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INTERPOLATION AND BALLS IN $\mathbb{C}^k$

VERN I. PAULSEN and JAMES P. SOLAZZO

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ABSTRACT. We compare and classify various types of Banach algebra norms on $\mathbb{C}^k$ through geometric properties of their unit balls. This study is motivated by various open problems in interpolation theory and in the isometric characterization of operator algebra norms.

KEYWORDS: *Interpolation bodies, k-idempotent operator algebras, Schur ideals.*

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1. INTRODUCTION

Given two points $v = (v_1, \dots, v_k)$ and $w = (w_1, \dots, w_k)$ in $\mathbb{C}^k$ we define $v \cdot w := (v_1 w_1, \dots, v_k w_k)$ so that $\mathbb{C}^k$ is a commutative algebra with unit $e = (1, \dots, 1)$. If in addition we have a norm $\| \cdot \|$ on $\mathbb{C}^k$ which satisfies $\|v \cdot w\| \leqslant \|v\| \cdot \|w\|$ for all v, w in $\mathbb{C}^k$, then the pair $(\mathbb{C}^k, \| \cdot \|)$ is a Banach algebra. We shall always require that $\|e\| = 1$. Given a norm $\| \cdot \|$ on $\mathbb{C}^k$ we will refer to the set $B := \{v \in \mathbb{C}^k : \|v\| \leqslant 1\}$ as the *unit ball* in $\mathbb{C}^k$ with respect to the given norm. We will denote the set of all balls determined by Banach algebra norms on $\mathbb{C}^k$ as $\mathcal{B}_k$.

In this paper we begin to compare and classify different properties that Banach algebra norms on $\mathbb{C}^k$ can possess in order to provide some insight into a number of problems. One of our motivations is to gain a deeper understanding of the geometry of the balls that arise as the solutions to various interpolation problems. The other problems include an attempt to characterize the Banach algebras that can be represented isometrically as algebras of operators on a Hilbert space, questions about the computability of various interpolation problems for uniform algebras, and questions about the failure of the multi-variable von Neumann inequality for three or more contractions on Hilbert space.

DEFINITION 1.1. Let $\mathcal{A}$ be a unital Banach algebra. Then $\mathcal{A}$ is said to *satisfy von Neumann's inequality* provided that for all a in $\mathcal{A}$ with $\|a\| \leqslant 1$ we have that $\|p(a)\| \leqslant \|p\|_\infty$ where p is a polynomial and $\|p\|_\infty = \sup\{|p(z)| : |z| \leqslant 1\}$.

Von Neumann proved that if H is a Hilbert space, then the algebra of operators on $H, B(H)$ satisfies von Neumann's inequality. Consequently, any unital Banach algebra that can be represented isometrically as an algebra of operators on a Hilbert space necessarily satisfies von Neumann's inequality. It is still unknown if the converse of this last statement holds. That is, can every unital Banach algebra that satisfies von Neumann's inequality be represented isometrically as an algebra of operators on a Hilbert space? It has been conjectured that the answer to this question is affirmative. For this reason we are interested in understanding the Banach algebra norms on $\mathbb{C}^k$ that satisfy von Neumann's inequality.

DEFINITION 1.2. The set of all balls determined by Banach algebra norms on $\mathbb{C}^k$ satisfying von Neumann's inequality will be denoted $\mathcal{V}_k$. The set of all balls determined by Banach algebra norms on $\mathbb{C}^k$ such that the resulting Banach algebra has a unital isometric representation as an algebra of operators on a Hilbert space will be denoted $\mathcal{O}_k$.

By von Neumann's result, we have that $\mathcal{O}_k \subseteq \mathcal{V}_k \subseteq \mathcal{B}_k$. If the answer to the above question is affirmative, then it must be the case that $\mathcal{O}_k = \mathcal{V}_k$. Conversely, if we were able to show, for some k, that these two sets are not equal, then that would provide a counterexample to the above conjecture.

The next subset of $\mathcal{B}_k$ that we wish to examine is motivated by general interpolation theory and by attempts to obtain multi-variable generalizations of von Neumann's inequality.

DEFINITION 1.3. A unital, commutative Banach algebra, $\mathcal{A}$, is said to *satisfy the multi-variable von Neumann inequality* provided that for every n and for every set of n elements $\{a_1, \ldots, a_n\}$ from the unit ball of $\mathcal{A}$ and for every polynomial p in n variables, we have that

$$\|p(a_1, \ldots, a_n)\| \leqslant \|p\|,$$

where $\|p\| = \sup\{|p(z_1, \ldots, z_n)| : |z_i| \leqslant 1, 1 \leqslant i \leqslant n\}$. A subset B of $\mathbb{C}^k$ is called *hyperconvex* if it is the unit ball of a Banach algebra norm on $\mathbb{C}^k$ that satisfies the multi-variable von Neumann inequality and we let $\mathcal{H}_k$ denote the collection of all such balls.

Let X be a compact Hausdorff space and let $C(X)$ denote the continuous complex-valued functions on X. We call $A \subseteq C(X)$ a *uniform algebra* provided that A is uniformly closed, contains the identity, and separates points in X. An algebra that is isometrically isomorphic to the quotient of a uniform algebra is called in the literature, we are sorry to say, an *IQ algebra*. Davie [7] proved that a unital, commutative Banach algebra with a unit of norm 1, is an IQ algebra if and only if it satisfies the multi-variable von Neumann inequality.

The term *hyperconvex* is due to Cole and Wermer [5], who introduced this concept because of the relationship between these sets, isometric quotients of uniform algebras and questions in interpolation theory for uniform algebras. We explain these connections below.

Ando [1] proved a 2-variable analogue of von Neumann's inequality, i.e., every commuting pair of contractions on a Hilbert space satisfies the 2-variable von Neumann inequality. Since then many mathematicians have given examples to show that commuting triples of contractions, even on a finite dimensional Hilbert space, need not satisfy the 3-variable von Neumann inequality. Thus, a commuting algebra of operators on a Hilbert space need not satisfy the multi-variable von Neumann inequality, i.e., need not be an IQ algebra. However, these examples generally consisted of commuting sets of nilpotent operators and examples of commuting sets of diagonalizable operators have been much harder to come by. In particular, it is still unknown if a simultaneously diagonalizable set of three commuting contractions on a three dimensional Hilbert space must satisfy the multi-variable von Neumann inequality. Lotto and Steger [10] constructed three commuting, diagonalizable contractions on a five dimensional Hilbert space that failed to satisfy von Neumann's inequality. Recently, Holbrook [8] constructed three commuting, diagonalizable contractions on a four dimensional Hilbert space that failed to satisfy von Neumann's inequality. We shall see that this problem is closely related to determining whether or not $\mathcal{O}_k = \mathcal{H}_k$.

Cole, Lewis, and Wermer in [4] make the following definition.

DEFINITION 1.4. Let $A \subseteq C(X)$ be a uniform algebra and let $x_1, \dots, x_k \in X$. The *interpolation body* associated with A and points $x_1, \dots, x_k$, is denoted as $\mathcal{D}(A; x_1, \dots, x_k)$, and is defined as the set $\{(w_1, \dots, w_k) : \forall \varepsilon > 0 \,\exists f \in A \ni \|f\|_\infty \leqslant 1 + \varepsilon \text{ and } f(x_i) = w_i\}$.

The connection between interpolation bodies and hyperconvex sets is due to Cole and Wermer [5], it uses the characterization of IQ algebras due to S. Davie [7].

Let I_x denote the ideal of functions vanishing at the points $x_1, \dots, x_k$ in X. Since A separates points, there exist functions $f_1, \dots, f_k$ in A with $f_i(x_j) = \delta_{ij}$. Thus, in A/I_x we have that $F_i = [f_i + I_x]$ is a set of k-commuting idempotents, satisfying $F_iF_j = \delta_{ij}F_j$, $F_1 + \cdots + F_k = I$, and which span A/I_x.

The following is immediate.

