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RANK-ONE COMMUTATORS ON INVARIANT SUBSPACES OF THE HARDY SPACE ON THE BIDISK. II

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ABSTRACT. As a continuation of the previous paper, we still study invariant subspaces of $H^2(\Gamma^2)$ with rank $[R_z^*, R_w] = 1$.

KEYWORDS: *Invariant subspace, Hardy space, cross commutator, rank-one operator.*

MSC (2000): Primary 47A15; Secondary 32A35.

1. INTRODUCTION

Let $L^2(\Gamma^2)$ be the Lebesgue space and $H^2(\Gamma^2)$ the Hardy space over $\Gamma^2 = \{(z,w) : |z| = |w| = 1\}$, and D be the open unit disk. A closed subspace M of $L^2(\Gamma^2)$ is called invariant if $zM \subset M$ and $wM \subset M$. We denote by $R_z = R_{z,M}$ and $R_w = R_{w,M}$ the operators on M defined by $R_z f = P_M z f$ and $R_w f = P_M w f$ for $f \in M$, where P_M is the orthogonal projection from $L^2(\Gamma^2)$ onto M. As usual, write $[R_z^*, R_w] = R_z^* R_w - R_w R_z^*$, where R_z^* is the adjoint operator of R_z on M. One easily sees that $[R_z^*, R_w] = 0$ if and only if $w(M \ominus zM) \subset M \ominus zM$. For an invariant subspace M of $H^2(\Gamma^2)$, Mandrekar [7] showed that $[R_z^*, R_w] = 0$ if and only if M is the Beurling type, that is, $M = \varphi H^2(\Gamma^2)$ for an inner function φ. Generally, in [8] Nakazi described all invariant subspaces M of $L^2(\Gamma^2)$ on which $[R_z^*, R_w] = 0$.

The problem discussed in this paper comes from Nakazi's conjecture: if $[R_z^*, R_w] = [R_z^*, R_w]^*$, then $[R_z^*, R_w] = 0$. In [5], Ohno and the first author showed that both $[R_z^*, R_w] = [R_z^*, R_w]^*$ and $[R_z^*, R_w] \neq 0$ hold if and only if

$$M = \varphi\Big(H^2(\Gamma^2) \oplus \frac{1}{w - rz} H^2(\Gamma_z)\Big), \tag{1.1}$$

where φ is a unimodular function on Γ^2, r is a real number with $0 < r < 1$, and $H^2(\Gamma_z)$ is the Hardy space on the unit circle Γ with variable z. In [4], the authors pointed out that there exists an inner function φ on Γ^2 such that $M \subset H^2(\Gamma^2)$

and rank $[R_z^*, R_w] = 1$ for M given in (1.1), and gave some examples of invariant subspaces in $H^2(\Gamma^2)$ with rank $[R_z^*, R_w] = 1$.

Since $[R_z^*, R_w] = 0$ on zM, generally a cross commutator $[R_z^*, R_w]$ is small. In Theorem 2.3 of [12], Yang showed that for an invariant subspace M of $H^2(\Gamma^2)$, $[R_z^*, R_w]$ is Hilbert–Schmidt under a mild condition on M. To understand the smallness of $[R_z^*, R_w]$, it is important to study when rank $[R_z^*, R_w] = 1$.

If M is an invariant subspace of $H^2(\Gamma^2)$ with rank $[R_z^*, R_w] = 1$, there exists a non-zero function $f \in M \ominus zM$ such that $wf \notin M \ominus zM$ and

$$w((M \ominus zM) \ominus \mathbb{C} \cdot f) \subset M \ominus zM.$$

It is known that $R_w^* f \in M \ominus zM$. In this paper, we concentrated on the case of $\overline{\operatorname{span}}\{R_w^{*n} f : n \geqslant 0\} = \mathbb{C} \cdot f$, where $\overline{\operatorname{span}}$ denotes the closed linear span. Under this condition, the function f is connected to non-extreme points in ball $H^\infty(\Gamma_z)$, the closed unit ball of $H^\infty(\Gamma_z)$. In Section 2, we prove that

$$f = \frac{c\varphi H(z)}{w - G(z)} \quad \text{a.e. on } \Gamma^2, \quad c \in \mathbb{C} \text{ with } c \neq 0,$$

where φ is an inner function on Γ^2 and functions $G(z), H(z)$ in ball $H^\infty(\Gamma_z)$ satisfy the following conditions;

(i) $G(z)$ is a non-extreme point in ball $H^\infty(\Gamma_z)$;
(ii) $|H(z)|^2 = 1 - |G(z)|^2$ a.e. on Γ_z.

So, f has a special form.

Conversely, suppose that the functions $G(z), H(z)$ in ball $H^\infty(\Gamma_z)$ satisfy (i)–(ii), and either $G(z)$ or $H(z)$ is not constant. If there is an inner function φ satisfying

$$\frac{\varphi H(z)}{w - G(z)} \in H^2(\Gamma^2), \tag{1.2}$$

then

$$M = \varphi H^2(\Gamma^2) \oplus \frac{\varphi H(z)}{w - G(z)} H^2(\Gamma_z)$$

is an invariant subspace of $H^2(\Gamma^2)$ and rank $[R_z^*, R_w] = 1$. It is a big problem whether there is an inner function φ satisfying (1.2) or not. When $G(z)$ is a function in the disk algebra, we prove the existence of an inner function φ satisfying (1.2).

