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# NON COMMUTATIVE SPHERES ASSOCIATED WITH THE HEXIC TRANSFORM AND THEIR K-THEORY

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ABSTRACT. Let $A_\theta$ be the rotation $C^*$-algebra generated by unitaries $U, V$ satisfying $VU = e^{2\pi i\theta}UV$ and let $\rho$ denote the hexic transform on $A_\theta$ defined by $\rho(U) = V$, $\rho(V) = e^{-\pi i\theta}U^{-1}V$. (It is the canonical order six automorphism of $A_\theta$.) It is shown that ten canonical classes in $K_0(A_\theta \rtimes_\rho \mathbb{Z}_6) \cong \mathbb{Z}^{10}$ yield a basis. The Connes-Chern character $K_0(A_\theta \rtimes_\rho \mathbb{Z}_6) \to H^{\mathrm{ev}}(A_\theta \rtimes_\rho \mathbb{Z}_6)^*$ is shown to be injective for each $\theta$, and its range is determined.

KEYWORDS: *$C^*$-algebras, K-theory, automorphisms, rotation algebras, unbounded traces, Chern characters.*

MSC (2000): 46L80, 46L40, 19K14.

## 1. INTRODUCTION

For $0 < \theta < 1$ let $A_\theta$ denote the rotation $C^*$-algebra generated by unitaries $U, V$ satisfying $VU = \lambda UV$, where $\lambda := e^{2\pi i\theta}$. Denote by $\rho$ the order six automorphism of $A_\theta$ defined by

$$\rho(U) = V, \quad \rho(V) = e^{-\pi i\theta}U^{-1}V.$$

We shall call it the *hexic* transform in accordance with our papers [3] and [15]. Throughout the paper, we shall denote the associated crossed product by $H_\theta := A_\theta \rtimes_\rho \mathbb{Z}_6$, where $\mathbb{Z}_6 = \mathbb{Z}/6\mathbb{Z}$, and call it the *hexic* $C^*$-algebra. It is the universal $C^*$-algebra generated by unitaries $U, V, W$ enjoying the commutation relations

$$(1.1) \qquad VU = \lambda UV, \quad WUW^{-1} = V, \quad WVW^{-1} = \lambda^{-1/2}U^{-1}V, \quad W^6 = I.$$

We shall also use $A_\theta$ to denote its canonical smooth dense $*$-subalgebra under the canonical toral action, and by $H_\theta$ the dense $*$-subalgebra of elements of the form $\sum_{j=0}^{5} a_j W^j$ where $a_j$ are smooth elements in $A_\theta$, and $W$ is the canonical order six unitary of the crossed product implementing $\rho$; so, $\rho(a) = WaW^{-1}$. (This identification is justified since both the $C^*$-algebra and its smooth $*$-subalgebra have the same $K$-theory, since the dense $*$-subalgebras are closed under the holomorphic functional calculus, and since it will be clear from the context which algebra is intended.)

In [3], we constructed ten canonical modules over $H_\theta$ and showed (using theta functions) that they give rise to independent positive classes in $K_0(H_\theta)$ for each $\theta$ (rational or irrational). (These modules are listed in Table 1 below.) This was done by examination of the Connes-Chern character $\mathrm{ch} : K_0(H_\theta) \to H^{\mathrm{ev}}(H_\theta)^*$ where $H^{\mathrm{ev}}(H_\theta)$ is Connes' even periodic cyclic cohomology group and $H^{\mathrm{ev}}(H_\theta)^*$ is its vector space dual ([5], III). (We prefer to view the codomain of ch as above instead of the usual cyclic homology group so as to readily use Connes' canonical pairing between $K_0$ and cyclic cohomology.) From ch a group homomorphism $\mathbf{T} : K_0(H_\theta) \to \mathbb{R}^{10}$ can be defined by taking the Connes-Chern character $\mathrm{ch}(x)$ of each element $x$ in $K_0(H_\theta)$ and restricting it to a certain 10-dimensional subspace of $H^{\mathrm{ev}}(H_\theta)$ spanned by the unbounded traces on the (smooth) algebra $H_\theta$ (as in [14]) and by Connes' canonical cyclic 2-cocycle (as in [4] or III.2.$\beta$ of [5]). In [3] we showed that $\mathbf{T}$ is injective when $\theta$ is rational. This suggests, presumably, that the subspace in question is all of $H^{\mathrm{ev}}(H_\theta)$ and that ch will in fact turn out to be, after tensoring with the complex plane, an isomorphism. (In view of this, we shall also refer to $\mathbf{T}$ as the Connes-Chern character.)

The main result of the present paper is to show that the ten canonical classes form a basis for $K_0(H_\theta)$ when $\theta$ is a special type of rational number (Proposition 5.1). This result allows us to prove that the range of $\mathbf{T}$ on $K_0(H_\theta)$ is equal to its range on the span of the ten classes. Combined with a recent result of Polishchuk [10] that $K_0(H_\theta) \cong \mathbb{Z}^{10}$ for all $\theta$ (which incidently used the independence of the ten classes [3]), this culminates with the following.

THEOREM 1.1. *For each $\theta > 0$ the following holds:*
(i) *The ten canonical modules form a basis for $K_0(H_\theta)$.*
(ii) *The Connes-Chern character* $\mathrm{ch} : K_0(H_\theta) \to H^{\mathrm{ev}}(H_\theta)^*$ *is injective.*
(iii) *The range of* $\mathbf{T} : K_0(H_\theta) \to \mathbb{R}^{10}$ *is the integral span of the rows in Table* 1.

Note that a basis for $K_0(H_\theta)$ is not given in [10], so our result gives a precise isomorphism. We comment briefly at the end that $K_1(H_\theta) = 0$ for a dense $G_\delta$ set of $\theta$'s, which in fact holds for all $\theta$ as shown in [6].

It is a well-known theorem of Bratteli and Kishimoto [2] (and independently in [13]) that the crossed product $A_\theta \rtimes \mathbb{Z}_2$ (under the flip) is approximately finite dimensional for any irrational $\theta$. In [16] it is shown that this holds for the Fourier transform for a dense $G_\delta$ set of irrational $\theta$. In quite recent work of Echterhoff, Lück, Phillips, and the author [6] the AF result is shown to be true for the Fourier, hexic, and cubic transforms (for all irrational $\theta$).

It is of historical interest to know that Hattori [9] and Stallings [12] have obtained (back in 1965) the trace of a finitely generated projective module. These are some of the earliest attempts to pair elements of $K$ theory of non-commutative algebras with trace-like functionals.

We shall write $e(t) := \mathrm{e}^{2\pi \mathrm{i} t}$, and $\delta_k^n$ is 1 if $k|n$ and 0 otherwise. We have $\sum_{j=0}^{q-1} e(nj/q) = q\delta_q^n$. Throughout, we shall assume that $0 < \theta < 1$. Since $\lambda = e(\theta)$, we shall also write $\lambda^t = e(t\theta)$. Denote by $\delta_{k,\ell}$ the usual $\delta$-function (1 if and only if $k = \ell$ and 0 otherwise).

## 2. $K$-CLASSES AND THEIR CONNES-CHERN CHARACTER

When considering the case that $\theta$ is rational, we shall tacitly assume throughout that $\theta = \frac{p}{q}$ where $p < q$ are positive relatively prime integers.

TEN $K_0$-CLASSES. As in [3], one has the following nine projections in $H_\theta$:

$$1, \quad p_j = \frac{1}{6}\sum_{i=0}^{5} \omega^{ij} W^i, \quad q_k = \frac{1}{3}\sum_{i=0}^{2} \omega^{2ik} \lambda^{i/6} \,(UW^2)^i, \quad r = \frac{1}{2}(I + UW^3),$$

where $j = 0, \ldots, 4, k = 0, 1$ and $\lambda^{1/6} UW^2$ is a unitary of order 3, $UW^3$ of order 2, and $\omega := e(1/6) = \frac{1}{2}(1 + \mathrm{i}\sqrt{3})$ (a primitive 6th root of 1).

One further has the hexic module $\mathcal{M}_6$ over $H_\theta$ ($0 < \theta < 1$) which we constructed in [3] from the Heisenberg $A_\theta$-module (see [4]) by equipping it with an action of $W$ represented by a suitable scaling of the hexic transform on the Schwartz space $S(\mathbb{R})$ (see [15] for how the hexic transform was obtained). The algebra $H_\theta$ has the canonical (bounded) trace $\tau$ given by $\tau\Big(\sum_{j=0}^{5} a_j W^j\Big) = \tau(a_0)$ for $a_j \in A_\theta$, where $\tau(a_0)$ is the canonical trace of $a_0$ in $A_\theta$ (relative to the unitaries $U, V$). (It is unique in the irrational case.) In [3] it was shown that one has the following unbounded traces on $H_\theta$ (the smooth $*$-subalgebra) given by:

$$T_{10}(U^m V^n W^5) = \lambda^{(m^2+n^2)/2}, \qquad T_{30}(U^m V^n W^3) = \lambda^{-mn/2}\, \delta_2^m \delta_2^n,$$

$$T_{20}(U^m V^n W^4) = \lambda^{(m-n)^2/6}\, \delta_3^{m-n}, \qquad T_{31}(U^m V^n W^3) = \lambda^{-mn/2},$$

$$T_{21}(U^m V^n W^4) = \lambda^{(m-n)^2/6},$$

where at generic elements $U^m V^n W^k$ for different $k$ they vanish.

Observe that $T_{3j}$ are self-adjoint trace functionals, but that $T_{10}$ and $T_{2k}$ are not. However, one can look at the real and imaginary parts of the latter. Let

$$\phi_0 = \tfrac{1}{2}(T_{10} + T_{10}^*), \quad \phi_0' = -\tfrac{\mathrm{i}}{2}(T_{10} - T_{10}^*)$$

be the real and imaginary parts of $T_{10}$, respectively, and

$$\phi_1 = \tfrac{1}{2}(T_{20} + T_{20}^*), \quad \phi_1' = -\tfrac{\mathrm{i}}{2}(T_{20} - T_{20}^*), \quad \phi_2 = \tfrac{1}{2}(T_{21} + T_{21}^*), \quad \phi_2' = -\tfrac{\mathrm{i}}{2}(T_{21} - T_{21}^*)$$

be those of $T_{20}$ and $T_{21}$ (where $T^*(x) := \overline{T(x^*)}$).