PROPOSITION 1.5. *Let $A \subseteq C(X)$ be a uniform algebra and let $x_1, \dots, x_k \in X$. Then $\mathcal{D}(A; x_1, \dots, x_k) = \{(w_1, \dots, w_k) \in \mathbb{C}^k : \|w_1F_1 + \cdots + w_kF_k\| \leqslant 1\}$.*

Hence, a point $(w_1, \dots, w_k)$ in $\mathbb{C}^k$ belongs to $\mathcal{D}(A; x_1, \dots, x_k)$ if and only if $\|w_1F_1 + \cdots + w_kF_k\| \leqslant 1$. Thus, an interpolation body $\mathcal{D}(A; x_1, \dots, x_k)$ is a natural coordinatization of the closed unit ball of the quotient algebra A/I_x. The isomorphism between A/I_x and $\mathbb{C}^k$ defined by sending a coset $[f + I_x]$ to the vector $(f(x_1), \ldots, f(x_k))$ endows $\mathbb{C}^k$ with a Banach algebra norm. The closed unit ball of this norm is the set $\mathcal{D}(A; x_1, \ldots, x_k)$.

Combining Proposition 1.5 with Davie's characterization of IQ algebras yields the following result of Cole-Wermer.

PROPOSITION 1.6 ([5], Theorem 5.2). *Let $A \subseteq C(X)$ be a uniform algebra and let $x_1, \ldots, x_k \in X$, then $\mathcal{D}(A; x_1, \ldots, x_k)$ is a hyperconvex subset of $\mathbb{C}^k$. Conversely, if B is any hyperconvex subset of $\mathbb{C}^k$, then there is a uniform algebra $A \subseteq C(X)$ and points $x_1, \ldots, x_k \in X$, such that $B = \mathcal{D}(A; x_1, \ldots, x_k)$.*

Thus, the hyperconvexity condition gives an abstract characterization of interpolation bodies. Unfortunately, given a ball B in $\mathbb{C}^k$ it is very difficult to determine if it is hyperconvex. One of the main difficulties is that to apply the hyperconvexity condition one needs analogues of the Nevanlinna-Pick interpolation theorem on polydisks, which is still an open problem for more than two variables.

By a result of Cole ([3], page 272), A/I_x is an operator algebra. Thus, we have that $\mathcal{H}_k \subseteq \mathcal{O}_k$ for all k.

Another collection of Banach algebra balls that we wish to consider is a well understood subset of $\mathcal{H}_k$. When the uniform algebra A is the disk algebra, denoted $A(\mathbb{D})$, the set $\mathcal{D}(A(\mathbb{D}); \alpha_1, \ldots, \alpha_k)$ is referred to as a *Pick body* and will be denoted as $\mathcal{P}(\alpha_1, \ldots, \alpha_k)$. By Pick's theorem [14] , if $\{\alpha_1, \ldots, \alpha_k\}$ is a subset of the open unit disk, then a point $(w_1, \ldots, w_k)$ belongs to $\mathcal{P}(\alpha_1, \ldots, \alpha_k)$ if and only if the matrix $\left(\frac{1-\overline{w}_i w_j}{1-\overline{\alpha}_i \alpha_j}\right)$ is positive semi-definite. When these points are all on the unit circle, then $\mathcal{P}(\alpha_1, \ldots, \alpha_k)$ can be shown to be the closed k-polydisk. The set of balls in $\mathbb{C}^k$ determined by interpolation bodies of the form $\mathcal{P}(\alpha_1, \ldots, \alpha_k)$ will be denoted as $\mathcal{P}_k$.

Thus, to summarize, we know that for any k in $\mathbb{N}$ we have that

$$\mathcal{P}_k \subseteq \mathcal{H}_k \subseteq \mathcal{O}_k \subseteq \mathcal{V}_k \subseteq \mathcal{B}_k.$$

Moreover, we have argued that proving equality between various pairs of these subsets will yield positive results on some open problems, while showing that certain of these subsets are not equal will yield counterexamples to certain open questions.

Thus, we are lead to study various properties of the balls belonging to these five families, to attempt to obtain more manageable characterizations of these five families, and to find means of generating balls that belong to these various families, in an attempt to distinguish between these five families. We will see that in some senses, focusing on the properties of the balls is an efficient way to produce examples of interpolation sets and operator algebras without actually needing to construct the uniform algebras or operators.

Here is a summary of what we can show about the relationships between these sets:

$$
\begin{aligned}
&k=2 \quad \mathcal{P}_2=\mathcal{H}_2=\mathcal{O}_2=\mathcal{V}_2 \subsetneq \mathcal{B}_2,\\
&k=3 \quad \mathcal{P}_k \subsetneq \mathcal{H}_k \subseteq^? \mathcal{O}_k \subseteq^? \mathcal{V}_k \subsetneq \mathcal{B}_k,\\
&k\geqslant 4 \quad \mathcal{P}_k \subsetneq \mathcal{H}_k \subsetneq \mathcal{O}_k \subseteq^? \mathcal{V}_k \subsetneq \mathcal{B}_k,
\end{aligned}
$$

Thus, for $k=2$, we will show that $\mathcal{P}_2=\mathcal{H}_2=\mathcal{O}_2=\mathcal{V}_2$ and provide an example of a Banach algebra norm in 2 dimensions that fails to satisfy von Neumann's inequality. For $k\geqslant 4$, we are able to show that the sets $\mathcal{P}_k, \mathcal{H}_k, \mathcal{O}_k$, and $\mathcal{B}_k$ are distinct. However, for $k=3$, the picture is not complete and we are still unable to resolve whether or not $\mathcal{O}_k=\mathcal{V}_k$ for all $k\geqslant 3$.

Some of these results are restatements of earlier results. In [4] (Theorem 5), Cole, Lewis, and Wermer show that every hyperconvex set Y in $\mathbb{C}^2$ is a Pick body, i.e., that $\mathcal{H}_2=\mathcal{P}_2$. In fact, we shall see that their proof actually implies $\mathcal{P}_2=\mathcal{V}_2$. They also show that for $k>2$, $\mathcal{H}_k\neq\mathcal{P}_k$. Similarly, the fact that $\mathcal{H}_k\neq\mathcal{O}_k$ for $k\geqslant 4$, is a restatement of a result of Holbrook [8].

There is one final concept that we shall study. Recall that a set is called *semi-algebraic* if it is defined by a finite set of polynomial inequalities. In a certain sense these are the most definable or computable subsets of $\mathbb{C}^k$, since determining whether or not a point belongs to such a set involves only finitely many algebraic operations, provided that the polynomials can be found.

It is easily seen that all the Pick bodies are semi-algebraic sets. In [6], Cole and Wermer show that $\mathcal{D}(A(\mathbb{D}^2); z_1,\ldots,z_k)$ is a semi-algebraic set. If A denotes the uniform algebra of all bounded, analytic functions on an annulus, then results of [15] show that $\mathcal{D}(A; x_1,\ldots,x_k)$ is semi-algebraic for any choice of points.

It is natural to wonder if every interpolation body is semi-algebraic. If interpolation bodies are known a priori to be semi-algebraic sets, then we would know that solving the corresponding interpolation problem can always be reduced to a finite set of algebraic operations. We conjecture that for a sufficiently pathological uniform algebra, there will exist interpolation bodies that are not semi-algebraic. The most interesting question is whether or not an interpolation body exists for some natural uniform algebra that is not semi-algebraic. However, since every two point interpolation body is a Pick body, we see that these interpolation problems are in some sense computable.

Although we have been unable to produce an example of an interpolation body that is not semi-algebraic, we will provide an example of a ball in $\mathcal{O}_3$ which we conjecture is not a semi-algebraic set. Thus, if the before mentioned conjecture is true, then either $\mathcal{H}_3\neq\mathcal{O}_3$ or there exist 3 point interpolation problems for which the interpolation body is not semi-algebraic.

2. CHARACTERIZATION AND GENERATION OF SETS IN THESE FAMILIES

In this section we present some basic results about the various classes of norms on $\mathbb{C}^k$ that we have introduced. We begin by gathering together some facts about $\mathcal{B}_k$.

Recall that a set $B \subseteq \mathbb{C}^k$ is the closed unit ball of some norm on $\mathbb{C}^k$ if and only if B is closed, bounded, absorbing and absolutely convex. We shall refer to such a set as a *ball.* Given a finite collection of balls, it is easily seen that their intersection will be closed, bounded, absorbing and absolutely convex. Thus, the intersection of a finite collection of balls is again a ball. However, for arbitrary collections of balls, their intersection will be closed, bounded and absolutely convex, but not necessarily absorbing. Thus, the intersection will be a ball if and only if it is absorbing. We shall show a similar result holds for the various classes of norms that we wish to consider.

Given a set $B \subseteq \mathbb{C}^k$ we set $B \cdot B = \{v \cdot w : v, w \in B\}$.

