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A GENERAL FACTORIZATION APPROACH TO THE EXTENSION THEORY OF NONNEGATIVE OPERATORS AND RELATIONS

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ABSTRACT. The Kreĭn-von Neumann and the Friedrichs extensions of a nonnegative linear operator or relation (i.e., a multivalued operator) are characterized in terms of factorizations. These factorizations lead to a novel approach to the transversality and equality of the Kreĭn-von Neumann and the Friedrichs extensions and to the notion of positive closability (the Kreĭn-von Neumann extension being an operator). Furthermore, all extremal extensions of the nonnegative operator or relation are characterized in terms of analogous factorizations. This approach for the general case of nonnegative linear relations in a Hilbert space extends the applicability of such factorizations. In fact, the extension theory of densely and nondensely defined nonnegative relations or operators fits in the same framework. In particular, all extremal extensions of a bounded nonnegative operator are characterized.

KEYWORDS: *Nonnegative relation, Friedrichs extension, Kreĭn-von Neumann extension, disjointness, transversality, positive closability, extremal extension.*

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INTRODUCTION

To illustrate the factorizations introduced in this paper consider the following simple completion problem. Let $\mathfrak{H}$ be a Hilbert space with the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$. Let S_{11} be a nonnegative bounded linear operator in $\mathfrak{H}_1$, let S_{21} be a bounded linear operator from $\mathfrak{H}_1$ to $\mathfrak{H}_2$, and let $S_{12} = S_{21}^*$. The usual form of the completion problem requires to determine all bounded linear operators S_{22} in $\mathfrak{H}_2$, such that

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & * \end{pmatrix} \tag{0.1}$$

becomes a nonnegative bounded linear operator in $\mathfrak{H}$, cf. [7], [13], [16], [28], [29], [33], [35]. This completion problem has a solution if and only if $\operatorname{ran} S_{21}^* \subset \operatorname{ran} S_{11}^{1/2}$.

To put this completion problem in a more general framework introduce the following linear relation

$$V = \{ \{h,k\} \in \mathfrak{H}_2 \times \overline{\operatorname{ran}}\, S_{11} : \; S_{11}^{1/2}k = S_{21}^* h \}, \tag{0.2}$$

as a subset of the Cartesian product $\mathfrak{H}_2 \times \overline{\operatorname{ran}}\, S_{11}$. Clearly, V is linear and closed. Furthermore, if $\{0,k\} \in V$ then by definition $k \in \ker S_{11} \cap \overline{\operatorname{ran}}\, S_{11} = \{0\}$, which means that V is the graph of a closed linear operator. The closed linear operator V gives rise to the following "solution" of (0.1):

$$\begin{pmatrix} S_{11}^{1/2} \\ V^* \end{pmatrix} \begin{pmatrix} S_{11}^{1/2} & V \end{pmatrix}, \tag{0.3}$$

where the product is in the sense of linear relations. If $\operatorname{dom} V = \mathfrak{H}_2$ (i.e., $\operatorname{ran} S_{21}^* \subset \operatorname{ran} S_{11}^{1/2}$), then $S_{11}^{1/2}V = S_{12}$, $S_{22} = V^*V$, and (0.3) gives the smallest solution of (0.1). If V is densely defined, then (0.3) still gives the smallest solution of (0.1), but now $S_{22} = V^*V$ is in general an unbounded operator. Moreover, if V is not densely defined then $S_{22} = V^*V$ is a nonnegative relation (i.e., a multivalued operator), and still (0.3) provides a smallest solution in the sense of relations, cf. [11], [16]. Furthermore, in the sense of relations, the completion problem (0.1) has always the following nonnegative solution:

$$\begin{pmatrix} S_{11}^{1/2} \\ O^* \end{pmatrix} \begin{pmatrix} S_{11}^{1/2} & O \end{pmatrix}, \tag{0.4}$$

where O is the trivial linear relation from $\mathfrak{H}_2$ to $\overline{\operatorname{ran}}\, S_{11}$ and its adjoint O^* is given by $O^* = \overline{\operatorname{ran}}\, S_{11} \times \mathfrak{H}_2$. In fact, in the sense of relations, (0.4) is the largest solution of (0.1). All the other solutions of (0.1) are between these extreme solutions. In particular, if R is an arbitrary restriction of the closed operator V in (0.2), then

$$\begin{pmatrix} S_{11}^{1/2} \\ R^* \end{pmatrix} \begin{pmatrix} S_{11}^{1/2} & R^{**} \end{pmatrix}, \tag{0.5}$$

is between (0.3) and (0.4), and hence a solution to the completion problem (0.1). For a proper interpretation of solutions to (0.1), introduce the operator S by

$$S = \begin{pmatrix} S_{11} \\ S_{21} \end{pmatrix} : \mathfrak{H}_1 \to \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix}. \tag{0.6}$$

Clearly, S is a nonnegative bounded operator from $\mathfrak{H}_1$ to $\mathfrak{H}$. The completion problem (0.1) can now be interpreted as an extension problem for S: the nonnegative solutions to (0.3) correspond to the nonnegative selfadjoint relation extensions of S. The factorizations of the extreme solutions of (0.1), i.e., of the extreme extensions of S in (0.6), persist in the general extension theory of nonnegative relations. It is the purpose of this paper to develop this approach of factorizations for the general case of the extension theory of nonnegative operators and relations and to study the consequences.

Let S be any nonnegative linear relation in a Hilbert space $\mathfrak{H}$. Then there are two nonnegative selfadjoint extensions of S in $\mathfrak{H}$, namely the Kreĭn-von Neumann extension S_{N} and the Friedrichs extension S_{F}, which are extreme in the sense that all other nonnegative selfadjoint extensions of S lie between them: if H is a nonnegative selfadjoint extension of S, then

$$S_{\mathrm{N}} \leqslant H \leqslant S_{\mathrm{F}}, \tag{0.7}$$

where the inequalities are in the sense of the corresponding resolvent operators:

$$(S_{\mathrm{F}}+a)^{-1} \leqslant (H+a)^{-1} \leqslant (S_{\mathrm{N}}+a)^{-1}, \quad a>0, \tag{0.8}$$

or, equivalently, in the sense of the corresponding closed nonnegative sesquilinear forms:

$$\mathfrak{t}_{S_{\mathrm{N}}} \leqslant \mathfrak{t}_H \leqslant \mathfrak{t}_{S_{\mathrm{F}}}. \tag{0.9}$$

When S is the nonnegative operator in (0.6) associated with the completion problem (0.1), then the Kreĭn-von Neumann extension S_{N} is given by (0.3) and the Friedrichs extension S_{F} is given by (0.4). The general theory of nonnegative selfadjoint extensions of densely defined nonnegative operators is due to M.G. Kreĭn [23], cf. also [36], [37], [38]. T. Ando and K. Nishio [1] have considered operator extensions in the case when S is a not necessarily densely defined operator. The general case involving relations goes back to [9], see also [17] for the interpretation of (0.9) in this case. For a review of Kreĭn's work (in the context of relations), see [16]. It will be shown in the present paper that the Kreĭn-von Neumann extension S_{N} in (0.7) has a factorization $S_{\mathrm{N}} = J^{**}J^{*}$, where J is a linear relation from an auxiliary space $\mathfrak{H}_S$ to $\mathfrak{H}$, and that the Friedrichs extension in (0.7) has a similar factorization. Such factorizations go back to J. Stochel, Z. Sebestyén, and coworkers (see [26], [27], [30], [31], [32]) for the case that S is a densely defined operator or under a condition which guarantees that the Kreĭn-von Neumann extension S_{N} is an operator. The factorizations of S_{N} and S_{F} in the general case provide a novel approach to notions such as disjointness, transversality, and equality of S_{N} and S_{F}; and to the notion of positive closability of S (S_{N} being an operator). A nonnegative selfadjoint extension H of S is called extremal when

$$\inf\{\, (f'-h', f-h) : \{h,h'\} \in S \,\} = 0 \quad \text{for all } \{f,f'\} \in H. \tag{0.10}$$

This definition goes back at least to Y.M. Arlinskiĭ and E.R. Tsekanovskiĭ [5]. In the densely defined case the factorization of the extremal extensions was studied in [4]. In the present paper the extremal extensions are characterized by factorizations in the general case. In particular, the extremal extensions of S in (0.6) are precisely the solutions of (0.1) given by (0.5). For another recent application of these factorizations, see [18].

The contents of this paper are now briefly listed. Section 1 contains some useful observations concerning linear relations and their adjoints in Cartesian products of Hilbert spaces and the product of a relation and a unitary operator. Moreover, Section 1 contains a simple treatment of the disjointness and transversality of two selfadjoint extensions of a symmetric relation. In Section 2 some results concerning the Kreĭn-von Neumann and the Friedrichs extensions are recalled. Furthermore this section provides a simple treatment of the disjointness and transversality for nonnegative selfadjoint extensions of a nonnegative relation. The generalization of the construction of the Kreĭn-von Neumann extension and the Friedrichs extensions in the sense of Sebestyén and Stochel can be found in Section 3. Disjointness, transversality and equality of the Kreĭn-von Neumann and Friedrichs extensions from the point of view of factorizations are discussed in Section 4. Also the question whether the Kreĭn-von Neumann extension is a (bounded) operator is discussed in this section. The treatment of extremal extensions and their factorizations can be found in Section 5. Finally the case of a nonnegative bounded operator and its extremal extensions is treated in Section 6. This provides the link with the completion problem discussed above.

1. PRELIMINARIES

This section contains some useful elementary observations concerning the adjoint of a linear relation. The following notations will be used. For the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$ the notation $[\mathfrak{H}, \mathfrak{K}]$ stands for the bounded linear operators from $\mathfrak{H}$ to $\mathfrak{K}$; moreover $[\mathfrak{H}] = [\mathfrak{H}, \mathfrak{H}]$. Furthermore $\mathfrak{H} \oplus \mathfrak{K}$ stands for the orthogonal sum of $\mathfrak{H}$ and $\mathfrak{K}$, i.e., the Cartesian product $\mathfrak{H} \times \mathfrak{K}$ provided with the usual inner product. The usual notations for linear relations are assumed throughout this paper, cf. [17].

1.1. DOMAIN DESCRIPTIONS. For the convenience of the reader the following general domain and range descriptions are recalled.

LEMMA 1.1. *Let T be a linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$. Then:*

(i) $g \in \operatorname{dom} T^*$ *if and only if* $g \in \mathfrak{K}$ *and there exists a nonnegative number* γ_g *such that*

$$|(h', g)_{\mathfrak{K}}| \leqslant \gamma_g \|h\|_{\mathfrak{H}} \quad \text{for all } \{h, h'\} \in T. \tag{1.1}$$

In this case the smallest γ_g *satisfying* (1.1) *is* $\gamma_g = \|g'\|_{\mathfrak{H}}$ *with* $\{g, g'\} \in T^*$ *and* $g' \in \overline{\operatorname{dom}}\, T$.

(ii) $g' \in \operatorname{ran} T^*$ *if and only if* $g' \in \mathfrak{K}$ *and there exists a nonnegative number* $\gamma_{g'}$ *such that*

$$|(h, g')_{\mathfrak{H}}| \leqslant \gamma_{g'} \|h'\|_{\mathfrak{K}} \quad \text{for all } \{h, h'\} \in T. \tag{1.2}$$

In this case the smallest $\gamma_{g'}$ *satisfying* (1.2) *is* $\gamma_{g'} = \|g\|_{\mathfrak{K}}$ *with* $\{g, g'\} \in T^*$ *and* $g \in \overline{\operatorname{ran}}\, T$.

The results in this lemma can be found in [14]; clearly the statements (i) and (ii) are dual (by interchanging domain and range). In the setting of densely defined operators such results go back at least to Y.L. Shmul'yan [34].

1.2. ADJOINT OPERATIONS. The operations to be considered concern linear relations of a special form (containing a bounded operator component). Let $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{K}_1$, and $\mathfrak{K}_2$ be Hilbert spaces, and let $\mathfrak{X}$ and $\mathfrak{Y}$ be closed linear subspaces of $\mathfrak{H}_2$. Let $A \in [\mathfrak{H}_1, \mathfrak{K}_1]$, $B \in [\mathfrak{K}_1, \mathfrak{H}_1]$, and let C and D be linear relations from $\mathfrak{H}_1$ to $\mathfrak{K}_2$ and from $\mathfrak{K}_2$ to $\mathfrak{H}_1$, respectively. Define the linear relation $\mathcal{A}$ from $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ to $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ by

$$\mathcal{A} = \left\{ \left\{ \begin{pmatrix} f \\ \eta \end{pmatrix}, \begin{pmatrix} Af \\ f' \end{pmatrix} \right\} : f \in \mathfrak{H}_1,\ \eta \in \mathfrak{X},\ \{f, f'\} \in C \right\}, \tag{1.3}$$

and define the linear relation $\mathcal{B}$ from $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ to $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ by

$$\mathcal{B} = \left\{ \left\{ \begin{pmatrix} h \\ \varphi \end{pmatrix}, \begin{pmatrix} Bh + \varphi' \\ \psi \end{pmatrix} \right\} : h \in \mathfrak{K}_1,\ \psi \in \mathfrak{Y},\ \{\varphi, \varphi'\} \in D \right\}. \tag{1.4}$$

LEMMA 1.2. *Let the linear relations $\mathcal{A}$ and $\mathcal{B}$ be given by* (1.3) *and* (1.4). *Then the adjoint $\mathcal{A}^*$ is the linear relation from $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ to $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ given by*

$$\mathcal{A}^* = \left\{ \left\{ \begin{pmatrix} h \\ \varphi \end{pmatrix}, \begin{pmatrix} A^*h + \varphi' \\ \psi \end{pmatrix} \right\} : h \in \mathfrak{K}_1,\ \psi \in \mathfrak{H}_2 \ominus \mathfrak{X},\ \{\varphi, \varphi'\} \in C^* \right\}, \tag{1.5}$$

and the double adjoint $\mathcal{A}^{**}$ is the relation from $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ to $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ given by

$$\mathcal{A}^{**} = \left\{ \left\{ \begin{pmatrix} f \\ \eta \end{pmatrix}, \begin{pmatrix} Af \\ f' \end{pmatrix} \right\} : f \in \mathfrak{H},\ \eta \in \mathfrak{X},\ \{f, f'\} \in C^{**} \right\}. \tag{1.6}$$

Furthermore, the adjoint $\mathcal{B}^*$ is the relation from $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ to $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ given by

$$\mathcal{B}^* = \left\{ \left\{ \begin{pmatrix} f \\ \eta \end{pmatrix}, \begin{pmatrix} B^*f \\ f' \end{pmatrix} \right\} : f \in \mathfrak{H}_1,\ \eta \in \mathfrak{H}_2 \ominus \mathfrak{Y},\ \{f, f'\} \in D^* \right\}, \tag{1.7}$$

and the double adjoint $\mathcal{B}^{**}$ is the relation from $\mathfrak{K}_1 \oplus \mathfrak{K}_2$ to $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ given by

$$\mathcal{B}^{**} = \left\{ \left\{ \begin{pmatrix} h \\ \varphi \end{pmatrix}, \begin{pmatrix} Bh + \varphi' \\ \psi \end{pmatrix} \right\} : h \in \mathfrak{K}_1,\ \psi \in \mathfrak{Y},\ \{\varphi, \varphi'\} \in D^{**} \right\}. \tag{1.8}$$

As to the proof of Lemma 1.2: one only needs to apply the definition of the adjoint of a linear relation to obtain (1.5) and (1.7). Clearly

$$\operatorname{mul} \mathcal{A}^* = \operatorname{mul} C^* \oplus (\mathfrak{H}_2 \ominus \mathfrak{X}), \quad \operatorname{mul} \mathcal{B}^* = \{0\} \oplus \operatorname{mul} D^*.$$

Observe that the closures of the relations in (1.5) and (1.7) are given by (1.6) and (1.8), respectively. In particular,

$$\operatorname{mul} \mathcal{A}^{**} = \{0\} \oplus \operatorname{mul} C^{**}, \quad \operatorname{mul} \mathcal{B}^{**} = \operatorname{mul} D^{**} \oplus \mathfrak{Y}, \tag{1.9}$$

so that the linear relation $\mathcal{A}$ is closable if and only if the linear relation C is closable, and the linear relation $\mathcal{B}$ is closable if and only if the linear relation D is closable and $\mathfrak{Y} = \{0\}$.