The remaining invariant we need is Connes' canonical cyclic 2-cocycle on the rotation algebra $A_\theta$:

$$\varphi(x^0, x^1, x^2) = \frac{1}{2\pi \mathrm{i}}\, \tau(x^0[\delta_1(x^1)\delta_2(x^2) - \delta_2(x^1)\delta_1(x^2)])$$

(see III.2.$\beta$ of [5]) where $\delta_j$, $j = 1, 2$, are the canonical derivations of $A_\theta$ under the canonical action of the 2-torus $\mathbb{T}^2$ (relative to $U, V$). The Chern character invariant that $\varphi$ induces is the group homomorphism $c_1 : K_0(A_\theta) \to \mathbb{Z}$ given by the cup product $c_1[E] := (\varphi\#\mathrm{Tr}_n)(E, E, E)$ for $E$ any smooth projection in $M_n(A_\theta)$. In Section 4 of [3] this invariant was extended to $H_\theta$ by taking the composition $C := c_1 \circ \Psi_* : K_0(H_\theta) \to \mathbb{Z}$ where $\Psi : H_\theta \to M_6(A_\theta)$ is the canonical injection given by $\Psi(a) = [\rho^{-i}(a_{i-j})]_{i,j=0}^5$ for $a = \sum_j a_j W^j \in H_\theta$, where $i - j$ is reduced mod 6 and where $a_j \in A_\theta$. (To clarify $\Psi_*$, if $E$ is a projection in some matrix algebra over $H_\theta$, then $\Psi(E)$ is a projection in some matrix algebra over $M_6(A_\theta)$, hence in a matrix algebra over $A_\theta$, and thus gives a class in $K_0(A_\theta)$ — e.g. $\Psi_*[1] = 6[1]_{K_0(A_\theta)}$.) For example (and we shall need this later), if $e_\theta$ is a smooth Powers-Rieffel projection in $A_\theta$ with trace $\theta$ ($0 < \theta < 1$ rational or irrational) then, viewing $e_\theta$ as an element of $H_\theta$ via the canonical inclusion $A_\theta \hookrightarrow H_\theta$, one has $C[e_\theta] = -6$. In fact, since $c_1[e_\theta] = -1$, $[\rho(e_\theta)] = [e_\theta]$ in $K_0(A_\theta)$, and $\Psi(e_\theta) = \mathrm{diag}(e_\theta, \rho^5(e_\theta), \rho^4(e_\theta), \rho^3(e_\theta), \rho^2(e_\theta), \rho(e_\theta))$, one has $\Psi_*[e_\theta]_{K_0(H_\theta)} = 6[e_\theta]_{K_0(A_\theta)}$, where $\Psi_* : K_0(H_\theta) \to K_0(A_\theta)$ is the induced map.

Consider the Connes-Chern character $\mathrm{ch} : K_0(H_\theta) \to HC^{\mathrm{ev}}(H_\theta)^*$ where $HC^{\mathrm{ev}}(H_\theta)^*$ is the complex vector space dual of the even periodic cyclic cohomology group ([5], III.1.$\alpha$). From this, one defines the map $\mathbf{T} : K_0(H_\theta) \to \mathbb{R}^{10}$ by the pairing

$$\begin{aligned}\mathbf{T}(x) &= \langle(\tau;\ \phi_0, \phi_0'; \phi_1, \phi_1', \phi_2, \phi_2';\ T_{30}, T_{31};\ C),\ \mathrm{ch}(x)\rangle \\ &= (\tau(x);\ \phi_0(x), \phi_0'(x);\ \phi_1(x), \phi_1'(x), \phi_2(x), \phi_2'(x);\ T_{30}(x), T_{31}(x);\ C(x)).\end{aligned}$$

All computations below will be done in terms of this map (as was done in [3]), so there is some justification for calling $\mathbf{T}$ the Connes-Chern character, since there is evidence that after tensoring with $\mathbb{C}$, one eventually has an isomorphism $K_0(H_\theta) \otimes\mathbb{C} \to HC^{\mathrm{ev}}(H_\theta)^*$ between vector spaces of dimension nine. The evidence for this comes from the fact proved in [3] (Corollary 3.2) that for irrational $\theta$ one has $HC^0(H_\theta) \cong \mathbb{C}^9$ and has as basis $\{\tau, \phi_0, \phi_0', \phi_1, \phi_1', \phi_2, \phi_2', T_{30}, T_{31}\}$. These, together with the class associated to Connes' cyclic 2-cocycle would presumably constitute a basis for $HC^{\mathrm{ev}}(H_\theta)$, which the authors suspect is $HC^0(H_\theta) \oplus HC^2(H_\theta)$ modulo identifications given by the periodicity operator after tensoring with the complex plane over the ring $HC^*(\mathbb{C})$. This further suggests that the Hochschild dimension of $H_\theta$ is two, as Connes showed to be the case for the rotation algebra. (Of course, for rational $\theta$, the group $HC^0(H_\theta)$ is infinite dimensional, but one would still expect that the periodic cohomology group $HC^{\mathrm{ev}}(H_\theta)$ to be finite dimensional — in fact, nine-dimensional.)

For the identity element and the Powers-Rieffel projection one clearly has

$$\mathbf{T}(1) = (1;\ 0,0;\ 0,0,0,0;\ 0,0;\ 0), \quad \mathbf{T}(e_\theta) = (\theta;\ 0,0;\ 0,0,0,0;\ 0,0;\ -6).$$

The main result of [3] is the following data of Connes-Chern character values for the above nine modules for any $\theta$. In this table we write $\omega = e(1/6) = \frac{1}{2}(1+\mathrm{i}\sqrt{3})$.

**Table 1.** Character table for the hexic transform

| $K_0$-class | $\tau$ | $C_6$ | $\phi_0$ | $\phi_0'$ | $\phi_1$ | $\phi_1'$ | $\phi_2$ | $\phi_2'$ | $T_{30}$ | $T_{31}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $[1]$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $[p_0]$ | $\frac{1}{6}$ | $0$ | $\frac{1}{6}$ | $0$ | $\frac{1}{6}$ | $0$ | $\frac{1}{6}$ | $0$ | $\frac{1}{6}$ | $\frac{1}{6}$ |
| $[p_1]$ | $\frac{1}{6}$ | $0$ | $\frac{1}{12}$ | $-\frac{\sqrt{3}}{12}$ | $-\frac{1}{12}$ | $-\frac{\sqrt{3}}{12}$ | $-\frac{1}{12}$ | $-\frac{\sqrt{3}}{12}$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ |
| $[p_2]$ | $\frac{1}{6}$ | $0$ | $-\frac{1}{12}$ | $-\frac{\sqrt{3}}{12}$ | $-\frac{1}{12}$ | $\frac{\sqrt{3}}{12}$ | $-\frac{1}{12}$ | $\frac{\sqrt{3}}{12}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |
| $[p_3]$ | $\frac{1}{6}$ | $0$ | $-\frac{1}{6}$ | $0$ | $\frac{1}{6}$ | $0$ | $\frac{1}{6}$ | $0$ | $-\frac{1}{6}$ | $-\frac{1}{6}$ |
| $[p_4]$ | $\frac{1}{6}$ | $0$ | $-\frac{1}{12}$ | $\frac{\sqrt{3}}{12}$ | $-\frac{1}{12}$ | $-\frac{\sqrt{3}}{12}$ | $-\frac{1}{12}$ | $-\frac{\sqrt{3}}{12}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |
| $[q_0]$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ |
| $[q_1]$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-\frac{1}{6}$ | $-\frac{\sqrt{3}}{6}$ | $0$ | $0$ |
| $[r]$ | $\frac{1}{2}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\frac{1}{2}$ |
| $[\mathcal{M}_6]$ | $\frac{\theta}{6}$ | $-1$ | $\frac{1}{12}$ | $\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $\frac{\sqrt{3}}{36}$ | $\frac{1}{4}$ | $\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $\frac{1}{3}$ |

This table yields the following.

THEOREM 2.1 ([3], Theorem 1.1). *For any $\theta > 0$, the ten classes $[1], [p_0], [p_1], [p_2], [p_3], [p_4], [q_0], [q_1], [r], [\mathcal{M}_6]$ are independent in $K_0(H_\theta)$. When $\theta$ is rational, the map $\mathbf{T}$ is injective on $K_0(H_\theta)$, and hence so is the Connes-Chern character* $\mathrm{ch} : K_0(H_\theta) \to HC^{\mathrm{ev}}(H_\theta)^*$.

NOTATION 2.2. We shall denote by $\mathcal{R}_\theta$ the subgroup of $K_0(H_\theta)$ generated by the ten classes listed in Table 1.

Consider the element of $K_0(H_{p/q})$ defined by (for relatively prime integers $p, q$)

$$\kappa_{p,q} = p[1] + q([p_0] - 4[p_1] - 3[p_2] - 2[p_3] - [p_4] + 2[q_0] - 2[q_1] + 3[r] - 6[\mathcal{M}_6]).$$

(Here, $p_j, q_j, r$, and $\mathcal{M}_6$ are evaluated at $\theta = \frac{p}{q}$.) It is easy to check that $\mathbf{T}(\kappa_{p,q}) = (0;0,0;0,0,0,0;0,0;6q)$ from Table 1. Since we have $\mathbf{T}(p[1] - q[e_\theta]) = (0;0,0;0,0,0,0;0,0;6q) = \mathbf{T}(\kappa_{p,q})$, the injectivity of $\mathbf{T}$ (in the rational case, Theorem 2.1) gives the equality $p[1] - q[e_\theta] = \kappa_{p,q}$ in $K_0(H_\theta)$. In fact, in the same manner one easily checks that the Powers-Rieffel projection $e_\theta$ is related to the nine modules as follows for rational $\theta$

$$[e_\theta] = -[p_0] + 4[p_1] + 3[p_2] + 2[p_3] + [p_4] - 2[q_0] + 2[q_1] - 3[r] + 6[\mathcal{M}_6]$$

in $K_0(H_\theta)$ (the right side evaluated at $\theta$). This shows that $[e_\theta] \in \mathcal{R}_\theta$ for rational $\theta$.

Define the *reduced* character $\mathbf{T}' : K_0(H_\theta) \to \mathbb{R}^9$ to be the degree zero part of the Connes-Chern character $\mathbf{T}$, namely, $\mathbf{T}' = (\tau(x);\ \phi_0, \phi_0';\ \phi_1, \phi_1', \phi_2, \phi_2';\ T_{30}, T_{31})$. Note that $\kappa_{p,q}$ is in $\mathrm{Ker}(\mathbf{T}')$. Two key steps in the proofs below is to show that in fact $\kappa_{p,q}$ generates $\mathrm{Ker}(\mathbf{T}')$ (Corollary 4.3) and that the range of $\mathbf{T}'$ on $K_0(H_\theta)$ is equal to its range on $\mathcal{R}_\theta$ for $\theta$ in a special dense set of rationals $\mathbb{P}$ described below (Proposition 4.1). These steps lead one to the equality $K_0(H_{p/q}) = \mathcal{R}_{p/q}$, from which it follows that the ten classes form a basis for $K_0(H_{p/q})$.