PROPOSITION 2.1. *Let $B \subseteq \mathbb{C}^k$. Then $B \in \mathcal{B}_k$ if and only if B is a ball, $e \in B$ and $B \cdot B \subseteq B$. Consequently, the intersection of a finite collection of sets in $\mathcal{B}_k$ is again in $\mathcal{B}_k$, but the intersection of an arbitrary collection of sets in $\mathcal{B}_k$ is in $\mathcal{B}_k$ if and only if the intersection is absorbing.*

Note that the fact that $B \subseteq \mathbb{C}^k$ is bounded and $B \cdot B \subseteq B$ implies that B must be a subset of the closed unit polydisk.

DEFINITION 2.2. Given any set $D \subseteq \mathbb{C}^k$ contained in the closed unit polydisk, we let $\mathcal{B}(D)$ denote the intersection of all elements of $\mathcal{B}_k$ containing D.

DEFINITION 2.3. Let $D \subseteq \mathbb{C}^k$. We say that D is *separating* provided that for every $i \neq j$ there exists $(w_1, \ldots, w_k) \in D$ such that $w_i \neq w_j$.

Note that $D = \{v_1, \ldots, v_n\} \subseteq \mathbb{C}^k$ is separating where $v_i = (w_{i,1}, \ldots, w_{i,k})$, if and only if setting $x_j = (w_{1,j}, \ldots, w_{n,j})$ defines k distinct points in $\mathbb{C}^n$.

PROPOSITION 2.4. *Let $D \subseteq \mathbb{C}^k$ be a separating set contained in the closed unit polydisk. Then $\mathcal{B}(D) \in \mathcal{B}_k$ and if $D \subseteq B_1$ with $B_1 \in \mathcal{B}_k$ then $\mathcal{B}(D) \subseteq B_1$.*

Proof. Clearly, if $B_1 \in \mathcal{B}_k$ contains D, then $\mathcal{B}(D) \subseteq B_1$. So it remains to show that $\mathcal{B}(D) \in \mathcal{B}_k$, which by the above is equivalent to proving that $\mathcal{B}(D)$ is absorbing. Since D is separating, $\mathcal{B}(D)$ is separating. Fix $i \neq j$, then by taking an absolute convex combination with e, we have an element of $\mathcal{B}(D)$ that is a strictly positive constant in the i-th coordinate and 0 in the j-th coordinate. Fixing i, choosing one such vector for every $j \neq i$ and taking the product of these vectors yields a vector in $\mathcal{B}(D)$ that is strictly positive in the i-th coordinate and is 0 in the remaining coordinates.

Repeating for each i, we see that $\mathcal{B}(D)$ contains a positive multiple of each basis vector. Since the absolutely convex hull of the set of such vectors will contain an open neighborhood of 0, we have that $\mathcal{B}(D)$ is absorbing. ∎

When D is separating and a subset of the closed unit polydisk, we shall refer to $\mathcal{B}(D)$ as the *Banach algebra ball generated by D*. Later, we shall prove that $\mathcal{B}(D)$ is a ball if and only if D is separating.

EXAMPLE 2.5. Let $D = \{e_1, e_2\} \subseteq \mathbb{C}^2$ where $e_1 = (1,0), e_2 = (0,1)$ which is a separating set. Clearly, $\mathcal{B}(D)$ must contain the absolutely convex hull of e_1, e_2 and $e = (1,1)$. But we claim that this latter set is closed under products and hence is in $\mathcal{B}_2$. To see this last claim, note that if $w = w_1e_1 + w_2e_2 + w_3e$ and $z = z_1e_1 + z_2e_2 + z_3e$ with $|w_1| + |w_2| + |w_3| \leqslant 1$ and $|z_1| + |z_2| + |z_3| \leqslant 1$, then

$$w \cdot z = (w_1z_1 + w_1z_3 + w_3z_1)e_1 + (w_2z_2 + w_2z_3 + w_3z_2)e_2 + (w_3z_3)e$$

and the sum of the absolute values of the coefficients of $w \cdot z$ are easily seen to be less than 1.

We now turn our attention to $\mathcal{V}_k$. Given a set $V \subset \mathbb{C}^k$ and a function f, we set $f(V) = \{(f(v_1), \ldots, f(v_k)) : (v_1, \ldots, v_k) \in V\}$.

PROPOSITION 2.6. *Let $V \subseteq \mathbb{C}^k$. Then $V \in \mathcal{V}_k$ if and only if $V \in \mathcal{B}_k$ and $f(V) \subseteq V$ for every $f \in A(\mathbb{D})$ with $\|f\| \leqslant 1$. Consequently, the intersection of a finite collection of sets in $\mathcal{V}_k$ is again in $\mathcal{V}_k$, but the intersection of an arbitrary collection of sets in $\mathcal{V}_k$ is in $\mathcal{V}_k$ if and only if the intersection is absorbing.*

DEFINITION 2.7. Given any set $D \subseteq \mathbb{C}^k$ contained in the closed unit polydisk, we let $\mathcal{V}(D)$ denote the intersection of all elements of $\mathcal{V}_k$ that contain D.

PROPOSITION 2.8. *Let $D \subseteq \mathbb{C}^k$ be a separating set contained in the closed unit polydisk. Then $\mathcal{V}(D) \in \mathcal{V}_k$ and if $D \subseteq V_1$ with $V_1 \in \mathcal{V}_k$ then $\mathcal{V}(D) \subseteq V_1$.*

Proof. As before it will be enough to prove that $\mathcal{V}(D)$ is absorbing. But note that $\mathcal{B}(D) \subseteq \mathcal{V}(D)$ and that $\mathcal{B}(D)$ is absorbing. Hence, $\mathcal{V}(D)$ is absorbing and so the result follows. ∎

When D is separating and contained in the closed unit polydisk, we shall refer to $\mathcal{V}(D)$ as the *von Neumann algebra ball generated by D.* We shall also see that $\mathcal{V}(D)$ is a ball if and only if D is separating.

We let $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$ denote the elementary Möbius map. Using these maps it is possible to reduce problems about $\mathcal{V}_k$ by one dimension.

Given a set $V \subset \mathbb{C}^k$, we set $\widehat{V} = \{(v_2, \ldots, v_k) \in \mathbb{C}^{k-1} : (0, v_2, \ldots, v_k) \in V\}$.

PROPOSITION 2.9. *Let $V_1, V_2 \in \mathcal{V}_k$. Then $V_1 = V_2$ if and only if $\widehat{V}_1 = \widehat{V}_2$.*

Proof. Assume that $\widehat{V}_1 = \widehat{V}_2$. If $(a, v_2, \ldots, v_k) \in V_1$, then $(0, \varphi_a(v_2), \ldots, \varphi_a(v_k)) \in V_1$ and hence in V_2. Applying φ_{-a} we find that $(a, v_2, \ldots, v_k) \in V_2$. Thus, $V_1 \subseteq V_2$, but by the symmetry of the argument, $V_1 = V_2$.

The other implication is obvious. ∎

THEOREM 2.10. *We have that* $\mathcal{P}_2 = \mathcal{H}_2 = \mathcal{O}_2 = \mathcal{V}_2 \neq \mathcal{B}_2$.

Proof. Let $V \in V_2$. Since V is a ball $\widehat{V} = \{z \in \mathbb{C} : |z| \leqslant r\}$ for some $r, 0 < r \leqslant 1$. Hence, $\widehat{V} = \widehat{\mathcal{P}(0,r)}$ and so $V = \mathcal{P}(0,r)$. Thus, $\mathcal{V}_2 \subseteq \mathcal{P}_2$, and the equalities follow.

To see that $\mathcal{V}_2 \neq \mathcal{B}_2$, we consider the Banach algebra ball $\mathcal{B}(D)$ for $D = \{e_1, e_2\}$ constructed in the above example. Assume $\mathcal{B}(D) \in \mathcal{V}_2$, then we see that $\widehat{\mathcal{B}(D)}$ is the closed unit disk and so necessarily, $\mathcal{P}(0,1) = \mathcal{B}(D)$. However, it is easily shown that $\mathcal{P}(0,1)$ is the closed unit bidisk and so $\mathcal{P}(0,1) \neq \mathcal{B}(D)$. Thus, $\mathcal{B}(D)$ is not in $\mathcal{V}_2$. ∎

The proof that $\mathcal{P}_2 = \mathcal{V}_2$ is essentially the one given by Cole and Wermer to prove that $\mathcal{P}_2 = \mathcal{H}_2$.

