

## CHARACTERIZATIONS OF COMPACT AND DISCRETE QUANTUM GROUPS THROUGH SECOND DUALS

VOLKER RUNDE

*Communicated by Kenneth R. Davidson*

ABSTRACT. A locally compact group  $G$  is compact if and only if  $L^1(G)$  is an ideal in  $L^1(G)^{**}$ , and the Fourier algebra  $A(G)$  of  $G$  is an ideal in  $A(G)^{**}$  if and only if  $G$  is discrete. On the other hand,  $G$  is discrete if and only if  $\mathcal{C}_0(G)$  is an ideal in  $\mathcal{C}_0(G)^{**}$ . We show that these assertions are special cases of results on locally compact quantum groups in the sense of J. Kustermans and S. Vaes. In particular, a von Neumann algebraic quantum group  $(\mathfrak{M}, \Gamma)$  is compact if and only if  $\mathfrak{M}_*$  is an ideal in  $\mathfrak{M}^*$ , and a (reduced)  $C^*$ -algebraic quantum group  $(\mathfrak{A}, \Gamma)$  is discrete if and only if  $\mathfrak{A}$  is an ideal in  $\mathfrak{A}^{**}$ .

KEYWORDS: *Compact quantum group, discrete quantum group, locally compact quantum group, second dual, weakly compact multiplication.*

MSC (2000): Primary 22C05, 22D35, 43A99, 46H10, 46L51, 46L65, 46L89; Secondary 47L50, 81R15, 81R50.

### INTRODUCTION

Recently, J. Kustermans and S. Vaes introduced a surprisingly simple set of axioms for what they call *locally compact quantum groups* ([9] and [10]). These axioms cover both the Kac algebras [4], and thus all locally compact groups, as well as the compact quantum groups in the sense of [21] and allow for a notion of duality that extends the Pontryagin duality for locally compact abelian groups.

What makes the locally compact quantum groups interesting from the point of view of abstract harmonic analysis is that many results in abstract harmonic analysis can easily be reformulated in the language of locally compact quantum groups. For instance, Leptin's theorem [12] asserts that a locally compact group  $G$  is amenable if and only if the Fourier algebra  $A(G)$  [5] has a bounded approximate identity. This statement can be rephrased as:  $G$  is amenable if and only if its quantum group dual is co-amenable (as defined in Definition 3.1 of [2]). Indeed, the "if" part is true for any locally compact quantum group ([2], Theorem 3.2)

whereas the converse is known to be true only in the discrete case ([18]; see also [15] and [2]).

In [20] (see also [6] and [7]), S. Watanabe proved that a locally compact group  $G$  is compact if and only if its group algebra  $L^1(G)$  is an ideal in its second dual. On the other hand, A.T.-M. Lau showed that  $A(G)$  is an ideal in  $A(G)^{**}$  if and only if  $G$  is discrete ([11], Theorem 3.7). We shall see that both results are just two facets of one theorem on locally compact quantum groups. To this end, note that, according to [10], one way of describing a locally compact quantum group is as a Hopf-von Neumann algebra  $(\mathfrak{M}, \Gamma)$  with additional structure. Hence, the predual  $\mathfrak{M}_*$  of  $\mathfrak{M}$  is a Banach algebra in a canonical way. Simultaneously extending the results by both Watanabe and Lau, we shall prove that the locally compact quantum group  $(\mathfrak{M}, \Gamma)$  is compact if and only if  $\mathfrak{M}_*$  is an ideal in its second dual.

If  $G$  is a discrete group, then  $\mathcal{C}_0(G) = c_0(G)$  is an ideal in  $\mathcal{C}_0(G)^{**} = \ell^\infty(G)$ . On the other hand, if  $\mathcal{C}_0(G)$  is an ideal in  $\mathcal{C}_0(G)^{**}$ , then multiplication in  $\mathcal{C}_0(G)$  is weakly compact ([14], Proposition 1.4.13) and thus compact (because  $\mathcal{C}_0(G)$  has the Dunford–Pettis property), so that  $G$  is discrete. We shall see that this result also extends to locally compact quantum groups: if  $(\mathfrak{A}, \Gamma)$  is a reduced  $C^*$ -algebraic quantum group, then  $(\mathfrak{A}, \Gamma)$  is discrete if and only if  $\mathfrak{A}$  is an ideal in  $\mathfrak{A}^{**}$ .

1. COMPACTNESS AND AMENABILITY FOR HOPF-VON NEUMANN ALGEBRAS

In this section, we formulate and briefly discuss the notions of compactness and amenability in a general Hopf-von Neumann algebra context.

We begin with recalling the definition of a Hopf-von Neumann algebra ( $\overline{\otimes}$  denotes the  $W^*$ -tensor product):

DEFINITION 1.1. A Hopf-von Neumann algebra is a pair  $(\mathfrak{M}, \Gamma)$ , where  $\mathfrak{M}$  is a von Neumann algebra and  $\Gamma : \mathfrak{M} \rightarrow \mathfrak{M} \overline{\otimes} \mathfrak{M}$  is a co-multiplication, i.e., a normal, unital  $*$ -homomorphism satisfying  $(\text{id} \otimes \Gamma) \circ \Gamma = (\Gamma \otimes \text{id}) \circ \Gamma$ .

The main examples we have in mind are from abstract harmonic analysis:

EXAMPLES. (i) For a locally compact group  $G$ , define  $\Gamma_G : L^\infty(G) \rightarrow L^\infty(G \times G)$  by letting

$$(\Gamma_G \phi)(x, y) := \phi(xy) \quad (\phi \in L^\infty(G), x, y \in G).$$

Then  $(L^\infty(G), \Gamma_G)$  is a Hopf-von Neumann algebra.

(ii) For a locally compact group  $G$ , let  $\lambda$  denote its left regular representation, and let  $\text{VN}(G) := \lambda(G)''$  denote the group von Neumann algebra of  $G$ . Then

$$\widehat{\Gamma}_G : \text{VN}(G) \rightarrow \text{VN}(G) \overline{\otimes} \text{VN}(G), \quad \lambda(x) \mapsto \lambda(x) \otimes \lambda(x)$$

defines a co-multiplication, so that  $(\text{VN}(G), \widehat{\Gamma}_G)$  is a Hopf-von Neumann algebra.