2.1. REALIZATION OF $A_{p/q}$ AS A DIMENSION-DROP ALGEBRA. Begin with the following realization of the rational rotation algebra as the subalgebra of $C([0,1]\times[0,1], M_q)$ given in [1], p. 64, by

$$A_{p/q} = \{f \in C([0,1]\times[0,1], M_q) : \ f(x,1) = \alpha_1(f(x,0)), \ \ f(1,y) = \alpha_2(f(0,y))\}$$

where $M_q := M_q(\mathbb{C})$ is generated by the unitaries

$$U_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda^{q-1} \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

satisfying $V_0U_0 = \lambda U_0V_0$, where $\lambda = e(p/q)$, and $\alpha_1, \alpha_2$ are the automorphisms of $M_q$ given by $\alpha_1(U_0) = U_0$, $\alpha_1(V_0) = wV_0$ and $\alpha_2(U_0) = wU_0$, $\alpha_2(V_0) = V_0$, where $w = e(1/q)$. With this realization, the canonical generators $U, V$ of $A_{p/q}$ are given by the functions $U(x,y) = e(x/q)U_0$, $V(x,y) = e(y/q)V_0$ and the hexic automorphism is given by

$$\rho(f)(x,y) = \eta_0(f(y, y - x - p\overline{q}/2))$$

where $\eta_0 \in \mathrm{Aut}(M_q)$ is given by $\eta_0(U_0) = V_0$, $\eta_0(V_0) = \lambda^{-(1/2)(1-\overline{q})}U_0^{-1}V_0$ where $\overline{q} = 0$ if $q$ is even, and 1 otherwise. In fact, with $W_0$ being the unitary

$$W_0 = \frac{1}{\sqrt{q}}\,[\lambda^{i(i+\overline{q})/2 - ij}]$$

where $i, j = 0, 1, \ldots, q-1$, one checks that $\eta_0(x) = W_0^* x W_0$ (see Sections 2 and 3 of [8]). Indeed, one checks the commutation relations

$$U_0W_0 = W_0V_0, \quad V_0W_0 = \lambda^{-(1/2)(1-\overline{q})}\,W_0U_0^{-1}V_0.$$

Consider the following self-adjoint $q \times q$ unitary matrix

$$\Gamma_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

It gives rise to the flip automorphism: $U_0\Gamma_0 = \Gamma_0 U_0^{-1}$, $V_0\Gamma_0 = \Gamma_0 V_0^{-1}$. The automorphisms $\alpha_1, \alpha_2$ are given by $\alpha_i(x) = W_i^* x W_i$, $i = 1, 2$ where

$$W_1 = U_0^{-p'} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & w & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & w^{q-1} \end{bmatrix}, \quad W_2 = V_0^{-p''} = \begin{bmatrix} \mathbf{0} & I_{p''} \\ I_{q-p''} & \mathbf{0} \end{bmatrix}$$

and $I_n$ is the $n \times n$ identity matrix, and $p'$, $p''$ are the unique integers in $[1, q-1]$ such that $pp' \equiv -1 \bmod q$ and $pp'' \equiv 1 \bmod q$. One has

$$W_1 W_0 = W_0 W_2^{-1}, \quad W_2 W_0 = w^{p''/2} W_0 W_2 W_1.$$

If $q$ is even (which is all we will need for our purposes) then one can check that

$$W_0^3 = \frac{G(p, 2q)}{2\sqrt{q}} \Gamma_0, \quad W_0^2 = \frac{G(p, 2q)}{2\sqrt{q}} Z_0,$$

where $(Z_0)_{ij} = \frac{1}{\sqrt{q}} \lambda^{-(j^2/2) - ij}$ for $i, j = 0, \ldots, q-1$, and $G(\cdot, \cdot)$ is the classical Gaussian sum (to be recalled below). One can therefore show that $W_0^6 = \mathrm{i} I$ for $\frac{p}{q} \in \mathbb{P}$, where $\mathbb{P}$ is a special dense set of rationals defined below.

Given positive relatively prime integers $p, q$, let $p', p''$ be the integers given above, and write $pp' = -1 + q\widetilde{p}$, $pp'' = 1 + q\widetilde{q}$ for some integers $\widetilde{p}$ and $\widetilde{q}$. One easily checks that $p = \widetilde{p} + \widetilde{q}$ and $q = p' + p''$. In the present paper we shall be interested in the following dense set of rational numbers in $(0, 1)$

$$\mathbb{P} := \left\{ \frac{2^{d+1}k + 1}{2^{2d-1}} : k = 3, 6, \ldots, 2^{d-2} - 1, \ k \equiv 0 \mod 3, \ d \geqslant 3 \right\}.$$

For such rationals, $p = 2^{d+1}k + 1$, $q = 2^{2d-1}$, and one can verify directly that

$$p' = 2^{d+1}k - 1, \quad p'' = 2^d(2^{d-1} - 2k) + 1, \quad \widetilde{p} = 8k^2, \quad \widetilde{q} = 8k(2^{d-2} - k) + 1.$$

2.2. GAUSSIAN SUMS. Recall the classical quadratic Gauss sum is given by

$$G(p, q) = \sum_{j=0}^{q-1} \lambda^{j^2}$$

where $p, q$ are relatively prime positive integers and $\lambda = e(p/q) = \mathrm{e}^{2\pi \mathrm{i} p/q}$. It is known that for odd $p$ and $q = 4^d$ the Gaussian sum takes the simpler form $G(p, 4^d) = 2^d(1 + \mathrm{i}^p)$. So for $\frac{p}{q} \in \mathbb{P}$ one has $G(p, 2q) = \sqrt{2q}(1 + \mathrm{i})$, since in this case $p$ is 1 mod 4, and $W_0^3 = \frac{1+\mathrm{i}}{\sqrt{2}} \Gamma_0$ and hence $W_0^6 = \mathrm{i} I$.

LEMMA 2.3. *Let $q = 2^{2d-1}$ where $d$ is a positive integer, let $p$ be an odd positive integer with $p < q$, and $\lambda = e(p/q)$. Then*

$$\sum_{k=0}^{q-1} \lambda^{(1/2)k^2 + ak} = \sqrt{q} \frac{1 + \mathrm{i}^p}{\sqrt{2}} \lambda^{-(1/2)a^2}, \quad \sum_{k=0}^{q-1} \lambda^{(3/2)k^2 + ak} = \sqrt{q} \frac{1 - \mathrm{i}^p}{\sqrt{2}} \lambda^{(1/2)a^2((2q-1)/3)},$$

*for any integer a (here, $\frac{2q-1}{3}$ is a positive integer).*

*Proof.* Note that since $q$ is even, the functions $\lambda^{(1/2)k^2}$ and $\lambda^{(3/2)k^2+ak}$ have period $q$ (so the sums are invariant under integer translations). Let $r = \frac{4^d-1}{3}$ (positive integer). Then $1 = 2q - 3r$. Letting $\mu = e(3p/2q) = \lambda^{3/2}$, we have

$$\begin{aligned}\sum_{k=0}^{q-1} \lambda^{(3/2)k^2+ak} &= \sum_{k=0}^{q-1} \lambda^{(3/2)k^2+a(2q-3r)k} = \sum_{k=0}^{q-1} \lambda^{(3/2)k^2-3ark} = \sum_{k=0}^{q-1} \mu^{k^2-2ark} \\ &= \frac{1}{2}\sum_{k=0}^{2q-1} \mu^{k^2-2ark} = \frac{1}{2}\mu^{-a^2r^2}\sum_{k=0}^{2q-1} \mu^{(k-ar)^2} = \frac{1}{2}\mu^{-a^2r^2}\sum_{k=0}^{2q-1} \mu^{k^2} \\ &= \frac{1}{2}\lambda^{-(3/2)a^2r^2} G(3p,2q).\end{aligned}$$

Now as $q = 2^{2d-1}$, $G(3p,2q) = 2^d(1-\mathrm{i}^p)$ and $\lambda^{-(3r/2)a^2r} = \lambda^{(1/6)a^2(4^d-1)}$, the second sum follows. To get the first sum, one has (by suitable substitution)

$$2^d(1+\mathrm{i}^p) = G(p,2q) = \sum_{k=0}^{2q-1} (\lambda^{1/2})^{k^2} = \sum_{k=0}^{q-1} \lambda^{(1/2)k^2} + \sum_{k=q}^{2q-1} \lambda^{(1/2)k^2} = 2\sum_{k=0}^{q-1} \lambda^{(1/2)k^2}.$$

The case for general $a$ (in the first sum in the lemma) follows from the case $a = 0$, by translation invariance. ∎

LEMMA 2.4. *For relatively prime $p, q$ with $q=2^{2d-1}$ ($d$ a positive integer), we have:*

$$\begin{aligned}\mathrm{Tr}(U_0^m V_0^n W_0) &= \frac{1-\mathrm{i}^p}{\sqrt{2}}\lambda^{(1/2)(m^2+n^2)}, \\ \mathrm{Tr}(U_0^m V_0^n W_0^2) &= \mathrm{i}^p \lambda^{(1/6)(m-n)^2}\omega^{-2p(m-n)^2}, \\ \mathrm{Tr}(U_0^m V_0^n W_0^3) &= \sqrt{2}(1+\mathrm{i}^p)\lambda^{-(1/2)mn}\delta_2^m \delta_2^n.\end{aligned}$$

*Proof.* Since $V_0^n = \begin{bmatrix} \mathbf{O} & I_{q-n} \\ I_n & \mathbf{O} \end{bmatrix}$ one decomposes $W_0$ into the following block form

$$W_0 = \begin{bmatrix} n \times (q-n) & n \times n \\ (q-n)\times(q-n) & (q-n)\times n \end{bmatrix} = \frac{1}{\sqrt{q}}\begin{bmatrix} * & X \\ Y & * \end{bmatrix},$$

where $X = [\lambda^{(1/2)i^2 - i(j+q-n)}]_{i,j=0,\dots,n-1}$ with relevant diagonal entries

$$X = \begin{bmatrix} 1 & * & * & \cdots & & \\ * & \lambda^{-(1/2)-(q-n)} & * & & & \\ * & * & \ddots & & & \\ \vdots & & * & \lambda^{-(1/2)j^2-j(q-n)} & & \\ & & & & \ddots & \\ & & & & & \lambda^{-(1/2)(n-1)^2-(n-1)(q-n)} \end{bmatrix}$$

where $j = 0, 1, \dots, n-1$, and $Y = [\lambda^{(1/2)(i+n)^2-(i+n)j}]$ with diagonals

$$Y = \begin{bmatrix} \lambda^{(1/2)n^2} & * & * & \cdots & & \\ * & \lambda^{(1/2)(n^2-1)} & * & & & \\ * & * & \ddots & & & \\ \vdots & & & \lambda^{(1/2)(n^2-j^2)} & & \\ & & & & \ddots & \\ & & & & & \lambda^{(1/2)(n^2-(q-n-1)^2)} \end{bmatrix}.$$