PROBLEM 2.11. Find necessary and sufficient conditions on a set $B \subseteq \mathbb{C}^{k-1}$ such that $B = \widehat{V}$ for some $V \in \mathcal{V}_k$.

Given $B \subseteq \mathbb{C}^{k-1}$, we set

$$\widetilde{B} = \{(a, v_2, \dots, v_k) : (\varphi_a(v_2), \dots, \varphi_a(v_k)) \in B\},$$

so the problem is to find conditions on B that guarantees $\widetilde{B} \in \mathcal{V}_k$.

The following gives some conditions that B must necessarily satisfy.

PROPOSITION 2.12. *If* $B = \widehat{V}$ *for some* $V \in \mathcal{V}_k$, *then:*

(i) *B is a closed subset of the closed polydisk;*

(ii) *B is a ball in* $\mathbb{C}^{k-1}$*;*

(iii) $B \cdot B \subseteq B$*;*

(iv) *for any* $f \in A(\mathbb{D})$ *with* $\|f\| \leqslant 1$ *and* $f(0) = 0$, *we have* $f(B) \subseteq B$.

Note that these conditions imply that B is the unit ball of a Banach algebra norm on $\mathbb{C}^k$, but one for which, possibly, $\|e\| > 1$.

Using the Nevanlinna-Pick theorem it is possible to replace the last condition on B by a matrix positivity condition that makes no reference to analytic functions. In fact, condition (iv) is equivalent to requiring that whenever, $(v_1, \dots, v_{k-1}) \in B$ and

$$\left(\frac{\overline{v}_i v_j - \overline{w}_i w_j}{1 - \overline{v}_i v_j}\right)$$

is positive semi-definite, then $(w_1, \dots, w_{k-1}) \in B$.

The following example shows that these conditions on B are not sufficient and illustrates some of the difficulties in determining even the elements of $\mathcal{V}_3$.

EXAMPLE 2.13. Fix $0 < r \leqslant 1$ and let $B = \{(w_1, w_2) : |w_1| + |w_2| \leqslant r\}$. We will show that B satisifes the conditions of the above proposition, but that $\widetilde{B} = V$ is not convex.

Clearly, B satisifes the first two conditions. If $w = (w_1, w_2)$ and $z = (z_1, z_2)$ are in B, then $|w_1 z_1| + |w_2 z_2| \leqslant |w_1| + |w_2| \leqslant r$ and so $w \cdot z \in B$. Also, if $f \in A(\mathbb{D})$, with $\|f\| \leqslant 1$ and $f(0) = 0$, then $f(z) = zg(z)$ with $\|g\| \leqslant 1$. From this it is easily seen that $|g(w_1)| + |g(w_2)| \leqslant r$.

Note that for any, $|a| \leqslant 1, (a, b, a) \in V$ if and only if $|\varphi_a(b)| \leqslant r$. Similarly, $(-a, -a, c) \in V$ if and only if $|\varphi_{-a}(c)| \leqslant r$. If V was convex, then $\frac{1}{2}[(a, b, a) + (-a, -a, c)] \in V$ and consequently, $|b - a| + |c + a| \leqslant 2r$. However, for $0 < a < r$, if we choose b, c satisfying $\varphi_a(b) = -r, \varphi_a(c) = +r$, then this latter inequality fails.

Similar calculations show that for $B = \{(w_1, w_2) : |w_1|^2 + |w_2|^2 \leqslant r^2\}$ one also has that $\widetilde{B}$ is not convex.

Note that in both of these examples the set B is *circled*, that is $(w_1, w_2) \in B$ implies that $(\mathrm{e}^{\mathrm{i}\theta_1} w_1, \mathrm{e}^{\mathrm{i}\theta_2} w_2) \in B$.

PROBLEM 2.14. Find necessary and sufficient conditions on a circled set $B \subseteq \mathbb{C}^2$ such that $B = \widehat{V}$ for some $V \in \mathcal{V}_3$.

We now examine $\mathcal{O}_k$. There is a dual object for sets in $\mathcal{O}_k$, called a Schur ideal, which we will find makes it much easier to determine if a set belongs to $\mathcal{O}_k$ than to $\mathcal{V}_k$ or $\mathcal{H}_k$. The following gives a convenient description of elements of $\mathcal{O}_k$.

DEFINITION 2.15. Let $E_1, \ldots, E_k$ be bounded operators on a Hilbert space H satisfying:

(i) $E_i E_j = \delta_{ij} E_i$ and

(ii) $E_1 + \cdots + E_k = I$.

Then the algebra $\mathcal{A} := \operatorname{span}\{E_i\}_{i=1}^k$ is called a *k-idempotent operator algebra*.

Each k-idempotent operator algebra determines a ball in $\mathbb{C}^k$ in the following way.

PROPOSITION 2.16. *Let $\mathcal{A}$ be a k-idempotent operator algebra. Then the set*

$$\mathcal{D}(\mathcal{A}) := \{(w_1, \ldots, w_k) \in \mathbb{C}^k : \|w_1 E_1 + \cdots + w_k E_k\| \leqslant 1\}$$

has the following properties:

(i) $\mathcal{D}(\mathcal{A}) \in \mathcal{O}_k$;

(ii) $\|v\|_{\mathcal{D}(\mathcal{A})} := \inf\{t : v \in t\mathcal{D}(\mathcal{A})\}$ *is a Banach algebra norm on* $\mathbb{C}^k$*; and*

(iii) $\|\cdot\|_{\mathcal{D}(\mathcal{A})}$ *satisfies von Neumann's inequality.*

Moreover, $B \in \mathcal{O}_k$ if and only if $B = \mathcal{D}(\mathcal{A})$ for some k-idempotent operator algebra $\mathcal{A}$.

PROPOSITION 2.17. *The intersection of a finite collection of sets in $\mathcal{O}_k$ is again in $\mathcal{O}_k$, but the intersection of an arbitrary collection of sets in $\mathcal{O}_k$ is in $\mathcal{O}_k$ if and only if the intersection is absorbing.*

Proof. Let B_α be a collection of sets in $\mathcal{O}_k$ and for each α choose a k-idempotent operator algebra $\mathcal{A}_\alpha$ acting on a Hilbert space $\mathcal{H}_\alpha$ and generated by $\{E_i^\alpha\}$ such that $B_\alpha = \mathcal{D}(\mathcal{A}_\alpha)$.

If the operators $E_i = \bigoplus_\alpha E_i^\alpha$ acting on the direct sum of the Hilbert spaces are bounded, then they will clearly generate a k-idempotent operator algebra $\mathcal{A}$ with $\mathcal{D}(\mathcal{A})$ equal to the intersection of the collection. Thus, all that remains is to note that the boundedness of the set of operators E_i is equivalent to the fact that the intersection is absorbing. ∎

DEFINITION 2.18. Given $D \subseteq \mathbb{C}^k$ contained in the closed unit polydisk, we let $\mathcal{O}(D)$ denote the intersection of all set in $\mathcal{O}_k$ that contain D.

PROPOSITION 2.19. *Let $D \subseteq \mathbb{C}^k$ be a subset of the closed polydisk. If D is separating, then $\mathcal{O}(D) \in \mathcal{O}_k$.*

Proof. Again, it is enough to prove that $\mathcal{O}(D)$ is absorbing, but since $\mathcal{B}(D) \subseteq \mathcal{O}(D)$, the result follows. ∎

When D is separating and a subset of the closed unit polydisk, then we refer to $\mathcal{O}(D)$ as the *operator algebra ball generated by D.*

In [11], the first author developed a dual object for operator norms, called *Schur ideals*. These will give us an alternate description of $\mathcal{O}(D)$.

Let $\mathcal{I}$ be a subset of M_k^+, the positive semi-definite $k \times k$ matrices satisfying:

(i) $P, Q \in \mathcal{I}$ then $P + Q \in \mathcal{I}$;
(ii) $P \in \mathcal{I}, Q$ positive, then $P * Q \in \mathcal{I}$;

where $(p_{ij}) * (q_{ij}) = (p_{ij}q_{ij})$, the Schur product, then $\mathcal{I}$ is called a *Schur ideal*. Note that the set of positive matrices is closed under $+$ and $*$, and that the matrix of all $1's$ acts as a unit for $*$. Given a set $\mathcal{S}$ of positive matrices we let $\langle \mathcal{S} \rangle$ denote the Schur ideal that it generates.