1.3. ADJOINTS OF PRODUCTS. Let T be a relation from $\mathfrak{H}$ to $\mathfrak{R}$ and let S be a relation from $\mathfrak{D}$ to $\mathfrak{H}$. Then TS is a relation from $\mathfrak{D}$ to $\mathfrak{R}$ and

$$(TS)^* \supset S^*T^*. \tag{1.10}$$

It is possible to give rather general statements concerning the equality. However, for the purposes of the present paper only the following situation will be considered. Let A be a linear relation from the Hilbert space $\mathfrak{H}$ to the Hilbert space $\mathfrak{K}$. Let U be a unitary operator from the Hilbert space $\mathfrak{D}$ onto $\mathfrak{H}$, and let V be a unitary operator from $\mathfrak{K}$ onto a Hilbert space $\mathfrak{R}$. Define the relation T from $\mathfrak{H}$ to $\mathfrak{R}$ and the relation S from $\mathfrak{D}$ to $\mathfrak{H}$, by

$$T = VA, \quad S = AU. \tag{1.11}$$

LEMMA 1.3. *The adjoints of T and S in* (1.11) *are given by*

$$T^* = A^*V^*, \quad S^* = U^*A^*, \tag{1.12}$$

and the double adjoints of T and S are given by

$$T^{**} = VA^{**}, \quad S^{**} = A^{**}U. \tag{1.13}$$

Proof. The inclusion $A^*V^* \subset T^*$ is clear. To show the reverse inclusion, let $\{f, f'\} \in T^*$. Then $(f', h) = (f, h')$ for all $\{h, h'\} \in T = VA$, or, in other words,

$$(f', h) = (f, V\psi) \quad \text{for all } \{h, \psi\} \in A.$$

For $f \in \operatorname{dom} T^* \subset \mathfrak{R}$ there is a unique $f_0 \in \mathfrak{K}$ such that $f = Vf_0$. Hence

$$(f', h) = (f_0, \psi) \quad \text{for all } \{h, \psi\} \in A,$$

which shows that $\{f_0, f'\} \in A^*$, while $\{f, f_0\} \in V^{-1} \subset V^*$. This implies that $\{f, f'\} \in A^*V^*$. Therefore the first identity in (1.12) holds.

The second identity in (1.12) is obtained from the first identity by applying it to the inverse of S.

The identities (1.13) are special cases of (1.12). ∎

In the context of relations the general inclusion result in (1.10) goes back to R. Arens [2]. Results as in Lemma 1.3 exist in various degrees of generality, cf. [19].

1.4. DISJOINTNESS AND TRANSVERSALITY. Let T be a linear relation in a Hilbert space $\mathfrak{H}$. For $\lambda \in \mathbb{C}$ the "eigenspace" $\widehat{\mathfrak{N}}_\lambda(T)$ associated to T is defined by

$$\widehat{\mathfrak{N}}_\lambda(T) = \{ \{f, f'\} \in T : \ f' = \lambda f \}.$$

If, in particular, T is closed, then also the eigenspace $\widehat{\mathfrak{N}}_\lambda(T)$ is closed for every $\lambda \in \mathbb{C}$. The following lemma can be seen as a motivation for this notion.

LEMMA 1.4. *Let T be a linear relation in $\mathfrak{H}$, let H be a restriction of T with a nonempty resolvent set, and assume that $\lambda \in \rho(H)$. Then H is closed and*

$$T = H \,\widehat{+}\, \widehat{\mathfrak{N}}_\lambda(T), \tag{1.14}$$

where $\widehat{+}$ *stands for the componentwise sum in* $\mathfrak{H} \times \mathfrak{H}$.

Proof. Closedness of H is guaranteed by the assumption $\rho(H) \neq \emptyset$, since $(H-\lambda)^{-1}$ as a bounded everywhere defined operator is automatically closed. Moreover, the inclusion $H \widehat{+} \widehat{\mathfrak{N}}_\lambda(T) \subset T$ is clear. It remains to prove the reverse inclusion. Assume that $\{f, f'\} \in T$. Then $\{f, f' - \lambda f\} \in T - \lambda$, and, since $\lambda \in \rho(H)$, there exists an element $h \in \mathfrak{H}$ such that $\{h, f' - \lambda f\} \in H - \lambda \subset T - \lambda$. This implies $\{h, f' - \lambda(f-h)\} \in H$ and $\{f-h, 0\} \in T - \lambda$, so that

$$\{f, f'\} = \{h, f' - \lambda(f-h)\} \widehat{+} \{f-h, \lambda(f-h)\} \in H \widehat{+} \widehat{\mathfrak{N}}_\lambda(T),$$

and (1.14) is proved. ∎

Let A, B, S, and T be linear relations in a Hilbert space $\mathfrak{H}$ such that

$$S \subset A \subset T, \quad S \subset B \subset T. \tag{1.15}$$

Then clearly $S \subset A \cap B$ and $A \widehat{+} B \subset T$. The next lemma gives simple criteria to check whether these inclusions are actually equalities.

LEMMA 1.5. *Let the linear relations* A, B, S, *and* T *in* $\mathfrak{H}$ *satisfy the inclusions* (1.15) *and, moreover, assume that* $\rho(A) \cap \rho(B) \neq \emptyset$. *Then:*

(i) $S = A \cap B$ *if and only if for some (equivalently for every)* $\lambda \in \rho(A) \cap \rho(B)$,

$$\operatorname{ran}(S-\lambda) = \ker\left((B-\lambda)^{-1} - (A-\lambda)^{-1}\right); \tag{1.16}$$

(ii) $T = A \widehat{+} B$ *if and only if for some (equivalently for every)* $\lambda \in \rho(A) \cap \rho(B)$,

$$\ker(T-\lambda) = \operatorname{ran}\left((B-\lambda)^{-1} - (A-\lambda)^{-1}\right). \tag{1.17}$$

Proof. (i) It suffices to show that $A \cap B \subset S$ if and only if with $\lambda \in \rho(A) \cap \rho(B)$

$$\ker\left((B-\lambda)^{-1} - (A-\lambda)^{-1}\right) \subset \operatorname{ran}(S-\lambda). \tag{1.18}$$

Assume that (1.18) holds for some $\lambda \in \rho(A) \cap \rho(B)$. If $\{h, h'\} \in A \cap B$, then $(A-\lambda)^{-1}(h' - \lambda h) = h$ and $(B-\lambda)^{-1}(h' - \lambda h) = h$, so that by (1.18) $h' - \lambda h \in \operatorname{ran}(S-\lambda)$, i.e., $\{h, h'\} \in S$. Hence $A \cap B \subset S$.

Now assume that $A \cap B \subset S$. Let $\varphi \in \ker\left((B-\lambda)^{-1} - (A-\lambda)^{-1}\right)$. Then with $\psi = (A-\lambda)^{-1}\varphi = (B-\lambda)^{-1}\varphi$ it follows that $\{\psi, \varphi + \lambda\psi\} \in A \cap B \subset S$. Consequently, $\varphi \in \operatorname{ran}(S-\lambda)$ and therefore (1.18) holds for every $\lambda \in \rho(A) \cap \rho(B)$.

(ii) It suffices to show that $T \subset A \widehat{+} B$ if and only if with $\lambda \in \rho(A) \cap \rho(B)$

$$\ker(T-\lambda) \subset \operatorname{ran}\left((B-\lambda)^{-1} - (A-\lambda)^{-1}\right). \tag{1.19}$$

Assume that (1.19) holds for some $\lambda \in \rho(A) \cap \rho(B)$. Then for each $f \in \ker(T-\lambda)$ there is $h \in \mathfrak{H}$ such that $f = \beta - \alpha$ with $\alpha = (A-\lambda)^{-1}h$ and $\beta = (B-\lambda)^{-1}h$. Then $\{\alpha, h + \lambda\alpha\} \in A$, $\{\beta, h + \lambda\beta\} \in B$, and $\{f, \lambda f\} = \{\beta - \alpha, \lambda(\beta - \alpha)\} \in A \widehat{+} B$. Therefore, $\widehat{\mathfrak{N}}_\lambda(T) \subset A \widehat{+} B$. Now (1.14) with $H = A$ or $H = B$ implies that $T \subset A \widehat{+} B$.

Conversely, assume that $T \subset A \widehat{+} B$. Let $f \in \ker\,(T-\lambda)$, then $\{f, \lambda f\} \in T$. Hence there exist $\{\alpha, \alpha'\} \in A$ and $\{\beta, \beta'\} \in B$ such that

$$\{f, \lambda f\} = \{\beta, \beta'\} - \{\alpha, \alpha'\},$$

which implies that

$$\beta' - \alpha' = \lambda(\beta - \alpha) \quad \text{or, equivalently,} \quad \beta' - \lambda\beta = \alpha' - \lambda\alpha.$$

It follows from $\{\alpha' - \lambda\alpha, \alpha\} \in (A-\lambda)^{-1}$ and $\{\beta' - \lambda\beta, \beta\} \in (B-\lambda)^{-1}$, that $f = \beta - \alpha = ((B-\lambda)^{-1} - (A-\lambda)^{-1})h$ with $h = \beta' - \lambda\beta = \alpha' - \lambda\alpha$. Therefore, $f \in \operatorname{ran}((B-\lambda)^{-1} - (A-\lambda)^{-1})$ and thus (1.19) holds for every $\lambda \in \rho(A) \cap \rho(B)$. ∎

The closed extensions A and B of S are said to be *disjoint* (with respect to S) if

$$S = A \cap B,$$

or equivalently, if S is closed and

$$S^* = \operatorname{clos}(A^* \widehat{+} B^*).$$

Furthermore, if A and B are intermediate extensions of S, i.e. $S \subset A \subset S^*$ and $S \subset B \subset S^*$, then A and B are said to be *transversal* (with respect to S) if

$$S = A \cap B \quad \text{and} \quad S^* = A \widehat{+} B,$$

cf. [11]. It is clear that disjointness and transversality of A and B can be characterized by the conditions (1.16) and (1.17) in Lemma 1.5. For this purpose the following corollary is often sufficient.

COROLLARY 1.6. *Let S be a symmetric relation in a Hilbert space $\mathfrak{H}$ and let A and B be selfadjoint extensions of S. Then:*

(i) *A and B are disjoint if and only if for some (equivalently for every) $\lambda \in \rho(A) \cap \rho(B)$,*

$$\operatorname{ran}(S-\lambda) = \ker\,((B-\lambda)^{-1} - (A-\lambda)^{-1});$$

(ii) *A and B are transversal if and only if for some (equivalently for every) $\lambda \in \rho(A) \cap \rho(B)$,*

$$\ker\,(S^*-\lambda) = \operatorname{ran}((B-\lambda)^{-1} - (A-\lambda)^{-1}).$$

Observe that if S is a symmetric relation in $\mathfrak{H}$, then $H := S \widehat{+} \widehat{\mathfrak{N}}_\lambda(S^*)$ is a restriction of S^* and, moreover, if S is closed,

$$\overline{\lambda} \in \rho(H), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Therefore, Lemma 1.4 shows that

$$S^* = S \widehat{+} \widehat{\mathfrak{N}}_\lambda(S^*) \widehat{+} \widehat{\mathfrak{N}}_{\overline{\lambda}}(S^*), \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

which is *von Neumann's decomposition* for the adjoint of a closed symmetric relation.

Originally, equivalent descriptions for disjointness and transversality have been established by means of boundary triplets using parameters in Kreĭn's formula, cf. [25] and [11]; for the case of sectorial linear relations, see also [3]. Disjointness and transversality play a role in the construction of boundary triplets, and more generally, of boundary relations, cf. [10].

2. THE KREĬN-VON NEUMANN AND THE FRIEDRICHS EXTENSIONS

This section contains general information concerning nonnegative selfadjoint extensions of a nonnegative relation S. In particular, the Kreĭn-von Neumann and the Friedrichs extensions are introduced. Furthermore there is a simple treatment of the transversality of any nonnegative selfadjoint extension of S and its Friedrichs extension.

2.1. SOME GENERAL REMARKS CONCERNING THE KREĬN-VON NEUMANN AND THE FRIEDRICHS EXTENSIONS. A linear relation S in a Hilbert space $\mathfrak{H}$ is said to be semibounded from below if there exists a number $a \in \mathbb{R}$ such that $(f', f) \geqslant a(f, f)$ for all $\{f, f'\} \in S$. Note that in this case the relation S is automatically symmetric and it has equal defect numbers. The largest number $a \in \mathbb{R}$ which serves this purpose is called the lower bound $m(S)$ of S. It is given by $m(S) = 0$ when S is purely multivalued and by

$$m(S) = \inf\{\, (f', f) : \{f, f'\} \in S,\ \|f\| = 1 \,\}$$

otherwise. Clearly, the lower bound of $\operatorname{clos} S$ is equal to the lower bound of S. When the lower bound is nonnegative the relation S is called nonnegative: $(f', f) \geqslant 0$, $\{f, f'\} \in S$. The fact that

$$(\lambda f' + \mu g', \lambda f + \mu g) \geqslant 0, \quad \{f, f'\}, \{g, g'\} \in S, \quad \lambda, \mu \in \mathbb{C},$$

leads to the Cauchy inequality for nonnegative relations:

$$|(f', g)|^2 \leqslant (f', f)(g', g), \quad \{f, f'\}, \{g, g'\} \in S. \tag{2.1}$$

Let S be a nonnegative linear relation and define on $\operatorname{dom} \mathfrak{t} = \operatorname{dom} S$

$$\mathfrak{t}[f, g] = (f', g), \quad \{f, f'\}, \{g, g'\} \in S. \tag{2.2}$$

Then (2.2) gives rise to a nonnegative form since

$$(f', g) = (f, g') = (f'', g) \geqslant 0, \quad \{f, f'\}, \{f, f''\}, \{g, g'\} \in S.$$

In fact, the form $\mathfrak{t}$ in (2.2) is closable, cf. [22]. The closure $\bar{\mathfrak{t}}$ of the form $\mathfrak{t}$ in (2.2) is nonnegative and induces a nonnegative selfadjoint relation $S_{\rm F}$ which is the orthogonal sum of the selfadjoint operator induced by the form $\bar{\mathfrak{t}}$ in $\overline{\operatorname{dom}}\, S$ (cf. [22]) and the multivalued part $\{0\} \times \operatorname{mul} S^*$ (cf. [11], [17]). The nonnegative selfadjoint relation $S_{\rm F}$ is an extension of S and has the same lower bound as S, cf. [8]. By construction $\operatorname{mul} S_{\rm F} = \operatorname{mul} S^*$, so that the Friedrichs extension is an operator if and only if S is densely defined (and necessarily an operator). For a nonnegative relation S introduce the space dom $[S]$ as the set of all $f \in \mathfrak{H}$ for which there exists a sequence $(\{f_n, f_n'\}) \subset S$ such that

$$f_n \to f, \quad (f_n' - f_m', f_n - f_m) \to 0 \quad m, n \to \infty.$$

It can be shown that dom $[S] =$ dom $[S_{\rm F}] =$ dom $S_{\rm F}^{1/2}$, and that

$$S_{\rm F} = \{ \{f, f'\} \in S^* : f \in \operatorname{dom}[S] \}. \tag{2.3}$$

Moreover, the Friedrichs extension is the only selfadjoint extension of S whose domain is contained in dom $[S]$.

If the relation S is nonnegative (selfadjoint), then likewise the formal inverse S^{-1} of S is nonnegative (selfadjoint). Hence the selfadjoint relation

$$S_{\rm N} = ((S^{-1})_F)^{-1} \tag{2.4}$$

is also a nonnegative selfadjoint extension of S; in fact it is the Kreĭn-von Neumann extension of S, cf. [23], [1], [9]. In particular, $S_{\rm N}$ is the only selfadjoint extension of S whose range is contained in ran $[S] :=$ dom $[S^{-1}]$ and the following description holds

$$S_{\rm N} = \{ \{f, f'\} \in S^* : f' \in \operatorname{ran}[S] \}.$$

Notice also that ker $S_{\rm N} =$ ker S^*, and that $f' \in$ ran $[S]$ if and only if there exists a sequence $(\{f_n, f_n'\}) \subset S$, such that

$$f_n' \to f', \quad (f_n' - f_m', f_n - f_m) \to 0, \quad m, n \to \infty.$$

The Kreĭn-von Neumann and the Friedrichs extensions are extreme nonnegative selfadjoint extensions of S: if H is any nonnegative selfadjoint extension of S, then (0.7) holds, where the inequalities are in the sense of resolvents (see (0.8)) or, equivalently, in the sense of the corresponding forms (see (0.9)), cf. [17]. The following proposition is just a reformulation of the definition of the form domain dom $[S]$ and of the form range ran $[S]$, respectively, used in the construction of $S_{\rm F}$ and $S_{\rm N}$.