We then have

$$\sqrt{q} U_0^m V_0^n W_0 = U_0^m \begin{bmatrix} \mathbf{O} & I_{q-n} \\ I_n & \mathbf{O} \end{bmatrix} \begin{bmatrix} * & X \\ Y & * \end{bmatrix} = U_0^m \begin{bmatrix} Y & * \\ * & X \end{bmatrix}$$

and since

$$U_0^m = \operatorname{diag}(1, \lambda^m, \dots, \lambda^{m(q-n-1)},\ \lambda^{m(q-n)}, \dots, \lambda^{m(q-1)})$$

we obtain

$$\sqrt{q}\,\operatorname{Tr}(U_0^m V_0^n W_0) = \sum_{j=0}^{q-n-1} \lambda^{mj} \cdot \lambda^{(1/2)(n^2-j^2)} + \sum_{j=0}^{n-1} \lambda^{m(q-n+j)} \cdot \lambda^{-(1/2)j^2-j(q-n)}.$$

Making the substitution $k = j + n$ in the first sum gives

$$\sum_{k=n}^{q-1} \lambda^{m(k-n)} \lambda^{(1/2)(n^2-(k-n)^2)} = \sum_{k=n}^{q-1} \lambda^{m(k-n)} \lambda^{-(1/2)k(k-2n)},$$

and using $\lambda^q = 1$ allows to write the second sum as $\sum_{j=0}^{n-1} \lambda^{m(j-n)} \lambda^{-(1/2)j(j-2n)}$. It follows, using Lemma 2.3, that

$$\sqrt{q}\operatorname{Tr}(U_0^m V_0^n W_0) = \lambda^{-mn} \sum_{k=0}^{q-1} \lambda^{-(1/2)k^2+(m+n)k} = \lambda^{-mn} \lambda^{(1/2)(m+n)^2} \sum_{k=0}^{q-1} \lambda^{-(1/2)(k-(m+n))^2}$$

$$= \lambda^{(1/2)(m^2+n^2)} \sum_{k=0}^{q-1} \lambda^{-(1/2)k^2} = \lambda^{(1/2)(m^2+n^2)} 2^{d-1}(1 - \mathrm{i}^p).$$

Using the relation $W_0^2 = \frac{1+\mathrm{i}^p}{\sqrt{2}} Z_0$, one gets $\operatorname{Tr}(U_0^m V_0^n W_0^2) = \frac{1+\mathrm{i}^p}{\sqrt{2}} \operatorname{Tr}(U_0^m V_0^n Z_0)$. As with $W_0$, we decompose $Z_0$ into the block form

$$Z_0 = \begin{bmatrix} n \times (q-n) & n \times n \\ (q-n) \times (q-n) & (q-n) \times n \end{bmatrix} = \frac{1}{\sqrt{q}} \begin{bmatrix} * & X' \\ Y' & * \end{bmatrix},$$

where $X' = [\lambda^{-(1/2)(j+q-n)^2 - i(j+q-n)}]$ with relevant diagonal entries

$$X' = \begin{bmatrix} \lambda^{-(1/2)(q-n)^2} & * & * & \cdots & & \\ * & * & * & & & \\ * & * & \ddots & & & \\ \vdots & & * & \lambda^{-(1/2)(i+q-n)^2 - i(i+q-n)} & & \\ & & & & \ddots & \\ & & & & & \lambda^{-(1/2)(q-1)^2-(n-1)(q-n)} \end{bmatrix}$$

and $Y' = [\lambda^{-(1/2)(i+n)^2-(i+n)j}]$ with relevant diagonal entries

$$Y' = \begin{bmatrix} 1 & * & \cdots & & \\ * & \ddots & & & \\ \vdots & & \lambda^{-(1/2)j(3j+2n)} & & \\ & & & \ddots & \\ & & & & \lambda^{-(1/2)(q-n-1)(3q-n-3)} \end{bmatrix}.$$

We then have

$$U_0^m V_0^n Z_0 = \tfrac{1}{\sqrt{q}} U_0^m \begin{bmatrix} Y' & * \\ * & X' \end{bmatrix}$$

hence

$$\sqrt{q}\operatorname{Tr}(U_0^m V_0^n Z_0) = \sum_{j=0}^{q-n-1} \lambda^{mj} \lambda^{-(3/2)j^2 - nj} + \sum_{j=0}^{n-1} \lambda^{m(q-n+j)} \lambda^{-(1/2)(j+q-n)^2 - j(j+q-n)}.$$

Making the substitution $k = j + n$ in the first sum gives

$$\sum_{k=n}^{q-1} \lambda^{m(k-n)} \lambda^{-(3/2)(k-n)^2 - n(k-n)} = \sum_{k=n}^{q-1} \lambda^{m(k-n)} \lambda^{-(3/2)k^2 + 2nk - (1/2)n^2},$$

and $\lambda^{(1/2)q^2} = 1$ allows us to write the second sum as

$$\sum_{j=0}^{n-1} \lambda^{m(j-n)} \lambda^{-(1/2)(j-n)^2 - j(j-n)} = \sum_{j=0}^{n-1} \lambda^{m(j-n)} \lambda^{-(3/2)j^2 + 2nj - (1/2)n^2}.$$

Using Lemma 2.3 again one has

$$\begin{aligned} \sqrt{q}\operatorname{Tr}(U_0^m V_0^n Z_0) &= \lambda^{-mn-(1/2)n^2} \sum_{k=0}^{q-1} \lambda^{-(3/2)k^2 + (m+2n)k} \\ &= \frac{\sqrt{q}(1 + \mathrm{i}^p)}{\sqrt{2}} \lambda^{-mn-(1/2)n^2} \lambda^{-(1/6)(m+2n)^2(2q-1)} \end{aligned}$$

and so $\operatorname{Tr}(U_0^m V_0^n W_0^2) = \mathrm{i}^p \lambda^{(1/6)(m-n)^2} \omega^{-2p(m-n)^2}$. (Recall $\omega = e(1/6)$.) From [14], and recalling that $q$ is even and $p$ is odd, we had $\operatorname{Tr}(U_0^m V_0^n \Gamma_0) = 2\lambda^{-(1/2)mn} \delta_2^n \delta_2^m$. Since $W_0^3 = \frac{1+\mathrm{i}^p}{\sqrt{2}} \Gamma_0$, we have $\operatorname{Tr}(U_0^m V_0^n W_0^3) = \sqrt{2}(1 + \mathrm{i}^p)\lambda^{-(1/2)mn} \delta_2^n \delta_2^m$. ∎

2.3. CONNES-CHERN′ CHARACTER ON $A_\theta$ (FOR RATIONAL $\theta$). Realizing $A_\theta$ as $M_q$-valued functions on the unit square as above, where $\theta = \frac{p}{q}$, the canonical trace is given by

$$\tau(F) = \frac{1}{q} \int_0^1\int_0^1 \mathrm{Tr}_q(F(x,y))\,\mathrm{d}x\mathrm{d}y$$

for $F \in A_\theta$, where $\mathrm{Tr}_q$ is the usual trace on $M_q(\mathbb{C})$. Also, the canonical derivations of $A_\theta$ are given by $\delta_1 = q\frac{\partial}{\partial x}$, $\delta_2 = q\frac{\partial}{\partial y}$. They are defined by

$$\delta_1(U^m V^n) = 2\pi\mathrm{i} m U^m V^n, \quad \delta_2(U^m V^n) = 2\pi\mathrm{i} n U^m V^n.$$

Connes' canonical cyclic 2-cocycle is given by (see III.2.$\beta$ of [5]):

$$\begin{aligned}\varphi_q(F^0, F^1, F^2) &= \frac{1}{2\pi\mathrm{i}}\,\tau(F^0[\delta_1(F^1)\delta_2(F^2) - \delta_2(F^1)\delta_1(F^2)]) \\ &= \frac{q}{2\pi\mathrm{i}} \int_0^1\int_0^1 \mathrm{Tr}_q\Big(F^0\Big[\frac{\partial F^1}{\partial x}\frac{\partial F^2}{\partial y} - \frac{\partial F^1}{\partial y}\frac{\partial F^2}{\partial x}\Big]\Big)\,\mathrm{d}x\mathrm{d}y\end{aligned}$$

where $F^j \in A_\theta$ (are smooth elements). The extension of $\varphi_q$ to $M_n(A_\theta)$ is given by the cup product

$$(\varphi_q \# \mathrm{Tr}_n)(F^0 \otimes a^0, F^1 \otimes a^1, F^2 \otimes a^2) = \varphi_q(F^0, F^1, F^2) \cdot \mathrm{Tr}_n(a^0 a^1 a^2)$$

where $F^j \in A_\theta$ and $a^j \in M_n(\mathbb{C})$. The Chern character invariant of Connes $c_1 : K_0(A_\theta) \to \mathbb{Z}$ is then given by $c_1[Q] = \langle [Q], \varphi_q \rangle = (\varphi_q \# \mathrm{Tr}_n)(Q, Q, Q)$, where $Q$ is a projection in $M_n(A_\theta)$. For $0 < \theta < 1$ the Powers-Rieffel projection $e_\theta$ has $c_1(e_\theta) = \varphi_q(e_\theta, e_\theta, e_\theta) = -1$ (as was shown by Connes). For $\theta = 1$, one can show that $c_1$ of the Bott projection is $\pm 1$, depending on the choices made for it (as in Section 5 of [14]).

## 3. UNBOUNDED TRACES AND SINGULAR SPHERE REALIZATION

In [8] it is proved that the crossed product $C^*$-algebra $H_\theta$, for rational $\theta = \frac{p}{q}$ (with $(p,q) = 1$), is isomorphic to a subalgebra of $C(\mathbb{S}^2, M_{6q})$ of continuous functions on the 2-sphere $\mathbb{S}^2$ with values in $M_{6q}$ that commute with certain projections at three points (normally referred to as "singularities"). Let $Q$ denote the quadrilateral shown below in Figures 1 and 2.

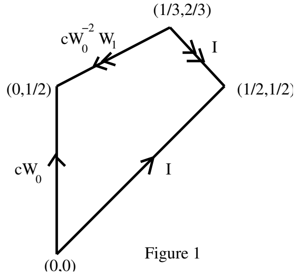

Figure 1

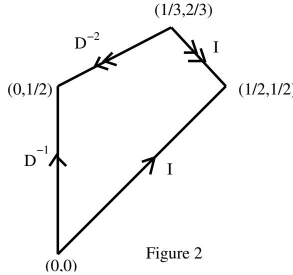

Figure 2

As in [8], the 2-sphere $\mathbb{S}^2$ shall be envisaged as $Q$ with the appropriate edges identified (as shown). For our purposes, we shall view this subalgebra as the set of all functions that commute with certain finite-order unitaries at the singular points.