In [6], deLeeuw and Rudin proved that for a function $G(z)$ in ball $H^\infty(\Gamma_z)$, $G(z)$ is a non-extreme point of ball $H^\infty(\Gamma_z)$ if and only if

$$-\infty < \int_0^{2\pi} \log(1 - |G(e^{i\theta})|) \frac{d\theta}{2\pi},$$

see also pp. 138–139 in [3]. In this case, there exists a function $H_1(z) \in H^\infty(\Gamma_z)$ satisfying $|H_1(z)| = 1 - |G(z)|$ a.e. on Γ_z. Let $f_1 = H_1(z)/(w - G(z))$. Then

$f_1 \in L^\infty(\Gamma^2) \setminus H^\infty(\Gamma^2)$ and

$$(w - G(z))f_1 = H_1(z) \in H^\infty(\Gamma_z).$$

For an invariant subspace M of $H^2(\Gamma^2)$, write

$$\mathcal{M}(M) = \{f \in L^\infty(\Gamma^2) : fM \subset H^2(\Gamma^2)\}.$$

Trivially $H^\infty(\Gamma^2) \subset \mathcal{M}(M)$. As seen in the above, $H^\infty(\Gamma^2) \subsetneqq \mathcal{M}([w - G(z)])$ for a non-extreme point $G(z)$ in ball $H^\infty(\Gamma_z)$, where $[w - G(z)]$ is the invariant subspace of $H^2(\Gamma^2)$ generated by a single function $w - G(z)$. K. Takahashi (unpublished) proved that for a function $G(z)$ in ball $H^\infty(\Gamma_z)$, $\mathcal{M}([w - G(z)]) = H^\infty(\Gamma^2)$ if and only if $G(z)$ is an extreme point in ball $H^\infty(\Gamma_z)$, see p. 495 in [9].

If a function $G(z)$ in ball $H^\infty(\Gamma_z)$ is a non-extreme point, there exists also a function $H_2(z)$ in ball $H^\infty(\Gamma_z)$ satisfying $|H_2(z)|^2 = 1 - |G(z)|^2$ a.e. on Γ_z. The function $H_2(z)/(w - G(z))$ is discussed in Section 2, so Takahashi's theorem is very close to our subject. Since we can not find its proof in references, in Section 3 we include an independent proof.

In [11], from another view point, Sarason studied the difference between extreme and non-extreme points in ball $H^\infty(\Gamma_z)$.

2. RANK-ONE COMMUTATORS

We start from the following lemma. Through this paper, we use the following facts in the sequel:

(i) $\ker R_z^* = M \ominus zM$;
(ii) $[R_z^*, R_w] = R_z^* R_w$ on $M \ominus zM$;
(iii) $[R_z^*, R_w] = 0$ on zM.

LEMMA 2.1. *Let M be an invariant subspace of $L^2(\Gamma^2)$. Then* rank $[R_z^*, R_w] = 1$ *if and only if there exists a non-zero function f in $M \ominus zM$ such that $wf \notin M \ominus zM$ and $wE \subset M \ominus zM$, where $E = (M \ominus zM) \ominus \mathbb{C} \cdot f$.*

Proof. Suppose that rank $[R_z^*, R_w] = 1$. Then there exist functions $f, \eta \in M$ satisfying $f \neq 0, \eta \neq 0$, and

$$[R_z^*, R_w]h = (\eta \otimes f)h = \langle h, f \rangle \eta \tag{2.1}$$

for every $h \in M$. Write $f = f_1 \oplus f_2$, where $f_1 \in M \ominus zM$ and $f_2 \in zM$. We have $0 = [R_z^*, R_w]f_2 = \|f_2\|^2 \eta$. Thus $f \in M \ominus zM$. Since

$$R_z^* wf = R_z^* R_w f = [R_z^*, R_w]f = \|f\|^2 \eta \neq 0,$$

we have $wf \notin M \ominus zM$. By (2.1), $[R_z^*, R_w] = 0$ on E. Hence $wE \subset M \ominus zM$.

Conversely, suppose that there exists a non-zero function $f \in M \ominus zM$ such that $wf \notin M \ominus zM$ and $wE \subset M \ominus zM$. One easily sees that

$$[R_z^*, R_w]M = [R_z^*, R_w](M \ominus zM) = \mathbb{C} \cdot R_z^* R_w f \neq \{0\}.$$