PROPOSITION 2.1. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let $\{f, f'\} \in S^*$. Then:*

(i) $\{f, f'\} \in S_{\rm F}$ *if and only if*

$$\inf\{\|f - h\|^2 + (f' - h', f - h) : \{h, h'\} \in S\} = 0;$$

(ii) $\{f, f'\} \in S_{\rm N}$ *if and only if*

$$\inf\{\|f' - h'\|^2 + (f' - h', f - h) : \{h, h'\} \in S\} = 0.$$

Sometimes the Kreĭn-von Neumann and the Friedrichs extensions can be given explicitly. If S is a nonnegative relation, then

$$S_{\rm N} = S \mathbin{\widehat{+}} (\ker S^* \times \{0\}), \tag{2.5}$$

if and only if

$$\operatorname{ran} S = \overline{\operatorname{ran}}\, S \cap \operatorname{ran} S^*, \tag{2.6}$$

and likewise, in view of (2.4),

$$S_{\mathrm{F}} = S \,\widehat{+}\, (\{0\} \times \operatorname{mul} S^*), \tag{2.7}$$

if and only if

$$\operatorname{dom} S = \overline{\operatorname{dom}}\, S \cap \operatorname{dom} S^*, \tag{2.8}$$

see [9], [21]. Observe that (2.6) is satisfied when $\operatorname{ran} S$ is closed, in which case (2.5) is valid. In particular, if S is closed and $m(S) > 0$, or if S^{-1} is a closed bounded operator, then (2.5) holds. Hence, if S is closed, then either $m(S_{\mathrm{N}}) = 0$ or S is selfadjoint, in which case $S_{\mathrm{N}} = S = S_{\mathrm{F}}$, cf. [9].

A review of the above facts can be found in [16]; some further facts concerning inequalities between nonnegative selfadjoint relations and the corresponding forms can be found in [17], see also [6].

2.2. DISJOINTNESS AND TRANSVERSALITY.

A concise treatment is given for the disjointness and the transversality of some nonnegative selfadjoint extension and the Friedrichs extension. The following observation is very useful.

LEMMA 2.2. *Let A and B be nonnegative selfadjoint relations in a Hilbert space $\mathfrak{H}$ and let $a > 0$. If $A \leqslant B$, then*

$$\operatorname{dom} A^{1/2} = \operatorname{ran} ((A+a)^{-1} - (B+a)^{-1})^{1/2} + \operatorname{dom} B^{1/2}. \tag{2.9}$$

Proof. Since $a > 0$ it follows that $-a$ is in the resolvent sets of A and B. Hence $R(a) = (A+a)^{-1} - (B+a)^{-1} \in [\mathfrak{H}]$. It follows from the inequality $A \leqslant B$ that $R(a)$ is nonnegative, cf. [9], [17]. Furthermore,

$$(A+a)^{-1} = R(a) + (B+a)^{-1} = \begin{pmatrix} R(a)^{1/2} & (B+a)^{-1/2} \end{pmatrix} \begin{pmatrix} R(a)^{1/2} \\ (B+a)^{-1/2} \end{pmatrix},$$

which leads to

$$\operatorname{ran} (A+a)^{-1/2} = \operatorname{ran} R(a)^{1/2} + \operatorname{ran} (B+a)^{-1/2},$$

cf. [12]. The last result coincides with the decomposition (2.9). ∎

Recall that if R is a nonnegative bounded linear operator in a Hilbert space $\mathfrak{H}$, then $\overline{\operatorname{ran}}\, R = \overline{\operatorname{ran}}\, R^{1/2}$, and $\operatorname{ran} R$ is closed if and only if $\operatorname{ran} R^{1/2}$ is closed.

PROPOSITION 2.3. *Let S be a nonnegative linear relation and let H be a nonnegative selfadjoint extension of S. Then, for every $a > 0$,*

$$\operatorname{ran} ((H+a)^{-1} - (S_{\mathrm{F}}+a)^{-1})^{1/2} = \ker\, (S^* + a) \cap \operatorname{dom} H^{1/2}, \tag{2.10}$$

and, furthermore,

$$\operatorname{dom} H^{1/2} = (\ker\, (S^*+a) \cap \operatorname{dom} H^{1/2}) + \operatorname{dom} S_{\mathrm{F}}^{1/2}, \quad \textit{direct sum}. \tag{2.11}$$

Proof. Recall that $H \leqslant S_{\rm F}$ via Kreĭn's inequalities. Hence, Lemma 2.2 may be applied with $A = H$ and $B = S_{\rm F}$. Define $R(a) = (H+a)^{-1} - (S_{\rm F}+a)^{-1}$, so that clearly $\operatorname{ran} R(a) \subset \ker\,(S^*+a)$. It suffices to show that

$$\operatorname{ran} R(a)^{1/2} = \ker\,(S^*+a) \cap \operatorname{dom} H^{1/2}.$$

Clearly, it follows from (2.9) that $\operatorname{ran} R(a)^{1/2} \subset \operatorname{dom} H^{1/2}$, since $\operatorname{dom} S_{\rm F}^{1/2} \subset \operatorname{dom} H^{1/2}$, cf. [17]. Since $\operatorname{ran} R(a) \subset \ker\,(S^*+a)$, it follows that $\operatorname{ran} R(a)^{1/2} \subset \ker\,(S^*+a)$. Hence, the lefthand side is contained in the righthand side. Next the reverse inclusion will be shown. Let $f \in \ker\,(S^*+a) \cap \operatorname{dom} H^{1/2}$, then $f \in \operatorname{dom} H^{1/2}$ implies that

$$f = h + k, \quad h \in \operatorname{dom} S_{\rm F}^{1/2}, \quad k \in \operatorname{ran} R(a)^{1/2},$$

cf. (2.9). Since $k \in \ker\,(S^*+a)$, it follows that $h \in \ker\,(S^*+a) \cap \operatorname{dom} S_{\rm F}^{1/2}$. Hence $h \in \ker\,(S_{\rm F}+a)$ by (2.3), so that $h = 0$. Therefore $f \in \operatorname{ran} R(a)^{1/2}$, which completes the proof of (2.10).

The identity (2.11) follows now from Lemma 2.2. The sum is direct since (2.3) implies $\ker\,(S^*+a) \cap \operatorname{dom} S_{\rm F}^{1/2} = \ker\,(S_{\rm F}+a)$ and, hence, this set is trivial. ∎

PROPOSITION 2.4. *Let S be a nonnegative linear relation and let H be a nonnegative selfadjoint extension of S. Then H and $S_{\rm F}$ are transversal if and only if*

$$\ker\,(S^*+a) \subset \operatorname{dom} H^{1/2}, \quad a > 0. \tag{2.12}$$

Furthermore, $H = S_{\rm F}$ if and only if

$$\ker\,(S^*+a) \cap \operatorname{dom} H^{1/2} = \{0\}, \quad a > 0. \tag{2.13}$$

Proof. Assume that (2.12) is satisfied. Then it follows from (2.10) that

$$\operatorname{ran}\,((H+a)^{-1} - (S_{\rm F}+a)^{-1})^{1/2} = \ker\,(S^*+a).$$

But then also

$$\operatorname{ran}\,((H+a)^{-1} - (S_{\rm F}+a)^{-1}) = \ker\,(S^*+a).$$

Hence by Corollary 1.6 H and $S_{\rm F}$ are transversal. The converse statement is obtained by retracing these steps. This proves the first assertion of the proposition.

Observe that $H = S_{\rm F}$ if and only if $\operatorname{dom} H^{1/2} = \operatorname{dom} S_{\rm F}^{1/2}$, since $S_{\rm F}$ is the only selfadjoint extension of S whose domain is contained in $\operatorname{dom}[S_{\rm F}] = \operatorname{dom} S_{\rm F}^{1/2}$. It follows from (2.11) that $\operatorname{dom} H^{1/2} = \operatorname{dom} S_{\rm F}^{1/2}$ if and only if (2.13) holds. This proves the last assertion of the proposition. ∎

COROLLARY 2.5 ([24]). *Let S be a nonnegative linear relation. Then $S_{\rm N}$ and $S_{\rm F}$ are transversal if and only if*

$$\operatorname{dom} S^* \subset \operatorname{dom} S_{\rm N}^{1/2}. \tag{2.14}$$

Proof. Clearly, if (2.14) holds then S_{F} and S_{N} are transversal, by Proposition 2.4.

Conversely, if S_{N} and S_{F} are transversal, then $\ker\,(S^*+a) \subset \operatorname{dom} S_{\mathrm{N}}^{1/2}$ by Proposition 2.4. If $a>0$, then $-a \in \rho(S_{\mathrm{N}})$, and hence by (1.14)

$$S^* = S_{\mathrm{N}} \,\widehat{+}\, \widehat{\mathfrak{N}}_{-a}(S^*).$$

In particular,

$$\operatorname{dom} S^* = \operatorname{dom} S_{\mathrm{N}} + \ker\,(S^*+a),$$

and, since $\ker\,(S^*+a) \subset \operatorname{dom} S_{\mathrm{N}}^{1/2}$, the inclusion (2.14) follows. ∎

The linear manifold $\ker\,(S^*+a) \cap \operatorname{dom} H^{1/2}$ in Proposition 2.3 is intimately connected with the so-called Kac subclass of the class of Nevanlinna functions, cf. [15], [20]. Observe that the decomposition in (2.11) is orthogonal when $\operatorname{dom} H^{1/2}$ is provided with the graph inner product relative to $H^{1/2}$. In the context of non-densely defined operators (or relations) this result goes back to [9]. Corollary 2.5 is due to M.M. Malamud [24], who gave a proof involving boundary triplets and Kreĭn's formula; see also Proposition 4.4.

3. A FACTORIZATION OF THE KREĬN-VON NEUMANN AND THE FRIEDRICHS EXTENSIONS

Let S be a nonnegative linear relation in a Hilbert space $\mathfrak{H}$. It is not assumed that S is closed or that its domain of definition $\operatorname{dom} S$ is dense in $\mathfrak{H}$. In this section the fundamental factorizations of the Kreĭn-von Neumann and the Friedrichs extensions of S are established.

Provide the linear space $\operatorname{ran} S$ with a semi-inner product $\langle\cdot,\cdot\rangle$ by

$$\langle f', g'\rangle := (f', g) = (f, g'), \quad \{f, f'\}, \{g, g'\} \in S. \tag{3.1}$$

Note that if also $\{f_0, f'\}, \{g_0, g'\} \in S$, then the symmetry of S implies that

$$(f', g) = (f, g') = (f', g_0) = (f_0, g'), \tag{3.2}$$

which shows that the inner product (3.1) is well defined. Define the linear space $\mathfrak{R}_0$ by

$$\mathfrak{R}_0 = \{\, f' \,:\, (f', f) = 0 \quad \text{for some } \{f, f'\} \in S \,\}. \tag{3.3}$$

Note that if $(f', f) = 0$ for $\{f, f'\} \in S$, then also $(f', f_0) = 0$ when $\{f_0, f'\} \in S$, cf. (3.2). In general, the space $\mathfrak{R}_0$ is nontrivial. Clearly the definition implies that $\operatorname{mul} S \subset \mathfrak{R}_0 \subset \operatorname{ran} S$.

LEMMA 3.1. *Let S be a nonnegative relation. Then*

$$\mathfrak{R}_0 = \operatorname{ran} S \cap \operatorname{mul} S^*. \tag{3.4}$$

Proof. Assume that $f' \in \operatorname{ran} S \cap \operatorname{mul} S^*$. Then $\{f, f'\} \in S$ for some $f \in \mathfrak{H}$, and $\{0, f'\} \in S^*$. This implies that $(f', f) = 0$ and therefore $f' \in \mathfrak{R}_0$.

Conversely, assume that $f' \in \mathfrak{R}_0$. Then $(f', f) = 0$ for some $\{f, f'\} \in S$. Clearly with $\{g, g'\} \in S$ the Cauchy-Schwarz inequality (2.1) gives

$$|(f', g)|^2 \leqslant (f', f)(g', g) = 0.$$

Hence $(f', g) = 0$, so that $f' \in (\operatorname{dom} S)^\perp = \operatorname{mul} S^*$. This implies that $f' \in \operatorname{ran} S \cap \operatorname{mul} S^*$. ∎

The quotient space $\operatorname{ran} S / \mathfrak{R}_0$ equipped with the inner product

$$\langle [f'], [g'] \rangle := (f', g) = (f, g'), \quad \{f, f'\}, \{g, g'\} \in S, \tag{3.5}$$

where $[f']$, $[g']$ denote the equivalence classes containing f' and g', is a pre-Hilbert space.

DEFINITION 3.2. The Hilbert space completion of $\operatorname{ran} S / \mathfrak{R}_0$ is denoted by $\mathfrak{H}_S$; its inner product is again denoted by $\langle \cdot, \cdot \rangle$. The linear relation Q from $\mathfrak{H}$ to $\mathfrak{H}_S$ is defined by

$$Q = \{ \{f, [f']\} : \{f, f'\} \in S \}. \tag{3.6}$$

The linear relation J from $\mathfrak{H}_S$ to $\mathfrak{H}$ is defined by

$$J = \{ \{[f'], f'\} : \{f, f'\} \in S \}. \tag{3.7}$$

Note that $\operatorname{dom} Q = \operatorname{dom} S$ and that $\operatorname{mul} Q = \{0\}$, i.e., Q is (the graph of) an operator. To see that $\operatorname{mul} Q = \{0\}$, assume that $f = 0$ in (3.6). Then $\{0, f'\} \in S$ and, by (3.3), clearly $f' \in \mathfrak{R}_0$, which shows that $[f'] = 0$. Moreover, note that J is multivalued with $\operatorname{mul} J = \mathfrak{R}_0$ and that $\operatorname{ran} J = \operatorname{ran} S$. The relation J is densely defined in $\mathfrak{H}_S$ and therefore J^* is an operator. The definitions (3.6) and (3.7) imply that

$$J \subset Q^*, \quad Q \subset J^*, \tag{3.8}$$

as follows from (3.5). In particular, $Q^{**} = \operatorname{clos} Q$ is a restriction of J^*, so that Q^{**}, the closure of the operator Q, is also an operator. Recall that the product relations $J^{**}J^*$ and Q^*Q^{**} are nonnegative and selfadjoint, cf. [17]. Since both J^* and Q^{**} are operators, the associated nonnegative forms are defined on $\operatorname{dom} J^*$ and $\operatorname{dom} Q^{**}$, respectively, cf. [17]. The next theorem extends Proposition 3.1 of [4]; here S need not be densely defined and is even allowed to be a nonnegative (not necessarily closed) relation.

THEOREM 3.3. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (3.6) *and* (3.7)*. Then the Kreĭn-von Neumann extension S_{N} of S is given by $S_{\mathrm{N}} = J^{**}J^*$ and the corresponding closed form $\mathfrak{t}_{\mathrm{N}}$ is given by*

$$\mathfrak{t}_{\mathrm{N}}[f, g] = \langle J^* f, J^* g \rangle, \quad f, g \in \operatorname{dom} J^* = \operatorname{dom} S_{\mathrm{N}}^{1/2}.$$

Furthermore, the Friedrichs extension $S_{\rm F}$ of S is given by $S_{\rm F} = Q^*Q^{**}$ and the corresponding closed form $\mathfrak{t}_{\rm F}$ is given by

$$\mathfrak{t}_{\rm F}[f,g] = \langle Q^{**}f, Q^{**}g\rangle, \quad f,\, g \in \operatorname{dom} Q^{**} = \operatorname{dom} S_{\rm F}^{1/2}.$$

Proof. First consider the case of the Kreĭn-von Neumann extension. Let $\{f,f'\} \in S$. Since $\{f,[f']\} \in Q \subset J^*$ and $\{[f'],f'\} \in J \subset J^{**}$ it follows that $\{f,f'\} \in J^{**}J^*$, i.e. $S \subset J^{**}J^*$. Hence the nonnegative selfadjoint relation $J^{**}J^*$ is an extension of S.