First, it is easy to check that by the universality of the crossed product $H_\theta$, there is a unique $C^*$-injection $H_\theta \to M_6(A_\theta)$ such that

$$f \;\mapsto\; T_f \;:=\; \begin{bmatrix} f & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho(f) & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho^2(f) & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho^3(f) & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho^4(f) & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho^5(f) \end{bmatrix},$$

$$W \;\mapsto\; Z \;:=\; \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $f \in A_\theta$ (understood by the realization mentioned in Section 2). (The "1" in the matrix entries here is the identity of $A_\theta$ which is $I_q$, the identity $q$ by $q$ matrix.) Now consider the $6 \times 6$ unitary matrix $E = \frac{1}{\sqrt{6}}[\omega^{-ij}]$, where $i, j = 0, 1, \ldots, 5$ and $\omega = e(1/6)$. One has

$$(ET_f\mathrm{E}^*)_{ij} = \frac{1}{6}\sum_{k,\ell=0}^{5} \omega^{-ik}\delta_{k,\ell}\rho^k(f)\omega^{j\ell} = \frac{1}{6}f_{j-i}$$

where $f_r = \sum_{k=0}^{5} \omega^{rk}\rho^k(f)$ (and $j-i$ is reduced mod 6). Further, it is easy to check that

$$E Z \mathrm{E}^* \;=\; D \;:=\; \mathrm{diag}(1, \omega, \omega^2, \omega^3, \omega^4, \omega^5).$$

Therefore, composing the above injection with the automorphism $E^*(\cdot)E$ (which is just a change of coordinates), one obtains the injection $\gamma : H_\theta \to M_6(A_\theta)$ given by

$$\gamma(f) = \frac{1}{6}\begin{bmatrix} f_0 & f_1 & f_2 & f_3 & f_4 & f_5 \\ f_5 & f_0 & f_1 & f_2 & f_3 & f_4 \\ f_4 & f_5 & f_0 & f_1 & f_2 & f_3 \\ f_3 & f_4 & f_5 & f_0 & f_1 & f_2 \\ f_2 & f_3 & f_4 & f_5 & f_0 & f_1 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_0 \end{bmatrix}, \quad \gamma(W) = I_q \otimes D.$$

Note that $f_r$ is in the eigenspace $A_\theta^\rho(\omega^{-r}) := \{g \in A_\theta : \rho(g) = \omega^{-r} g\}$. Fix $g$ in this eigenspace. Then

$$\omega^{-r} g(x,y) = \eta_0(g(y, y-x)) \tag{3.1}$$

for all $x, y \in \mathbb{R}$. Along the left edge of $Q$ one gets $\omega^{-r} g(0,y) = \eta_0(g(y,y))$ for $0 \leqslant y \leqslant \frac{1}{2}$. Evaluation of (3.1) at $(z, 1-z)$, for $\frac{1}{3} \leqslant z \leqslant \frac{1}{2}$, one gets (upon reapplying (3.1) and using the fact that $\eta_0^2 = \zeta_0$ as $q$ is even):

$$\begin{aligned} \omega^{-r} g(z, 1-z) &= \eta_0(g(1-z, 1-2z)) = \omega^r \eta_0^2(g(1-2z, -z)) \\ &= \omega^r \zeta_0 \alpha_1^{-1} \alpha_1(g(1-2z, -z)) = \omega^r \zeta_0 \alpha_1^{-1}(g(1-2z, 1-z)). \end{aligned}$$

Thus, $g(z, 1-z) = \omega^{2r} \zeta_0 \alpha_1^{-1}(g(1-2z, 1-z))$. This gives

$$\begin{aligned} &A_\theta^\rho(\omega^{-r}) \\ &= \left\{ g \in C(Q, M_q) : \begin{array}{rr} g(0,y) = \omega^r \eta_0(g(y,y)), & 0 \leqslant y \leqslant \frac{1}{2}, \\ g(z, 1-z) = \omega^{2r} \zeta_0 \alpha_1^{-1}(g(1-2z, 1-z)), & \frac{1}{3} \leqslant z \leqslant \frac{1}{2} \end{array} \right\}. \end{aligned}$$

For $r = 0$ this is the realization obtained in Section 4.4 of [8]. This shows that $H_\theta$ is isomorphic to the $C^*$-algebra

$$\begin{aligned} \mathcal{T}_\theta := \Big\{ & F \in C(Q, M_q \otimes M_6) : \\ & \begin{array}{rr} F(0,y) = (\eta_0 \otimes \mathrm{Ad}_{D^{-1}})(F(y,y)), & 0 \leqslant y \leqslant \frac{1}{2}, \\ F(z, 1-z) = (\zeta_0 \alpha_1^{-1} \otimes \mathrm{Ad}_{D^{-2}})(F(1-2z, 1-z)), & \frac{1}{3} \leqslant z \leqslant \frac{1}{2} \end{array} \Big\}, \end{aligned}$$

where $\mathrm{Ad}_C(\cdot) = C(\cdot)C^*$. As has been done before (in the Fourier case [14]) and still carries through in our case, there is an isomorphism $\beta : \mathcal{T}_\theta \to S_\theta$ where

$$S_\theta := \left\{ F \in C(\mathbb{S}^2, M_q \otimes M_6) : \begin{array}{ccc} F(s_0) & \leftrightarrow & W_0 \otimes D \\ F(s_1) & \leftrightarrow & U_0^{p'} \Gamma_0 \otimes D^3 \\ F(s_2) & \leftrightarrow & U_0^{p'} W_0^2 \otimes D^2 \end{array} \right\}$$

where $s_0 = (0,0), s_1 = (0, 1/2), s_2 = (1/3, 2/3)$ are the singular points and inserting $W_1 = U_0^{-p'}$. (Here, "$A \leftrightarrow B$" means $AB = BA$.) For $g \in \mathcal{T}_\theta$ one defines $\beta(g)$ to be the continuous function on $Q$ such that

$$\beta(g)(s) := (R_s \otimes D_s) \cdot g(s) \cdot (R_s \otimes D_s)^{-1}$$

for $s \in Q - \{s_0, s_1, s_2\}$, where $s \mapsto R_s$ and $s \mapsto D_s$ are unitary-valued maps on $Q$, with respective values in $M_q$ and $M_6$, that are continuous on $Q - \{s_0, s_1, s_2\}$ and have edge-limits as indicated in Figures 1 and 2. (See [8].) The mapping $D_s$ can be chosen to be diagonal-valued (since the edge limits are all diagonal), a fact used below. These maps have jump discontinuities at the singular points, but they are carefully chosen so that $\beta(g)(s)$ is well-defined, continuous on $Q$, and has the same values on the corresponding edges, so that it extends to a continuous function on $\mathbb{S}^2$. Composing $\beta$ with the isomorphism $H_\theta \to \mathcal{T}_\theta$ described above one obtains the isomorphism $\beta\gamma : H_\theta \to S_\theta$ that gives the singular sphere realization of the crossed product (in the case $\theta$ is rational).

It is easy to see that the canonical (normalized) trace on $S_\theta$, which arises from that of $A_\theta$ given in Section 2, is given by

$$\tau(F) = \frac{1}{q} \iint_Q \mathrm{Tr}_{6q}(F(x,y))\, \mathrm{d}x\mathrm{d}y. \tag{3.2}$$

Consider the following trace functionals

$$\tau_{0k}(F) = \mathrm{Tr}(F(s_0)\,(W_0 \otimes D)^k), \quad k = 0,1,2,3,4,5;$$

$$\tau_{1k}(F) = \mathrm{Tr}(F(s_1)\,(U_0^{p'} \Gamma_0 \otimes D^3)^k), \quad k = 0,1;$$

$$\tau_{2k}(F) = \mathrm{Tr}(F(s_2)\,(U_0^{p'} W_0^2 \otimes D^2)^k), \quad k = 0,1,2.$$

(These are in fact tracial maps on $S_\theta$.) To simplify, denote the underlying unitaries in each case by $w_j \otimes D_j$, $j = 0,1,2$, so that all these traces can all be written as

$$\tau_{jk}(F) = \mathrm{Tr}(F(s_j)(w_j \otimes D_j)^k).$$

Let $Y := \{s_0, s_1, s_2\}$. Fixing $f \in A_\theta$ and expanding $\gamma(f)$ as

$$\gamma(f) = \frac{1}{6}\Big(f_0 \otimes I_6 + \sum_{j=1}^{3} f_j \otimes (\text{matrices with zero diagonal})\Big)$$

one has, for $s$ in $Q - Y$,

$$\begin{aligned}\beta(\gamma(f))(s) &= (R_s \otimes D_s) \cdot \gamma(f)(s) \cdot (R_s \otimes D_s)^{-1} \\ &= \frac{1}{6}(R_s f_0(s) R_s^*) \otimes I_6 + \frac{1}{6}\sum_{j=1}^{5}(R_s f_j(s) R_s^*) \otimes (\text{matrices with zero diagonal})\end{aligned}$$

and since $\beta(\gamma(W)) = \beta(I_q \otimes D) = I_q \otimes D$ (viewed as a constant function on $Q$) and $D_j$ are all diagonal, then using the same idea as in [14] one gets

$$\tau_{0k}(\beta(\gamma(f)\gamma(W^r))) = \delta_6^{r+k}\, \mathrm{Tr}(f_0(0,0) W_0^k), \quad k = 0, \dots, 5.$$

(Note: $\mathrm{Tr}(D^n) = 6\delta_6^n$.) Similarly, for the other two singularities one gets

$$\tau_{1k}(\beta(\gamma(f)\,\gamma(W^r))) = \delta_6^{r+3k}\, \mathrm{Tr}(f_0(0,1/2)(U_0^{p'} \Gamma_0)^k), \quad k = 0,1;$$

$$\tau_{2k}(\beta(\gamma(f)\,\gamma(W^r))) = \delta_6^{r+2k}\, \mathrm{Tr}(f_0(1/3,2/3)\,(U_0^{p'} W_0^2)^k), \quad k = 0,1,2.$$

There is no confusion in denoting by $U, V, W$ the unitaries in $S_\theta$ corresponding to the original unitaries $U, V, W$ in $B_\theta$ under the isomorphism $\beta\gamma$. With $f = U^m V^n$ these yield

$$\tau_{0k}(U^m V^n W^r) = \delta_6^{r+k}\,\mathrm{Tr}(f_0(0,0)W_0^k), \quad k = 0,\ldots,5;$$
$$\tau_{1k}(U^m V^n W^r) = \delta_6^{r+3k}\,\mathrm{Tr}(f_0(0,1/2)(U_0^{p'}\Gamma_0)^k), \quad k = 0,1;$$
$$\tau_{2k}(U^m V^n W^r) = \delta_6^{r+2k}\,\mathrm{Tr}(f_0(1/3,2/3)(U_0^{p'}W_0^2)^k), \quad k = 0,1,2.$$

We are now ready to relate the traces $\{\tau_{jk}\}$ with the original traces $\{T_{jk}\}$.