Thus rank $[R_z^*, R_w] = 1$. ∎

Now, we assume that M is an invariant subspace of $H^2(\Gamma^2)$ satisfying rank $[R_z^*, R_w] = 1$. By Lemma 2.1, there exists a non-zero function $f \in M \ominus zM$ such that $wf \notin M \ominus zM$ and

$$wE \subset M \ominus zM, \tag{2.2}$$

where $E = (M \ominus zM) \ominus \mathbb{C} \cdot f$. One easily sees that $R_w^*(M \ominus zM) \subset M \ominus zM$. Let

$$E_0 = \overline{\mathrm{span}}\{R_w^{*n} f : n \geqslant 0\}. \tag{2.3}$$

Then $E_0 \subset M \ominus zM$. Let $h \in (M \ominus zM) \ominus E_0$. Then $h \perp R_w^{*n} f$ for every $n \geqslant 0$. Hence $wh \perp R_w^{*n} f$ for every $n \geqslant 0$. By (2.2), $wh \in (M \ominus zM) \ominus E_0$. So

$$w((M \ominus zM) \ominus E_0) \subset (M \ominus zM) \ominus E_0. \tag{2.4}$$

Let $E_1 = (M \ominus zM) \ominus E_0$. Here we assume that $E_1 \neq \{0\}$. By the Wold decomposition theorem,

$$M = \sum_{n=0}^{\infty} \bigoplus z^n (M \ominus zM) = \Big(\sum_{n=0}^{\infty} \bigoplus z^n E_1 \Big) \oplus \Big(\sum_{n=0}^{\infty} \bigoplus z^n E_0 \Big).$$

By (2.4), $wE_1 \subset E_1$, so that

$$M_1 := \sum_{n=0}^{\infty} \bigoplus z^n E_1$$

is an invariant subspace of $H^2(\Gamma^2)$ with $[R_{z,M_1}^*, R_{w,M_1}] = 0$. By the Mandrekar theorem [7], $M_1 = \varphi H^2(\Gamma^2)$ for some inner function φ. Thus we get

$$w\overline{\varphi} M = wH^2(\Gamma^2) \oplus w\overline{\varphi} \Big(\sum_{n=0}^{\infty} \bigoplus z^n E_0 \Big)$$

and

$$w\overline{\varphi} M \ominus zw\overline{\varphi} M = wH^2(\Gamma_w) \oplus w\overline{\varphi} E_0. \tag{2.5}$$

Note that $w\overline{\varphi} M$ is an invariant subspace of $L^2(\Gamma^2)$. In this section, we shall study the case of $\dim E_0 = 1$.

THEOREM 2.2. *Let $F \in L^2(\Gamma^2)$ with $F \neq 0$, and M be an invariant subspace of $L^2(\Gamma^2)$ generated by $wH^2(\Gamma_w)$ and F. Then*

$$M \ominus zM = wH^2(\Gamma_w) \oplus \mathbb{C} \cdot F$$

if and only if there exist functions $G(z), H(z)$ in ball $H^\infty(\Gamma_z)$ *satisfying the following conditions:*

(i) *$G(z)$ is a non-extreme point in* ball $H^\infty(\Gamma_z)$;

(ii) *$|H(z)|^2 = 1 - |G(z)|^2$ a.e. on Γ^2;*

(iii) *$F(z,w) = cH(z)/(1 - G(z)\overline{w})$ a.e. on Γ^2, $c \in \mathbb{C}$ with $c \neq 0$.*

Proof. Suppose that $M \ominus zM = wH^2(\Gamma_w) \oplus \mathbb{C} \cdot F$. Then

$$M = \sum_{n=0}^{\infty} \bigoplus z^n (M \ominus zM) = wH^2(\Gamma^2) \oplus FH^2(\Gamma_z). \tag{2.6}$$

Since $F \perp wH^2(\Gamma^2)$, we have $wF = wf(z) + FG(z)$ for some functions $f(z), G(z) \in H^2(\Gamma_z)$. Then $(w - G(z))F = wf(z)$. We note that $f(z) \neq 0$. We shall prove that

$$G(D) \cap \partial D = \emptyset. \tag{2.7}$$

To prove this by contradiction, suppose that $|G(z_1)| = 1$ for some $z_1 \in D$. If $G(z)$ is constant, say $G(z) = e^{i\theta_1}$, then

$$\frac{wf(z)}{w - e^{i\theta_1}} = F \in L^2(\Gamma^2).$$

Since $1/(w - e^{i\theta_1}) \notin L^2(\Gamma_w)$, $f(z) = 0$, and this is a contradiction. Hence $G(z)$ is non-constant, so that $G(z)$ is an open mapping. Since $|G(z_1)| = 1$ and $z_1 \in D$, $G(D)$ contains an open subarc I of $\partial D = \Gamma$ with $G(z_1) \in I$. So, there exists a curve J in D such that $z_1 \in J$ and $|G(z)| = 1$ on J. By (2.6),

$$F \in \sum_{n=0}^{\infty} \bigoplus z^n L^2(\Gamma_w).$$

Hence $F(z, e^{it}) \in H^2(\Gamma_z)$ for almost every $e^{it} \in \Gamma_w$, so that

$$\frac{f(z)}{1 - G(z)e^{-it}} \in H^2(\Gamma_z)$$

for almost all $e^{it} \in \Gamma_w$. Since $|G(z)| = 1$ on J, we get $f(z) = 0$ on J. Hence $f(z) = 0$, and this is a contradiction. Thus we get (2.7).