Now let $\{f,f'\} \in J^{**}J^*$. Then $\{f,J^*f\} \in J^*$ and $\{J^*f,f'\} \in J^{**}$. The identity $J^{**} = \operatorname{clos} J$ implies the existence of a sequence $(\{[f_n'],f_n'\}) \subset J$, where $\{f_n,f_n'\} \in S$, such that

$$[f_n'] \to J^*f \text{ in } \mathfrak{H}_S, \quad f_n' \to f' \text{ in } \mathfrak{H}. \tag{3.9}$$

It follows from $\{[f_n'],f_n'\} \in J$ and $\{f,J^*f\} \in J^*$ that

$$\langle [f_n'], J^*f\rangle = (f_n', f). \tag{3.10}$$

Likewise, it follows from $\{f,J^*f\} \in J^*$ and $\{J^*f,f'\} \in J^{**}$ that

$$\langle J^*f, J^*f\rangle = (f, f'). \tag{3.11}$$

Finally note that $\{f_n,f_n'\} \in S \subset J^{**}J^*$ and $\{f,f'\} \in J^{**}J^*$ imply that

$$(f', f_n) = (f, f_n'). \tag{3.12}$$

This leads to the following identity

$$\begin{aligned}\langle [f_n'] - J^*f, [f_n'] - J^*f\rangle &= \langle [f_n'],[f_n']\rangle - \langle [f_n'],J^*f\rangle - \langle J^*f,[f_n']\rangle + \langle J^*f,J^*f\rangle \\ &= (f_n',f_n) - (f_n',f) - (f,f_n') + (f,f') \\ &= (f_n',f_n) - (f_n',f) - (f',f_n) + (f',f) = (f'-f_n', f-f_n),\end{aligned}$$

where (3.10), (3.11), and (3.12) have been used, respectively. Therefore (3.9) implies that

$$f_n' \to f' \text{ in } \mathfrak{H}, \quad (f'-f_n', f-f_n) \to 0.$$

Since $\{f_n,f_n'\} \in S$, this shows that $\{f,g\} \in S_{\rm N}$, cf. Proposition 2.1. Hence, $J^{**}J^* \subset S_{\rm N}$, and since $J^{**}J^*$ and $S_{\rm N}$ are both selfadjoint, the identity $J^{**}J^* = S_{\rm N}$ follows. The statement concerning the associated form $\mathfrak{t}_{\rm N}$ follows from Proposition 5.2 in [17].

Next consider the case of the Friedrichs extension. Let $\{f,f'\} \in S$. Since $\{f,[f']\} \in Q \subset Q^{**}$ and $\{[f'],f'\} \in J \subset Q^*$ it follows that $\{f,f'\} \in Q^*Q^{**}$, i.e. $S \subset Q^*Q^{**}$. Thus the nonnegative selfadjoint relation Q^*Q^{**} is an extension of S.

Now let $\{f,f'\} \in Q^*Q^{**}$. Then $\{f,Q^{**}f\} \in Q^{**}$ and $\{Q^{**}f,f'\} \in Q^*$. The identity $Q^{**} = \operatorname{clos} Q$ implies the existence of a sequence $(\{f_n,[f_n']\}) \subset Q$ where $\{f_n,f_n'\} \in S$, such that

$$f_n \to f \text{ in } \mathfrak{H}, \quad [f_n'] \to Q^{**}f \text{ in } \mathfrak{H}_S. \tag{3.13}$$

It follows from $\{[f_n'], f_n'\} \in J \subset Q^*$, and $\{f, Q^{**}f\} \in Q^{**}$ that

$$\langle [f_n'], Q^{**}f \rangle = (f_n', f). \tag{3.14}$$

Likewise, it follows from $\{f, Q^{**}f\} \in Q^{**}$ and $\{Q^{**}f, f'\} \in Q^*$ that

$$\langle Q^{**}f, Q^{**}f \rangle = (f, f'). \tag{3.15}$$

Finally, note that $\{f_n, f_n'\} \in S \subset Q^*Q^{**}$, and $\{f, f'\} \in Q^*Q^{**}$ imply that

$$(f', f_n) = (f, f_n'). \tag{3.16}$$

This leads to the following identity

$$\begin{aligned} \langle [f_n'] - Q^{**}f, [f_n'] - Q^{**}f \rangle &= \langle [f_n'], [f_n'] \rangle - \langle [f_n'], Q^{**}f \rangle - \langle Q^{**}f, [f_n'] \rangle + \langle Q^{**}f, Q^{**}f \rangle \\ &= (f_n', f_n) - (f_n', f) - (f, f_n') + (f, f') \\ &= (f_n', f_n) - (f_n', f) - (f', f_n) + (f', f) = (f' - f_n', f - f_n), \end{aligned}$$

where (3.14), (3.15), and (3.16) have been used, respectively. Therefore (3.13) implies that

$$f_n \to f \text{ in } \mathfrak{H}, \quad (f' - f_n', f - f_n) \to 0.$$

Since $\{f_n, f_n'\} \in S$, this shows that $\{f, f'\} \in S_{\mathrm{F}}$, cf. Proposition 2.1. Hence, $Q^*Q^{**} \subset S_{\mathrm{F}}$, and since Q^*Q^{**} and S_{F} are both selfadjoint, the identity $Q^*Q^{**} = S_{\mathrm{F}}$ follows. Again the statement concerning the associated form $\mathfrak{t}_{\mathrm{F}}$ follows from Proposition 5.2 in [17]. ∎

The factorizations of the Kreĭn-von Neumann and the Friedrichs extensions in Theorem 3.3, in the case where these extensions are operators, go back to Z. Sebestyén, J. Stochel, and coworkers, see [26], [27], [30], [31], [32]. The case of densely defined nonnegative operators was treated in [4], where also the connection with the corresponding closed nonnegative forms was given. The context of nondensely defined operators and, more generally, of relations requires the introduction of equivalence classes and the space $\mathfrak{R}_0$ in order to define the relation J. However, it is implicit in the construction that the adjoint J^* is (the graph of) an operator. This fact makes it possible to describe the forms associated with the Kreĭn-von Neumann and the Friedrichs extensions in terms of J^* and its restriction $Q^{**} \subset J^*$.

4. SOME CONSEQUENCES OF THE FACTORIZATIONS

The factorizations of the Kreĭn-von Neumann and the Friedrichs extensions in Theorem 3.3 appear to be natural tools to study various questions concerning nonnegative selfadjoint extensions of nonnegative operators or relations. In particular, in this section the disjointness, the transversality, and the equality of the Kreĭn-von Neumann and the Friedrichs extensions are characterized by means of these factorizations. Also criteria for the Kreĭn-von Neumann extension to be an operator will be developed in terms of its factorization.

4.1. SOME GENERAL OBSERVATIONS ABOUT FACTORIZATIONS. The intersection of the Kreĭn-von Neumann extension S_{N} and the Friedrichs extension S_{F} is a closed nonnegative relation which extends S:

$$S \subset S_{\mathrm{N}} \cap S_{\mathrm{F}}. \tag{4.1}$$

Clearly, it follows from (4.1) that

$$S_{\mathrm{N}} \hat{+} S_{\mathrm{F}} \subset \operatorname{clos}(S_{\mathrm{N}} \hat{+} S_{\mathrm{F}}) \subset S^*. \tag{4.2}$$

It will be of interest to characterize the situation when there is equality instead of inclusion in (4.1) and (4.2). Furthermore, the other extremal case, where the Kreĭn-von Neumann extension and the Friedrichs extension coincide, will be characterized.

LEMMA 4.1. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (3.6) *and* (3.7). *Then*

$$S_{\mathrm{N}} \cap S_{\mathrm{F}} = J^{**}Q^{**}, \tag{4.3}$$

and, in particular,

$$\operatorname{mul}(S_{\mathrm{N}} \cap S_{\mathrm{F}}) = \operatorname{mul} J^{**}. \tag{4.4}$$

Proof. It follows from (3.8) that

$$Q \subset Q^{**} \subset J^*, \quad J \subset J^{**} \subset Q^*,$$

and therefore Theorem 3.3 shows that

$$J^{**}Q^{**} \subset Q^*Q^{**} \cap J^{**}J^* = S_{\mathrm{N}} \cap S_{\mathrm{F}}.$$

Thus, $J^{**}Q^{**} \subset S_{\mathrm{N}} \cap S_f$. To prove the reverse inclusion, let $\{f, f'\} \in S_{\mathrm{F}} \cap S_{\mathrm{N}}$. This implies that $\{f, h\} \in Q^{**}$, $\{h, f'\} \in Q^*$ for some $h \in \mathfrak{H}_S$, and that $\{f, k\} \in J^*$, $\{k, f'\} \in J^{**}$ for some $k \in \mathfrak{H}_S$. Observe that $\{f, h\} \in Q^{**} \subset J^*$ and since J^* is an operator, one concludes that $k = h$. Hence, $\{f, f'\} \in J^{**}Q^{**}$.

Since $\operatorname{mul} Q^{**} = \{0\}$, $\operatorname{mul} J^{**}Q^{**} = \operatorname{mul} J^{**}$ and hence the equality (4.4) follows immediately from (4.3). ∎

Closely related to the closed linear relation $J^{**}Q^{**}$ in $\mathfrak{H}$ is the relation Q^*J^*, also in $\mathfrak{H}$, which is not necessarily closed.

LEMMA 4.2. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (3.6) *and* (3.7). *Then*

$$S_{\mathrm{N}} \hat{+} S_{\mathrm{F}} \subset Q^*J^* \subset S^*, \tag{4.5}$$

and, in particular,

$$S \subset (Q^*J^*)^* \subset S_{\mathrm{N}} \cap S_{\mathrm{F}}. \tag{4.6}$$

Proof. Recall that $Q^{**} \subset J^*$ and $J^{**} \subset Q^*$. Hence

$$S_{\rm F} = Q^* Q^{**} \subset Q^* J^*, \quad S_{\rm N} = J^{**} J^* \subset Q^* J^*,$$

so that $S_{\rm N} \,\widehat{+}\, S_{\rm F} \subset Q^* J^*$. Next it is shown that $Q^* J^* \subset S^*$. Assume that $\{f, f'\} \in Q^* J^*$, so that $\{f, \varphi\} \in J^*$ and $\{\varphi, f'\} \in Q^*$ for some $\varphi \in \mathfrak{H}_S$. By definition, for any $\{h, h'\} \in S$ one has

$$\{[h'], h'\} \in J, \quad \{h, [h']\} \in Q,$$

cf. (3.6) and (3.7). This leads to

$$(f, h') = \langle \varphi, [h'] \rangle, \quad (f', h) = \langle \varphi, [h'] \rangle,$$

which implies $(f', h) = (f, h')$, and hence $\{f, f'\} \in S^*$. This completes the proof of (4.5). The inclusions in (4.6) follow by taking adjoints in (4.5). ∎

4.2. DISJOINTNESS AND TRANSVERSALITY OF THE KREĬN-VON NEUMANN AND THE FRIEDRICHS EXTENSIONS. The operator $Q^* J^*$ plays an important role in the description of the disjointness and transversality of $S_{\rm N}$ and $S_{\rm F}$.

PROPOSITION 4.3. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (3.6) *and* (3.7). *Then the following statements are equivalent:*

(i) *$S_{\rm N}$ and $S_{\rm F}$ are disjoint;*
(ii) *$S = J^{**} Q^{**}$;*
(iii) *$S = (Q^* J^*)^*$;*
(iv) *S is closed and $S^* = \operatorname{clos}(Q^* J^*)$.*

Proof. The equivalence between (i) and (ii) is clear from Lemma 4.1. Furthermore, the equivalence between (iii) and (iv) is obvious.

(i) ⇒ (iii) Since $S_{\rm N}$ and $S_{\rm F}$ are disjoint, so that $S_{\rm N} \cap S_{\rm F} = S$, the statement follows from (4.6).

(iii) ⇒ (ii) Assume that $S = (Q^* J^*)^*$, then by (1.10) and by Lemma 4.1 it follows that

$$S = (Q^* J^*)^* \supset J^{**} Q^{**} = S_{\rm N} \cap S_{\rm F} \supset S,$$

so that S is disjoint. ∎

PROPOSITION 4.4. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (3.6) *and* (3.7). *Then the following statements are equivalent:*

(i) *$S_{\rm N}$ and $S_{\rm F}$ are transversal;*
(ii) *$S^* = Q^* J^*$;*
(iii) *$\operatorname{dom} S^* \subset \operatorname{dom} J^*$;*
(iv) *$\ker (S^* + a) \subset \operatorname{dom} J^*$ for some (and hence for all) $a > 0$.*

Proof. (i) ⇒ (ii) Assume that $S_{\rm N}$ and $S_{\rm F}$ are transversal. Then it follows from (4.5) that $S^* = Q^* J^*$, i.e., (ii) is valid.

(ii) ⇒ (iii) & (iii) ⇒ (iv) These implications are trivial.

(iv) ⇒ (i) Assume that $\ker (S^* + a) \subset \operatorname{dom} J^*$. By Theorem 3.3 $\operatorname{dom} J^* = \operatorname{dom} S_{\rm N}^{1/2}$ and the transversality of $S_{\rm N}$ and $S_{\rm F}$ follows from Proposition 2.4. ∎

4.3. EQUALITY OF THE KREĬN-VON NEUMANN AND THE FRIEDRICHS EXTENSIONS. The factorizations can also be used to determine when the Kreĭn-von Neumann and the Friedrichs extensions coincide. In the present context one simple criterion is immediate.

PROPOSITION 4.5. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by* (3.6) *and* (3.7). *Then $S_{\mathrm{N}} = S_{\mathrm{F}}$ if and only if $J^* = Q^{**}$.*

Proof. If $J^* = Q^{**}$, then $J^{**} = Q^*$, so that $J^{**}J^* = Q^*Q^{**}$, which is equivalent to $S_{\mathrm{F}} = S_{\mathrm{N}}$ by Theorem 3.3.

Conversely, assume that $S_{\mathrm{N}} = S_{\mathrm{F}}$, or equivalently, $J^{**}J^* = Q^*Q^{**}$. Then $\operatorname{dom} J^* = \operatorname{dom} Q^{**}$ and since $Q^{**} \subset J^*$ and J^* is an operator, one obtains that $J^* = Q^{**}$. ∎

In order to obtain an analytical uniqueness criterion the following observation is useful.

PROPOSITION 4.6. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$. Then*

$$\operatorname{dom} J^* = \{\, g \in \mathfrak{H} : |(f',g)|^2 \leqslant C_g(f',f) \text{ for all } \{f,f'\} \in S \text{ and some } C_g < \infty \,\}.$$

Proof. Recall that J is the relation from $\mathfrak{H}_S$ to $\mathfrak{H}$ defined in (3.7). Apply the description (1.1) in Lemma 1.1 to J to obtain a description of $\operatorname{dom} J^*$: $g \in \operatorname{dom} J^*$ if and only if $g \in \mathfrak{H}$ and there exists a nonnegative number γ_g such that

$$|(f',g)| \leqslant \gamma_g \|[f']\|_{\mathfrak{H}_S} \quad \text{for all } \{[f'],f'\} \in J. \tag{4.7}$$

By (3.7) and (3.5) the estimate in (4.7) can be rewritten as

$$|(f',g)|^2 \leqslant \gamma_g^2 (f',f) \quad \text{for all } \{f,f'\} \in S.$$

This gives the description of $\operatorname{dom} J^*$. ∎

The description of $\operatorname{dom} J^*$ leads to the uniqueness criterion due to Kreĭn [23].