Given a set $\mathcal{S}$ of positive $k \times k$ matrices and a subset $\mathcal{D}$ of the closed k-polydisk containing 0, the first author in [11] defines:

$$\mathcal{S}^\perp := \{(w_1, \ldots, w_k) \in \mathbb{C}^k : ((1 - \overline{w}_i w_j)p_{ij}) \geqslant 0 \text{ for all } (p_{ij}) \in \mathcal{S}\},$$
$$\mathcal{D}^\perp := \{(p_{ij}) : ((1 - \overline{w}_i w_j)p_{ij}) \geqslant 0 \text{ for all } (w_1, \ldots, w_k) \in \mathcal{D}\}.$$

Note that $\mathcal{S}^\perp = \langle \mathcal{S} \rangle^\perp$.

In [11] it is shown that, if $\mathcal{D} \in \mathcal{O}_k$, then $\mathcal{D}^\perp$ is a Schur ideal with the following two properties:

(i) for each $i = 1, \ldots, k$ there exists $P \in \mathcal{D}^\perp$ such that $p_{ii} \neq 0$ and
(ii) there exists a $\delta > 0$ such that for all $P \in \mathcal{D}^\perp$, we have that $P \geqslant \delta^2 \mathrm{Diag}(P)$.

A Schur ideal satisfying (i) is said to be *non-trivial*. A Schur ideal satisfying (ii) is said to be *bounded*.

Given an arbitrary Schur ideal (which is *non-trivial* and *bounded*) contained in M_k^+, say $\mathcal{I}$, one can construct an "affiliated" k-idempotent operator algebra, say $\mathcal{A}_\mathcal{I}$. The following proposition, which is a non-matricial version of Theorem 3.2

in [12] provides a way to compute the ball $\mathcal{D}(\mathcal{A}_{\mathcal{I}})$ in terms of the "perp" of the Schur ideal $\mathcal{I}$.

PROPOSITION 2.20. *Let $\mathcal{I}$ be a non-trivial and bounded Schur ideal contained in the positive $k \times k$ matrices. Then there exists a k-idempotent operator algebra $\mathcal{A}_{\mathcal{I}}$ such that*

$$\mathcal{D}(\mathcal{A}_{\mathcal{I}}) = \mathcal{I}^{\perp}.$$

Proof. First we will construct the k-idempotent operator algebra $\mathcal{A}_{\mathcal{I}}$. Let $\mathcal{I}^{-1}$ denote the set of invertible elements in $\mathcal{I}$ and E_{ii} denote the canonical matrix units. Note that since $\mathcal{I}$ is non-trivial the set of invertible elements $\mathcal{I}^{-1}$ are dense in $\mathcal{I}$. This can be seen by observing the identity matrix I belongs to $\mathcal{I}$ and we have that for any $Q \in \mathcal{I}$, $(Q + \varepsilon I) \in \mathcal{I}$ and is invertible. Now set $E_i := \bigoplus_{Q \in \mathcal{I}^{-1}} Q^{1/2} E_{ii} Q^{-1/2}$ for $i = 1, \ldots, k$. These are operators acting on $\bigoplus_{P \in \mathcal{I}^{-1}} M_k$ and we let

$$\mathcal{A}_{\mathcal{I}} := \operatorname{span}\{E_1, \ldots, E_k\}.$$

Now since $\mathcal{I}$ is bounded, by definition we have that there exists a $\delta > 0$ such that $(q_{ij}) \geqslant \delta^2 \operatorname{Diag}(q_{ii})$ for all $(q_{ij}) \in \mathcal{I}^{-1}$. Thus, $\delta^2 q_{mm} E_{mm} \leqslant (q_{ij})$ for each $1 \leqslant m \leqslant k$ and for each $(q_{ij}) \in \mathcal{I}^{-1}$. We have that $E_i^* E_i = \bigoplus_{Q \in \mathcal{I}^{-1}} Q^{-1/2} q_{ii} E_{ii} Q^{-1/2} \leqslant \delta^{-2} \bigoplus_{Q \in \mathcal{I}^{-1}} I_{k \times k}$ and we have that $\|E_i\| \leqslant \delta^{-1}$. Thus each E_i is bounded and $\mathcal{A}_{\mathcal{I}}$ is a k-idempotent operator algebra.

We now have the following logical equivalences:

$$\begin{aligned}
&(w_1, \ldots, w_k) \in \mathcal{D}(\mathcal{A}_{\mathcal{I}}) \iff \Big\| \sum_{i=1}^{k} w_i E_i \Big\| \leqslant 1 \\
&\iff \Big\| \bigoplus_{Q \in \mathcal{I}^{-1}} Q^{1/2} \operatorname{Diag}(w_i) Q^{-1/2} \Big\| \leqslant 1 \quad \text{for all } Q \in \mathcal{I}^{-1} \\
&\iff ((1 - \overline{w}_i w_j) q_{ij}) \geqslant 0 \quad \text{for all } Q \in \mathcal{I}^{-1} \\
&\iff (w_1, \ldots, w_k) \in \mathcal{I}^{\perp}. \qquad \blacksquare
\end{aligned}$$

The following is often a useful way to generate non-trivial, bounded Schur ideals.

PROPOSITION 2.21. *Let $\mathcal{S}$ be a uniformly bounded collection of positive, invertible matrices whose inverses are also uniformly bounded. Then $\langle \mathcal{S} \rangle$ is a non-trivial, bounded Schur ideal and consequently, $\mathcal{S}^{\perp} \in \mathcal{O}_k$.*

Proof. Since each matrix is invertible, non-triviality follows. The fact that the matrices and their inverses are uniformly bounded implies that there are positive constants c and d such that $cI \leqslant P \leqslant dI$ for all $P \in \mathcal{S}$. Hence, $\operatorname{Diag}(P) \leqslant dI \leqslant c^{-1} dP$ for any $P \in \mathcal{S}$. Thus, there is a $\delta > 0$ such that $P \geqslant \delta \operatorname{Diag}(P)$ for all $P \in \mathcal{S}$.

Now if $P_1, \dots, P_m \in \mathcal{S}$ and $Q_1, \dots, Q_m \in M_k^+$, then

$$\begin{aligned} P_1 * Q_1 + \cdots + P_m * Q_m &\geqslant \delta \mathrm{Diag}(P_1) * Q_1 + \cdots + \delta \mathrm{Diag}(P_m) * Q_m \\ &= \delta \mathrm{Diag}(P_1 * Q_1 + \cdots + P_m * Q_m). \end{aligned}$$

Since $P_1 * Q_1 + \cdots + P_m * Q_m$ is a typical element of the Schur ideal generated by $\mathcal{S}$, the result follows. ∎

These results allow us to give a complete characterization of elements of $\mathcal{O}_k$.

THEOREM 2.22. *Let $D \subseteq \mathbb{C}^k$ be a subset of the closed polydisk. Then $D \in \mathcal{O}_k$ if and only if $D = D^{\perp\perp}$. Consequently, if $O \in \mathcal{O}_k$ and $D \subseteq O$, then $D^{\perp\perp} \subseteq O$.*

Proof. The first statement is Theorem 5.8 in [11]. If $D \subseteq O$, then $O^{\perp} \subseteq D^{\perp}$ and hence $D^{\perp\perp} \subseteq O^{\perp\perp} = O$. ∎

PROPOSITION 2.23. *Let $D \subset \mathbb{C}^k$ be a subset of the closed polydisk. Then $D^{\perp\perp} = \mathcal{O}(D)$.*

Proof. By the above we have that $D^{\perp\perp}$ is contained in the intersection.

Assume that $w = (w_1, \dots, w_k)$ in the polydisk is not in $D^{\perp\perp}$. Then there exists $P = (p_{i,j}) \in D^{\perp}$ such that $((1 - \overline{w}_i w_j) p_{ij})$ is not positive semi-definite. By continuity, we may replace P by P plus a small positive multiple of the identity matrix and the above matrix will still not be positive definite. Since the identity matrix is in $D^{\perp}$, this new matrix will be in $D^{\perp}$. Thus, we may assume that P is invertible. Now as in the proof of Proposition 2.20, if we let $\mathcal{A}$ be the k-idempotent operator algebra generated by $\{P^{1/2} E_{ii} P^{-1/2}\}$, then $\mathcal{D}(\mathcal{A}) \in \mathcal{O}_k, D \subseteq \mathcal{D}(\mathcal{A})$ and w will not be in $\mathcal{D}(\mathcal{A})$. ∎

By the above results to determine if $\mathcal{V}_k = \mathcal{O}_k$, it is enough to decide whether or not $V^{\perp\perp} = V$ for every $V \in \mathcal{V}_k$.