PROPOSITION 3.1. *With $\frac{p}{q} \in \mathbb{P}$, one has*

$$\tau_{01} = 3\sqrt{2}(1-\mathrm{i})T_{10}, \quad \tau_{02} = 6\mathrm{i}[(1+\omega)T_{20} - \omega T_{21}],$$
$$\tau_{03} = 6\sqrt{2}(1+\mathrm{i})T_{30}, \quad \tau_{11} = 4(T_{31} - T_{30}), \quad \tau_{21} = 3\mathrm{i}\omega\lambda^{(p')^2/6}[(\omega-2)T_{20} - \omega T_{21}].$$

*Proof.* We shall make free use of the results obtained in Lemma 2.4. We take $f = U^m V^n$ so that

$$\begin{aligned} f_0 &= \sum_{j=0}^{5} \rho^j(U^m V^n) \\ &= U^m V^n + U^{-m}V^{-n} + \lambda^{-n^2/2-mn}(U^{-n}V^{m+n} + U^n V^{-(m+n)}) \\ &\quad + \lambda^{-m^2/2-mn}(U^{-(m+n)}V^m + U^{m+n}V^{-m}) \end{aligned}$$

or

$$\begin{aligned} f_0(x,y) &= e((mx+ny)/q)U_0^m V_0^n + e(-(mx+ny)/q)U_0^{-m}V_0^{-n} \\ &+ \lambda^{-n^2/2-mn}(e(((m+n)y-nx)/q)U_0^{-n}V_0^{m+n} + e((nx-(m+n)y)/q)U_0^n V_0^{-(m+n)}) \\ &+ \lambda^{-m^2/2-mn}(e((my-(m+n)x)/q)U_0^{-(m+n)}V_0^m + e(((m+n)x-my)/q)U_0^{m+n}V_0^{-m}). \end{aligned}$$

For $\tau_{01}$ one takes $r = 5$ and obtains

$$\begin{aligned} &\tau_{01}(U^m V^n W^5) \\ &\quad = \mathrm{Tr}(f_0(0,0)W_0) \\ &\quad = 2\mathrm{Tr}(U_0^m V_0^n W_0) + 2\lambda^{-n^2/2-mn}\mathrm{Tr}(U_0^{-n}V_0^{m+n}W_0) + 2\lambda^{-m^2/2-mn}\mathrm{Tr}(U_0^{m+n}V_0^{-m}W_0) \\ &\quad = \frac{2(1-\mathrm{i})}{\sqrt{2}}(\lambda^{(m^2+n^2)/2} + \lambda^{-n^2/2-mn}\lambda^{(n^2+(m+n)^2)/2} + \lambda^{-m^2/2-mn}\lambda^{((m+n)^2+m^2)/2}) \\ &\quad = 3\sqrt{2}(1-\mathrm{i})\lambda^{(m^2+n^2)/2} = 3\sqrt{2}(1-\mathrm{i})T_{10}(U^m V^n W^5). \end{aligned}$$

For $\tau_{02}$ one takes $r = 4$ and obtains (recalling that $p \equiv 1 \bmod 3$)

$$\begin{aligned} &\tau_{02}(U^m V^n W^4) = \mathrm{Tr}(f_0(0,0)W_0^2) \\ &\quad = 2\mathrm{Tr}(U_0^m V_0^n W_0^2) + 2\lambda^{-n^2/2-mn}\mathrm{Tr}(U_0^{-n}V_0^{m+n}W_0^2) + 2\lambda^{-m^2/2-mn}\mathrm{Tr}(U_0^{m+n}V_0^{-m}W_0^2) \end{aligned}$$

$$= 2\mathrm{i}\omega^{-2p(m-n)^2}\lambda^{(m-n)^2/6} + 2\mathrm{i}\omega^{-2p(m+2n)^2}\lambda^{-n^2/2-mn}\lambda^{(m+2n)^2/6}$$
$$+ 2\mathrm{i}\omega^{-2p(2m+n)^2}\lambda^{-m^2/2-mn}\lambda^{(2m+n)^2/6}$$
$$= 6\mathrm{i}\omega^{-2p(m-n)^2}\lambda^{(m-n)^2/6} = 6\mathrm{i}[(1+\omega)T_{20}(U^mV^nW^4) - \omega T_{21}(U^mV^nW^4)].$$

For $\tau_{03}$ one takes $r = 3$ and obtains

$$\tau_{03}(U^mV^nW^3)$$
$$= \mathrm{Tr}(f_0(0,0)W_0^3)$$
$$= 2\mathrm{Tr}(U_0^mV_0^nW_0^3) + 2\lambda^{-n^2/2-mn}\mathrm{Tr}(U_0^{-n}V_0^{m+n}W_0^3) + 2\lambda^{-m^2/2-mn}\mathrm{Tr}(U_0^{m+n}V_0^{-m}W_0^3)$$
$$= 2\sqrt{2}(1+\mathrm{i})(\lambda^{-mn/2}\delta_2^m\delta_2^n + \lambda^{-n^2/2-mn}\lambda^{(m+n)n/2}\delta_2^n\delta_2^{m+n} + \lambda^{-m^2/2-mn}\lambda^{(m+n)m/2}\delta_2^{m+n}\delta_2^n)$$
$$= 6\sqrt{2}(1+\mathrm{i})T_{30}(U^mV^nW^3).$$

For $\tau_{11}$ one observes that $pp' \equiv -1 \bmod 2q$ which allows us to write $e(\alpha/2q) = \lambda^{-\alpha p'/2}$, where $\alpha$ is a linear combination of $m$ and $n$. One then takes $r = 3$ and obtains

$$\tau_{11}(U^mV^nW^3) = \mathrm{Tr}(f_0(0,(1/2))U_0^{p'}\Gamma_0) = \frac{\sqrt{2}}{(1+\mathrm{i})}\mathrm{Tr}(f_0(0,(1/2))U_0^{p'}W_0^3)$$
$$= \frac{\sqrt{2}}{(1+\mathrm{i})}[e(n/2q)\mathrm{Tr}(U_0^mV_0^nU_0^{p'}W_0^3) + e(-n/2q)\mathrm{Tr}(U_0^{-m}V_0^{-n}U_0^{p'}W_0^3)$$
$$+\lambda^{-n^2/2-mn}(e((m+n)/2q)\mathrm{Tr}(U_0^{-n}V_0^{m+n}U_0^{p'}W_0^3)$$
$$+e(-(m+n)/2q)\mathrm{Tr}(U_0^nV_0^{-(m+n)}U_0^{p'}W_0^3))$$
$$+\lambda^{-m^2/2-mn}(e(m/2q)\mathrm{Tr}(U_0^{-(m+n)}V_0^mU_0^{p'}W_0^3) + e(-m/2q)\mathrm{Tr}(U_0^{m+n}V_0^{-m}U_0^{p'}W_0^3))]$$
$$= \frac{\sqrt{2}}{(1+\mathrm{i})}[\lambda^{np'/2}\mathrm{Tr}(U_0^{m+p'}V_0^nW_0^3) + \lambda^{-np'/2}\mathrm{Tr}(U_0^{-m+p'}V_0^{-n}W_0^3)$$
$$+\lambda^{-n^2/2-mn}(\lambda^{(m+n)p'/2}\mathrm{Tr}(U_0^{-n+p'}V_0^{m+n}W_0^3) + \lambda^{-(m+n)p'/2}\mathrm{Tr}(U_0^{n+p'}V_0^{-(m+n)}W_0^3))$$
$$+\lambda^{-m^2/2-mn}(\lambda^{mp'/2}\mathrm{Tr}(U_0^{-(m+n)+p'}V_0^mW_0^3) + \lambda^{-mp'/2}\mathrm{Tr}(U_0^{m+n+p'}V_0^{-m}W_0^3))]$$
$$= 2\lambda^{np'/2}\lambda^{-(m+p')n/2}\delta_2^{m+p'}\delta_2^n + 2\lambda^{-np'/2}\lambda^{(-m+p')n/2}\delta_2^{-m+p'}\delta_2^n$$
$$+ 2\lambda^{-n^2/2-mn}[\lambda^{(m+n)p'/2}\lambda^{-(-n+p')(m+n)/2}\delta_2^{-n+p'}\delta_2^{m+n}$$
$$+ \lambda^{-(m+n)p'/2}\lambda^{(n+p')(m+n)/2}\delta_2^{n+p'}\delta_2^{m+n}]$$
$$+2\lambda^{-m^2/2-mn}[\lambda^{mp'/2}\lambda^{(m+n-p')m/2}\delta_2^{-(m+n)+p'}\delta_2^m + \lambda^{-mp'/2}\lambda^{(m+n+p')m/2}\delta_2^{m+n+p'}\delta_2^m]$$
$$= 2\lambda^{-mn/2}(2\delta_2^{m-1}\delta_2^n + 2\delta_2^{n-1}\delta_2^{m+n} + 2\delta_2^{m+n-1}\delta_2^m)$$
$$= 4\lambda^{-mn/2}(1 - \delta_2^m\delta_2^n) = 4(T_{31}(U^mV^nW^3) - T_{30}(U^mV^nW^3)).$$