Given a Hopf–von Neumann algebra  $(\mathfrak{M}, \Gamma)$ , one can define a product  $*$  on  $\mathfrak{M}_*$ , the unique predual of  $\mathfrak{M}$ , turning it into a Banach algebra:

$$(1.1) \quad (f * g)(x) := (f \otimes g)(\Gamma x) \quad (f, g \in \mathfrak{M}_*, x \in \mathfrak{M}).$$

EXAMPLE. Let  $G$  be a locally compact group. Then applying (1.1) to  $(L^\infty(G), \Gamma_G)$  yields the usual convolution product on  $L^1(G)$  whereas, for  $(VN(G), \widehat{\Gamma}_G)$ , we obtain pointwise multiplication on  $A(G)$  [5].

For any Banach algebra  $\mathfrak{A}$ , there are two canonical ways to extend the product to the second dual: the two *Arens products* (see [3] or [14]). These two products need not coincide; they are identical, however, whenever one of the two factors involved is from  $\mathfrak{A}$ . Given a Hopf–von Neumann algebra  $(\mathfrak{M}, \Gamma)$ , we will write  $*$  for any of the two Arens products on  $\mathfrak{M}_*^{**} = \mathfrak{M}^*$ .

We now define what it means for a Hopf–von Neumann algebra to be (left) amenable and compact, respectively:

DEFINITION 1.2. Let  $(\mathfrak{M}, \Gamma)$  be a Hopf–von Neumann algebra. Then we call  $(\mathfrak{M}, \Gamma)$  *left amenable* if there is a *left invariant* state  $M \in \mathfrak{M}^*$ , i.e., satisfying

$$f * M = f(1)M \quad (f \in \mathfrak{M}_*).$$

If  $M$  can be chosen to be in  $\mathfrak{M}_*$ , we call  $(\mathfrak{M}, \Gamma)$  *left compact*.

EXAMPLE. If  $G$  is a locally compact group, then  $(L^\infty(G), \Gamma_G)$  is left amenable if and only if  $G$  is amenable and left compact if and only if  $G$  is compact.

Of course one can equally well define right amenable (compact) Hopf–von Neumann algebras, through the existence of right invariant (normal) states, as well as amenable (compact) Hopf–von Neumann algebras by demanding the existence of a (normal) state that is both left and right invariant.

The following is well known, but for convenience, we include a proof:

PROPOSITION 1.3. *Let  $(\mathfrak{M}, \Gamma)$  be a Hopf–von Neumann algebra. Then  $(\mathfrak{M}, \Gamma)$  is amenable (compact) if and only if  $(\mathfrak{M}, \Gamma)$  is both left and right amenable (compact).*

*Proof.* If  $M_l \in M^*$  is a left invariant state and  $M_r \in \mathfrak{M}^*$  is a right invariant state, then  $M_l * M_r$  is an invariant state (no matter which Arens product is chosen on  $\mathfrak{M}^*$ ). Moreover, if  $M_l$  and  $M_r$  are both normal, then so is  $M_l * M_r$ . ■

We conclude this section with recalling the notion of a co-involution.

DEFINITION 1.4. Let  $(\mathfrak{M}, \Gamma)$  be a Hopf–von Neumann algebra. A *co-involution* for  $(\mathfrak{M}, \Gamma)$  is a  $*$ -antihomomorphism  $R: \mathfrak{M} \rightarrow \mathfrak{M}$  with  $R^2 = \text{id}$  satisfying

$$(R \otimes R) \circ \Gamma = \sigma \circ \Gamma \circ R,$$

where  $\sigma$  is the flip map on  $\mathfrak{M} \overline{\otimes} \mathfrak{M}$ .

A co-involution  $R$  for a Hopf–von Neumann algebra  $(\mathfrak{M}, \Gamma)$  is necessarily normal and can be used to define an involution  $\flat$  on the Banach algebra  $\mathfrak{M}_*$ . For

$f \in \mathfrak{M}_*$ , define  $\bar{f} \in \mathfrak{M}_*$  by letting

$$\bar{f}(x) := \overline{f(x^*)} \quad (x \in \mathfrak{M}),$$

and set  $f^\natural := \bar{f} \circ R$ . It is easily seen that  $^\natural$  is indeed an involution.

In view of Proposition 1.3, we obtain:

**COROLLARY 1.5.** *The following are equivalent for a Hopf-von Neumann algebra  $(\mathfrak{M}, \Gamma)$  with co-involution:*

- (i)  $(\mathfrak{M}, \Gamma)$  is left amenable (compact);
- (ii)  $(\mathfrak{M}, \Gamma)$  is right amenable (compact);
- (iii)  $(\mathfrak{M}, \Gamma)$  is amenable (compact).

## 2. LOCALLY COMPACT QUANTUM GROUPS

Compact quantum groups in the  $C^*$ -algebraic setting were defined by S.L. Woronowicz (see [21] and [13] for accounts). In [9], J. Kustermans and S. Vaes introduced a comparatively simple set of axioms to define general locally compact quantum groups in a  $C^*$ -algebraic context. Alternatively, locally compact quantum groups can also be described as Hopf-von Neumann algebras with additional structure (see [10] and [19]): both approaches are equivalent.

In this section, we give an outline of the von Neumann algebraic approach to locally compact quantum groups. For details, see [9], [10], and also [8].

We start with recalling some notions about weights on von Neumann algebras (see [17], for instance).

Let  $\mathfrak{M}$  be a von Neumann algebra, and let  $\mathfrak{M}^+$  denote its positive elements. A *weight* on  $\mathfrak{M}$  is an additive map  $\phi: \mathfrak{M}^+ \rightarrow [0, \infty]$  such that  $\phi(tx) = t\phi(x)$  for  $t \in [0, \infty)$  and  $x \in \mathfrak{M}^+$ . We let

$$\mathcal{M}_\phi^+ := \{x \in \mathfrak{M}^+ : \phi(x) < \infty\}, \quad \mathcal{M}_\phi := \text{the linear span of } \mathcal{M}_\phi^+,$$

and

$$\mathcal{N}_\phi := \{x \in \mathfrak{M} : x^*x \in \mathcal{M}_\phi\}.$$

Then  $\phi$  extends to a linear map on  $\mathcal{M}_\phi$ , and  $\mathcal{N}_\phi$  is a left ideal of  $\mathfrak{M}$ . Using the GNS-construction ([17], p. 42), we obtain a representation  $\pi_\phi$  of  $\mathfrak{M}$  on some Hilbert space  $\mathfrak{H}_\phi$ ; we denote the canonical map from  $\mathcal{N}_\phi$  into  $\mathfrak{H}_\phi$  by  $\Lambda_\phi$ . Moreover, we call  $\phi$  *finite* if  $\mathcal{M}_\phi^+ = \mathfrak{M}^+$ , *semi-finite* if  $\mathcal{M}_\phi$  is  $w^*$ -dense in  $\mathfrak{M}$ , *faithful* if  $\phi(x) = 0$  for  $x \in \mathfrak{M}^+$  implies that  $x = 0$ , and *normal* if  $\sup_\alpha \phi(x_\alpha) = \phi\left(\sup_\alpha x_\alpha\right)$  for each bounded, increasing net  $(x_\alpha)_\alpha$  in  $\mathfrak{M}^+$ . If  $\phi$  is faithful and normal, then the corresponding representation  $\pi_\phi$  is faithful and normal, too ([17], Proposition VII.1.4).