Although the above results make it relatively easy to produce sets in $\mathcal{O}_k$ we have no clear way to determine whether or not they are hyperconvex.

PROBLEM 2.24. Find necessary and sufficient conditions on a Schur ideal $\mathcal{I}$ so that $\mathcal{I}^{\perp}$ is hyperconvex.

The following example illustrates the difficulty.

PROBLEM 2.25. Let $a, c > 0$ and set

$$P_{a,c} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a+1 & 1 \\ 1 & 1 & c+1 \end{pmatrix}.$$

Then the set $\{P_{a,c}\}^{\perp} \in \mathcal{O}_3$, but is it hyperconvex?

We would like to note that one other reason that we are interested in studying k-idempotent operator algebra balls is that often one wants to study interpolation for some operator algebra A of functions on a set X, that is not a uniform algebra. For example, the algebra $\mathcal{M}(X)$ of functions that act as multipliers on some reproducing kernel Hilbert space $\mathcal{H}$ on X is a subalgebra of $B(\mathcal{H})$. This algebra equipped with the operator norm is sometimes, but not always, a uniform algebra. One still wishes to study the corresponding interpolation bodies, $\{(f(x_1), \ldots, f(x_k)) : \|M_f\| \leqslant 1\}$ in this situation. Since it is known by a result of [2] that A/I_x is an operator algebra generated by k-commuting idempotents as above, then these more general interpolation bodies will be in $\mathcal{O}_k$ but not necessarily in $\mathcal{H}_k$.

We now turn our attention to hyperconvex sets.

PROPOSITION 2.26. *The intersection of a finite collection of sets in $\mathcal{H}_k$ is again in $\mathcal{H}_k$, but the intersection of an arbitrary collection of sets in $\mathcal{H}_k$ is in $\mathcal{H}_k$ if and only if the intersection is absorbing.*

Proof. The result follows because the intersection of any collection of sets that satisfy the multi-variable von Neumann inequality will again satisfy the multi-variable von Neumann inequality. ∎

PROPOSITION 2.27. *Let $k \geqslant 2$ be an integer. Then $\mathcal{H}_k = \mathcal{O}_k$ if and only if for every positive, invertible matrix $P \in M_k$, the set $\{P\}^\perp$ is hyperconvex.*

Proof. If $\mathcal{H}_k = \mathcal{O}_k$, then since $\{P\}^\perp \in \mathcal{O}_k$ it is hyperconvex.

Conversely, if $D \in \mathcal{O}_k$, then by Theorem 2.22, $D = D^{\perp\perp}$ and, hence, D is an intersection of sets of the form $\{P\}^\perp$, each of which is hyperconvex and consequently, D is hyperconvex by the above result. Thus, $\mathcal{O}_k \subseteq \mathcal{H}_k$ and so the two sets are equal. ∎

DEFINITION 2.28. Let $D \subseteq \mathbb{C}^k$ be a subset of the closed unit polydisk, then we let $\mathcal{H}(D)$ denote the intersection of all elements in $\mathcal{H}_k$ that contain D.

PROPOSITION 2.29. *Let $D \subseteq \mathbb{C}^k$ be a separating set contained in the closed unit polydisk. Then $\mathcal{H}(D) \in \mathcal{H}_k$ and if $H_1 \in \mathcal{H}_k$ with $D \subseteq H_1$, then $\mathcal{H}(D) \subseteq H_1$.*

When D is a subset of the closed unit polydisk that is separating then we refer to $\mathcal{H}(D)$ as the *IQ algebra ball generated by D*. Again, we shall show that when D is not separating, then $\mathcal{H}(D)$ is not absorbing.

The following gives another way to realize $\mathcal{H}(D)$.

DEFINITION 2.30. Let $D \subseteq \mathbb{C}^k$ be a subset of the closed unit polydisk. For each m, for each polynomial p in m variables with $\|p\| \leqslant 1$ and for each choice of m points, $w_1, \ldots, w_m$ in D, the vector $p(w_1, \ldots, w_m) \in \mathbb{C}^k$ will lie in the closed unit polydisk. We call the closure of the set of all such vectors in $\mathbb{C}^k$, the *hyperconvex hull of D* and we denote it by $\mathrm{HC}(D)$.

PROPOSITION 2.31. *Let $D \subseteq \mathbb{C}^k$ be contained in the closed unit polydisk. Then* $\mathrm{HC}(D)$ *is closed, absolutely convex,* $\mathrm{HC}(D) \cdot \mathrm{HC}(D) \subseteq \mathrm{HC}(D)$ *and is hyperconvex. If D is also separating, then* $\mathrm{HC}(D) \in \mathcal{H}_k$.

Proof. An absolute convex combination of points in $\mathrm{HC}(D)$ is the image of points in D under the corresponding absolute convex combination of the polynomials. Similarly, the product of two points in $\mathrm{HC}(D)$ is the image under the product of the corresponding polynomials. Finally, the fact that $\mathrm{HC}(D)$ is hyperconvex follows since the image of a collection of points in $\mathrm{HC}(D)$ under a polynomial is the image of their predecessors under a composition of polynomials.

It remains to show that if D is separating, then $\mathrm{HC}(D) \in \mathcal{H}_k$. It will be enough to show that $\mathrm{HC}(D)$ is absorbing. Arguing as in the proof of Proposition 2.4, we can show that a positive multiple of each basis vector is in $\mathrm{HC}(D)$ and then it follows that $\mathrm{HC}(D)$ is absorbing. ∎

THEOREM 2.32. *Let $D \subseteq \mathbb{C}^k$ be contained in the closed unit polydisk. Then $\mathrm{HC}(D) = \mathcal{H}(D)$.*

Proof. If $D \subseteq H$ and $H \in \mathcal{H}_k$, then necessarily $\mathrm{HC}(D) \subseteq H$. Thus, $\mathrm{HC}(D) \subseteq \mathcal{H}(D)$.

If D is separating, then it follows that $\mathrm{HC}(D) \in \mathcal{H}_k$ and so $\mathcal{H}(D) \subseteq \mathrm{HC}(D)$. Now assume that D is arbitrary. Without loss of generality, we may assume that $0 \in D$. Let $\varepsilon > 0$ be arbitrary and pick a vector $w = (w_1, \ldots, w_k)$ with $w_i \neq w_j$ for $i \neq j$ and $w_i \leqslant \varepsilon$ for all i. Let $D_1 = D \cup \{w\}$, so that D_1 is separating. Consequently, $\mathcal{H}(D) \subseteq \mathcal{H}(D_1) = \mathrm{HC}(D_1)$.

Finally, note that the only elements that are in $\mathrm{HC}(D_1)$ but not in $\mathrm{HC}(D)$ are polynomials that involve the vector w and other elements of D. Freezing all of the variables except the one involving w we see that we obtain a polynomial $p(z)$ in a single variable, with $\|p\| \leqslant 1$. Consequently, $|p(w_i) - p(0)| \leqslant \varepsilon$ and we deduce that every vector in $\mathrm{HC}(D_1)$ is at most distance $k\varepsilon$ from a vector in $\mathrm{HC}(D)$.

Now assume that v is not in $\mathrm{HC}(D)$. Then for a small enough ε, v will not be in $\mathrm{HC}(D_1)$. But this latter set is the intersection of all balls in $\mathcal{H}_k$ that contains D_1. Hence, there will exist $H \in \mathcal{H}_k$, such that v is not in H, and $D \subseteq D_1 \subseteq H$. Thus, v is not in $\mathcal{H}(D)$ and so $\mathcal{H}(D) \subseteq \mathrm{HC}(D)$. ∎

COROLLARY 2.33. *Let $D \subseteq \mathbb{C}^k$ be contained in the closed unit polydisk. If D is not separating, then each of the sets $\mathcal{B}(D), \mathcal{V}(D), \mathcal{O}(D)$ and $\mathcal{H}(D)$ is not absorbing.*

Proof. Since $\mathcal{B}(D) \subseteq \mathcal{V}(D) \subseteq \mathcal{O}(D) \subseteq \mathcal{H}(D)$, it will be enough to show that $\mathcal{H}(D)$ is not absorbing. But $\mathcal{H}(D) = \mathrm{HC}(D)$ and if there exists $i \neq j$ such that $w_i = w_j$ for every vector $w = (w_1, \ldots, w_k) \in D$, then it is easily seen that every vector in $\mathrm{HC}(D)$ will be equal in the i-th and j-th components and so $\mathrm{HC}(D)$ can not be absorbing. ∎

We close this section with another way to describe the sets $\mathcal{H}(D)$ and $\mathcal{O}(D)$ in the special case that $D = \{v_1, \ldots, v_n\} \subseteq \mathbb{C}^k$ is a separating set consisting of n points. Recall that in this case setting $x_j = (w_{1,j}, \dots, w_{n,j})$ where $v_i = (w_{i,1}, \dots, w_{i,k})$ defines k distinct points in the closed polydisk in $\mathbb{C}^n$.