THEOREM 4.7 (Kreĭn's uniqueness criterion). *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$. Then $S_{\mathrm{N}} = S_{\mathrm{F}}$ if and only if for some (and hence for all) $a > 0$*

$$\sup \left\{ \frac{|(f,\varphi)|^2}{(f',f)} : \{f,f'\} \in S \right\} = \infty \quad \text{for all } \varphi \in \ker\,(S^*+a) \setminus \{0\}. \tag{4.8}$$

Proof. By Proposition 2.4 with $H = S_{\mathrm{N}}$, one has $S_{\mathrm{N}} = S_{\mathrm{F}}$ if and only if

$$\ker\,(S^*+a) \cap \operatorname{dom} S_{\mathrm{N}}^{1/2} = \{0\}.$$

In other words $S_{\mathrm{N}} = S_{\mathrm{F}}$ if and only if

$$\varphi \in \ker\,(S^*+a) \setminus \{0\} \Rightarrow \varphi \notin \operatorname{dom} S_{\mathrm{N}}^{1/2}.$$

By Proposition 4.6 and Theorem 3.3 it follows that $\varphi \in \operatorname{dom} S_{\mathrm{N}}^{1/2} = \operatorname{dom} J^*$ if and only if there is a nonnegative number C_φ such that

$$|(f',\varphi)|^2 \leqslant C_\varphi (f',f) \quad \text{for all } \{f,f'\} \in S.$$

Since $\varphi \in \ker\,(S^* + a)$ it is clear that $(f', \varphi) = a(f, \varphi)$. Hence, $\varphi \in \operatorname{dom} S_{\mathrm{N}}^{1/2}$ if and only if

$$|(f,\varphi)|^2 \leqslant \frac{C_\varphi}{a}(f',f) \quad \text{for all } \{f,f'\} \in S,$$

and this leads to the description (4.8). ∎

The description of the form domain $\operatorname{dom} \mathfrak{t}_{\mathrm{N}} = \operatorname{dom} J^*$ in Theorem 3.3 is made explicit in Proposition 4.6. This description coincides with a result of Ando and Nishio [1] for the class of positively closable operators. Kreĭn's uniqueness criterion was originally stated for the case of densely defined operators; its formulation for nonnegative relations can be found in [16].

4.4. POSITIVE CLOSABILITY. The Friedrichs extension S_{F} is an operator if and only if S is densely defined. In this case, each nonnegative selfadjoint extension H of S is an operator. However, when S is not densely defined, there may be nonnegative selfadjoint extensions of S which are operators, in which case S is automatically an operator. In what follows, a relation in a Hilbert space $\mathfrak{H}$ is called *closable* if its closure in the Cartesian product $\mathfrak{H} \times \mathfrak{H}$ is (the graph of) an operator. Of course, a closable relation is itself already an operator.

PROPOSITION 4.8. *Let S be a nonnegative relation. Then*

$$\operatorname{mul} S_{\mathrm{N}} = \operatorname{mul} J^{**} \subset \overline{\operatorname{ran}}\, S \cap \operatorname{mul} S^*. \tag{4.9}$$

Furthermore, the following statements are equivalent:

(i) *S_{N} is an operator;*

(ii) *S has a nonnegative selfadjoint operator extension;*

(iii) *the relation J is closable;*

(iv) *$S_{\mathrm{N}} \cap S_{\mathrm{F}}$ is an operator.*

In this case S is a nonnegative operator.

Proof. By Theorem 3.3 $S_{\mathrm{N}} = J^{**}J^*$, which gives the identity in (4.9). For the inclusion in (4.9), recall that $J \subset Q^*$ and $\operatorname{dom} Q = \operatorname{dom} S$, so that

$$\operatorname{mul} J^{**} \subset \operatorname{mul} Q^* = \operatorname{mul} S^*.$$

Furthermore, observe that $\operatorname{mul} J^{**} \subset \operatorname{ran} J^{**} \subset \overline{\operatorname{ran}}\, J$ and $\operatorname{ran} J = \operatorname{ran} S$, so that

$$\operatorname{mul} J^{**} \subset \overline{\operatorname{ran}}\, S.$$

Hence also the inclusion in (4.9) has been proved.

(i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (i) Let H be a nonnegative selfadjoint operator extension of S (so that also S is an operator), then it follows from (0.7) that S_{N} is an operator.

(i) $\Leftrightarrow$ (iii) According to (4.9) S_{N} is an operator if and only if $\operatorname{mul} J^{**}$ is trivial.

(i) $\Leftrightarrow$ (iv) This follows immediately from (4.4), which leads to $\operatorname{mul}(S_{\mathrm{N}} \cap S_{\mathrm{F}}) = \operatorname{mul} S_{\mathrm{N}}$. ∎

COROLLARY 4.9. *Let S be a nonnegative relation in $\mathfrak{H}$. Then*

$$\operatorname{mul} S_{\mathrm{N}} = \{\, f' \in \mathfrak{H} : \{f_n, f_n'\} \in S,\ f_n' \to f',\ (f_n', f_n) \to 0 \,\}. \tag{4.10}$$

In particular, S_{N} is an operator if and only if

$$\{f_n, f_n'\} \in S,\ \lim_{n\to\infty} (f_n', f_n) = 0, \text{ and } \lim_{n\to\infty} f_n' = f' \text{ imply } f' = 0. \tag{4.11}$$

Proof. By Proposition 4.8 one has $\operatorname{mul} S_{\mathrm{N}} = \operatorname{mul} J^{**}$ and hence the definitions of $\mathfrak{H}_S$ and J, see (3.1), (3.3), (3.7), show that the multivalued part of the closure J^{**} of J is given by (4.10). Therefore, S_{N} is an operator if and only if (4.11) is satisfied. This completes the proof. ∎

A nonnegative relation S is said to be *positively closable* if it satisfies the property (4.11). In this case, S is automatically closable (as $S \subset S_{\mathrm{N}} = S_{\mathrm{N}}^*$) and thus an operator. Hence the Kreĭn-von Neumann extension S_{N} is an operator if and only if S is positively closable.

COROLLARY 4.10. *Let S be a nonnegative relation in $\mathfrak{H}$. Then:*
(i) *if* $\operatorname{ran} S$ *is closed, then* $S_{\mathrm{N}} = S \,\widehat{+}\, (\ker S^* \times \{0\})$ *and* $\operatorname{mul} S_{\mathrm{N}} = \operatorname{mul} J$;
(ii) *if* $m(S) > 0$, *then* $\operatorname{mul} S_{\mathrm{N}} = \operatorname{mul} \operatorname{clos} S$.

Proof. (i) Since $\operatorname{ran} S$ is closed it follows from (2.6) that S_{N} has the form (2.5). By Proposition 4.8 and (3.4) one has

$$\operatorname{mul} S_{\mathrm{N}} \subset \overline{\operatorname{ran}}\, S \cap \operatorname{mul} S^* = \operatorname{ran} S \cap \operatorname{mul} S^* = \operatorname{mul} J,$$

and, since $\operatorname{mul} J \subset \operatorname{mul} S_{\mathrm{N}}$, this gives $\operatorname{mul} S_{\mathrm{N}} = \operatorname{mul} J$.

(ii) The inclusion $\operatorname{mul} \operatorname{clos} S \subset \operatorname{mul} S_{\mathrm{N}}$ is clear. The converse follows from (4.10) since $(f_n', f_n) \geqslant m(S) \|f_n\|^2$ for $\{f_n, f_n'\} \in S$. ∎

In other words, if $\operatorname{ran} S$ is closed, then S is positively closable if and only if $\operatorname{mul} J = \{0\}$, and if $m(S) > 0$, then S is positively closable if and only if S is closable.

PROPOSITION 4.11. *Let S be a nonnegative operator. Then:*
(i) *if S_{N} and S_{F} are disjoint, then S_{N} is an operator;*
(ii) *if S_{N} is an operator and if S satisfies the property* (2.8), *then S_{N} and S_{F} are disjoint. In particular, if S is a nonnegative closed bounded operator, then S_{N} is an operator if and only if S_{N} and S_{F} are disjoint.*

Proof. (i) Assume that $S_{\mathrm{N}} \cap S_{\mathrm{F}} = S$. Let $\{0, \varphi\} \in S_{\mathrm{N}}$. Then clearly $\{0, \varphi\} \in S_{\mathrm{F}}$, so that $\{0, \varphi\} \in S_{\mathrm{N}} \cap S_{\mathrm{F}} \subset S$. Therefore $\varphi = 0$, and S_{N} is an operator.

(ii) Let S be a nonnegative operator which satisfies the property (2.8) and let S_{N} be an operator. Assume that $\{f, f'\} \in S_{\mathrm{N}} \cap S_{\mathrm{F}}$. Since $\{f, f'\} \in S_{\mathrm{F}}$, it follows from (2.7) that $\{f, f'\} \in S \,\widehat{+}\, (\{0\} \times \operatorname{mul} S^*)$. Hence $\{f, f'\} = \{h, Sh\} \,\widehat{+}\, \{0, \varphi\}$ with $h \in \operatorname{dom} S$ and $\varphi \in \operatorname{mul} S^*$. Therefore $\{0, \varphi\} \in S_{\mathrm{N}} \cap S_{\mathrm{F}}$ and since S_{N} is an operator, one concludes that $\varphi = 0$. Hence $\{f, f'\} = \{h, Sh\} \in S$, which shows $S_{\mathrm{N}} \cap S_{\mathrm{F}} \subset S$.

If S is a bounded closed nonnegative operator, then $\operatorname{dom} S$ is closed, so that the condition (2.8) is satisfied. ∎

Observe that the conditions in (ii) of Proposition 4.11 automatically imply that S is closed. Furthermore, if S_{N} is an operator and the condition (2.8) is not satisfied, then S_{N} and S_{F} need not be disjoint: there are densely defined nonnegative nonselfadjoint operators S for which S_{N} and S_{F} coincide, cf. Theorem 4.7.

Now the problem arises to determine, parallel to Proposition 4.8, the nonnegative linear relations S whose Kreĭn-von Neumann extension S_{N} is a bounded operator.

PROPOSITION 4.12. *Let S be a nonnegative relation. Then the following statements are equivalent:*

(i) *S_{N} belongs to $[\mathfrak{H}]$;*
(ii) *S has at least one nonnegative selfadjoint extension in $[\mathfrak{H}]$;*
(iii) *$\|f'\|^2 \leqslant M(f',f)$ for all $\{f,f'\} \in S$ and for some $M \geqslant 0$;*
(iv) *J is a bounded operator.*

In this case S is a nonnegative bounded operator.

Proof. (i) ⇒ (ii) This implication is clear.

(ii) ⇒ (iii) Let H be a nonnegative operator in $[\mathfrak{H}]$ which extends S. Then clearly $\|H^{1/2}(H^{1/2}f)\| \leqslant \|H^{1/2}\|\,\|H^{1/2}f\|$, so that

$$\|Hf\|^2 \leqslant \|H\|\,(Hf,f), \quad f \in \operatorname{dom} H.$$

Restricting f to $\operatorname{dom} S$ one gets the inequality in (iii) with $M = \|H\|$.

(iii) ⇒ (iv) In view of (3.3) one has $\mathfrak{R}_0 = \{0\}$ and hence the relation J is (the graph of) an operator. The boundedness of J is also a consequence of the inequality.

(iv) ⇒ (i) Let J be a bounded operator. Then its closure J^{**} is a bounded operator with the same norm. Furthermore, since J is densely defined, the bounded operator J^{**} is defined everywhere. Hence $S_{\mathrm{N}} = J^{**}J^* \in [\mathfrak{H}]$. ∎

Furthermore the following proposition is parallel to Proposition 4.11.

PROPOSITION 4.13. *Let S be a bounded nonnegative operator. Then $S_{\mathrm{N}} \in [\mathfrak{H}]$ if and only if S_{N} and S_{F} are transversal.*

Proof. Since the operator S is bounded, it is closable, i.e., $\operatorname{mul}\operatorname{clos} S = \{0\}$. Hence $\operatorname{dom} S^* = (\operatorname{mul}\operatorname{clos} S)^{\perp} = \mathfrak{H}$.

Assume that $S_{\mathrm{N}} \in [\mathfrak{H}]$. Then J is a bounded operator, densely defined, so that J^* is a bounded operator, cf. Proposition 4.12. Therefore $\operatorname{dom} S^* = \mathfrak{H} = \operatorname{dom} J^*$. Hence S_{N} and S_{F} are transversal by Proposition 4.4.

Conversely, assume that S_{N} and S_{F} are transversal. Hence $\mathfrak{H} = \operatorname{dom} S^* \subset \operatorname{dom} J^*$, which implies that J^* is a bounded operator. Therefore J is a bounded operator, and hence $S_{\mathrm{N}} \in [\mathfrak{H}]$. ∎

The notion of positive closability, introduced by Ando and Nishio [1] for closed nonnegative operators, is inherent in the representation of the Kreĭn-von Neumann extension S_{N} in Theorem 3.3. The connection between the identities (4.10) and (4.11) was first noticed in [16]. The inequality in part (iii) of Proposition 4.12 guarantees that S is an operator. Hence it can be rewritten as $\|Sf\|^2 \leqslant M(Sf,f)$, $f \in \operatorname{dom} S$, and in this form it goes back to Ando and Nishio [1]. As the proof of Proposition 4.12 shows, this inequality implies that the mapping J from $\mathfrak{H}_S$ to $\mathfrak{H}$ is bounded with the norm $\|J\| \leqslant \sqrt{M}$, cf. [19]. As to Propositions 4.11 and 4.13, see also the treatment in Derkach and Malamud [11].

5. A FACTORIZATION OF THE EXTREMAL EXTENSIONS OF NONNEGATIVE RELATIONS

In this section the characterization of the extremal extensions of an arbitrary nonnegative relation in terms of factorizations is established. This extends the treatment for densely defined nonnegative operators in [4]. In Section 6 the present results will be made explicit when the nonnegative symmetric relation is actually a bounded nonnegative operator.

Let S be a nonnegative relation in $\mathfrak{H}$ and let $\mathfrak{L}$ be any subspace such that

$$\operatorname{dom} S \subset \mathfrak{L} \subset \operatorname{dom} J^* = \operatorname{dom} S_{\mathrm{N}}^{1/2}. \tag{5.1}$$

Associate with $\mathfrak{L}$ the restriction operator $R_{\mathfrak{L}}$ from $\mathfrak{H}$ to $\mathfrak{H}_S$ by

$$R_{\mathfrak{L}} := J^* \upharpoonright \mathfrak{L} = \{\, \{f,f'\} \in J^* \,:\, f \in \mathfrak{L} \,\}. \tag{5.2}$$

Since J^* is a closed operator from $\mathfrak{H}$ to $\mathfrak{H}_S$, it is clear that $R_{\mathfrak{L}}$ is a closable operator. The definition of $R_{\mathfrak{L}}$ and Theorem 3.3 lead to

$$\langle R_{\mathfrak{L}}f, R_{\mathfrak{L}}g\rangle = \langle J^*f, J^*g\rangle = \mathfrak{t}_{\mathrm{N}}[f,g], \quad f,\, g \in \mathfrak{L}.$$

Hence, $R_{\mathfrak{L}}$ is closed if and only if the restriction of the form $\mathfrak{t}_{\mathrm{N}}[\cdot,\cdot]$ to $\mathfrak{L}$ is closed, cf. [22]. Clearly, operators of the form $R_{\mathfrak{L}}$ induce nonnegative selfadjoint relations $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ and corresponding closed nonnegative forms $\mathfrak{t}_{\mathfrak{L}}$ defined by

$$\mathfrak{t}_{\mathfrak{L}}[f,g] = \langle R_{\mathfrak{L}}^{**}f, R_{\mathfrak{L}}^{**}g\rangle \quad f,g \in \operatorname{dom} R_{\mathfrak{L}}^{**}.$$

Assume that the linear, not necessarily closed, subspaces $\mathfrak{L}$ and $\mathfrak{M}$ satisfy

$$\operatorname{dom} S \subset \mathfrak{L} \subset \mathfrak{M} \subset \operatorname{dom} S_{\mathrm{N}}^{1/2}.$$

Then the closed forms induced by $\mathfrak{L}$ and $\mathfrak{M}$ satisfy the inclusion $\mathfrak{t}_{\mathfrak{L}} \subset \mathfrak{t}_{\mathfrak{M}}$. In particular, $\mathfrak{t}_{\mathfrak{L}} \geqslant \mathfrak{t}_{\mathfrak{M}}$ which leads to the following monotonicity property (cf. [4], [17]):

$$R_{\mathfrak{M}}^* R_{\mathfrak{M}}^{**} \leqslant R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}.$$

The nonnegative selfadjoint extensions of S which are extremal can be characterized in terms of nonnegative selfadjoint factorizations $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ induced by the operators $R_{\mathfrak{L}}$ in (5.2).

THEOREM 5.1. *Let S be a nonnegative linear relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ *for some $\mathfrak{L}$ such that* $\operatorname{dom} S \subset \mathfrak{L} \subset \operatorname{dom} S_{\rm N}^{1/2}$;

(ii) $\widetilde{A}$ *is a nonnegative selfadjoint extremal extension of S;*

(iii) $\widetilde{A}$ *is a nonnegative selfadjoint extension of S whose associated closed form $\widetilde{\mathfrak{t}}$ satisfies $\widetilde{\mathfrak{t}} \subset \mathfrak{t}_{\rm N}$.*

Proof. (i) $\Rightarrow$ (ii) Let $\{f,f'\} \in S$. Since $\{f,[f']\} \in Q \subset J^* \upharpoonright \mathfrak{L} = R_{\mathfrak{L}} \subset R_{\mathfrak{L}}^{**}$ and $\{[f'],f'\} \in J \subset J^{**} \subset R_{\mathfrak{L}}^*$, it follows that $\{f,f'\} \in R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$, i.e., $S \subset R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$. Hence, the nonnegative selfadjoint relation $\widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ is an extension of S.