**DEFINITION 2.1.** A *locally compact quantum group* is a Hopf-von Neumann algebra  $(\mathfrak{M}, \Gamma)$  such that:

(i) there is a normal, semifinite, faithful weight  $\phi$  on  $\mathfrak{M}$ , a *left Haar weight*, which is left invariant, i.e., satisfies

$$\phi((f \otimes \text{id})(\Gamma x)) = f(1)\phi(x) \quad (f \in \mathfrak{M}_*, x \in \mathcal{M}_\phi);$$

(ii) there is a normal, semifinite, faithful weight  $\psi$  on  $\mathfrak{M}$ , a *right Haar weight*, which is right invariant, i.e., satisfies

$$\psi((\text{id} \otimes f)(\Gamma x)) = f(1)\psi(x) \quad (f \in \mathfrak{M}_*, x \in \mathcal{M}_\psi).$$

EXAMPLE. Let  $G$  be a locally compact group. Then the Hopf-von Neumann algebra  $(L^\infty(G), \Gamma_G)$  is a locally compact quantum group:  $\phi$  and  $\psi$  can be chosen as left and right Haar measure, respectively.

Even though only the existence of a left and a right Haar weight, respectively, is presumed, both weights are actually unique up to a positive scalar multiple (see [9] and [10]).

For each locally compact quantum group  $(\mathfrak{M}, \Gamma)$ , there is a unique unitary, the *multiplicative unitary*,  $W \in \mathcal{B}(\mathfrak{H}_\phi \otimes_2 \mathfrak{H}_\phi)$ , where  $\otimes_2$  stands for the Hilbert space tensor product, such that

$$W^*(\Lambda_\phi(x) \otimes \Lambda_\phi(y)) = (\Lambda_\phi \otimes \Lambda_\phi)((\Gamma y)(x \otimes 1)) \quad (x, y \in \mathcal{N}_\phi)$$

([10], Theorem 1.2). The unitary  $W$  lies in  $\mathfrak{M} \overline{\otimes} \mathcal{B}(\mathfrak{H}_\phi)$  and implements the co-multiplication via

$$\Gamma x = W^*(1 \otimes x)W \quad (x \in \mathfrak{M})$$

(see the discussion following Theorem 1.2 of [10]). As discussed in [9] and [10], locally compact quantum groups can equivalently be described in  $C^*$ -algebraic terms. The corresponding  $C^*$ -algebra is obtained as

$$\mathfrak{A} := \overline{\{(\text{id} \otimes \nu)(W) : \nu \in \mathcal{B}(\mathfrak{H}_\phi)_*\}}^{\|\cdot\|}.$$

To emphasize the parallels between locally compact quantum groups and groups, we shall use the following notation (which was suggested by Z.-J. Ruan): a locally compact quantum group  $(\mathfrak{M}, \Gamma)$  is denoted by the symbol  $\mathbb{G}$ , and we write  $L^\infty(\mathbb{G})$  for  $\mathfrak{M}$ ,  $L^1(\mathbb{G})$  for  $\mathfrak{M}_*$ ,  $L^2(\mathbb{G})$  for  $\mathfrak{H}_\phi$ , and  $\mathcal{C}_0(\mathbb{G})$  for  $\mathfrak{A}$ . If  $L^\infty(\mathbb{G}) = L^\infty(G)$  for a locally compact group  $G$  and  $\Gamma = \Gamma_G$ , we say that  $\mathbb{G}$  actually is a locally compact group.

The *left regular* representation of a locally compact quantum group  $\mathbb{G}$  is defined as

$$\lambda: L^1(\mathbb{G}) \rightarrow \mathcal{B}(L^2(\mathbb{G})), \quad f \mapsto (f \otimes \text{id})(W).$$

(If  $\mathbb{G}$  is a locally compact group  $G$ , this is just the usual left regular representation of  $L^1(G)$  on  $L^2(G)$ .) It is a faithful, contractive representation of the Banach algebra  $L^1(\mathbb{G})$  on  $L^2(\mathbb{G})$ .

Locally compact quantum groups allow for the development of a duality theory that extends Pontryagin duality for locally compact abelian groups.

Set

$$L^\infty(\widehat{\mathbb{G}}) := \overline{\lambda(L^1(\mathbb{G}))}^{\sigma\text{-strongly}^*}.$$

Then  $L^\infty(\widehat{\mathbb{G}})$  is a von Neumann algebra, and

$$\widehat{\Gamma}: L^\infty(\widehat{\mathbb{G}}) \rightarrow L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} L^\infty(\widehat{\mathbb{G}}), \quad x \mapsto \sigma W(x \otimes 1) W^* \sigma$$

is a co-multiplication (here,  $\sigma$  is the flip map on  $L^2(\mathbb{G}) \otimes_2 L^2(\mathbb{G})$ ). One can also define a left Haar weight  $\widehat{\varphi}$  and a right Haar weight  $\widehat{\psi}$  for  $(L^\infty(\widehat{\mathbb{G}}), \widehat{\Gamma})$  turning it into a locally compact quantum group again, the *dual quantum group* of  $\mathbb{G}$ , which we denote by  $\widehat{\mathbb{G}}$ . Generally, if  $X$  is an object associated with  $\mathbb{G}$ , we convene to denote the corresponding object associated with  $\widehat{\mathbb{G}}$  by  $\widehat{X}$ . Finally, a Pontryagin duality theorem holds, i.e.,  $\widehat{\widehat{\mathbb{G}}} = \mathbb{G}$ .