Since $F \in L^2(\Gamma^2)$, $F(e^{is}, w) \in L^2(\Gamma_w)$ for almost all $e^{is} \in \Gamma_z$. If $|G(z)| = 1$ on some subset E of Γ_z with $d\theta(E) > 0$, then there exists a point $e^{is} \in E$ such that $f(e^{is}) \neq 0$ and $F(e^{is}, w) \in L^2(\Gamma_w)$. Since $F = wf(z)/(w - G(z))$,

$$\frac{w}{w - G(e^{is})} \in L^2(\Gamma_w).$$

Since $|G(e^{is})| = 1$, this leads to a contradiction. Thus we get

$$|G(z)| \neq 1 \quad \text{a.e. on } \Gamma_z. \tag{2.8}$$

By (2.7), either $G(D) \subset D$ or $G(D) \cap D = \emptyset$. Suppose that $G(D) \cap D = \emptyset$. Then $1/G(z) \in H^\infty(\Gamma_z)$, by (2.8) $|1/G(z)| < 1$ a.e. on Γ_z, and

$$F = \frac{wf(z)}{G(z)\left(\frac{w}{G(z)} - 1\right)} = -\sum_{n=0}^{\infty} \frac{f(z)}{G(z)^{n+1}} w^{n+1} \quad \text{a.e. on } \Gamma^2.$$

By (2.6), $F \perp wH^2(\Gamma^2)$, hence $f(z) = 0$, and this is a contradiction. Thus we get $G(D) \subset D$ and $|G(z)| < 1$ a.e. on Γ_z. Therefore

$$F = \frac{f(z)}{1 - G(z)\overline{w}} = \sum_{n=0}^{\infty} f(z)G(z)^n \overline{w}^n \quad \text{a.e. on } \Gamma^2.$$

Since $F \in L^2(\Gamma^2)$,

$$\infty > \|F\|^2 = \sum_{n=0}^{\infty} \|f(z)G(z)^n\|^2 = \int_0^{2\pi} \frac{|f(\mathrm{e}^{\mathrm{i}\theta})|^2}{1 - |G(\mathrm{e}^{\mathrm{i}\theta})|^2} \frac{\mathrm{d}\theta}{2\pi}.$$

Hence

$$\frac{|f(z)|^2}{1 - |G(z)|^2} \in L^1(\Gamma_z).$$

Since $z^k F \perp F$ for every $k \geqslant 1$,

$$0 = \langle z^k F, F \rangle = \int_0^{2\pi} \frac{|f(\mathrm{e}^{\mathrm{i}\theta})|^2}{1 - |G(\mathrm{e}^{\mathrm{i}\theta})|^2} \mathrm{e}^{\mathrm{i}k\theta} \frac{\mathrm{d}\theta}{2\pi}.$$

This shows that

$$\frac{|f(\mathrm{e}^{\mathrm{i}\theta})|^2}{1 - |G(\mathrm{e}^{\mathrm{i}\theta})|^2} = a$$

for some constant $a > 0$. Therefore $f(z) = cH(z)$ for some function $H(z)$ in ball $H^\infty(\Gamma_z)$ with $|H(\mathrm{e}^{\mathrm{i}\theta})|^2 = 1 - |G(\mathrm{e}^{\mathrm{i}\theta})|^2$ a.e. on Γ_z, and $G(z)$ is not an extreme point in ball $H^\infty(\Gamma_z)$.

Next, suppose that conditions (i)–(iii) hold. Note that $F \neq 0$. We shall prove that $M \ominus zM = wH^2(\Gamma_w) \oplus \mathbb{C} \cdot F$. By (i) and (iii)

$$F(z, w) = c \sum_{n=0}^{\infty} H(z)G^n(z)\overline{w}^n \quad \text{a.e. on } \Gamma^2,$$

so that $z^k F \perp wH^2(\Gamma^2)$ for every $k \geqslant 0$. By (iii), $wF = cwH(z) + G(z)F$. This shows that $M = wH^2(\Gamma^2) \oplus \overline{FH^\infty(\Gamma_z)}$. Since $F \in L^2(\Gamma^2)$,

$$\frac{|H(z)|^2}{1 - |G(z)|^2} \in L^1(\Gamma_z).$$

By (ii), we have

$$\langle z^k F, F \rangle = \int_0^{2\pi} \frac{|H(\mathrm{e}^{\mathrm{i}\theta})|^2}{1 - |G(\mathrm{e}^{\mathrm{i}\theta})|^2} \mathrm{e}^{\mathrm{i}k\theta} \frac{\mathrm{d}\theta}{2\pi} = 0$$

for every $k \geqslant 1$. Hence $M = wH^2(\Gamma^2) \oplus FH^2(\Gamma_z)$ and

$$M \ominus zM = wH^2(\Gamma_w) \oplus \mathbb{C} \cdot F.$$

This completes the proof. ∎

COROLLARY 2.3. *Let M be an invariant subspace of $L^2(\Gamma^2)$ satisfying $M \ominus zM = wH^2(\Gamma_w) \oplus \mathbb{C} \cdot F$ for some function $F \in L^2(\Gamma^2)$ with $F \neq 0$. Then* rank $[R_z^*, R_w] = 1$ *if and only if $F \neq a/(1 - b\overline{w})$ for every $a \in \mathbb{C}$ and $b \in D$.*