Finally, for $\tau_{21}$ we observe that $pp' \equiv -1 \bmod 6q$ which allows us to write $e(\alpha/3q) = \lambda^{-\alpha p'/3}$, where again $\alpha$ is a linear combination of $m$ and $n$. One takes $r = 4$ and obtains

$$\begin{aligned}
&\tau_{21}(U^mV^nW^4) = \mathrm{Tr}(f_0(1/3,2/3)(U_0^{p'}W_0^2))\\
&= e((m+2n)/3q)\lambda^{np'}\mathrm{Tr}(U_0^{m+p'}V_0^nW_0^2)\\
&\quad + e(-(m+2n)/3q)\lambda^{-np'}\mathrm{Tr}(U_0^{-m+p'}V_0^{-n}W_0^2)\\
&\quad + \lambda^{-n^2/2-mn}e((2m+n)/3q)\lambda^{(m+n)p'}\mathrm{Tr}(U_0^{-n+p'}V_0^{m+n}W_0^2)\\
&\quad + \lambda^{-n^2/2-mn}e(-(2m+n)/3q)\lambda^{-(m+n)p'}\mathrm{Tr}(U_0^{n+p'}V_0^{-(m+n)}W_0^2)\\
&\quad + \lambda^{-m^2/2-mn}e((m-n)/3q)\lambda^{mp'}\mathrm{Tr}(U_0^{-(m+n)+p'}V_0^{m}W_0^2)\\
&\quad + \lambda^{-m^2/2-mn}e((n-m)/3q)\lambda^{-mp'}\mathrm{Tr}(U_0^{m+n+p'}V_0^{-m}W_0^2)\\
&= \frac{\mathrm{i}\lambda^{p'^2/6}\lambda^{(m-n)^2/6}}{\omega^{2pp'^2}\omega^{2p(m-n)^2}}[\omega^{2pp'(m-n)}\lambda^{-(m+2n)p'/3}\lambda^{3np'/3}\lambda^{(m-n)p'/3}\\
&\quad + \omega^{2pp'(n-m)}\lambda^{(m+2n)p'/3}\lambda^{-3np'/3}\lambda^{(n-m)p'/3}\\
&\quad + \omega^{4pp'(m+2n)}\lambda^{-(2m+n)p'/3}\lambda^{3(m+n)p'/3}\lambda^{-(m+2n)p'/3}\\
&\quad + \omega^{2pp'(m+2n)}\lambda^{(2m+n)p'/3}\lambda^{-3(m+n)p'/3}\lambda^{(m+2n)p'/3}\\
&\quad + \omega^{4pp'(2m+n)}\lambda^{-(m-n)p'/3}\lambda^{3mp'/3}\lambda^{-(2m+n)p'/3}\\
&\quad + \omega^{2pp'(2m+n)}\lambda^{(m-n)p'/3}\lambda^{-3mp'/3}\lambda^{(2m+n)p'/3}]\\
&= \mathrm{i}\lambda^{p'^2/6}\omega^{-2}\omega^{4(m-n)^2}\lambda^{(m-n)^2/6}\\
&\qquad \cdot[\omega^{4(m-n)}+\omega^{2(m-n)}+\omega^{2(m+2n)}+\omega^{4(m+2n)}+\omega^{2(2m+n)}+\omega^{4(2m+n)}]\\
&= -3\mathrm{i}\lambda^{p'^2/6}\omega\omega^{4(m-n)^2}\lambda^{(m-n)^2/6}(\omega^{4(m-n)}+\omega^{2(m-n)})\\
&= 3\mathrm{i}\lambda^{p'^2/6}\omega[(\omega-2)T_{20}(U^mV^nW^4)-\omega T_{21}(U^mV^nW^4)],
\end{aligned}$$

since $\omega^{4k^2}(\omega^{4k}+\omega^{2k}) = (2-\omega)\delta_3^k+\omega$. This completes the proof. ∎

## 4. AN AUXILIARY BASIS FOR $K_0(H_{p/q})$

As a step toward showing that the ten modules generate $K_0(H_\theta)$ (for rational $\theta$), we consider in this section an auxiliary basis for $K_0(H_\theta)$ that arises naturally from the realization of $H_\theta$ as a sphere with singularities, as obtained in the previous section. This will enable one to show that the range of the reduced character $\mathbf{T}'$ on $K_0(H_\theta)$ (as defined in Section 2) is equal to its range on $\mathcal{R}_\theta$. To do this, we shall assume that $\theta$ is in the dense set of rationals $\mathbb{P}$, as defined in Section 2.

Let $\theta = \frac{p}{q}$ be any rational in $(0,1)$. Let $F_0$ be a rank one subprojection of the spectral projection of $\omega^{-1/4}W_0$ (which has order six) corresponding to the eigenvalue 1 (corresponding to the singularity $s_0 = (0,0)$). Similarly, let $F_1$ be such a projection for $U_0^{p'}\Gamma_0$, and $F_2$ for $\mathrm{i}^{-1/3}\lambda^{-(1/6)(p'')^2}U_0^{p'}W_0^2$. These are all projections in $M_q(\mathbb{C})$, and we think of them as being associated with the singular points $s_0, s_1, s_2$, respectively (cf. definition of $S_\theta$ in Section 3). Thus, by definition, one has

$$W_0F_0 = \omega^{1/4}F_0, \quad U_0^{p'}\Gamma_0F_1 = F_1, \quad U_0^{p'}W_0^2F_2 = \mathrm{i}^{1/3}\lambda^{(1/6)(p'')^2}F_2.$$

Now consider the rank one projection $e_k^j := F_j \otimes E_k$ for $j = 0,1,2$ and $k = 1,2,3,4,5,6$, where $E_k \in M_6(\mathbb{C})$ is the diagonal matrix that has 1 at the $k$-th diagonal entry and zeros elsewhere. It will be convenient to introduce the following notation. If $e, f, g$ are matrix projections of equal rank, we denote by $[e,f,g]$ a smooth projection-valued function on $\mathbb{S}^2$ such that

$$[e,f,g](s_0) = e, \quad [e,f,g](s_1) = f, \quad [e,f,g](s_2) = g.$$

(Such a function clearly exists since the projections have equal rank.) So $[e,f,g]$ defines a projection in $S_\theta$, and hence a unique positive class in $K_0(S_\theta)$. Now consider the following nine projections in $S_\theta$:

$$\begin{aligned}(4.1)\qquad &[e_1^0,e_1^1,e_1^2], \quad [e_4^0,e_4^1,e_4^2], \quad [e_1^0,e_2^1,e_2^2], \quad [e_2^0,e_2^1,e_2^2], \quad [e_5^0,e_5^1,e_5^2],\\ &[e_1^0,e_2^1,e_3^2], \quad [e_3^0,e_3^1,e_3^2], \quad [e_6^0,e_6^1,e_6^2], \quad [e_1^0,e_3^1,e_3^2].\end{aligned}$$

We claim that these projections, together with one other class in the kernel of $\mathbf{T}'$, which is $\kappa_{p,q}$, form a basis for $K_0(S_\theta) \cong K_0(H_\theta)$.

Since $W_0 \otimes D^{-1}$ has order six, let $n_k$ be the dimension of its eigenspace corresponding to the eigenvalue $\omega^k$, $k = 1,\ldots,6$. (So, $\sum_k n_k = 6q$.) Similarly, let $k, 6q-k$ be the spectral dimensions of $\Gamma_0W_1 \otimes D^3$ (which has order 2), and $m_1, m_2, m_3$ those of $W_0^2W_1 \otimes D^{-2}$ (which has order 3). The commutant of $W_0 \otimes D^{-1}$ in $M_q \otimes M_6$ is isomorphic to $\bigoplus_{k=1}^{6} M_{n_k}$. For $\Gamma_0W_1 \otimes D^3$ the commutant algebra is isomorphic to $M_k \oplus M_{6q-k}$, and for $W_0^2W_1 \otimes D^{-2}$ it is $\bigoplus_{k=1}^{3} M_{m_j}$. (Although these dimensions are known from [8] and [1], their exact values will not be needed here.) Identifying each commutant in this way with its corresponding matrix algebra direct sum, one has the surjective map obtained by evaluations

$$\mathcal{E}: \; S_\theta \;\longrightarrow\; \mathbb{F} := \Big(\bigoplus_{k=1}^{6} M_{n_k}\Big) \oplus \Big(\bigoplus_{j=1}^{3} M_{m_j}\Big) \oplus (M_k \oplus M_{6q-k})$$

$$(4.2)\qquad \mathcal{E}(F) = (F(s_0);\; F(s_2);\; F(s_1))$$

where $F(s_1) \in M_k \oplus M_{6q-k}$. Letting $J$ denote the kernel of $\mathcal{E}$, one has the short exact sequence

$$(4.3)\qquad 0 \longrightarrow J \xrightarrow{\;j\;} S_\theta \xrightarrow{\;\mathcal{E}\;} \mathbb{F} \longrightarrow 0$$

where $j : J \hookrightarrow S_\theta$ is inclusion. Under the induced map

$$\mathcal{E}_* : K_0(S_\theta) \to K_0(\mathbb{F}) \cong \mathbb{Z}^6 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}^2,$$

one gets (since $F_j$ has rank one)

$$\begin{aligned}
[e^0_1, e^1_1, e^2_1] &\mapsto (1,0,0,0,0,0);\ (1,0,0);\ (1,0)\\
[e^0_2, e^1_2, e^2_2] &\mapsto (0,1,0,0,0,0);\ (0,1,0);\ (0,1)\\
[e^0_3, e^1_3, e^2_3] &\mapsto (0,0,1,0,0,0);\ (0,0,1);\ (1,0)\\
[e^0_4, e^1_4, e^2_4] &\mapsto (0,0,0,1,0,0);\ (1,0,0);\ (0,1)\\
[e^0_5, e^1_5, e^2_5] &\mapsto (0,0,0,0,1,0);\ (0,1,0);\ (1,0)\\
[e^0_6, e^1_6, e^2_6] &\mapsto (0,0,0,0,0,1);\ (0,0,1);\ (0,1)\\
[e^0_1, e^1_2, e^2_2] &\mapsto (1,0,0,0,0,0);\ (0,1,0);\ (0,1)\\
[e^0_1, e^1_2, e^2_3] &\mapsto (1,0,0,0,0,0);\ (0,1,0);\ (1,0)\\
[e^0_1, e^1_3, e^2_3] &\mapsto (1,0,0,0,0,0);\ (0,0,1);\ (1,0).
\end{aligned} \tag{4.4}$$

Since $J$ is the ideal of all functions $\mathbb{S}^2 \to M_{6q}$ vanishing at the three singular points $s_j$, it is isomorphic to $R_0 \otimes M_{6q}$ where

$$R_0 := \{f \in C(\mathbb{S}^2, \mathbb{C}) :\ f(s_0) = f(s_1) = f(s_2) = 0\}. \tag{4.5}$$

Hence $K_0(J) \cong K_0(R_0) \cong \mathbb{Z}$ and $K_1(J) \cong K_1(R_0) \cong \mathbb{Z}^2$. Now consider the following part of the six-term exact $K$-theory sequence associated with (4.3)

$$\mathbb{Z} \cong K_0(J) \xrightarrow{\ j_*\ } K_0(S_\theta) \xrightarrow{\ \mathcal{E}_*\ } K_0(\mathbb{F}) = \mathbb{Z}^{11} \xrightarrow{\ \delta_0\ } K_1(J) \cong \mathbb{Z}^2 \longrightarrow 0 \tag{4.6}$$

where $\delta_0$, the connecting homomorphism, is surjective (as $K_1(S_\theta) = 0$, by Theorems 3 and 4 of [7]). Since $K_0(S_\theta) \cong \mathbb{Z}^{10}$, and since the nine elements in $\mathbb{Z}^{11}$ given by the right sides of (4.4) together with

$$(0,0,0,0,0,0);\ (0,0,0);\ (0,1) \quad \text{and} \quad (1,0,0,0,0,0);\ (0,0,0);\ (0,0) \tag{4.7}$$

constitute an $11 \times 11$ matrix whose determinant is $\pm 1$, it follows that $\mathcal{E}_*(K_0(S_\theta))$ is spanned by the images of the nine projections in (4.1). These, together with the image under $j_*$ of a generator $\xi$ of $K_0(J)$, constitute a basis for $K_0(S_\theta)$. The remaining basis element $j_*(\xi)$ will be shown to be $\pm\kappa_{p,q}$ (see Corollary 4.4).