EXAMPLE. If  $\mathbb{G}$  is a locally compact group  $G$ , then  $\widehat{\mathbb{G}} = (\text{VN}(G), \widehat{\Gamma}_G)$ , and  $\widehat{\varphi} = \widehat{\psi}$  is the Plancherel weight on  $\text{VN}(G)$  ([17], Definition VII.3.2).

Each locally compact quantum group  $\mathbb{G}$  is equipped with a canonical co-involution, the *unitary antipode*, so that  $L^1(\mathbb{G})$  can be equipped with an involution  $\natural$ . Unlike for groups, however, and more generally for Kac algebras (see [4]), the left regular representation of  $L^1(\mathbb{G})$  need not be a  $*$ -representation with respect to this involution.

The *antipode* of  $\mathbb{G}$  is a  $\sigma$ -strongly\*-closed operator  $S$  on  $L^\infty(\mathbb{G})$  whose domain  $\mathcal{D}(S)$  is  $\sigma$ -strongly\*-dense. In general,  $S$  is not totally defined. Letting

$$L^1_*(\mathbb{G}) := \{f \in L^1(\mathbb{G}) : \text{there is } g \in L^1(\mathbb{G}) \text{ such that } g(x) = \overline{f}(Sx) \text{ for } x \in \mathcal{D}(x)\},$$

we obtain a dense subalgebra of  $L^1(\mathbb{G})$ . On  $L^1_*(\mathbb{G})$ , we can then define an involution by letting

$$f^*(x) := \overline{f}(Sx) \quad (x \in \mathcal{D}(S)).$$

Defining a norm  $\| \cdot \|$  on  $L^1_*(\mathbb{G})$  via

$$\|f\| := \max\{\|f\|, \|f^*\|\} \quad (f \in L^1_*(\mathbb{G})),$$

we turn  $L^1_*(\mathbb{G})$  into a Banach  $*$ -algebra (in fact, with isometric involution). The restriction of  $\lambda$  to  $L^1_*(\mathbb{G})$  is then a  $*$ -representation of  $L^1_*(\mathbb{G})$ . Furthermore, as a Banach  $*$ -algebra,  $L^1_*(\mathbb{G})$  has an enveloping  $C^*$ -algebra, which we denote by  $C^*_0(\widehat{\mathbb{G}})$ . For more details, see [8].

### 3. COMPACT QUANTUM GROUPS

We shall call a locally compact quantum group  $\mathbb{G}$  *compact* if the Hopf-von Neumann algebra  $(L^\infty(\mathbb{G}), \Gamma)$  is (left) compact in the sense of Definition 1.2.

Other characterizations of compactness for locally compact quantum groups are collected below:

PROPOSITION 3.1. *The following are equivalent for a locally compact quantum group  $\mathbb{G}$ :*

- (i)  $\mathbb{G}$  is compact;
- (ii) the  $C^*$ -algebra  $C_0(\mathbb{G})$  is unital;
- (iii) the left Haar weight of  $\mathbb{G}$  is finite.

*Proof.* (i)  $\iff$  (ii) is Proposition 3.1 of [2] (compare also Remark 3.8(5) of [18]).

(ii)  $\implies$  (iii): Let  $\phi$  denote the left Haar weight of  $\mathbb{G}$ . Since  $C_0(\mathbb{G}) \cap \mathcal{N}_\phi$  is a norm dense left ideal of  $C_0(\mathbb{G})$  it must contain the identity, so that  $\phi$  is finite.

(iii)  $\implies$  (i) is trivial.  $\blacksquare$

In this section, we shall prove another, less straightforward characterization of the compact quantum groups: a locally compact quantum group  $\mathbb{G}$  is compact if and only if  $L^1(\mathbb{G})$  is an ideal in  $L^1(\mathbb{G})^{**}$ .

We first prove a lemma for general locally compact quantum groups:

LEMMA 3.2. *Let  $\mathbb{G}$  be a locally compact quantum group, and let  $f \in L^1_*(\mathbb{G})$ . Then we have*

$$\lambda(f^*)\Lambda_\phi(x) = \Lambda_\phi((\bar{f} \otimes \text{id})(\Gamma x)) \quad (x \in \mathcal{N}_\phi).$$

*Proof.* Fix  $x \in \mathcal{N}_\phi$ , and let  $\nu \in \mathcal{B}(L^2(\mathbb{G}))_*$  be arbitrary. Define  $x\nu \in \mathcal{B}(L^2(\mathbb{G}))_*$  by letting  $(x\nu)(T) = \nu(Tx)$  for  $T \in \mathcal{B}(L^2(\mathbb{G}))$ . We obtain that

$$\begin{aligned} \nu(\lambda(f^*)x) &= (f^* \otimes \nu)(W(1 \otimes x)) = (f^* \otimes x\nu)(W) = \bar{f}(S(\text{id} \otimes x\nu)(W)) \\ &= \bar{f}((\text{id} \otimes x\nu)(W^*)), \quad \text{by Proposition 8.3 of [9],} \\ &= (\bar{f} \otimes \nu)(W^*(1 \otimes x)) = \nu((\bar{f} \otimes \text{id})(W^*)x). \end{aligned}$$

Since  $\nu \in \mathcal{B}(L^2(\mathbb{G}))_*$  is arbitrary, this means that

$$\lambda(f^*)x = (\bar{f} \otimes \text{id})(W^*)x.$$

In view of the fact that  $x \in \mathcal{N}_\phi$ , it follows from (8.3) in [9] that

$$\lambda(f^*)\Lambda_\phi(x) = (\bar{f} \otimes \text{id})(W^*)\Lambda_\phi(x) = \Lambda_\phi((\bar{f} \otimes \text{id})(\Gamma x)),$$

as claimed.  $\blacksquare$

Suppose now that  $\mathbb{G}$  is a compact quantum group. Since the left Haar weight  $\phi$  is then finite by Proposition 3.1, and can be chosen to be a state,  $\Lambda_\phi : \mathcal{N}_\phi \rightarrow L^2(\mathbb{G})$  is a contraction defined on all of  $L^\infty(\mathbb{G})$ . Define  $\iota : L^2(\mathbb{G}) \rightarrow L^1(\mathbb{G})$  by letting

$$\iota(\xi)(x) = \langle \xi, \Lambda_\phi(x^*) \rangle \quad (\xi \in L^2(\mathbb{G}), x \in L^\infty(\mathbb{G})).$$

Clearly,  $\iota$  is a linear, injective contraction with dense range.

