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C^*-ALGEBRAS WITH MULTIPLE SUBNORMAL GENERATORS

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ABSTRACT. If A is an irreducible essentially normal operator, then we prove that the C^*-algebra generated by A has a finite number of irreducible subnormal operators as generators if and only if the essential spectrum of A is uncountable. It is shown that, in general, at most eight irreducible subnormal generators are required. Additionally, it is shown that frequently two irreducible subnormal operators will suffice and that, in many instances, the subnormal operators can be taken to be unilateral shifts of multiplicity one or unitarily equivalent to the dual of the Bergman shift.

KEYWORDS: *C^*-algebra, subnormal operator, generator, irreducible, essentially normal.*

MSC (2000): Primary 47B20; Secondary 47C15.

INTRODUCTION

If $A_1, \dots, A_n$ are bounded linear operators on a separable complex Hilbert space, then $C^*(A_1, \dots, A_n)$ will denote the C^*-algebra of operators generated by $A_1, \dots, A_n$ and the identity operator. If $S_1, \dots, S_n$ belong to the singly generated algebra $C^*(A)$, then $S_1, \dots, S_n$ generate $C^*(A)$ if $C^*(S_1, \dots, S_n) = C^*(A)$. We are interested in the question of which operators A have the property that $C^*(A)$ has a finite number of irreducible subnormal generators. Subnormal and hyponormal generators of von Neumann algebras have been studied by Wogen [17] and Behncke [2]. In 1984 Putnam [15] showed that certain hyponormal operators have C^*-algebras generated by a unilateral shift, and raised a related question. This question and more was answered by Conway and McGuire [6] where they characterized the operators whose C^*-algebra is generated by a unilateral shift. In 1988, McGuire [12] extended that result to operators whose C^*-algebras are generated by a single subnormal operator whose essential spectrum is a finite union of *disjoint* Jordan curves. In 2006, Feldman and McGuire [10] considered the question of which irreducible essentially normal operators A have their C^*-algebra generated by a subnormal (or hyponormal) operator. The answer depends only on the spectral picture of A, that is the essential spectrum of A and the values of the Fredholm index function, $\operatorname{ind}(A - \lambda I)$, off the essential spectrum. It was shown that many such operators A do have subnormal generators for $C^*(A)$, and yet many do not. In this paper we consider the question of just how many irreducible subnormal operators are necessary to generate $C^*(A)$ when A is an irreducible essentially normal operator. One of the principal results is Theorem 3.11, which asserts that every irreducible essentially normal operator with an uncountable essential spectrum has its C^*-algebra generated by at most eight irreducible subnormal operators. Additional results are obtained that provide conditions on A in order that $C^*(A)$ is generated by two irreducible subnormal operators of a special type. In particular Theorem 3.7 establishes that if the essential spectrum of A is an interval, then $C^*(A)$ is generated by two subnormal operators unitarily equivalent to the dual of the Bergman shift. This is striking in that the dual of the Bergman shift is an operator whose spectrum and essential spectrum are both equal to the closed unit disk. Also, it is shown in Corollary 2.7 that if the Fredholm index of A takes on either the value 1 or -1, then $C^*(A)$ is generated by a pair of subnormal operators each unitarily equivalent to the unilateral shift of multiplicity one.