Proof. By Theorem 2.2,

$$F = \frac{cH(z)}{1 - G(z)\overline{w}}, \quad c \neq 0,$$

where $G(z) \in \operatorname{ball} H^\infty(\Gamma_z)$ is a non-extreme point and $H(z) \in \operatorname{ball} H^\infty(\Gamma_z)$ with $|H(z)|^2 = 1 - |G(z)|^2$. We have

$$wF = \frac{cwH(z)}{1 - G(z)\overline{w}} = cwH(z) + \frac{cG(z)H(z)}{1 - G(z)\overline{w}} = cwH(z) + G(z)F.$$

Hence $wF \notin M \ominus zM$ if and only if either $H(z)$ or $G(z)$ is not constant, that is, $F \neq a/(1 - b\overline{w})$ for every $a \in \mathbb{C}$ and $b \in D$. By Lemma 2.1, we get the assertion. ∎

Replacing the variable w by $\overline{w}$ in Theorem 2.2, we have the following.

COROLLARY 2.4. *Let $F \in H^2(\Gamma^2)$ with $F \neq 0$, and N be the smallest closed subspace of $H^2(\Gamma^2)$ satisfying $F \in N, zN \subset N$, and $T_w^* N \subset N$, where $T_w^* f = P_{H^2(\Gamma^2)}\overline{w} f$ for $f \in H^2(\Gamma^2)$. Then $N \ominus zN = \mathbb{C} \cdot F$ if and only if there exist functions $G(z), H(z)$ in* ball $H^\infty(\Gamma_z)$ *satisfying the following conditions:*

(i) $G(z)$ *is a non-extreme point in* ball $H^\infty(\Gamma_z)$;

(ii) $|H(z)|^2 = 1 - |G(z)|^2$ *a.e. on* Γ^2;

(iii) $F(z, w) = cH(z)/(1 - G(z)\overline{w})$ *a.e. on* Γ^2, $c \in \mathbb{C}$ *with* $c \neq 0$.

One easily sees the following lemma.

LEMMA 2.5. *Let M_1 and M_2 be invariant subspaces of $L^2(\Gamma^2)$. If $M_2 = \varphi M_1$ for some unimodular function φ on Γ^2, then* rank $[R_{z,M_1}^*, R_{w,M_1}] =$ rank $[R_{z,M_2}^*, R_{w,M_2}]$.

Now we study invariant subspaces in $H^2(\Gamma^2)$.

THEOREM 2.6. *Let M be an invariant subspace of $H^2(\Gamma^2)$ satisfying* rank$[R_z^*, R_w] = 1$. *Let $[R_z^*, R_w] = \eta \otimes f$ for functions $f, \eta \in M$ with $f \neq 0$ and $\eta \neq 0$. Suppose that $\overline{\operatorname{span}}\{R_w^{*n} f : n \geqslant 0\} = \mathbb{C} \cdot f$. Then there exist functions $G(z), H(z)$ in* ball $H^\infty(\Gamma_z)$ *and an inner function φ on Γ^2 satisfying the following conditions:*

(i) $G(z)$ *is a non-extreme point in* ball $H^\infty(\Gamma_z)$;

(ii) $|H(z)|^2 = 1 - |G(z)|^2$ *a.e. on* Γ^2;

(iii) $f = c\varphi H(z)/(w - G(z))$ *a.e. on* Γ^2, $c \in \mathbb{C}$ *with* $c \neq 0$.

Moreover,

$$M = \varphi H^2(\Gamma^2) \oplus \frac{\varphi H(z)}{w - G(z)} H^2(\Gamma_z).$$

Conversely, suppose that $G(z), H(z) \in H^\infty(\Gamma_z)$ *satisfy conditions* (i) *and* (ii), *and there exists an inner function* φ *such that*

$$\frac{\varphi H(z)}{w - G(z)} \in H^2(\Gamma^2).$$

Moreover suppose that either $H(z)$ *or* $G(z)$ *is not constant. Then*

$$M = \varphi H^2(\Gamma^2) \oplus \frac{\varphi H(z)}{w - G(z)} H^2(\Gamma_z)$$

is an invariant subspace of $H^2(\Gamma^2)$ *and* $\operatorname{rank}[R_z^*, R_w] = 1$.

Proof. We shall prove the first assertion. By the argument below Lemma 2.1, see (2.3) and (2.5), there exists an inner function φ satisfying

$$w\overline{\varphi}M \ominus zw\overline{\varphi}M = wH^2(\Gamma_w) \oplus \mathbb{C} \cdot w\overline{\varphi}f.$$

By Theorem 2.2, there exist functions $G(z), H(z)$ in ball $H^\infty(\Gamma_z)$ satisfying (i), (ii), and

$$w\overline{\varphi}f = \frac{cH(z)}{1 - G(z)\overline{w}}, \quad c \neq 0.$$

Hence

$$f = \frac{c\varphi H(z)}{w - G(z)}$$

and

$$M = \sum_{n=0}^{\infty} \bigoplus z^n (M \ominus zM) = \varphi H^2(\Gamma^2) \oplus \frac{\varphi H(z)}{w - G(z)} H^2(\Gamma_z).$$