The following compatibility result holds:

LEMMA 3.3. *Let  $\mathbb{G}$  be a compact quantum group. Then we have*

$$(3.1) \quad f * \iota(\xi) = \iota(\lambda(f)\xi) \quad (f \in L^1(\mathbb{G}), \xi \in L^2(\mathbb{G})).$$

*Proof.* By Lemma 3.2 and the fact that  $\lambda$  is a  $*$ -representation of  $L^1_*(\mathbb{G})$ , we have

$$(3.2) \quad \lambda(f)^* \Lambda_\phi(x) = \Lambda_\phi((\bar{f} \otimes \text{id})(\Gamma x)) \quad (f \in L^1_*(\mathbb{G}), x \in L^\infty(\mathbb{G})).$$

Hence, we obtain

$$\begin{aligned} \iota(\lambda(f)\bar{\xi})(x) &= \langle \lambda(f)\bar{\xi}, \Lambda_\phi(x^*) \rangle = \langle \bar{\xi}, \lambda(f)^* \Lambda_\phi(x^*) \rangle \\ &= \langle \bar{\xi}, \Lambda_\phi((\bar{f} \otimes \text{id})(\Gamma x^*)) \rangle, \quad \text{by (3.2),} \\ &= \iota(\bar{\xi})(((\bar{f} \otimes \text{id})(\Gamma x^*))^*) = \iota(\bar{\xi})((f \otimes \text{id})(\Gamma x)) = (f \otimes \iota(\bar{\xi}))(\Gamma x) \\ &= (f * \iota(\bar{\xi}))(x) \quad (f \in L^1_*(\mathbb{G}), \bar{\xi} \in L^2(\mathbb{G}), x \in L^\infty(\mathbb{G})) \end{aligned}$$

and thus

$$\iota(\lambda(f)\bar{\xi}) = f * \iota(\bar{\xi}) \quad (f \in L^1_*(\mathbb{G}), \bar{\xi} \in L^2(\mathbb{G})).$$

Since  $L^1_*(\mathbb{G})$  is dense in  $L^1(\mathbb{G})$ , this yields (3.1). ■

Let  $E$  and  $F$  be Banach spaces, and let  $T : E \rightarrow F$  be linear. Recall that  $T$  is said to be *weakly compact* if it maps the unital ball of  $E$  onto a relatively weakly compact subset of  $F$ .

As a consequence of Lemma 3.3, we obtain:

**PROPOSITION 3.4.** *Let  $\mathbb{G}$  be a compact quantum group. Then multiplication in  $L^1(\mathbb{G})$  is weakly compact.*

*Proof.* Let

$$I := \{g \in L^1(\mathbb{G}) : L^1(\mathbb{G}) \ni f \mapsto f * g \text{ is weakly compact}\}.$$

We claim that  $I = L^1(\mathbb{G})$ . To see this, note first that, due to the reflexivity of  $L^2(\mathbb{G})$ , the map

$$L^1(\mathbb{G}) \rightarrow L^2(\mathbb{G}), \quad f \mapsto \lambda(f)\bar{\xi}$$

is trivially weakly compact for each  $\bar{\xi} \in L^2(\mathbb{G})$ . Since  $\iota : L^2(\mathbb{G}) \rightarrow L^1(\mathbb{G})$  is continuous, Lemma 3.3 yields that  $\iota(L^2(\mathbb{G})) \subset I$ . Since  $I$  is closed and  $\iota(L^2(\mathbb{G}))$  is dense in  $L^1(\mathbb{G})$ , we conclude that  $I = L^1(\mathbb{G})$ .

Since

$$f * g = (g^\natural * f^\natural)^\natural \quad (f, g \in L^1(\mathbb{G})),$$

it follows that multiplication, both from the left and from the right, is indeed weakly compact in  $L^1(\mathbb{G})$ . ■

It is a well known Banach algebraic fact that a Banach algebra is an ideal in its second dual if and only if multiplication in  $\mathfrak{A}$  is weakly compact ([14], Proposition 1.4.13). Hence, we obtain:

**COROLLARY 3.5.** *Let  $\mathbb{G}$  be a compact quantum group. Then  $L^1(\mathbb{G})$  is an ideal in  $L^1(\mathbb{G})^{**}$ .*



To prove the converse of Corollary 3.5, we first prove two lemmas, which are patterned on Lemmas 4.2 and 4.3 of [13]. (We are grateful to S. Vaes for outlining this argument.)

LEMMA 3.6. *Let  $\mathbb{G}$  be a locally compact quantum group such that  $L^1(\mathbb{G})$  is an ideal in  $L^1(\mathbb{G})^{**}$ . Then, for each state  $f \in L^1(\mathbb{G})$ , there is a state  $g \in L^1(\mathbb{G})$  such that  $f * g = g * f = g$ .*

*Proof.* Let  $g \in L^1(\mathbb{G})^{**}$  be a  $\sigma(L^1(\mathbb{G})^{**}, L^\infty(\mathbb{G}))$ -accumulation point of the sequence  $\left(\frac{1}{n} \sum_{k=1}^n f^{*k}\right)_{n=1}^\infty$ , where  $f^{*k}$  stands for the  $k$ -th power of  $f$  with respect to the product  $*$  in  $L^1(\mathbb{G})$ . An inspection of the proof of Lemma 4.2 of [13] yields that  $g \in L^1(\mathbb{G})^{**}$  is a state satisfying  $f * g = g * f = g$ . Since  $L^1(\mathbb{G})$  is an ideal in  $L^1(\mathbb{G})^{**}$ , we see that  $g \in L^1(\mathbb{G})$ . ■

LEMMA 3.7. *Let  $\mathbb{G}$  be a locally compact quantum group, and let  $f, g \in L^1(\mathbb{G})$  be states such that  $f * g = g * f = g$ . If  $h \in L^1(\mathbb{G})$  is such that  $0 \leq h \leq f$ , then  $h * g = h(1)g$  holds.*

*Proof.* The proof of Lemma 4.3 in [13] carries over verbatim. ■

We can now state and prove the first main result of this paper:

THEOREM 3.8. *The following are equivalent for a locally compact quantum group  $\mathbb{G}$ :*

- (i)  $\mathbb{G}$  is compact;
- (ii)  $L^1(\mathbb{G})$  is an ideal in  $L^1(\mathbb{G})^{**}$ .