PROPOSITION 2.34. *Let $D = \{v_1, \dots, v_n\} \subseteq \mathbb{C}^k$ be a separating set consisting of n points and let $x_1, \dots, x_k$ be defined as above, then $\mathcal{H}(D) = \mathcal{D}(A(\mathbb{D}^n); x_1, \dots, x_k)$.*

Proof. Given a polynomial p in n variables the i-th component of the vector $p(v_1, \dots, v_n)$ is the number $p(x_i)$. Thus, $\mathcal{D}(A(\mathbb{D}^n); x_1, \dots, x_k) \subseteq \mathrm{HC}(D) = \mathcal{H}(D)$. However, evaluating the n coordinate functions at $x_1, \dots, x_k$ shows that $D \subset \mathcal{D}(A(\mathbb{D}^n); x_1, \dots, x_k)$ and since this latter set is hyperconvex, we have that $\mathcal{H}(D) \subseteq \mathcal{D}(A(\mathbb{D}^n); x_1, \dots, x_k)$ and the result follows. ∎

A similar result holds for $\mathcal{O}(D)$ if one first introduces the *universal operator algebra for n commuting contractions* $A_{\mathrm{u}}(\mathbb{D}^n)$, as in [13]. This algebra is the completion of the algebra of polynomials in n variables where the norm of a polynomial is defined by taking the supremum of the norms of the operators defined by evaluating the polynomial at an arbitrary n tuple of commuting operators on a Hilbert space. By the fact that von Neumann's inequality fails for 3 or more commuting contractions, we see that in general the norm of a polynomial in $A_{\mathrm{u}}(\mathbb{D}^n)$ will generally be larger than its supremum norm over the polydisk.

Just as for a uniform algebra, given $x_1, \dots, x_k$ in the closed polydisk, one may form the set

$$\mathcal{D}(A_{\mathrm{u}}(\mathbb{D}^n); x_1, \dots, x_k) = \{(f(x_1), \dots, f(x_k)) : \|f\|_{\mathrm{u}} \leqslant 1\}.$$

Applying either Cole's theorem [3] or the theorem of Blecher-Ruan-Sinclair, one finds that this set is the unit ball of an operator algebra norm on $\mathbb{C}^k$.

LEMMA 2.35. *Let $B \in \mathcal{O}_k$, let $v_1, \dots, v_n \in B$ and let $f \in A_{\mathrm{u}}(\mathbb{D}^n)$ with $\|f\|_{\mathrm{u}} \leqslant 1$, then $f(v_1, \dots, v_n) \in B$.*

Proof. It is enough to consider the case that f is a polynomial, in which case the result follows from the factorization theory for universal operator algebra norms. See for example, Corollary 18.2 of [13]. ∎

PROPOSITION 2.36. *Let $D = \{v_1, \dots, v_n\} \subseteq \mathbb{C}^k$ be a separating set consisting of n points and let $x_1, \dots, x_k$ be defined as above, then $\mathcal{O}(D) = \mathcal{D}(A_{\mathrm{u}}(\mathbb{D}^n); x_1, \dots, x_k)$.*

Proof. Let $f \in A_{\mathrm{u}}(\mathbb{D}^n)$ with $\|f\|_{\mathrm{u}} \leqslant 1$. By the lemma, $(f(x_1), \dots, f(x_k)) = f(v_1, \dots, v_n) \in \mathcal{O}(D)$ and hence, $\mathcal{D}(A_{\mathrm{u}}(\mathbb{D}^n); x_1, \dots, x_k) \subseteq \mathcal{O}(D)$.

Conversely, $\mathcal{D}(A_{\mathrm{u}}(\mathbb{D}^n); x_1, \dots, x_k) \in \mathcal{O}_k$ and contains D and hence contains $\mathcal{O}(D)$. ∎

COROLLARY 2.37. *Let $D \subseteq \mathbb{C}^k$ be contained in the closed unit polydisk. If D consists of two or fewer points, then $\mathcal{O}(D) = \mathcal{H}(D)$.*

Proof. By Ando's theorem [1], we have that $A(\mathbb{D}^2) = A_{\mathrm{u}}(\mathbb{D}^2)$ isometrically and hence $\mathcal{H}(D) = \mathcal{D}(A(\mathbb{D}^2); x_1, x_2) = \mathcal{D}(A_{\mathrm{u}}(\mathbb{D}^2); x_1, x_2) = \mathcal{O}(D)$. ∎

3. EXAMPLES

In this section we will present two examples. For the first example we will show that an example of Holbrook [8] yields a 4-idempotent operator algebra acting on $\mathbb{C}^4$ whose unit ball is not a hyperconvex set, i.e., an element of $\mathcal{O}_4$ not in $\mathcal{H}_4$.

Recall that Cole and Wermer in [6] show that $\mathcal{D}(A; x_1, \ldots, x_k)$ is a semi-algebraic set when A is the bidisk algebra. This leads naturally to the question of whether or not there exists a uniform algebra A such that $\mathcal{D}(A; x_1, \ldots, x_k)$ is not a semi-algebraic set. For the second example we will construct a 3-idempotent operator algebra and conjecture that the unit ball determined by this algebra is not a semi-algebraic set. We have included a heuristic argument for why we believe the conjecture to be true.

EXAMPLE 3.1. In this example we will use a result of J. Holbrook to show that $\mathcal{H}_4$ is a proper subset of $\mathcal{O}_4$ and consequently that $\mathcal{H}_k \subsetneq \mathcal{O}_k$ for all $k \geqslant 4$.

Recall that in 1951 J. von Neumann proved that if T is a contraction on a complex Hilbert space and p is polynomial in one variable (with complex coefficients), then

$$\|p(T)\| \leqslant \sup\{|p(z)| : |z| \leqslant 1\}.$$

T. Ando extended this result in 1963 by showing that if S and T are commuting contractions and p is a polynomial in two variables, then

$$\|p(S,T)\| \leqslant \sup\{|p(z,w)| : |z| \leqslant 1,\ |w| \leqslant 1\}.$$

In 1973, S. Kaijser and N.T. Varopoulos explicitly describe 3 commuting contractions T_1, T_2, and T_3 acting on a 5-dimensional Hilbert space and a polynomial p in three variables such that

$$\|p(T_1, T_2, T_3)\| > \|p\|_\infty.$$

In 1991, Lotto and Steger proved a diagonalizable set of such contractions exist. Holbrook [8] improved this result by lowering the dimension to 4.

THEOREM 3.2 (Lotto-Steger and Holbrook). *There are three commuting, diagonalizable contractions T_1, T_2, and T_3 on $\mathbb{C}^4$ and a polynomial p in three variables such that $\|p(T_1, T_2, T_3)\| > \|p\|_\infty$.*

We use this fact to show that there exists a 4-idempotent operator algebra which is not an interpolation body (hyperconvex set).

Now let T_1, T_2, T_3, and p be as in Theorem 3.2 acting on $\mathbb{C}^4$ and choose an invertible 4×4 matrix Q such that QT_jQ^{-1} for $j=1,2,3$ are the diagonal matrices

$$QT_jQ^{-1} = \begin{pmatrix} w_1^j & 0 & 0 & 0 \\ 0 & w_2^j & 0 & 0 \\ 0 & 0 & w_3^j & 0 \\ 0 & 0 & 0 & w_4^j \end{pmatrix}.$$

Let E_{ij} denote the canonical matrix units and define idempotents

$$E_i := QE_{ii}Q^{-1}$$

for $i=1,2,3,4$ and let $\mathcal{A} := \operatorname{span}\{E_1,E_2,E_3,E_4\}$. Next observe that $(w_1^j, w_2^j, w_3^j, w_4^j) \in \mathcal{D}(\mathcal{A})$ for $j=1,2,3$. However,

$$p((w_1^1,\dots,w_4^1),(w_1^2,\dots,w_4^2),(w_1^3,\dots,w_4^3)) \notin \mathcal{D}(\mathcal{A}).$$

Therefore, $\mathcal{D}(\mathcal{A})$ fails to be hyperconvex and we have that $\mathcal{H}_k \subsetneq \mathcal{O}_k$ for all $k \geqslant 4$.