1. PRELIMINARIES

In what follows $\mathcal{H}$ will denote a separable infinite dimensional complex Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, $\mathcal{B}_0(\mathcal{H})$ the ideal of all compact operators on $\mathcal{H}$, and $\mathcal{B}/\mathcal{B}_0$ the Calkin algebra. An operator $S \in \mathcal{B}(\mathcal{H})$ is *subnormal* if there is a normal operator $N \in \mathcal{B}(\mathcal{K})$, $\mathcal{H} \subset \mathcal{K}$, such that $\mathcal{H}$ is invariant for N and S is the restriction of N to $\mathcal{H}$. We will assume N is the minimal normal extension of S. The restriction T of N^* to $(\mathcal{H})^\perp = \mathcal{K} \ominus \mathcal{H}$ is also a subnormal operator and is known as the *dual of S*. One can show that S is irreducible if and only if T is irreducible. An operator is essentially normal if its self-commutator is compact. The kernel of an operator A is $\ker(A) = \{x : Ax = 0\}$. An operator A is *Fredholm* if it has closed range, $\dim\ker(A) < \infty$, and $\dim\ker(A^*) < \infty$. When A is Fredholm, then its (Fredholm) index is defined as $\operatorname{ind}(A) = \dim\ker(A) - \dim\ker(A^*)$. The essential spectrum of A is $\sigma_{\mathrm{e}}(A) = \{\lambda \in \mathbb{C} : (A - \lambda I) \text{ is not Fredholm}\}$. The (Fredholm) index function for A is the integer-valued continuous function $\lambda \mapsto \operatorname{ind}(A - \lambda I)$ defined on $\mathbb{C} \setminus \sigma_{\mathrm{e}}(A)$.

The term *spectral picture* of an operator A generally refers to the essential spectrum of A, $\sigma_{\mathrm{e}}(A)$, and the values of its index function on the components of $\mathbb{C} \setminus \sigma_{\mathrm{e}}(A)$, and also perhaps to the spectrum of A and some other subsets of the spectrum. *In this paper*, the term *spectral picture* of an operator will mean the essential spectrum and the index function of that operator. If S is subnormal and essentially normal, then T, the dual of S, is also essentially normal and $\sigma_{\mathrm{e}}(N) =$

$\sigma_e(T) \cup \sigma_e(S)$. For more on subnormal operators and their duals see Conway [4] and Conway [5].

With the terminology above, the well known Brown-Douglas-Fillmore Theorem [3] takes the following form:

THEOREM 1.1 (BDF Theorem). *Two essentially normal operators A and B are unitarily equivalent modulo the compact operators if and only if A and B have the same spectral picture.*

That is, there exists a unitary operator U and a compact operator C such that $U^*AU = B + C$ if and only if $\sigma_e(A) = \sigma_e(B)$ and $\operatorname{ind}(A - \lambda I) = \operatorname{ind}(B - \lambda I)$ for all $\lambda \in \mathbb{C} \setminus K$, where $K = \sigma_e(A) = \sigma_e(B)$.

1.1. SOME TOPOLOGICAL TOOLS. For a set $E \subseteq \mathbb{C}$, $\operatorname{int}(E)$ and $\operatorname{cl}(E)$ will denote the interior and closure of E, respectively. For a Jordan curve γ in $\mathbb{C}$, $\operatorname{inside}(\gamma)$ will denote the bounded component of $\mathbb{C} \setminus \gamma$ and $\operatorname{outside}(\gamma)$ will denote the unbounded component of $\mathbb{C} \setminus \gamma$. For a compact set $K \subseteq \mathbb{C}$, the outer boundary of K will be the boundary of the unbounded component of $\mathbb{C} \setminus K$.

1.1.1. JORDAN REGIONS AND WINDING NUMBERS. If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable continuous closed curve in the complex plane and λ is a point not on the curve, then $n(\gamma, \lambda) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\gamma} \frac{1}{z-\lambda} \mathrm{d}z$ is the winding number of γ about λ. The winding number is well known to be a homotopy invariant. If γ is only a continuous curve, then one can approximate it by rectifiable curves and use the homotopy invariance to define the winding number of γ. Alternatively, if $\gamma : [a, b] \to \mathbb{C}$ is a continuous closed curve and, say, 0 is a point not on γ, then let $\theta : [a, b] \to \mathbb{R}$ be a continuous branch of the argument of γ. So, $\theta(t)$ is a continuous function and $\gamma(t) = |\gamma(t)|\mathrm{e}^{\mathrm{i}\theta(t)}$ for $t \in [a, b]$. Then $n(\gamma, 0) := \frac{1}{2\pi}(\theta(b) - \theta(a))$. If Γ is a finite system of closed curves $\gamma_1, \ldots, \gamma_n$, then $n(\Gamma, \lambda) := \sum\limits_{k=1}^{n} n(\gamma_k, \lambda)$ for $\lambda \notin \bigcup\limits_{k} \gamma_k$. The inside and outside of a system Γ of closed curves are defined by $\operatorname{inside}(\Gamma) = \{\lambda \in \mathbb{C} : n(\Gamma, \lambda) = 1\}$ and $\operatorname{outside}(\Gamma) = \{\lambda \in \mathbb{C} : n(\Gamma, \lambda) = 0\}$.