*Proof.* (i)  $\implies$  (ii) is Corollary 3.5.  
 (ii)  $\implies$  (i): For each positive  $f \in L^1(\mathbb{G})$ , set

$$K_f := \{g \in L^1(\mathbb{G}) : g \text{ is a state such that } f * g = f(1)g\}.$$

By Lemma 3.6,  $K_f$  is non-empty, and since multiplication with  $f$  from the left is weakly compact, we see that  $K_f$  is a weakly compact subset of  $L^1(\mathbb{G})$  whenever  $f \neq 0$ . Let  $h, f \in L^1(\mathbb{G})$  be states with  $0 \leq h \leq f$ . Then Lemma 3.7 yields that  $K_f \subset K_h$ . As a consequence, we have  $K_{f_1+f_2} \subset K_{f_1} \cap K_{f_2}$  for any positive  $f_1, f_2 \in L^1(\mathbb{G})$ . Fix a non-zero state  $f_0 \in L^1(\mathbb{G})$ . Then  $K_f \cap K_{f_0}$  is weakly compact for each positive  $f \in L^1(\mathbb{G})$  and non-empty (because it contains  $K_{f+f_0}$ ). Consequently,  $\bigcap\{K_f : f \in L^1(\mathbb{G}) \text{ is positive}\}$  is non-empty. Any element of this intersection is a normal, left invariant state, so that  $\mathbb{G}$  is compact. ■

Let  $G$  be a locally compact group, and let  $\mathbb{G} = (L^\infty(G), \Gamma_G)$ . Since  $\mathbb{G}$  is compact if and only if  $G$  is compact, the main result of [20] is a particular case of Theorem 3.8. On the other hand,  $\widehat{\mathbb{G}} = (\text{VN}(G), \widehat{\Gamma}_G)$  is compact if and only if  $G$  is discrete. From Theorem 3.8, we thus recover Theorem 3.7 of [11]:

COROLLARY 3.9. *Let  $G$  be a locally compact group. Then  $A(G)$  is an ideal in  $A(G)^{**}$  if and only if  $G$  is discrete.*

4. DISCRETE QUANTUM GROUPS

There are various ways to define discrete quantum groups. Following [18], we say that the locally compact quantum group  $\mathbb{G}$  is *discrete* if  $\widehat{\mathbb{G}}$  is compact. As for compactness, we collect a few equivalent characterizations:

PROPOSITION 4.1. *The following are equivalent for a locally compact quantum group  $\mathbb{G}$ :*

- (i)  $\mathbb{G}$  is discrete;
- (ii) there is a central minimal projection in  $\mathcal{C}_0(\mathbb{G})$ ;
- (iii)  $L^1(\mathbb{G})$  has an identity.

*Proof.* (i)  $\implies$  (ii) is noted in Remarks 3.8(2) of [18].

(ii)  $\implies$  (i): Let  $p \in \mathcal{C}_0(\mathbb{G})$  be a central minimal projection. Then  $p$  is also central and minimal in  $L^\infty(\mathbb{G})$ . Let  $\varepsilon \in L^1(\mathbb{G})$  denote the corresponding character. From the definition of  $\lambda$ , it is clear that  $\lambda(\varepsilon) \in \mathcal{C}_0(\widehat{\mathbb{G}})$  is a unitary operator on  $L^2(\widehat{\mathbb{G}})$ . Consequently,  $\mathcal{C}_0(\widehat{\mathbb{G}})$  has an identity, so that  $\widehat{\mathbb{G}}$  is compact.

(i)  $\implies$  (iii): In Remarks 3.8(2) of [18], it is observed that the character  $\varepsilon \in L^1(\mathbb{G})$ , as in (ii)  $\implies$  (i), can also be chosen to satisfy  $(\varepsilon \otimes \text{id}) \circ \Gamma = (\text{id} \otimes \varepsilon) \circ \Gamma$  and thus is an identity for  $L^1(\mathbb{G})$ .

(iii)  $\implies$  (i): If  $L^1(\mathbb{G})$  has an identity, then so has  $\mathcal{C}_0(\widehat{\mathbb{G}}) = \overline{\lambda(L^1(\mathbb{G}))}$ , so that  $\widehat{\mathbb{G}}$  is compact. ■

As a trivial consequence of Theorem 3.8, a locally compact quantum group  $\mathbb{G}$  is discrete if and only if  $L^1(\widehat{\mathbb{G}})$  is an ideal in  $L^1(\widehat{\mathbb{G}})^{**}$ . In this section, we give another characterization of discrete quantum groups in terms of second duals. For the proof, we require a general fact on Banach  $*$ -algebras, which may be of independent interest.

We say that a  $*$ -representation  $\pi$  of a Banach  $*$ -algebra  $\mathfrak{A}$  on a Hilbert space  $\mathfrak{H}$  is *non-degenerate* if the closed linear span of  $\{\pi(a)\zeta : a \in \mathfrak{A}, \zeta \in \mathfrak{H}\}$  is all of  $\mathfrak{H}$ .

In our next lemma, let  $\mathcal{K}(\mathfrak{H})$  denote the compact operators on a Hilbert space  $\mathfrak{H}$ .

LEMMA 4.2. *Let  $\mathfrak{A}$  be a Banach  $*$ -algebra, let  $\mathfrak{H}$  be a Hilbert space, and let  $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathfrak{H})$  be a faithful, non-degenerate  $*$ -representation of  $\mathfrak{A}$  such that  $\pi(\mathfrak{A}) \subset \mathcal{K}(\mathfrak{H})$ . Then, for each self-adjoint  $a \in \mathfrak{A}$ , the spectra of  $a$  in  $\mathfrak{A}$  and of  $\pi(a)$  in  $\mathcal{B}(\mathfrak{H})$  coincide.*

*Proof.* The claim is straightforward if  $\dim \mathfrak{H} < \infty$ . We thus limit ourselves to the case where  $\dim \mathfrak{H} = \infty$ .

Let  $a \in \mathfrak{A}$  be self-adjoint, and let  $\text{Sp}(a)$  and  $\text{Sp}(\pi(a))$  denote the spectra of  $a$  and  $\pi(a)$  (in  $\mathfrak{A}$  and  $\mathcal{B}(\mathfrak{H})$ , respectively). Since  $\pi(a) \in \mathcal{K}(\mathfrak{H})$  and since  $\dim \mathfrak{H} = \infty$ ,

we have  $0 \in \text{Sp}(\pi(a))$ . Also, as  $\mathfrak{A}$  cannot have an identity (again because  $\pi(\mathfrak{A}) \subset \mathcal{K}(\mathfrak{H})$ ), we have  $0 \in \text{Sp}(a)$  as well. From Proposition 1.5.28 of [3], we conclude that  $\text{Sp}(\pi(a)) \subset \text{Sp}(a)$ .