Let $\{f,f'\} \in \widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$. Then $\{f,J^*f\} \in R_{\mathfrak{L}}^{**}$, and $\{J^*f,f'\} \in R_{\mathfrak{L}}^*$. Therefore,

$$(f',f) = \langle J^*f, J^*f \rangle. \tag{5.3}$$

Let $\{h,h'\} \in S$. It follows from $\{J^*f,f'\} \in R_{\mathfrak{L}}^*$ and $\{h,[h']\} \in Q \subset R_{\mathfrak{L}}$, that

$$(f',h) = \langle J^*f, [h'] \rangle. \tag{5.4}$$

Furthermore, it follows from $\{f,J^*f\} \in J^*$ and $\{[h'],h'\} \in J$, that

$$(h',f) = \langle [h'], J^*f \rangle. \tag{5.5}$$

Finally, note that by definition,

$$(h',h) = \langle [h'], [h'] \rangle. \tag{5.6}$$

The identities (5.3)–(5.6) lead to

$$\begin{aligned} &(f'-h',f-h) \\ &\quad = (f',f)-(f',h)-(h',f)+(h',h) \\ &\quad = \langle J^*f,J^*f\rangle - \langle J^*f,[h']\rangle - \langle [h'],J^*f\rangle + \langle [h'],[h']\rangle = \|J^*f-[h']\|_{\mathfrak{H}_S}^2. \end{aligned} \tag{5.7}$$

The assumption $\{f,f'\} \in \widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ implies that $f \in \operatorname{dom} J^*$, so that $J^*f \in \mathfrak{H}_S$. By definition, $\mathfrak{H}_S$ is the completion of $\operatorname{ran} S/\mathfrak{N}_0$ with respect to the norm $\|\cdot\|_{\mathfrak{H}_S}$. Therefore,

$$\inf\{\|J^*f-[h']\|_{\mathfrak{H}_S} : \{h,h'\} \in S\} = 0,$$

so that (5.7) leads to

$$\inf\{\, (f'-h',f-h) : \{h,h'\} \in S \,\} = 0.$$

This shows that the nonnegative selfadjoint extension $\widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ is extremal.

(ii) $\Rightarrow$ (iii) Assume that $\widetilde{A}$ is a nonnegative selfadjoint extension of S and let $\widetilde{\mathfrak{t}}$ be the associated nonnegative closed form. Observe that the inequality $S_{\rm N} \leqslant \widetilde{A}$ is equivalent to $\mathfrak{t}_{\rm N} \leqslant \widetilde{\mathfrak{t}}$, in other words

$$\operatorname{dom} \widetilde{\mathfrak{t}} \subset \operatorname{dom} \mathfrak{t}_{\rm N}, \quad \mathfrak{t}_{\rm N}[f,f] \leqslant \widetilde{\mathfrak{t}}[f,f], \quad f \in \operatorname{dom} \widetilde{\mathfrak{t}}, \tag{5.8}$$

cf. Theorem 4.3 of [17]. Let $\{f,f'\} \in \widetilde{A}$ and $\{h,h'\} \in S$. It follows from

$$\operatorname{dom} \widetilde{A} \subset \operatorname{dom} \widetilde{A}^{1/2} = \operatorname{dom} \widetilde{\mathfrak{t}} \subset \operatorname{dom} \mathfrak{t}_{\rm N} = \operatorname{dom} J^*,$$

that $\{f, J^*f\} \in J^*$. Together with $\{[h'], h'\} \in J$ this leads to

$$(h', f) = \langle [h'], J^*f \rangle. \tag{5.9}$$

Furthermore, by definition (see (3.1))

$$(h', h) = \langle [h'], [h'] \rangle. \tag{5.10}$$

Finally, note that $\{h, h'\}$, $\{f, f'\} \in \widetilde{A}$ imply that

$$(h', f) = (h, f'). \tag{5.11}$$

The identities (5.9)–(5.11) show that for all $\{f, f'\} \in \widetilde{A}$ and $\{h, h'\} \in S$

$$(f' - h', f - h) - \|J^*f - [h']\|^2_{\mathfrak{H}_S} = (f', f) - \langle J^*f, J^*f \rangle. \tag{5.12}$$

Now assume in addition that the nonnegative selfadjoint extension $\widetilde{A}$ of S is extremal. Let $\{f, f'\} \in \widetilde{A}$ and let $\varepsilon > 0$. By (0.10) there exists an element $\{h, h'\} \in S$ such that

$$(f' - h', f - h) < \varepsilon. \tag{5.13}$$

Clearly $\{f - h, f' - h'\} \in \widetilde{A}$ and by (5.8) it follows that

$$\mathfrak{t}_{\mathrm{N}}[f - h, f - h] \leqslant \widetilde{\mathfrak{t}}[f - h, f - h] = (f' - h', f - h). \tag{5.14}$$

Recall that $\{h, h'\} \in S$ implies that $\{h, [h']\} \in Q \subset J^*$, so that according to Theorem 3.3

$$\mathfrak{t}_{\mathrm{N}}[f - h, f - h] = \langle J^*(f - h), J^*(f - h) \rangle = \|J^*f - [h']\|^2_{\mathfrak{H}_S}. \tag{5.15}$$

A combination of (5.13), (5.14), and (5.15) leads to

$$0 \leqslant (f' - h', f - h) - \|J^*f - [h']\|^2_{\mathfrak{H}_S} < \varepsilon. \tag{5.16}$$

Now combine the inequality (5.16) with the identity (5.12) to obtain

$$0 \leqslant (f', f) - \langle J^*f, J^*f \rangle < \varepsilon,$$

where $\{f, f'\} \in \widetilde{A}$ and $\varepsilon > 0$ is arbitrary. Hence

$$(f', f) = \langle J^*f, J^*f \rangle = \mathfrak{t}_{\mathrm{N}}[f, f],$$

for all $\{f, f'\} \in \widetilde{A}$, and, by polarization,

$$(f', g) = \mathfrak{t}_{\mathrm{N}}[f, g], \quad \{f, f'\}, \{g, g'\} \in \widetilde{A}.$$

Therefore the restriction to $\operatorname{dom} \widetilde{A}$ of the form $\widetilde{\mathfrak{t}}$ is a restriction of the form $\mathfrak{t}_{\mathrm{N}}$. Since the form $\mathfrak{t}_{\mathrm{N}}$ is closed, the inclusion $\widetilde{\mathfrak{t}} \subset \mathfrak{t}_{\mathrm{N}}$ holds too.

(iii) $\Rightarrow$ (i) Assume that $\widetilde{A}$ is a nonnegative selfadjoint extension of S such that the closed form $\widetilde{\mathfrak{t}}$ associated to $\widetilde{A}$ satisfies the inclusion $\widetilde{\mathfrak{t}} \subset \mathfrak{t}_{\mathrm{N}}$. Define the subspace $\mathfrak{L}$ by $\mathfrak{L} = \operatorname{dom} \widetilde{\mathfrak{t}}$. Then $\mathfrak{L}$ satisfies the relation (5.1). Let $R_{\mathfrak{L}}$ be the operator given by (5.2). Then

$$\widetilde{\mathfrak{t}}[f, g] = \mathfrak{t}_{\mathrm{N}}[f, g] = \langle J^* \restriction_{\mathfrak{L}} f, J^* \restriction_{\mathfrak{L}} g \rangle = \langle R_{\mathfrak{L}} f, R_{\mathfrak{L}} g \rangle = \langle R_{\mathfrak{L}}^{**} f, R_{\mathfrak{L}}^{**} g \rangle, \quad f, g \in \mathfrak{L},$$

and thus the linear relation $\widetilde{A}$ coincides with $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$, cf. [22], and Proposition 5.2 of [17]. ∎

COROLLARY 5.2. *There is a one-to-one correspondence between the closed restrictions* $\widetilde{\mathfrak{t}}$ *of* $\mathfrak{t}_{\mathrm{N}}$ *with* $\operatorname{dom} S \subset \operatorname{dom} \widetilde{\mathfrak{t}}$*, i.e.,*

$$\widetilde{\mathfrak{t}}[f,g] = \langle R_{\mathfrak{L}} f, R_{\mathfrak{L}} g \rangle, \quad f,g \in \mathfrak{L} = \operatorname{dom} \widetilde{\mathfrak{t}},$$

and the extremal nonnegative selfadjoint extensions $\widetilde{A}$ *of* S*, given by*

$$\widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}, \quad \mathfrak{L} = \operatorname{dom} \widetilde{A}^{1/2}.$$

Proof. Let $\widetilde{\mathfrak{t}}$ be a closed restriction of the form $\mathfrak{t}_{\mathrm{N}}$ with $\operatorname{dom} S \subset \operatorname{dom} \widetilde{\mathfrak{t}} =: \mathfrak{L}$. Then

$$\widetilde{\mathfrak{t}}[f,g] = \mathfrak{t}_{\mathrm{N}}[f,g] = \langle J^* f, J^* g \rangle = \langle R_{\mathfrak{L}} f, R_{\mathfrak{L}} g \rangle, \quad f, g \in \mathfrak{L},$$

and the closedness of the form $\widetilde{\mathfrak{t}}$ implies that the operator $R_{\mathfrak{L}}$ is closed. Thus, $R_{\mathfrak{L}} = R_{\mathfrak{L}}^{**}$ and by Theorem 5.1 $\widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}$ is an extremal extension of S with $\mathfrak{L} = \operatorname{dom} \widetilde{A}^{1/2}$.

The mapping $\widetilde{\mathfrak{t}} \to \widetilde{A}$ is surjective, since if $\widetilde{A}$ is an extremal nonnegative selfadjoint extension of S, then with $\mathfrak{L} = \operatorname{dom} \widetilde{A}^{1/2}$ one has $\operatorname{dom} R_{\mathfrak{L}} = \mathfrak{L} = \operatorname{dom} \widetilde{\mathfrak{t}} = \operatorname{dom} R_{\mathfrak{L}}^{**}$ and $R_{\mathfrak{L}}^{**} = R_{\mathfrak{L}}$. Moreover, $\widetilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}$ and $\widetilde{\mathfrak{t}}[f,g] = \langle R_{\mathfrak{L}} f, R_{\mathfrak{L}} g \rangle$ is closed since $R_{\mathfrak{L}}$ is closed.

To see that the mapping is injective, let $\widetilde{\mathfrak{t}}_1$ and $\widetilde{\mathfrak{t}}_2$ be closed restrictions of $\mathfrak{t}_{\mathrm{N}}$ such that $\operatorname{dom} S \subset \operatorname{dom} \widetilde{\mathfrak{t}}_i$, $i = 1,2$, for which the corresponding selfadjoint extensions coincide, i.e.,

$$R_{\mathfrak{L}_1}^* R_{\mathfrak{L}_1} = R_{\mathfrak{L}_2}^* R_{\mathfrak{L}_2}.$$

This implies that $\operatorname{dom} \widetilde{\mathfrak{t}}_1 = \operatorname{dom} \widetilde{\mathfrak{t}}_2$ and since $\widetilde{\mathfrak{t}}_i \subset \mathfrak{t}_{\mathrm{N}}$, $i = 1,2$, the equality $\widetilde{\mathfrak{t}}_1 = \widetilde{\mathfrak{t}}_2$ follows. ∎

The one-to-one correspondence in Corollary 5.2 is between the extremal nonnegative selfadjoint extensions $\widetilde{A}$ of S and the closed restrictions $\widetilde{\mathfrak{t}}$ of the form $\mathfrak{t}_{\mathrm{N}}$ to the subspaces $\mathfrak{L}\,(= \operatorname{dom} \widetilde{A}^{1/2})$ which satisfy

$$\operatorname{dom} S \subset \mathfrak{L} \subset \operatorname{dom} S_{\mathrm{N}}^{1/2}$$

or, equivalently,

$$\operatorname{dom} S_{\mathrm{F}}^{1/2} \subset \mathfrak{L} \subset \operatorname{dom} S_{\mathrm{N}}^{1/2},$$

and which are closed subspaces in the form topology of $\mathfrak{t}_{\mathrm{N}}$. If, for instance, $\mathfrak{L}$ is any subspace satisfying

$$\operatorname{dom} S \subset \mathfrak{L} \subset \operatorname{dom} S_{\mathrm{F}}^{1/2}$$

and $R_{\mathfrak{L}} = J^* \upharpoonright \mathfrak{L}$, then $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**} = S_{\mathrm{F}}$, since $R_{\mathfrak{L}}$ has the same closure as $Q = J^* \upharpoonright \operatorname{dom} S$. In particular, the choice $\mathfrak{L} = \operatorname{dom} S_{\mathrm{F}}^{1/2}$, $R_{\mathfrak{L}} = J^* \upharpoonright \operatorname{dom} S_{\mathrm{F}}^{1/2}$, gives the closure of the operator Q.

The next result reflects some further similar facts, cf. Proposition 4.5 of [4].

THEOREM 5.3. *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$, let $\widetilde{A}$ be a nonnegative selfadjoint extension of S, and let $\mathfrak{L} := \operatorname{dom} \widetilde{A}$. Then:*

(i) $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**} \leqslant \widetilde{A}$;

(ii) *if H is an extremal nonnegative selfadjoint extension of S, such that*

$$R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**} \leqslant H \leqslant \widetilde{A}, \tag{5.17}$$

then $H = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$;

(iii) $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**} = \widetilde{A}$ *if and only if $\widetilde{A}$ is an extremal nonnegative selfadjoint extension of S.*

Proof. (i) Let $\widetilde{A}_{\mathfrak{L}} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$, let $\widetilde{\mathfrak{t}}_{\mathfrak{L}}$ be the closed form corresponding to the selfadjoint relation $\widetilde{A}_{\mathfrak{L}}$, and let $\widetilde{\mathfrak{t}}$ be the closed form corresponding to $\widetilde{A}$. Then $\operatorname{dom} \widetilde{A} = \mathfrak{L} \subset \operatorname{dom} \widetilde{\mathfrak{t}}_{\mathfrak{L}}$ and the inequality $S_{\mathrm{N}} \leqslant \widetilde{A}$ leads to

$$\widetilde{\mathfrak{t}}_{\mathfrak{L}}[f,f] = \|R_{\mathfrak{L}}^{**} f\|_{\mathfrak{H}_S}^2 = \mathfrak{t}_{\mathrm{N}}[f,f] \leqslant \widetilde{\mathfrak{t}}[f,f], \quad f \in \operatorname{dom} \widetilde{A}. \tag{5.18}$$

Therefore $\widetilde{A}_{\mathfrak{L}} \leqslant \widetilde{A}$ by Theorem 4.3 of [17].

(ii) Let H be an extremal extension of S such that (5.17) holds and let $\widetilde{\mathfrak{t}}_H$ be the closed form associated to H. Then $\operatorname{dom} \widetilde{A} \subset \operatorname{dom} \widetilde{\mathfrak{t}}_H$ and according to Theorem 5.1 one has $\widetilde{\mathfrak{t}}_H \subset \mathfrak{t}_{\mathrm{N}}$. Therefore (5.18) gives

$$\widetilde{\mathfrak{t}}_{\mathfrak{L}}[f,g] = \widetilde{\mathfrak{t}}_H[f,g], \quad f, g \in \operatorname{dom} \widetilde{A}.$$

Since $\mathfrak{L} = \operatorname{dom} \widetilde{A}$ is a core for both $\widetilde{\mathfrak{t}}_{\mathfrak{L}}$ and $\widetilde{\mathfrak{t}}_H$, the equality $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**} = H$ follows.

(iii) Let $\widetilde{A}$ be an extremal extension of S. Then (i) and (ii) with $H = \widetilde{A}$ imply that $R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**} = \widetilde{A}$. The converse statement is clear by Theorem 5.1. ∎

In the context of closed sectorial relations the equivalence (ii) ⇔ (iii) in Theorem 5.1 goes back to Arlinskiĭ [3]. The factorization of the extremal extensions of S in Theorem 5.1 leads to an explicit representation of the closed forms associated with the extremal extensions of S along the lines of the densely defined case in [4].