A Jordan region is a region bounded by a finite number of disjoint rectifiable Jordan curves. A Jordan region G is positively oriented if each Jordan curve in the boundary of G is oriented such that $\operatorname{inside}(\partial G) = G$ and $\operatorname{outside}(\partial G) = \mathbb{C} \setminus \operatorname{cl}G$.

1.2. THE SPECTRAL PICTURE OF $f(A)$. For a compact set K in the complex plane, $C(K)$ will denote the set of all continuous complex valued functions on K. If A is an essentially normal operator and $f \in C(\sigma_e(A))$, and $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}/\mathcal{B}_0$ is the natural projection into the Calkin algebra, then $\pi(A)$ is a normal element of the C^*-algebra $\mathcal{B}/\mathcal{B}_0$, thus $f(\pi(A))$ is a well-defined element of $\mathcal{B}/\mathcal{B}_0$. Since any two operators in $\pi^{-1}(f(\pi(A))$ differ by a compact operator, they must have the same spectral picture, hence we may define the spectral picture of $f(A)$ to be the spectral picture of any operator in $\pi^{-1}(f(\pi(A)))$. In general we will use $f(A)$ to denote any operator in $\pi^{-1}(f(\pi(A))$ and anything done with $f(A)$ will be invariant under compact perturbations. The C^*-algebra generated by A is equal to $C^*(A) = \pi^{-1}(C^*(\pi(A)) = \{f(A) + K : f \in \mathcal{C}(\sigma_e(A)), K \in C^*(A) \cap \mathcal{B}_0\}$. If the operator A is irreducible, then $C^*(A)$ contains all the compact operators and $C^*(A) = \{f(A) + K : f \in \mathcal{C}(\sigma_e(A)), K \in \mathcal{B}_0\}$.

The next result is proved in Feldman and McGuire [9].

THEOREM 1.2 (Functions of Spectral Pictures). *If A is an essentially normal operator and $f \in C(\sigma_e(A))$, then the following hold:*

(i) $\sigma_e(f(A)) = f(\sigma_e(A))$.

(ii) *Let $\{G_n\}_{n=1}^{\infty}$ be the bounded components of $\mathbb{C} \setminus \sigma_e(A)$ and let $\widehat{f}$ denote any continuous extension of f to $\sigma(A)$. For each $n \geqslant 1$, let $a_n \in G_n$. If $\lambda \in \mathbb{C} \setminus \sigma_e(f(A))$, then there exists an integer $0 < N < \infty$ and a compact set $K \subseteq \bigcup_{n=1}^{N} \operatorname{int}(\operatorname{cl}G_n)$ such that if $\{\Omega_n\}_{n=1}^{N}$ is any finite collection of positively oriented Jordan regions satisfying $K \subseteq \bigcup_{n=1}^{N} \Omega_n \subseteq \bigcup_{n=1}^{N} \operatorname{cl}\Omega_n \subseteq \bigcup_{n=1}^{N} \operatorname{int}(\operatorname{cl}G_n)$, then*

$$\operatorname{ind}(f(A) - \lambda I) = \sum_{n=1}^{N} n(\widehat{f}(\partial\Omega_n), \lambda)\operatorname{ind}(A - a_n I).$$