For the converse inclusion, let  $\lambda \in \text{Sp}(\pi(a))$ , and suppose without loss of generality that  $\lambda \neq 0$ . From the spectral theory of compact operators on Hilbert space it is then well known that  $\lambda$  is an eigenvalue of  $\pi(a)$ , i.e., there is  $\xi \in \mathfrak{H} \setminus \{0\}$  such that  $(\lambda - \pi(a))\xi = 0$ . Let  $\mathfrak{A}^\#$  denote the *unitization* of  $\mathfrak{A}$ , i.e., the algebra  $\mathfrak{A}$  with an identity adjoined, and assume that  $\lambda \notin \text{Sp}(a)$ . By the definition of  $\text{Sp}(a)$  (see p. 78 of [3], for instance), there is  $x \in \mathfrak{A}^\#$  such that  $x(\lambda - a) = 1$ . The representation  $\pi$  extends canonically to a unital  $*$ -representation  $\pi^\# : \mathfrak{A}^\# \rightarrow \mathcal{B}(\mathfrak{H})$ , so that we obtain

$$\xi = \pi^\#(1)\xi = \pi^\#(x)(\lambda - \pi(a))\xi = 0,$$

which is a contradiction. ■

By a  $C^*$ -norm on a Banach  $*$ -algebra  $\mathfrak{A}$  we mean a submultiplicative norm  $\gamma$  on  $\mathfrak{A}$  satisfying  $\gamma(a^*a) = \gamma(a)^2$  for  $a \in \mathfrak{A}$ . We denote the completion of  $\mathfrak{A}$  with respect to  $\gamma$  by  $C_\gamma^*(\mathfrak{A})$ , which is a  $C^*$ -algebra. We say that  $\mathfrak{A}$  has a *unique  $C^*$ -norm* if there is only one  $C^*$ -norm on  $\mathfrak{A}$ .

We also require the notion of a regular Banach algebra ([3], Definition 4.1.16): a commutative Banach algebra  $\mathfrak{A}$  with character space  $\Phi_\mathfrak{A}$  is called *regular* if, for each closed  $F \subset \Phi_\mathfrak{A}$  and  $\phi \in \Phi_\mathfrak{A} \setminus F$ , there is  $a \in \mathfrak{A}$  with  $\widehat{a}|_F \equiv 0$  and  $\widehat{a}(\phi) = 1$ , where  $\widehat{a}$  stands for the Gelfand transform of  $a$ .

**PROPOSITION 4.3.** *Let  $\mathfrak{A}$  be a Banach  $*$ -algebra, and suppose that there is a  $C^*$ -norm  $\gamma$  on  $\mathfrak{A}$  such that  $C_\gamma^*(\mathfrak{A})$  is an ideal in  $C_\gamma^*(\mathfrak{A})^{**}$ . Then  $\mathfrak{A}$  has a unique  $C^*$ -norm.*

*Proof.* Suppose without loss of generality that  $\mathfrak{A}$  is infinite-dimensional.

Let  $a \in \mathfrak{A}$  be self-adjoint, and let  $\langle a \rangle$  denote the closed subalgebra of  $\mathfrak{A}$  generated by  $a$ . We claim that the Banach algebra  $\langle a \rangle$  is regular.

Since  $C_\gamma^*(\mathfrak{A})$  is an ideal in  $C_\gamma^*(\mathfrak{A})^{**}$ , it follows from Exercises III.5.3 and III.5.4 of [16], that there is a family  $(\mathfrak{H}_\alpha)_\alpha$  of Hilbert spaces such that

$$C_\gamma^*(\mathfrak{A}) \cong c_0\text{-}\bigoplus_\alpha \mathcal{K}(\mathfrak{H}_\alpha) \subset \mathcal{K}(\mathfrak{H}),$$

where  $\mathfrak{H} := \ell^2\text{-}\bigoplus_\alpha \mathfrak{H}_\alpha$ . This yields a faithful, non-degenerate  $*$ -representation  $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathfrak{H})$  with  $\pi(\mathfrak{A}) \subset \mathcal{K}(\mathfrak{H})$ . From Lemma 4.2 and the spectral theory of compact operators on Hilbert space we conclude that the spectrum of  $a$  in  $\mathfrak{A}$  is of the form  $\{\lambda_n : n \in \mathbb{N}\} \cup \{0\}$ , where  $\lambda_n \neq 0$  for  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} \lambda_n = 0$ . Consequently, the character space of  $\langle a \rangle$  is canonically homeomorphic to the discrete subset  $\{\lambda_n : n \in \mathbb{N}\}$  of  $\mathbb{C}$ . With the help of the holomorphic functional calculus, it is then straightforward to see that  $\langle a \rangle$  is regular, as claimed.

Since  $a \in \mathfrak{A}$  was an arbitrary self-adjoint element,  $\mathfrak{A}$  is therefore *locally regular* in the sense of Definition 4.1 in [1] and thus has a unique  $C^*$ -norm ([1], Lemma 4.2). ■

THEOREM 4.4. *The following are equivalent for a locally compact quantum group  $\mathbb{G}$ :*

- (i)  $\mathbb{G}$  is discrete;
- (ii)  $\mathcal{C}_0(\mathbb{G})$  is an ideal in  $\mathcal{C}_0(\mathbb{G})^{**}$ .