6. THE CASE OF BOUNDED NONNEGATIVE OPERATORS

The constructions and the results in Section 3 and Section 5 can be made more explicit if the underlying nonnegative relation S is a bounded operator. In particular, this leads to an interpretation of the completion problem (0.1) as an extension problem as announced in the introduction.

6.1. SOME GENERAL REMARKS. Let S be a closed bounded nonnegative operator in the Hilbert space $\mathfrak{H}$. Decompose the Hilbert space $\mathfrak{H}$ as $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, where $\operatorname{dom} S = \mathfrak{H}_1$. Then the operator S has the block decomposition

$$S = \begin{pmatrix} S_{11} \\ S_{21} \end{pmatrix} : \mathfrak{H}_1 \to \begin{pmatrix} \mathfrak{H}_1 \\ \mathfrak{H}_2 \end{pmatrix}, \tag{6.1}$$

where $S_{11} \in [\mathfrak{H}_1]$ is nonnegative and $S_{21} \in [\mathfrak{H}_1, \mathfrak{H}_2]$. By Lemma 1.2 the adjoint of S in $\mathfrak{H}$ is the closed linear relation given by

$$S^* = \left\{ \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} S_{11}h_1 + S_{21}^* h_2 \\ \beta \end{pmatrix} \right\} : h_1 \in \mathfrak{H}_1,\, h_2,\, \beta \in \mathfrak{H}_2 \right\}, \tag{6.2}$$

and, in particular, it follows from (6.2) that

$$\operatorname{mul} S^* = \{0\} \oplus \mathfrak{H}_2.$$

The results from the earlier sections will be completely described in terms of the operators S_{11} and S_{21}. Since the operator S is nonnegative, it follows from Lemma 3.1 that the space $\mathfrak{R}_0$ in (3.3) is given by

$$\mathfrak{R}_0 = S(\ker\, S_{11}) = \{0\} \oplus \operatorname{ran}(S_{21} \upharpoonright \ker\, S_{11}).$$

The linear operator Q from $\mathfrak{H}$ to $\mathfrak{H}_S$ in (3.6) is given by

$$Q = \left\{ \left\{ \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \left[\begin{pmatrix} S_{11}h_1 \\ S_{21}h_1 \end{pmatrix} \right] \right\} : h_1 \in \mathfrak{H}_1 \right\}, \tag{6.3}$$

and the linear relation J from $\mathfrak{H}_S$ to $\mathfrak{H}$ in (3.7) is given by

$$J = \left\{ \left\{ \left[\begin{pmatrix} S_{11}h_1 \\ S_{21}h_1 \end{pmatrix} \right], \begin{pmatrix} S_{11}h_1 \\ S_{21}h_1 \end{pmatrix} \right\} : h_1 \in \mathfrak{H}_1 \right\}, \tag{6.4}$$

so that $\operatorname{mul} J = \mathfrak{R}_0 = \{0\} \oplus \operatorname{ran}(S_{21} \upharpoonright \ker\, S_{11})$. Introduce the linear spaces

$$\mathfrak{M}_0 := \operatorname{ran} S_{11}^{1/2}, \quad \mathfrak{M} := \overline{\operatorname{ran}}\, S_{11}^{1/2} = \overline{\operatorname{ran}}\, S_{11},$$

so that $\mathfrak{M}$ is a closed subspace of $\mathfrak{H}_1$ (with the original topology). Define the linear relation T_0 from $\mathfrak{M}$ to $\mathfrak{H}_S$ by

$$T_0 = \left\{ \left\{ S_{11}^{1/2} h_1, \left[\begin{pmatrix} S_{11}h_1 \\ S_{21}h_1 \end{pmatrix} \right] \right\} : h_1 \in \mathfrak{H}_1 \right\},$$

so that

$$\|S_{11}^{1/2} h_1\| = \left\| \left[\begin{pmatrix} S_{11}h_1 \\ S_{21}h_1 \end{pmatrix} \right] \right\|_{\mathfrak{H}_S}, \quad h_1 \in \mathfrak{H}_1,$$

where the norm in the righthand side is induced by the inner product (3.5). Thus, the relation T_0 is isometric from $\mathfrak{M}_0$ onto $\operatorname{ran} S/\mathfrak{R}_0$; and, in fact, T_0 is an operator. Hence, the closure T of T_0 is a closed isometric operator from the Hilbert space $\mathfrak{M}$ onto the Hilbert space $\mathfrak{H}_S$. Define the linear operator Q_1 from $\mathfrak{H}$ to $\mathfrak{M}$ by

$$Q_1 = \left\{ \left\{ \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, S_{11}^{1/2} h_1 \right\} : h_1 \in \mathfrak{H}_1 \right\}, \tag{6.5}$$

so that Q_1 is not densely defined in $\mathfrak{H}$; in fact, $\operatorname{dom} Q_1 = \mathfrak{H}_1$. Define the linear relation J_1 from $\mathfrak{M}$ to $\mathfrak{H}$ by

$$J_1 = \left\{ \left\{ S_{11}^{1/2} h_1, \begin{pmatrix} S_{11}h_1 \\ S_{21}h_1 \end{pmatrix} \right\} : h_1 \in \mathfrak{H}_1 \right\}, \tag{6.6}$$

so that $\mathrm{mul}\, J_1 = \{0\} \oplus \mathrm{ran}\,(S_{21} \restriction \ker S_{11})$. Comparison of the definitions (6.3) and (6.5) shows that the linear operators Q_1 and Q are connected by

$$Q_1 = T^* Q,$$

and comparison of the definitions (6.4) and (6.6) shows that the linear relations J_1 and J are connected by

$$J_1 = JT.$$

The adjoints of Q and J have to be taken in terms of the Hilbert spaces $\mathfrak{H}_S$ and $\mathfrak{H}$. However, the relations Q_1 and J_1 are in the original Hilbert space and, hence, so are their adjoints.

LEMMA 6.1. *The operator Q_1 in* (6.5) *is a closed operator, so that $Q_1^{**} = Q_1$. The adjoint Q_1^* of Q_1 is given by*

$$Q_1^* = \left\{ \left\{ g_1, \begin{pmatrix} S_{11}^{1/2} g_1 \\ g_2 \end{pmatrix} \right\} : g_1 \in \mathfrak{M},\, g_2 \in \mathfrak{H}_2 \right\}, \tag{6.7}$$

and $\mathrm{mul}\, Q_1^* = \mathfrak{H}_2$.

Proof. Apply Lemma 1.2 to (1.4) with $\mathfrak{K}_1 = \mathfrak{H}_1$, $\mathfrak{K}_2 = \mathfrak{H}_2$, $\mathfrak{H}_1 = \mathfrak{M}$, $\mathfrak{Y} \subset \mathfrak{H}_2 = \emptyset$, $D = \{0, 0\}$, and $B = S_{11}^{1/2}$. Then clearly $D^* = \mathfrak{M} \times \mathfrak{H}_2$ and (1.7) leads to (6.7). ∎

Define the linear relation W from $\mathfrak{M}$ to $\mathfrak{H}_2$ as follows

$$W = \{ \{S_{11}^{1/2} g, S_{21} g\} : g \in \mathfrak{H}_1 \},$$

so that W is densely defined and $\{0\} \oplus \mathrm{mul}\, W = \mathfrak{N}_0$. The adjoint $V = W^*$ is a closed operator from $\mathfrak{H}_2$ to $\mathfrak{M}$ given by

$$V = \{ \{h, k\} : S_{11}^{1/2} k = S_{21}^* h \}.$$

The definition of $\mathfrak{M}$ shows directly that

$$\mathrm{dom}\, V = \{ h : S_{21}^* h \in \mathrm{ran}\, S_{11}^{1/2} \}, \tag{6.8}$$

and $S_{11}^{1/2} V h = S_{21}^* h$, $h \in \mathrm{dom}\, V$. In fact, the operator V is given by

$$V = S_{11}^{(-1/2)} S_{21}^*,$$

where $S_{11}^{(-1/2)}$ is the *pseudo-inverse* of $S_{11}^{1/2}$ defined as the operator that assigns to an element in $\mathrm{ran}\, S_{11}^{1/2}$ its uniquely defined original in $\mathfrak{H}_1 \ominus \ker S_{11} = \overline{\mathrm{ran}}\, S_{11} = \mathfrak{M}$. Observe that

$$S_{11}^{1/2} V = S_{21}^* \restriction \mathrm{dom}\, V. \tag{6.9}$$

The identity $V^* = W^{**}$ shows that W is closable if and only if V is densely defined. It is clear from (6.6) that J_1 can be rewritten in the form

$$J_1 = \left\{ \left\{ h_1, \begin{pmatrix} S_{11}^{1/2} h_1 \\ k_1 \end{pmatrix} \right\} : \{h_1, k_1\} \in W \right\}. \tag{6.10}$$

LEMMA 6.2. *Let the relation J_1 from $\mathfrak{M}$ to $\mathfrak{H}$ be defined by* (6.6)*. Then J_1^* is an operator from $\mathfrak{H}$ to $\mathfrak{M}$ given by*

$$J_1^* = \left\{ \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, S_{11}^{1/2} g_1 + V g_2, \right\} : g_1 \in \mathfrak{H}_1, \, g_2 \in \operatorname{dom} V \right\}. \tag{6.11}$$

Its adjoint relation J_1^{**} from $\mathfrak{M}$ to $\mathfrak{H}$ is given by

$$J_1^{**} = \left\{ \left\{ \alpha, \begin{pmatrix} S_{11}^{1/2}\alpha \\ \beta \end{pmatrix} \right\} : \{\alpha, \beta\} \in V^* \right\}, \tag{6.12}$$

and $\operatorname{mul} J_1^{**} = \operatorname{mul} V^*$.

Proof. Apply Lemma 1.2 with $\mathfrak{X} \subset \mathfrak{H}_2 = \emptyset$ and $\mathfrak{H} = \mathfrak{H}_1$ to obtain (6.11) from (6.10). In fact, (6.12) is a direct consequence of (6.10). The description of $\operatorname{mul} J_1^{**}$ is a consequence of (6.12), see also (1.9). ∎

With the above identification of the spaces $\mathfrak{H}_S$ and $\mathfrak{M}$ under the isometry T, it is possible to translate the results involving Q and J in terms of Q_1 and J_1. It will be helpful to use the following notations

$$Q_1 = \begin{pmatrix} S_{11}^{1/2} & O \end{pmatrix}, \quad Q_1^* = \begin{pmatrix} S_{11}^{1/2} \\ O^* \end{pmatrix},$$

where O stands for the trivial linear relation from $\mathfrak{H}_2$ to $\mathfrak{M}$, so that $O^* = \mathfrak{M} \times \mathfrak{H}_2$, and

$$J_1^* = \begin{pmatrix} S_{11}^{1/2} & V \end{pmatrix}, \quad J_1^{**} = \begin{pmatrix} S_{11}^{1/2} \\ V^* \end{pmatrix}.$$

6.2. THE KREĬN-VON NEUMANN AND THE FRIEDRICHS EXTENSIONS. The following result is a straightforward translation of Theorem 3.3.

THEOREM 6.3. *Let S in* (6.1) *be a closed bounded nonnegative operator in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ with* $\operatorname{dom} S = \mathfrak{H}_1$*. Then the Kreĭn-von Neumann extension $S_{\rm N}$ of S is given by*

$$S_{\rm N} = \begin{pmatrix} S_{11}^{1/2} \\ V^* \end{pmatrix} \begin{pmatrix} S_{11}^{1/2} & V \end{pmatrix}, \tag{6.13}$$

and the corresponding closed form $\mathfrak{t}_{\rm N}$ is given by

$$\mathfrak{t}_{\rm N} \left[\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right] = \| S_{11}^{1/2} h_1 + V h_2 \|^2, \quad h_1 \in \mathfrak{H}_1, \, h_2 \in \operatorname{dom} V. \tag{6.14}$$

Furthermore, the Friedrichs extension $S_{\rm F}$ of S is given by

$$S_{\rm F} = \begin{pmatrix} S_{11}^{1/2} \\ O^* \end{pmatrix} \begin{pmatrix} S_{11}^{1/2} & O \end{pmatrix}, \tag{6.15}$$

and the corresponding closed form $\mathfrak{t}_{\rm F}$ is given by

$$\mathfrak{t}_{\rm F} \left[\begin{pmatrix} h_1 \\ 0 \end{pmatrix} \right] = \| S_{11}^{1/2} h_1 \|^2, \quad h_1 \in \mathfrak{H}_1. \tag{6.16}$$

Proof. First consider the Kreĭn-von Neumann extension. Lemma 1.3 and (6.6) lead to

$$J_1^* = T^* J^*, \quad J_1^{**} = J^{**} T. \tag{6.17}$$

By multiplying the relations in (6.17) it follows that

$$J_1^{**} J_1^* = J^{**} T T^* J^* = J^{**} J^*,$$

and hence $J_1^{**} J_1^* = J^{**} J^* = S_{\mathrm{N}}$ by Theorem 3.3. Hence (6.14) is clear, see Proposition 5.2 in [17].

Now consider the Friedrichs extension. Lemma 1.3 and (6.5) lead to

$$Q_1^* = Q^* T, \quad Q_1^{**} = T^* Q^{**}. \tag{6.18}$$

By multiplying the relations in (6.18) one obtains

$$Q_1^* Q_1^{**} = Q^* T T^* Q^{**} = Q^* Q^{**},$$

and hence $Q_1^* Q_1^{**} = Q^* Q^{**} = S_{\mathrm{F}}$ by Theorem 3.3. Hence (6.16) is clear, see Proposition 5.2 in [17]. ∎

The factorizations in (6.13) and (6.15) may be rewritten more explicitly. It follows from (6.13) that

$$\begin{aligned} S_{\mathrm{N}} = \Bigg\{ \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} S_{11} h_1 + S_{11}^{1/2} V h_2 \\ \beta \end{pmatrix} \right\} : \\ h_1 \in \mathfrak{H}_1,\, h_2 \in \operatorname{dom} V,\, \{ S_{11}^{1/2} h_1 + V h_2, \beta \} \in V^* \Bigg\}. \end{aligned} \tag{6.19}$$

Observe that according to (6.9) one has $S_{11}^{1/2} V \subset S_{21}^*$, which shows that the righthand side of (6.19) is indeed a restriction of the righthand side of (6.2). Furthermore, note that it also follows from (6.9) that $\{ S_{11}^{1/2} h_1, S_{21} h_1 \} \in V^*$. Hence S_{N} may be also written as

$$S_{\mathrm{N}} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & V^* V \end{pmatrix}.$$

It follows from (6.15) that

$$S_{\mathrm{F}} = \left\{ \left\{ \begin{pmatrix} h_1 \\ 0 \end{pmatrix}, \begin{pmatrix} S_{11} h_1 \\ \varphi \end{pmatrix} \right\} : h_1 \in \mathfrak{H}_1,\, \varphi \in \mathfrak{H}_2 \right\}.$$

As a consequence of (6.13) one obtains that

$$\operatorname{mul} S_{\mathrm{N}} = \{0\} \oplus \operatorname{mul} V^*.$$

Hence the following corollary is straightforward.

COROLLARY 6.4. *Let S in* (6.1) *be a bounded nonnegative operator in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ with* $\operatorname{dom} S = \mathfrak{H}_1$. *Then the following conditions are equivalent:*

(i) *S_{N} is an operator;*

(ii) *W is a closable operator from $\mathfrak{M}$ to $\mathfrak{H}_2$;*

(iii) *V is a densely defined operator from $\mathfrak{H}_2$ to $\mathfrak{M}$.*

Furthermore, the following conditions are equivalent:

(iv) $S_{\rm N} \in [\mathfrak{H}]$;
(v) $W \in [\mathfrak{M}, \mathfrak{H}_2]$;
(vi) $V \in [\mathfrak{H}_2, \mathfrak{M}]$.

The conditions (i), (ii), and (iii) are equivalent to $S_{\rm N}$ and $S_{\rm F}$ being disjoint, cf. Proposition in 4.11. The conditions (iv), (v), and (vi) are equivalent to $S_{\rm N}$ and $S_{\rm F}$ being transversal, cf. Proposition in 4.13.