Next, we shall prove the second assertion. Let M_1 be the invariant subspace of $L^2(\Gamma^2)$ generated by $wH^2(\Gamma_w)$ and a function $H(z)/(1-G(z)\overline{w})$. By Theorem 2.2,

$$M_1 \ominus zM_1 = wH^2(\Gamma_w) \oplus \mathbb{C} \cdot \frac{H(z)}{1 - G(z)\overline{w}}.$$

By Corollary 2.3, $\operatorname{rank}[R_{z,M_1}^*, R_{w,M_1}] = 1$. We have

$$\overline{w}\varphi M_1 \ominus z\overline{w}\varphi M_1 = \varphi H^2(\Gamma_w) \oplus \mathbb{C} \cdot \frac{\varphi H(z)}{w - G(z)},$$

so that

$$\overline{w}\varphi M_1 = \varphi H^2(\Gamma^2) \oplus \frac{\varphi H(z)}{w - G(z)} H^2(\Gamma_z) = M.$$

Hence M is an invariant subspace of $H^2(\Gamma^2)$, and by Lemma 2.5, $\operatorname{rank}[R_z^*, R_w] = 1$. ∎

Let $G(z) \in \operatorname{ball} H^\infty(\Gamma_z)$ be a non-extreme point. Then there exists an outer function $H_0(z)$ in ball $H^\infty(\Gamma_z)$ such that

$$|H_0(z)|^2 = 1 - |G(z)|^2 \quad \text{a.e. on } \Gamma_z.$$

Let

$$F(z,w) = \frac{H_0(z)}{w - G(z)} \quad \text{on } \Gamma^2.$$

Then

$$\|F\|^2 = \int_0^{2\pi} \frac{|H_0(e^{i\theta})|^2}{1 - |G(e^{i\theta})|^2} \frac{d\theta}{2\pi} = 1,$$

so that $F \in L^2(\Gamma^2)$. Here we have a problem; for which $G(z)$, is there an inner function φ on Γ^2 satisfying

$$\varphi F = \frac{\varphi H_0(z)}{w - G(z)} \in H^2(\Gamma^2)\,?$$

We denote by $H(D^2)$ the space of analytic functions in the bidisk D^2. The space $A(D^2)$ is the class of all continuous functions on the closure $\overline{D}^2$ of D^2 whose restriction to D^2 is analytic there. This is a so-called polydisk algebra. Similarly we can define the disk algebra $A(D)$. Let $N(D^2)$ be the class of all functions $f \in H(D^2)$ which satisfy

$$\sup_{0 \leqslant r < 1} \int_0^{2\pi}\int_0^{2\pi} \log^+ |f_r(e^{is}, e^{it})| \frac{ds\,dt}{(2\pi)^2} < \infty,$$

where $f_r(z,w) = f(rz, rw)$. We denote by $N_*(D^2)$ the class of functions $f \in N(D^2)$ for which the functions $\log^+ |f_r|$ form a uniformly integrable family. For each $f \in N(D^2)$ with $f \neq 0$, $\log |f|$ has a least 2-harmonic majorant which is denoted by $u(f)$. By Theorem 3.3.5 of [10], every f in $N(D^2)$ has radial limit f^* a.e. on Γ^2. Moreover there is a real singular measure σ_f on Γ^2 determined by f such that $u(f)$ is given by

$$u(f)(Z) = P_Z(\log |f^*| + d\sigma_f), \quad Z \in D^2,$$

where P_Z denotes the Poisson integral. In particular, $f \in N_*(D^2)$ if and only if $d\sigma_f \leqslant 0$. The following lemma is proved in Theorem 5.4.5 of [10].

LEMMA 2.7. *Suppose that $f \in H^\infty(\Gamma^2)$, $f \neq 0$, and $|f|$ is upper semi-continuous on Γ^2. Then for some function $h \in H(D^2)$ with $|h| > 0$ on D^2, fh is an inner function.*

We need the following lemma.

LEMMA 2.8. *Let $G(z) \in \operatorname{ball} H^\infty(\Gamma_z)$ and $h \in H(D^2)$. If $(w - G(z))h = g \in N_*(D^2)$, then $h \in N_*(D^2)$.*

Proof. We follow the proof given by Chen and Guo in Proposition 4.1.1 of [1]. By the assumption, we have $h \in N(D^2)$. Write $F(z,w) = w - G(z)$. For $\lambda = (\lambda_1, \lambda_2) \in \Gamma^2$, let $F_\lambda(\zeta) = F(\lambda_1\zeta, \lambda_2\zeta)$ for $\zeta \in D$. Then

$$F_\lambda(\zeta) = \lambda_2\zeta - G(\lambda_1\zeta).$$

We shall prove that $F_\lambda(\zeta)$ has no singular inner factor. If $\|G\|_\infty < 1$, clearly $F_\lambda(\zeta)$ has no singular inner factor. So, we assume that $\|G\|_\infty = 1$. We have

$$\lambda_2\zeta - F_\lambda(\zeta) = G(\lambda_1\zeta) \in \text{ball}\, H^\infty(\Gamma_z) \quad \text{and} \quad |\lambda_2\zeta - F_\lambda(\zeta)| \leqslant 1 \quad \text{a.e. on } \Gamma.$$