REMARK 4.1. By showing that $\delta_0$ maps the two $K_0$-elements corresponding to (4.7) are mapped onto generators of $K_1(J)$ one obtains another proof that $K_0(S_{p/q}) \cong \mathbb{Z}^{10}$ and $K_1(S_{p/q}) = 0$.

Now let us calculate the traces $T_{10}, T_{20}, T_{21}, T_{30}, T_{31}$ on these nine projections. In view of Proposition 3.1 (with $\theta = p/q \in \mathbb{P}$), for $k = 1, \ldots, 6$ one has (with "$*$" denoting any value)

$$3\sqrt{2}(1-\mathrm{i})T_{10}[e^0_k, e^1_*, e^2_*] = \tau_{01}[e^0_k, e^1_*, e^2_*] = \mathrm{Tr}(e^0_k(W_0 \otimes D)) = \mathrm{Tr}(F_0 W_0)\,\mathrm{Tr}(E_k D)$$

since $W_0F_0 = \omega^{1/4}F_0$, $\mathrm{Tr}(F_0W_0) = \omega^{1/4}$, and $\mathrm{Tr}(E_kD) = \omega^{k-1}$, one gets

$$T_{10}[e^0_k, e^1_*, e^2_*] = \frac{(1+\mathrm{i})}{6\sqrt{2}}\omega^{1/4}\omega^{k-1} = \frac{1}{6}\omega^k$$

($k = 1, \dots, 6$). This gives the values for $T_{10}$ in Table 2. Similar calculations for the other traces yields the following equalities and the remaining values in the table,

$$T_{21}[e^0_k, e^1_*, e^2_*] = -\frac{1}{6}\omega^{2k}, \quad T_{30}[e^0_k, e^1_*, e^2_*] = \frac{1}{12}(-1)^{k-1},$$
$$T_{31}[e^0_k, e^1_\ell, e^2_*] = \frac{1}{12}(-1)^{k-1} + \frac{1}{4}(-1)^{\ell-1}, \quad T_{20}[e^0_k, e^1_*, e^2_m] = -\frac{1}{9}\omega^{2(m-1)} - \frac{1}{18}\omega^{2k},$$
$$\omega T_{21}[e^0_k, e^1_*, e^2_m] = (1+\omega)T_{20}[e^0_k, e^1_*, e^2_m] + \frac{\mathrm{i}}{6}\omega^{1/2}\omega^{2(k-1)}.$$

(To facilitate the computations, one uses the equalities $1+\omega = \mathrm{i}\sqrt{3}\omega^{-1}$, $\mathrm{i}\omega^{1/2} = \omega^2$, $\omega^2 = \omega - 1$.)

**Table 2.** Values of $\mathbf{T}'$ for $p/q \in \mathbb{P}$

| $K_0$-class | $\tau$ | $\phi_0$ | $\phi_0'$ | $\phi_1$ | $\phi_1'$ | $\phi_2$ | $\phi_2'$ | $T_{30}$ | $T_{31}$ |
|---|---|---|---|---|---|---|---|---|---|
| $[e^0_1, e^1_1, e^2_1]$ | $\frac{1}{6q}$ | $\frac{1}{12}$ | $\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $\frac{\sqrt{3}}{36}$ | $\frac{1}{4}$ | $\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $\frac{1}{3}$ |
| $[e^0_2, e^1_2, e^2_2]$ | $\frac{1}{6q}$ | $-\frac{1}{12}$ | $\frac{\sqrt{3}}{12}$ | $-\frac{1}{12}$ | $\frac{\sqrt{3}}{36}$ | $-\frac{1}{4}$ | $\frac{\sqrt{3}}{12}$ | $-\frac{1}{12}$ | $-\frac{1}{3}$ |
| $[e^0_3, e^1_3, e^2_3]$ | $\frac{1}{6q}$ | $-\frac{1}{6}$ | $0$ | $0$ | $-\frac{\sqrt{3}}{18}$ | $0$ | $-\frac{\sqrt{3}}{6}$ | $\frac{1}{12}$ | $\frac{1}{3}$ |
| $[e^0_4, e^1_4, e^2_4]$ | $\frac{1}{6q}$ | $-\frac{1}{12}$ | $-\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $\frac{\sqrt{3}}{36}$ | $\frac{1}{4}$ | $\frac{\sqrt{3}}{12}$ | $-\frac{1}{12}$ | $-\frac{1}{3}$ |
| $[e^0_5, e^1_5, e^2_5]$ | $\frac{1}{6q}$ | $\frac{1}{12}$ | $-\frac{\sqrt{3}}{12}$ | $-\frac{1}{12}$ | $\frac{\sqrt{3}}{36}$ | $-\frac{1}{4}$ | $\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $\frac{1}{3}$ |
| $[e^0_6, e^1_6, e^2_6]$ | $\frac{1}{6q}$ | $\frac{1}{6}$ | $0$ | $0$ | $-\frac{\sqrt{3}}{18}$ | $0$ | $-\frac{\sqrt{3}}{6}$ | $-\frac{1}{12}$ | $-\frac{1}{3}$ |
| $[e^0_1, e^1_2, e^2_2]$ | $\frac{1}{6q}$ | $\frac{1}{12}$ | $\frac{\sqrt{3}}{12}$ | $-\frac{1}{12}$ | $-\frac{\sqrt{3}}{36}$ | $-\frac{1}{12}$ | $\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $-\frac{1}{6}$ |
| $[e^0_1, e^1_2, e^2_3]$ | $\frac{1}{6q}$ | $\frac{1}{12}$ | $\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $-\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $-\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $-\frac{1}{6}$ |
| $[e^0_1, e^1_3, e^2_3]$ | $\frac{1}{6q}$ | $\frac{1}{12}$ | $\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $-\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $-\frac{\sqrt{3}}{12}$ | $\frac{1}{12}$ | $\frac{1}{3}$ |

(The canonical trace values are immediate from the expression for $\tau$ in (3.2).)

One is now in a position to check that each row of Table 2 is in the $\mathbb{Z}$-span of the rows of Table 1, and vice versa (a simple computer program can be used to verify this quickly). (Recall that in Table 1, $\phi_k, \phi_k'$ are the real and imaginary components of $T_{ij}$.) In checking this, however, it is helpful to use the fact that $\frac{p}{q}$ is in $\mathbb{P}$, that $1 + 2^{2k+1}$ and $2^{2k} - 1$ are divisible by 3, and that $q - 2$ is divisible by 6.

We have therefore proved the following.

PROPOSITION 4.2. *For any $\theta \in \mathbb{P}$, one has $\mathbf{T}'(K_0(H_\theta)) = \mathbf{T}'(\mathcal{R}_\theta)$.*

By the same proof as in [14] (Section 5) one obtains, almost mutatis mutandis, the following result.

PROPOSITION 4.3. *For any positive rational $\theta = \frac{p}{q} < 1$, the class $\kappa_{p,q} \in K_0(S_\theta)$ is the image of a generator of $K_0(J) \cong \mathbb{Z}$ under the canonical map $j_* : K_0(J) \to K_0(S_\theta)$.*

Combining this with what we have just shown one obtains:

COROLLARY 4.4. *For* $\theta \in \mathbb{P}$, *one has* $\mathrm{Ker}(\mathbf{T}') = \mathbb{Z} j_*(\xi) = \mathbb{Z}\kappa_{p,q}$.

## 5. CONCLUSIONS

PROPOSITION 5.1. *For* $\theta \in \mathbb{P}$, *the classes* $[1], [p_0], [p_1], [p_2], [p_3], [p_4], [q_0], [q_1], [r], [\mathcal{M}_6]$ *form a basis for the group* $K_0(H_\theta) = \mathbb{Z}^{10}$.

*Proof.* In view of Theorem 2.1, these classes are already independent (for each $\theta$), so it is enough to show that they generate. Pick any $x$ in $K_0(S_\theta)$. From Proposition 4.2 (since $\theta \in \mathbb{P}$) one has $\mathbf{T}'(x) = \mathbf{T}'(y)$ for some $y \in \mathcal{R}_\theta$. Therefore, by the Corollary 4.4, $x - y = m\kappa_{p,q}$ for some integer $m$ (where $\theta = \frac{p}{q}$). Since $\kappa_{p,q}$ is already in $\mathcal{R}_\theta$, the result follows. ∎

Using the exact same techniques of [14] one obtains the following result.

THEOREM 5.2. (Range of the Connes-Chern character.) *For any* $0 < \theta < 1$ *one has the range of the Connes-Chern character:* $\mathbf{T}(K_0(H_\theta)) = \mathbf{T}(\mathcal{R}_\theta)$, *where* $\mathcal{R}_\theta$ *is the subgroup of* $K_0(H_\theta)$ *generated by the ten classes in Table* 1. *More specifically, the range is spanned by the rows in Table* 1.

THEOREM 5.3. *For each* $\theta > 0$ *the ten canonical classes form a basis for the group* $K_0(H_\theta) = \mathbb{Z}^{10}$.

*Proof.* We use the result of Polishchuk [10] that $K_0(H_\theta) \cong \mathbb{Z}^{10}$. Since $\mathbf{T}$ is injective on $\mathcal{R}_\theta$, whose rank is equal to the rank of $K_0(H_\theta)$, it follows that $\mathbf{T}$ is injective on all of $K_0(H_\theta)$. Now the result follows from Theorem 5.2 since the ten classes are already known to be independent by Theorem 2.1. ∎

The result for $K_1$ can be obtained at this point for a dense $G_\delta$ set of $\theta$'s using essentially the same Baire category argument used in Theorem 7.2-B of [14]. One gets

THEOREM 5.4. *There is a dense* $G_\delta$ *set of parameters* $\theta$ *in* $(0, 1)$ *(containing the rationals) for which* $K_1(H_\theta) = 0$.

Of course, this result will follow from [6] for all $\theta$ since it is shown there that $H_\theta$ is an AF-algebra.

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