COROLLARY 6.5. *Let S in* (6.1) *be a bounded nonnegative operator in $\mathfrak{H}$. Then the following statements are equivalent:*
(i) $S_{\rm F} = S_{\rm N}$, *i.e. S has a unique nonnegative selfadjoint extension;*
(ii) $\operatorname{dom} V = \{0\}$;
(iii) $\operatorname{ran} S_{11}^{1/2} \cap \operatorname{ran} S_{21}^* = \{0\}$ *and* $\ker S_{21}^* = \{0\}$.

Proof. (i) $\Leftrightarrow$ (ii) Comparing (6.14) and (6.16) in Theorem 6.3 it is clear that $S_{\rm N} = S_{\rm F}$ if and only if $\operatorname{dom} V = \{0\}$.
(ii) $\Leftrightarrow$ (iii) This follows from (6.8). ∎

The equivalence of the items (i), (ii), and (iii) in Corollary 6.4 can be found in [16] under the technical condition that $\ker S_{11} = \{0\}$. Items (iv), (v), and (vi) in Corollary 6.4 go back to [1]. The condition (vi) is equivalent to the condition $\operatorname{ran} S_{21}^* \subset \operatorname{ran} S_{11}^{1/2}$.

6.3. EXTREMAL EXTENSIONS. In order to consider the extremal extensions of S, let $\mathfrak{L}_2$ be any (not necessarily closed) subspace of $\mathfrak{H}_2$ such that $\mathfrak{L}_2 \subset \operatorname{dom} V$, let $R_{\mathfrak{L}}$ be the operator from $\mathfrak{H}$ to $\mathfrak{H}_S$ given by

$$R_{\mathfrak{L}} = J^* \upharpoonright \mathfrak{H}_1 \oplus \mathfrak{L}_2, \tag{6.20}$$

and let $R_{1\mathfrak{L}}$ be the operator from $\mathfrak{H}$ to $\mathfrak{M}$, defined by

$$R_{1\mathfrak{L}} = \left\{ \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, S_{11}^{1/2} g_1 + V g_2, \right\} : g_1 \in \mathfrak{H}_1, g_2 \in \mathfrak{L}_2 \right\}. \tag{6.21}$$

Comparison of the definitions (6.20) and (6.21) shows that the linear operators $R_{1\mathfrak{L}}$ and $R_{\mathfrak{L}}$ are connected by $R_{1\mathfrak{L}} = T^* R_{\mathfrak{L}}$, so that

$$R_{1\mathfrak{L}}^* R_{1\mathfrak{L}}^{**} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}. \tag{6.22}$$

The following notation turns out to be useful: the restriction $V \upharpoonright \mathfrak{L}_2$ will be denoted by $V_{\mathfrak{L}_2}$.

LEMMA 6.6. *Let the operator $R_{1\mathfrak{L}}$ from $\mathfrak{H}$ to $\mathfrak{M}$ be defined by* (6.21). *Then the relation $R_{1\mathfrak{L}}^*$ from $\mathfrak{M}$ to $\mathfrak{H}$ is given by*

$$R_{1\mathfrak{L}}^* = \left\{ \left\{ \alpha, \begin{pmatrix} S_{11}^{1/2}\alpha \\ \beta \end{pmatrix} \right\} : \{\alpha, \beta\} \in V_{\mathfrak{L}_2}^* \right\}, \tag{6.23}$$

and the operator $R_{1\mathfrak{L}}^{**}$ from $\mathfrak{H}$ to $\mathfrak{M}$ is given by

$$R_{1\mathfrak{L}}^{**} = \left\{ \left\{ \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, S_{11}^{1/2} g_1 + V_{\mathfrak{L}_2}^{**} g_2, \right\} : g_1 \in \mathfrak{H}_1,\ g_2 \in \operatorname{dom} V_{\mathfrak{L}_2}^{**} \right\}. \tag{6.24}$$

Proof. Again the statements in (6.23) and (6.24) are obtained from Lemma 1.2. Observe that $V_{\mathfrak{L}_2}^{**}$ is an operator as a restriction of the closed operator V. The identity (6.24) follows also directly from (6.21). ∎

Now, taking into account (6.22) and Lemma 6.6, a characterization of all nonnegative extremal extensions of S can be easily obtained via Theorem 5.1. It will be helpful to use the following notation:

$$R_{1\mathfrak{L}}^{*} = \begin{pmatrix} S_{11}^{1/2} \\ V_{\mathfrak{L}_2}^{*} \end{pmatrix}, \quad R_{1\mathfrak{L}}^{**} = \begin{pmatrix} S_{11}^{1/2} & V_{\mathfrak{L}_2}^{**} \end{pmatrix}.$$

PROPOSITION 6.7. *Let S in* (6.1) *be a bounded nonnegative operator in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ with* $\operatorname{dom} S = \mathfrak{H}_1$ *and let $\widetilde{A}$ be a nonnegative selfadjoint extension of S. Then the following statements are equivalent:*

(i) *$\widetilde{A}$ is an extremal extension of S;*

(ii) *$\widetilde{A}$ is of the following form, with a subspace $\mathfrak{L}_2$ of $\mathfrak{H}_2$, $\mathfrak{L}_2 \subset \operatorname{dom} V$:*

$$\widetilde{A} = \begin{pmatrix} S_{11}^{1/2} \\ V_{\mathfrak{L}_2}^{*} \end{pmatrix} \begin{pmatrix} S_{11}^{1/2} & V_{\mathfrak{L}_2}^{**} \end{pmatrix},$$

(iii) *the closed form $\widetilde{\mathfrak{t}}$ associated to $\widetilde{A}$ is given by*

$$\widetilde{\mathfrak{t}}\left[\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right] = \|S_{11}^{1/2} h_1 + V_{\mathfrak{L}_2}^{**} h_2\|^2, \quad h_1 \in \mathfrak{H}_1,\ h_2 \in \operatorname{dom} V_{\mathfrak{L}_2}^{**}, \tag{6.25}$$

with a subspace $\mathfrak{L}_2$ of $\mathfrak{H}_2$, $\mathfrak{L}_2 \subset \operatorname{dom} V$.
Furthermore, (6.25) *establishes a one-to-one correspondence between the extremal nonnegative selfadjoint extensions $\widetilde{A}$ of S and the closed restrictions $V_{\mathfrak{L}_2}$ of V, or equivalently, those subspaces $\mathfrak{L}_2 \subset \operatorname{dom} V$ for which the restriction $V_{\mathfrak{L}_2}$ is a closed operator.*

Proof. The equivalence of the statements (i)–(iii) follows immediately from Theorem 5.1 and Lemma 6.6. As to the last statement observe, that $\widetilde{\mathfrak{t}}$ in (6.25) is closed if and only if the block operator $\begin{pmatrix} S_{11}^{1/2} & V_{\mathfrak{L}_2} \end{pmatrix}$ is closed. Since here $S_{11}^{1/2}$ is a closed bounded operator, this block operator is closed precisely when $V_{\mathfrak{L}_2}$ is closed. ∎

Again, the factorization in Proposition 6.7 may be rewritten explicitly. The form of S^* in (6.2) leads to the following representation:

$$\widetilde{A} = \left\{ \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} S_{11} h_1 + S_{11}^{1/2} V_{\mathfrak{L}_2}^{**} h_2 \\ \beta \end{pmatrix} \right\} : \right.$$
$$\left. h_1 \in \mathfrak{H}_1,\ h_2 \in \operatorname{dom} V_{\mathfrak{L}_2}^{**},\ \{S_{11}^{1/2} h_1 + V_{\mathfrak{L}_2}^{**} h_2, \beta\} \in V_{\mathfrak{L}_2}^{*} \right\}.$$

It is an immediate consequence of (6.9) that $\{S_{11}^{1/2}h_1, S_{21}h_1\} \in V_{\mathfrak{L}_2}^*$. Therefore also $\{V_{\mathfrak{L}_2}^{**}h_2, \beta - S_{21}h_1\} \in V_{\mathfrak{L}_2}^*$. Hence $\widetilde{A}$ may also be rewritten as

$$\widetilde{A} = \begin{pmatrix} S_{11} & S_{21}^* \\ S_{21} & V_{\mathfrak{L}_2}^* V_{\mathfrak{L}_2}^{**} \end{pmatrix}.$$

Note that $\operatorname{mul} V_{\mathfrak{L}_2}^* V_{\mathfrak{L}_2}^{**} = \operatorname{mul} V_{\mathfrak{L}_2}^* = \mathfrak{H}_2 \ominus (\operatorname{clos} \mathfrak{L}_2)$.

REMARK 6.8. Clearly, Proposition 6.7 implies that an extremal extension $\widetilde{A}$ of the bounded nonnegative operator S is an operator if and only if

$$\overline{\operatorname{dom}}\, V_{\mathfrak{L}_2} = \overline{\operatorname{dom}}\, V_{\mathfrak{L}_2}^{**} = \mathfrak{H}_2.$$

Now assume that $\widetilde{A} \in [\mathfrak{H}]$ is a nonnegative extremal extension of S. Then, in particular, $S_{\rm N} \in [\mathfrak{H}]$ by Proposition 4.12. Furthermore, $\operatorname{dom} V_{\mathfrak{L}_2}^{**} = \mathfrak{H}_2$ and, since $V_{\mathfrak{L}_2}^{**}$ is a closed restriction of the closed operator V, the equality $V_{\mathfrak{L}_2}^{**} = V$ follows, so that $\widetilde{A} = S_{\rm N}$. Hence, if $S_{\rm N} \in [\mathfrak{H}]$, then $S_{\rm N}$ is the only nonnegative extremal selfadjoint extension of S which belongs to $[\mathfrak{H}]$.

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REFERENCES

[1] T. ANDO, K. NISHIO, Positive selfadjoint extensions of positive symmetric operators, *Tôhoku Math. J.* **22**(1970), 65–75.

[2] R. ARENS, Operational calculus of linear relations, *Pacific J. Math.* **9**(1961), 9-23.

[3] Y.M. ARLINSKIĬ, Extremal extensions of sectorial linear relations, *Math. Stud.* **7**(1997), 81–96.

[4] Y.M. ARLINSKIĬ, S. HASSI, Z. SEBESTYÉN, H.S.V. DE SNOO, On the class of extremal extensions of a nonnegative operator, in *Recent Advances in Operator Theory and Related Topics (Szeged, 1999)*, Oper. Theory Adv. Appl., vol. 127, Birkhäuser, Basel 2001, pp. 41–81.

[5] Y.M. ARLINSKIĬ, E.R. TSEKANOVSKIĬ, Quasi selfadjoint contractive extensions of Hermitian contractions, *Teor. Funktsii Funktsional Anal. i Prilozhen* **50**(1988), 9–16.

[6] Y.M. ARLINSKIĬ, E.R. TSEKANOVSKIĬ, The von Neumann problem for nonnegative symmetric operators, *Integral Equations Operator Theory* **51**(2005), 319–356.

[7] G. ARSENE, A. GHEONDEA, Completing matrix contractions, *J. Operator Theory* **7**(1982), 179–189.

[8] E.A. CODDINGTON, Extension theory of formally normal and symmetric subspaces, *Mem. Amer. Math. Soc.* **134**(1973).

[9] E.A. CODDINGTON, H.S.V. DE SNOO, Positive selfadjoint extensions of positive symmetric subspaces, *Math. Z.* **159**(1978), 203–214.

[10] V.A. DERKACH, S. HASSI, M.M. MALAMUD, H.S.V. DE SNOO, Boundary relations and their Weyl families, *Trans. Amer. Math. Soc.* **358**(2006), 5351–5400.

[11] V.A. DERKACH, M.M. MALAMUD, The extension theory of Hermitian operators and the moment problem, *J. Math. Sci.* **73**(1995), 1–95.

[12] P. FILLMORE, J. WILLIAMS, On operator ranges, *Adv. Math.* **7**(1971), 254–281.

[13] P.R. HALMOS, Subnormal suboperators and the discrete topology, in *Anniversary Volume on Approximation Theory and Functional Analysis (Oberwolfach, 1983)*, Internat. Schriftenreihe Numer. Math., vol. 65, Birkhäuser, Basel 1984, pp. 49–65.

[14] S. HASSI, On the Friedrichs and the Kreĭn-von Neumann extension of nonnegative relations, in *Contributions to Management Science, Mathematics and Modelling*, Acta Wasaensia, vol. 122, Vaasan Yliopisto, Vaasa 2004, pp. 37–54.

[15] S. HASSI, M. KALTENBÄCK, H.S.V. DE SNOO, Triplets of Hilbert spaces and Friedrichs extensions associated with the subclass $\mathbf{N}_1$ of Nevanlinna functions, *J. Operator Theory* **37**(1997), 155–181.

[16] S. HASSI, M.M. MALAMUD, H.S.V. DE SNOO, On Kreĭn's extension theory of nonnegative operators, *Math. Nachr.* **274/275**(2004), 40–73.

[17] S. HASSI, A. SANDOVICI, H.S.V. DE SNOO, H. WINKLER, Form sums of nonnegative selfadjoint operators, *Acta Math. Hungar.* **111**(2006), 81–105.

[18] S. HASSI, A. SANDOVICI, H.S.V. DE SNOO, H. WINKLER, Extremal extensions for the sum of nonnegative selfadjoint relations, *Proc. Amer. Math. Soc.* **135**(2007), 3193–3204.

[19] S. HASSI, Z. SEBESTYÉN, H.S.V. DE SNOO, On the nonnegativity of operator products, *Acta Math. Hungar.* **109**(2005), 1–15.

[20] S. HASSI, H.S.V. DE SNOO, One-dimensional graph perturbations of selfadjoint relations, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **22**(1997), 123–164.

[21] S. HASSI, H.S.V. DE SNOO, F.H. SZAFRANIEC, Normal extensions of symmetric relations, in preparation.

[22] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin 1980.

[23] M.G. KREĬN, Theory of selfadjoint extensions of semibounded operators and its applications. I, II, *Mat. Sb.* **20**, **21**(1947), 431–495, 365–404.

[24] M.M. MALAMUD, Some classes of extensions of a Hermitian operator with gaps [Russian], *Ukrain. Mat. Zh.* **44**(1992), 215–233; English *Ukrainian Math. J.* **44**(1992), 190–204.

[25] M.M. MALAMUD, On a formula of the generalized resolvents of a nondensely defined Hermitian operator [Russian], *Ukrain. Mat. Zh.* **44**(1992), 1658–1688; English *Ukrainian Math. J.* **44**(1992), 1522–1547.

[26] V. PROKAJ, Z. SEBESTYÉN, On Friedrichs extensions of operators, *Acta Sci. Math. (Szeged)* **62**(1996), 243–246.

[27] V. PROKAJ, Z. SEBESTYÉN, On extremal positive operator extensions, *Acta Sci. Math. (Szeged)* **62**(1996), 458–491.

[28] Z. SEBESTYÉN, Restrictions of positive operators, *Acta Sci. Math. (Szeged)* **46**(1983), 299–301.

[29] Z. SEBESTYÉN, Operator extensions on Hilbert space, *Acta Sci. Math. (Szeged)* **57**(1993), 233–248.

[30] Z. SEBESTYÉN, E. SIKOLYA, On Kreĭn-von Neumann and Friedrichs extensions, *Acta Sci. Math. (Szeged)* **69**(2003), 323–336.

[31] Z. SEBESTYÉN, J. STOCHEL, Restrictions of positive self-adjoint operators, *Acta Sci. Math. (Szeged)* **55**(1991), 149–154.

[32] Z. SEBESTYÉN, J. STOCHEL, On products of unbounded operators, *Acta Math. Hungar.* **100**(2003), 105–129.

[33] Y.L. SHMUL'YAN, A Hellinger operator integral, *Mat. Sb.* **49**(1959), 381–430.

[34] Y.L. SHMUL'YAN, Two-sided division in a ring of operators, *Math. Notes* **1**(1967), 400–403.

[35] Y.L. SHMUL'YAN, R.N. YANOVSKAYA, On matrices whose entries are contractions, *Izv. Vissh. Ucheb. Zaved. Matematica* **7**(1981), 72–75.

[36] O.G. STOROZH, Extremal extensions of a nonnegative operator and accretive boundary problems, *Ukrainian Math. J.* **42**(1990), 857–860.

[37] A.V. ŠTRAUS, On extensions of semibounded operators, *Dokl. Akad. Nauk SSSR* **211**(1973), 543–546.

[38] A.V. ŠTRAUS, On the theory of extremal extensions of a bounded positive operator, *Funkts. Anal.* **18**(1982), 115–126.

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