*Proof.* (i)  $\implies$  (ii): Let

$$I := \{a \in \mathcal{C}_0(\mathbb{G}) : \mathcal{C}_0(\mathbb{G}) \ni b \mapsto ba \text{ is weakly compact}\},$$

and note that  $I$  is closed. Since  $\mathbb{G}$  is discrete,  $\widehat{\mathbb{G}}$  is compact, so that we have the map  $\iota : L^2(\widehat{\mathbb{G}}) \rightarrow L^1(\widehat{\mathbb{G}})$  as defined immediately prior to Lemma 3.3; moreover, we have

$$\widehat{\lambda}(f)\widehat{\lambda}(\iota(\xi)) = \widehat{\lambda}(f * \iota(\xi)) = \widehat{\lambda}(\iota(\widehat{\lambda}(f)\xi)) \quad (f \in L^1(\widehat{\mathbb{G}}), \xi \in L^2(\widehat{\mathbb{G}})).$$

Since  $L^2(\widehat{\mathbb{G}})$  is reflexive, it follows, as in the proof of Proposition 3.4, that  $\widehat{\lambda}(\iota(L^2(\widehat{\mathbb{G}})))$  is contained in  $I$ . Since  $\widehat{\lambda}(\iota(L^2(\widehat{\mathbb{G}})))$  is dense in  $\mathcal{C}_0(\mathbb{G})$ ,  $I$  must equal all of  $\mathcal{C}_0(\mathbb{G})$ , i.e., right multiplication in  $\mathcal{C}_0(\mathbb{G})$  is weakly compact. Using the involution on  $\mathcal{C}_0(\mathbb{G})$ , we see that left multiplication in  $\mathcal{C}_0(\mathbb{G})$  is also weakly compact, so that  $\mathcal{C}_0(\mathbb{G})$  is an ideal in  $\mathcal{C}_0(\mathbb{G})^{**}$ .

(ii)  $\implies$  (i): Let  $\gamma$  denote the  $C^*$ -norm on  $L^1_*(\widehat{\mathbb{G}})$  induced by  $\widehat{\lambda}$ , so that  $C^*_\gamma(L^1_*(\widehat{\mathbb{G}})) = \mathcal{C}_0(\mathbb{G})$ . From Proposition 4.3, we conclude that  $L^1_*(\widehat{\mathbb{G}})$  has a unique  $C^*$ -norm; in particular,  $\mathcal{C}^u_0(\mathbb{G}) \cong \mathcal{C}_0(\mathbb{G})$  holds. Consequently,  $\mathcal{C}_0(\mathbb{G})$  has a character, say  $\varepsilon$ , by Theorem 3.1 of [2]. Let  $p \in \mathcal{C}_0(\mathbb{G})^{**}$  be the corresponding central minimal projection. Choose  $a \in \mathcal{C}_0(\mathbb{G})$  with  $\varepsilon(a) = 1$ . It follows that  $p = \varepsilon(a)p = ap \in \mathcal{C}_0(\mathbb{G})$ . By Proposition 4.1, this means that  $\mathbb{G}$  is discrete. ■

Since compactness and discreteness are dual to each other, we finally obtain as a consequence of both Theorems 3.8 and 4.4:

COROLLARY 4.5. *The following are equivalent for a locally compact quantum group  $\mathbb{G}$ :*

- (i)  $\mathbb{G}$  is compact;
- (ii)  $L^1(\mathbb{G})$  is an ideal in  $L^1(\mathbb{G})^{**}$ ;
- (iii)  $\mathcal{C}_0(\widehat{\mathbb{G}})$  is an ideal in  $\mathcal{C}_0(\widehat{\mathbb{G}})^{**}$ ;
- (iv)  $\widehat{\mathbb{G}}$  is discrete.

REFERENCES

[1] B.A. BARNES, Ideal and representation theory of the  $L^1$ -algebra of a group with polynomial growth, *Colloq. Math.* **45**(1981), 301–315.  
 [2] E. BÉDOS, L. TUSET, Amenability and co-amenability for locally compact quantum groups, *Internat. J. Math.* **14**(2003), 865–884.

- [3] H.G. DALES, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monographs N.S., vol. 24, Oxford Sci. Publ., Clarendon Press, Oxford Univ. Press, New York 2000.
- [4] M. ENOCK, J.-M. SCHWARTZ, *Kac Algebras and Duality of Locally Compact Groups*, Springer Verlag, Berlin 1992.
- [5] P. EYMARD, L'algèbre de Fourier d'un groupe localement compact, *Bull. Soc. Math. France* **92**(1964), 181–236.
- [6] M. GROSSER,  $L_1(G)$  as an ideal in its second dual space, *Proc. Amer. Math. Soc.* **73**(1979), 363–364.
- [7] D.L. JOHNSON, A characterization of compact groups, *Proc. Amer. Math. Soc.* **74**(1979), 381–382.
- [8] J. KUSTERMANS, Locally compact quantum groups in the universal setting, *Internat. J. Math.* **12**(2001), 289–338.
- [9] J. KUSTERMANS, S. VAES, Locally compact quantum groups, *Ann. Sci. École Norm. Sup. (4)* **33**(2000), 837–934.
- [10] J. KUSTERMANS, S. VAES, Locally compact quantum groups in the von Neumann algebraic setting, *Math. Scand.* **92**(2003), 68–92.
- [11] A.T.-M. LAU, The second conjugate algebra of the Fourier algebra of a locally compact group, *Trans. Amer. Math. Soc.* **267**(1981), 53–63.
- [12] H. LEPTIN, Sur l'algèbre de Fourier d'un groupe localement compact, *C. R. Acad. Sci. Paris Sér. A* **266**(1968), 1180–1182.
- [13] A. MAES, A. VAN DAELE, Notes on compact quantum groups, *Nieuw Arch. Wisk.* **16**(1998), 73–112.
- [14] T.W. PALMER, *Banach Algebras and the General Theory of \*-Algebras. I. Algebras and Banach Algebras*, Encyclopedia Math. Appl., vol. 49, Cambridge Univ. Press, Cambridge 1994.
- [15] Z.-J. RUAN, Amenability of Hopf von Neumann and Kac algebras, *J. Funct. Anal.* **139**(1996), 466–499.
- [16] M. TAKESAKI, *Theory of Operator Algebras. I*, Encyclopedia Math. Sci., vol. 124, Springer Verlag, Berlin 2002.
- [17] M. TAKESAKI, *Theory of Operator Algebras. II*, Encyclopedia Math. Sci., vol. 125, Springer Verlag, Berlin 2002.
- [18] R. TOMATSU, Amenable discrete quantum groups, *J. Math. Soc. Japan* **58**(2006), 949–964.
- [19] A. VAN DAELE, Locally compact quantum groups. A von Neumann algebra approach, ArXiv: math.OA/0602212.
- [20] S. WATANABE, A Banach algebra which is an ideal in the second dual space, *Sci. Rep. Niigata Univ. Ser. A* **11**(1974), 95–101.
- [21] S.L. WORONOWICZ, Compact quantum groups, in *Symétries Quantiques (Les Houches, 1995)*, North-Holland, Amsterdam 1998, pp. 845–884.

VOLKER RUNDE, DEPT. OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA T6G 2G1

*E-mail address:* vrunde@ualberta.ca

Received May 16, 2006.