EXAMPLE 3.3. In this example we will inductively construct a Schur ideal (non-trivial and bounded) and we conjecture that the "perp" of this Schur ideal is not a semi-algebraic subset of $\mathbb{R}^3$. Hence, if the conjecture is true, then by Proposition 2.20 we will have that there is a 3-idempotent operator algebra such that the ball of this operator algebra is not a semi-algebraic set.

First we will inductively construct the appropriate Schur ideal.

LEMMA 3.4. *Let $a,c>0$. Then for $x,y\in\mathbb{R}$ we have that $(0,x,y)\in\{P_{a,c}\}^\perp$ if and only if*

$$x^2 \leqslant \frac{a}{a+1} \text{ and} \tag{3.1}$$

$$y^2 \leqslant \frac{ac-(ac+c)x^2}{(ac+a)-(ac+a+c)x^2}, \tag{3.2}$$

where $P_{a,c}$ is the following 3×3 positive definite matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & a+1 & 1 \\ 1 & 1 & c+1 \end{pmatrix}.$$

Proof. By definition $(0,x,y)\in\{P_{a,c}\}^\perp$ if and only if the following matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1-x^2 & 1-xy \\ 1 & 1-xy & 1-y^2 \end{pmatrix} * P_{a,c} \tag{3.3}$$

is positive semi-definite. The matrix in (3.3) is positive semi-definite if and only if

$$(a-(a+1)x^2)(c-(c+1)y^2)-x^2y^2 \geqslant 0 \text{ and} \tag{3.4}$$

$$x^2 \leqslant \frac{a}{a+1} \quad \text{and} \quad y^2 \leqslant \frac{c}{c+1}, \tag{3.5}$$

by applying the Cholesky algorithm [9].

Solving (3.4) for y^2 yields

$$y^2 \leqslant \frac{ac-(ac+c)x^2}{(ac+a)-(ac+a+c)x^2}.$$

Thus, if the matrix is positive semidefinite, then the two inequalities hold.

Conversely, if the two inequalities hold, then we have by (3.1) that $ac-(ac+c)x^2 \geqslant 0$ and since $a-ax^2 \geqslant 0$, it follows that $ac+a-(ac+a+c)x^2 \geqslant 0$. Since $y^2 \geqslant 0$, it follows that at any point (x,y) satisfying the two inequalities, both the numerator and denominator of the fraction in inequality (3.2) must be non-negative. From this last statement, one can now see that (3.2) implies (3.4). Using the positivity of both the numerator and denominator and cross-multiplying yields,

$$\frac{ac-(ac+c)x^2}{(ac+a)-(ac+a+c)x^2} \leqslant \frac{c}{c+1},$$

and hence, $y^2 \leqslant \frac{c}{c+1}$. Thus, (3.1) and (3.2) imply that (3.4) and (3.5) hold and hence the matrix is positive semidefinite. ∎

Let $u = x^2, v = y^2$ and let $f_{a,c}(u) := \frac{ac-(ac+c)u}{(ac+a)-(ac+a+c)u}$. Note that $f_{a,c}$ is decreasing with $f_{a,c}(0) = \frac{c}{c+1}$ and $f_{a,c}\left(\frac{a}{a+1}\right) = 0$.

Next observe that if $a_1 < a$ and $c_1 > c$, then $\frac{a_1}{a_1+1} < \frac{a}{a+1}$ and $\frac{c_1}{c_1+1} > \frac{c}{c+1}$. Thus the graphs of $f_{a,c}$ and f_{a_1,c_1} intersect at some point $\mu_1 < \frac{a_1}{a_1+1}$.

For each function in the family, $f'_{a,c}(u) = \frac{-ac}{(ac-(ac+c)u)^2}$ is also a monotone decreasing function of u. Since, $\lim_{c_1 \to +\infty} f'_{a_1,c_1}(0) = -\infty$, by chosing c_1 sufficiently large we can guarantee that at μ_1, we have $f'_{a_1,c_1}(\mu_1) < f'_{a_1,c_1}(0) < f'_{a,c}(\mu_1)$.

Now choose any $a_2 < a_1$ such that $\mu_1 < \frac{a_2}{a_2+1}$ and then choose $c_2 > c_1$ such that the point of intersection, μ_2, of f_{a_1,c_1} and f_{a_2,c_2} satisfies $\mu_1 < \mu_2$. To see that this can be done, note that for fixed a_2, as c_2 increases the point of intersection of the two curves moves to the right continuously and approaches $\frac{a_2}{a_2+1}$ in the limit. This also allows us to choose c_2 so that $|\frac{a_2}{a_2+1} - \mu_2| \leqslant \frac{1}{2}|\frac{a_1}{a_1+1} - \mu_1|$.

Now if we look at the set

$$\{(u,v) : v \leqslant f_{a,c}(u),\, v \leqslant f_{a_1,c_1}(u),\, v \leqslant f_{a_2,c_2}(u)\}$$

it will have two non-differentiable corners.

Now inductively choose $\{a_n\}$ and $\{c_n\}$ such that the $a_n \searrow$ and $c_n \nearrow$ and so that if μ_n is the point of intersection of f_{a_n,c_n} with $f_{a_{n+1},c_{n+1}}$, then we have that $\mu_n \nearrow$, with $|\frac{a_{n+1}}{a_{n+1}+1} - \mu_{n+1}| \leqslant \frac{1}{2}|\frac{a_n}{a_n+1} - \mu_n|$ and $\mu_n < \frac{a_m}{a_m+1}$ for all $m,n \in \mathbb{N}$. Note that $\lim_{n\to\infty}\mu_n = \lim_{n\to\infty}\dfrac{a_n}{a_n+1}$ and we call this common limit μ. Set $a_0 = a, b_0 = b$ and $\mu_0 = 0$.

CONJECTURE 3.5. Let a_n, $c_n > 0$ be chosen as above. Then $\{P_{a_n,c_n} : n \geqslant 0\}^{\perp}$ is a non-algebraic subset of $\mathcal{O}_4$.

Our only obstruction to proving the above conjecture is a problem concerning semi-algebraic sets that seems likely to be true, but which we have been unable to prove or find in the literature.

CONJECTURE 3.6. Let $f : [0,1] \to \mathbb{R}$ be continuous, non-negative and set $C = \{(x,y) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant f(x)\}$. If f is non-differentiable at infinitely many points, then C is not semi-algebraic.

PROPOSITION 3.7. *If the above conjecture concerning semi-algebraic sets is true, then there exists $B \in \mathcal{O}_3$ that is not semi-algebraic.*

Proof. We let $B = \{P_{a_n,c_n} : n \geqslant 0\}^{\perp}$ denote the set in the earlier conjecture.

By our earlier results, it is clear that B is in $\mathcal{O}_4$. So it remains to show that this set is not semi-algebraic.

Now for x and y real, $(0,x,y) \in B$, if and only if $y^2 \leqslant f(x^2)$ and $x^2 \leqslant \frac{\mu}{\mu+1}$ where

$$f(u) := \inf\{f_{a_n,c_n}(u) : n \in \mathbb{N}\}.$$

Also, for $\mu_n \leqslant u \leqslant \mu_{n+1}$, we have that $f(u) = f_{a_n,c_n}(u)$.

Now if B was semi-algebraic, then $C = \{(x,y) \in \mathbb{R}^2 : (0,x,y) \in B\}$ is semi-algebraic, by the Tarski-Seidenberg theorem [6]. Loosely speaking, Tarski-Seidenberg says that if a subset $X \subseteq \mathbb{R}^{n+1}$ is a semi-algebraic set, then any coordinate projection of X onto $\mathbb{R}^n$ is also a semi-algebraic set.

But by our construction, C is the region under the graph of a function f that has a countable collection of points of non-differentiability. By the second conjecture, this set can not be semi-algebraic. ∎

Cole and Wermer [6] has an appendix which includes many of the important theorems on semi-algebraic sets and serves as a nice introduction to this area.

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VERN I. PAULSEN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX, 77204, U.S.A.

E-mail address: vern@math.uh.edu

JAMES P. SOLAZZO, DEPARTMENT OF MATHEMATICS, COASTAL CAROLINA UNIVERSITY, CONWAY, S.C., 29526 U.S.A.

E-mail address: jsolazzo@coastal.edu

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