\|_{\varphi_2}$.

Finally, by hypothesis, there are two orthogonal norm-one elements $x, y$ in $B$ such that $u_1 \leq x$ and $w \leq y$. In particular $0 < \|w\|_{\varphi_2} \leq \|y\|_{\varphi_2}$ and $\varphi_1(x) = 1$. Therefore,

$$|\varphi_2(x)|^2 \leq \|x\|_{\varphi_2}^2 \leq \|x\|_{\varphi_2}^2 + \|y\|_{\varphi_2}^2 = \|x + y\|_{\varphi_2}^2 \leq \|x + y\|^2 = 1,$$

which proves the desired statements. 

Since every minimal tripotent in the bidual of a $\text{JB}^*$-triple is compact (see Theorem 3.4 of [16]) we have:
Corollary 2.6. Let B be a JB*-subtriple of a JB*-triple E. Suppose that B has the SW-property with respect to E. Then B separates $\partial_e(E^*_1) \cup \{0\}$.

The significant results obtained by B. Sheppard on the Stone-Weierstrass theorem for JB*-triples in [39] allow us to get the following result connecting the SW-property and the Stone-Weierstrass Theorem for postliminal JB*-triples.

Corollary 2.7. Let B a JB*-subtriple of a JB*-triple E. Suppose that B has the SW-property with respect to E, and E or B is postliminal. Then $B = E$.

Proof. This follows from Theorems 2.5 and 2.2 (see Theorem 5.7 of [39]).

Remark 2.8. Let $A$ be a C*-algebra regarded as a JB*-triple and let $p$ be a projection in $A^{**}$. Let $\circ$ denote the Jordan product on $A$. Suppose that $x$ is a norm-one element in $A$ such that $L(p, p)x = p$ (that is, $p \leq x$ in $A^{**}$ regarded the latter as a JB*-triple), and hence $x = p + P_0(p)(x)$. In this case $L(p, p)(x \circ x^*) = p$. This shows that $p \leq x \circ x^*$.

Now, the proof given in Theorem 2.5 can be literally adapted, via Remark 2.8, to show that the assumption of $B$ separating the pure states of $A$ and zero can be dropped in Theorem 2.1 (see also Theorem II.7 of [2]).

Corollary 2.9. Let $B$ be a C*-subalgebra of a C*-algebra $A$. Suppose that for every pair of orthogonal projections $p, q$ in $A^{**}$ with $q$ minimal and $p$ compact relative to $A$, there exists orthogonal (positive) elements $x, y$ in $B$ such that $\|y\| = 1$, $\|x\| \in \{0, 1\}$, $p \leq x$ and $q \leq y$. Then $B = A$.

Proof. The proof of Theorem 2.5 can be literally followed up to its last part. To finish, in this case, we note that the element $w$ can be chosen as a minimal projection, for example $ww^*$ or $w^*w$.

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References


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