Hence

$$Re\,\overline{\lambda}_2\overline{\zeta}F_\lambda(\zeta) \geqslant 0 \quad \text{a.e. on } \Gamma.$$

Let

$$f(\zeta) = \frac{\overline{\lambda}_2 F_\lambda(\zeta)}{\zeta}, \quad \zeta \in D \setminus \{0\}.$$

Then $f(\zeta)$ is analytic in $D \setminus \{0\}$. Let I be an open arc in Γ. We may assume that $f(\zeta)$ has nontangential limits at the end points $e^{i\theta_1}, e^{i\theta_2}$ of I and that $\mathrm{Re}\, f(e^{i\theta_j}) > 0, j = 1, 2$. Let J be a circular arc in D jointing $e^{i\theta_1}$ to $e^{i\theta_2}$. We may further assume that $\inf_J |f(\zeta)| > 0$. Let U be the domain bounded by $I \cup J$, and let τ be a conformal mapping from D onto U. We may assume that $0 \notin \overline{U}$. Then $f \circ \tau \in H^\infty(\Gamma)$ and $\mathrm{Re}\, f \circ \tau \geqslant 0$ a.e. on $\tau^{-1}(I)$. By pp. 96–97 in [2],

$$\begin{aligned}\lim_{r\to 1} \int_{I_1} \log |F_r(e^{i\theta})| \frac{d\theta}{2\pi} &= \lim_{r\to 1} \int_{I_1} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_{I_1} \log |f(e^{i\theta})| \frac{d\theta}{2\pi} \\ &= \int_{I_1} \log |F_\lambda(e^{i\theta})| \frac{d\theta}{2\pi}\end{aligned}$$

for any compact subarc I_1 of I, which means that the inner factor of $F_\lambda(\zeta)$ has no singularities on I. Hence $F_\lambda(\zeta)$ has no singular inner factor.

Since $F_\lambda(\zeta)$ has no singular inner factor for every $\lambda \in \Gamma^2$, by Theorem 3.3.6 of [10] we have $d\sigma_F = 0$. Since $g = hF, d\sigma_g = d\sigma_h + d\sigma_F$. Since $g \in N_*(D^2)$, $d\sigma_g \leqslant 0$. Thus we get $d\sigma_h \leqslant 0$, and then $h \in N_*(D^2)$. ∎

Suppose that $G(z) \in A(D) \cap \text{ball}\, H^\infty(\Gamma_z)$. By Lemma 2.7, there is a function $h \in H(D^2)$ such that $(w - G(z))h$ is an inner function. Write $\varphi = (w - G(z))h$. Let $\psi(z)$ be a non-constant inner function. Then

$$\varphi(\psi(z), w) = (w - (G \circ \psi)(z))h(\psi(z), w).$$

Note that $\varphi(\psi(z), w)$ is inner and $(G \circ \psi)(z) \in \text{ball}\, H^\infty(\Gamma_z)$. By Lemma 2.8, $h(\psi(z), w) \in N_*(D^2)$. Suppose that $(G \circ \psi)(z)$ is a non-extreme point in $\text{ball}\, H^\infty(\Gamma_z)$. Note that if $G(z)$ is non-extreme, then so is $(G \circ \psi)(z)$. Then there exists an outer function $H_0(z)$ in $\text{ball}\, H^\infty(\Gamma_z)$ satisfying

$$|H_0(z)|^2 = 1 - |(G \circ \psi)(z)|^2 \quad \text{a.e. on } \Gamma_z.$$

Then

$$\frac{H_0(z)}{w - (G \circ \psi)(z)} \in L^2(\Gamma^2).$$

Hence

$$\frac{\varphi(\psi(z),w)H_0(z)}{w-(G\circ\psi)(z)} = H_0(z)h(\psi(z),w) \in N_*(D^2)\cap L^2(\Gamma^2) = H^2(\Gamma^2).$$

Combining with Theorem 2.6, we have the following theorem. A similar discussion is given in [4].

We denote by $\mathcal{A}(D)$ the set of all $(G\circ\psi)(z)$, where $G(z)\in A(D)\cap \mathrm{ball}\, H^\infty(\Gamma_z)$ and $\psi(z)$ are non-constant inner functions. Then $A(D)\subset \mathcal{A}(D)\subset \mathrm{ball}\, H^\infty(\Gamma_z)$.

THEOREM 2.9. *Let $G(z)\in\mathcal{A}(D)$. Suppose that $G(z)$ is not an extreme point in* $\mathrm{ball}\, H^\infty(\Gamma_z)$. *Let $H_0(z)\in H^\infty(\Gamma_z)$ be an outer function with $|H_0(z)|^2 = 1-|G(z)|^2$ a.e. on Γ_z. Let $H(z)\in \mathrm{ball}\, H^\infty(\Gamma_z)$ with $|H(z)| = |H_0(z)|$ a.e. on Γ_z. Assume that either $G(z)$ or $H(z)$ is non-constant. Then there exists an inner function φ on Γ^2 such that*

$$\frac{\varphi H_0(z)}{w-G(z)}\in H^2(\Gamma^2) \quad \text{and} \quad M = \varphi H^2(\Gamma^2)\oplus \frac{\varphi H(z)}{w-G(z)}H^2(\Gamma_z)$$

is an invariant subspace of $H^2(\Gamma^2)$ and $\mathrm{rank}\,[R_z^*, R_w] = 1$.

3. TAKAHASHI'S THEOREM

We prove an $H^2(\Gamma^2)$-version of Takahashi's theorem.

THEOREM 3.1. *Let $g(z)$ be a function in* $\mathrm{ball}\, H^\infty(\Gamma_z)$. *Then $g(z)$ is a non-extreme point in* $\mathrm{ball}\, H^\infty(\Gamma_z)$ *if and only if there is a function F in $L^2(\Gamma^2)\setminus H^2(\Gamma^2)$ such that $(w-g(z))F\in H^2(\Gamma^2)$.*

Proof. Suppose that $g(z)$ is not an extreme point in $\mathrm{ball}\, H^\infty(\Gamma_z)$. Then there exists a function $h(z)$ in $H^\infty(\Gamma_z)$ with $|h(z)| = 1-|g(z)|$ a.e. on Γ_z. Let

$$F(z,w) = \frac{h(z)}{w-g(z)}.$$

Then $(w-g(z))F\in H^\infty(\Gamma^2)$. Since

$$\Big|\frac{h(z)}{w-g(z)}\Big| \leqslant \frac{|h(z)|}{1-|g(z)|} = 1 \quad \text{a.e. on } \Gamma_z,$$

we have $F\in L^\infty(\Gamma^2)$. Since

$$F(z,w) = \sum_{n=0}^\infty h(z)g^n(z)\overline{w}^{(n+1)} \quad \text{a.e. on } \Gamma_z,$$

$F(z,w)\in H^\infty(\Gamma^2)$ if and only if $h(z)=0$. Since $|h(z)|>0$ a.e. on Γ_z, $F\notin H^\infty(\Gamma^2)$.

Next, suppose that $(w-g(z))F\in H^2(\Gamma^2)$ for some $F\in L^2(\Gamma^2)\setminus H^2(\Gamma^2)$. We have $(\xi-g(z))F(z,\xi)\in H^2(\Gamma_z)$ for almost all $\xi\in\Gamma_w$. Since $\xi-g(z)\in H^2(\Gamma_z)$

is outer for all $\xi \in \Gamma_w$, $F(z,\xi) \in H^2(\Gamma_z)$ for almost all $\xi \in \Gamma_w$. This implies that

$$F \in \sum_{n=-\infty}^{\infty} \bigoplus w^n H^2(\Gamma_z).$$

Write

$$F = \sum_{n=-\infty}^{\infty} f_n(z) w^n, \quad f_n(z) \in H^2(\Gamma_z).$$

Since $(w - g(z))F \in H^2(\Gamma^2)$, $f_{n-1}(z) - g(z) f_n(z) = 0$ for every $n \leqslant -1$. Hence

$$f_{-k}(z) = f_{-1}(z) g^{k-1}(z), \quad k \geqslant 1.$$

Write

$$F'(z,w) = \sum_{n=-\infty}^{-1} f_n(z) w^n.$$

Then $F'(z,w) = \sum_{k=1}^{\infty} f_{-k}(z)\overline{w}^k = f_{-1}(z) \sum_{k=1}^{\infty} g(z)^{k-1}\overline{w}^k$. Since $F \notin H^2(\Gamma^2)$, we have $f_{-1}(z) \neq 0$. Since $F' \in L^2(\Gamma^2)$,

$$\infty > \|F'\|^2 = \int_0^{2\pi} |f_{-1}(e^{i\theta})|^2 \sum_{k=1}^{\infty} |g(e^{i\theta})|^{2(k-1)} \frac{d\theta}{2\pi}.$$

Hence $|g| < 1$ a.e. on Γ. Thus we get

$$\int_0^{2\pi} \frac{|f_{-1}(e^{i\theta})|^2}{1 - |g(e^{i\theta})|^2} \frac{d\theta}{2\pi} < \infty.$$

Let

$$G(e^{i\theta}) = \frac{|f_{-1}(e^{i\theta})|^2}{1 - |g(e^{i\theta})|^2}.$$

Then $G \in L^1(\Gamma_z)$ and

$$\int_0^{2\pi} \log G(e^{i\theta}) \frac{d\theta}{2\pi} + \int_0^{2\pi} \log(1 - |g(e^{i\theta})|^2) \frac{d\theta}{2\pi} = 2 \int_0^{2\pi} \log |f_{-1}(e^{i\theta})| \frac{d\theta}{2\pi}.$$

We have $\int_0^{2\pi} \log G(e^{i\theta}) \frac{d\theta}{2\pi} < \infty$. Since $f_{-1}(z) \in H^2(\Gamma_z)$, by Jensen's inequality, see p. 52 in [3],

$$-\infty < \int_0^{2\pi} \log |f_{-1}(e^{i\theta})| \frac{d\theta}{2\pi}.$$

Hence

$$\int_0^{2\pi} \log(1 - |g(e^{i\theta})|^2) \frac{d\theta}{2\pi} > -\infty.$$

Therefore $g(z)$ is not an extreme point in ball $H^\infty(\Gamma_z)$. ∎

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