(iii) *Keeping the notation from* (ii), *if each component G_n is bounded by a (positively oriented) Jordan curve, then for each $\lambda \in \mathbb{C} \setminus \sigma_e(f(A))$ there exists an integer $0 < N < \infty$ such that*

$$\operatorname{ind}(f(A) - \lambda I) = \sum_{n=1}^{N} n(f(\partial G_n), \lambda)\operatorname{ind}(A - a_n I).$$

In fact, $n(f(\partial G_n), \lambda) = 0$ for all $n > N$, so

$$\operatorname{ind}(f(A) - \lambda I) = \sum_{n=1}^{\infty} n(f(\partial G_n), \lambda)\operatorname{ind}(A - a_n I).$$

We will need the following proposition which follows from Theorem 1.2 and appeared as Proposition 6.5 on page 476 of Feldman and McGuire [10].

PROPOSITION 1.3. *Let A be an irreducible essentially normal operator, G_0 a bounded component of $\mathbb{C} \setminus \sigma_e(A)$, and $N = \operatorname{ind}(A - \lambda I)$ for $\lambda \in G_0$. Also, let Δ be a closed disk contained in G_0. If ψ is a Mobius transformation such that $\psi(\mathbb{C} \setminus \Delta) = \Delta$, then the spectral picture of $\psi(A)$ is as follows: $\sigma_e(\psi(A)) = \psi(\sigma_e(A))$ and $\operatorname{ind}(\psi(A) - \psi(\lambda)I) = \operatorname{ind}(A - \lambda I) - N$, $\lambda \in \mathbb{C} \setminus \sigma_e(A)$.*

2. RESULTS

DEFINITION 2.1. If $K \subseteq \mathbb{C}$ and $f_1, f_2, \ldots, f_n \in C(K)$, then $(f_1, f_2, \ldots, f_n)$ is a *one-to-one* n-tuple if $F : K \to \mathbb{C}^n$ given by $F(z) = (f_1(z), f_2(z), \ldots, f_n(z))$ is a one-to-one map. Equivalently, $\{f_1, \ldots, f_n\}$ separate the points of K.

THEOREM 2.2. *If $K \subset \mathbb{C}$ is a compact set and $f_1, f_2, \ldots, f_n \in C(K)$, then the C^*-algebra generated by $f_1, f_2, \ldots, f_n$ equals $C(K)$ if and only if $(f_1, f_2, \ldots, f_n)$ is a one-to-one n-tuple.*

Proof. This is just the Stone-Weierstrass Theorem. ∎

THEOREM 2.3. *If A is an irreducible essentially normal operator and $B_1, \ldots, B_n$ belong to $C^*(A)$, then $C^*(A) = C^*(B_1, \ldots, B_n)$ if and only if*

(i) *there are continuous functions $f_1, \ldots, f_n : \sigma_e(A) \to \mathbb{C}$ and compact operators $K_1, \ldots, K_n$ such that $B_i = f_i(A) + K_i$, for $i = 1, \ldots, n$;*

(ii) *the n-tuple $(B_1, \ldots, B_n)$ is irreducible;*

(iii) *$(f_1, \ldots, f_n)$ is a one-to-one n-tuple.*

Proof. Since $B_i \in C^*(A)$, then $B_i = f_i(A) + K_i$ for some $f_i \in C(\sigma_e(A))$ and $K_i \in \mathcal{B}_0$. Also, $\pi(C^*(A)) \cong C(\sigma_e(A))$ and $\pi(C^*(B_1, \ldots, B_n)) \cong C^*(f_1, \ldots, f_n)$.

Since A is irreducible, then $C^*(A)$ contains all the compact operators, thus we have $C^*(A) = \pi^{-1}(\pi(C^*(A))$. In order for $C^*(A)$ to equal $C^*(B_1, \ldots, B_n)$, we need $C^*(B_1, \ldots, B_n)$ to contain all the compact operators and $\pi(C^*(B_1, \ldots, B_n)) = \pi(C^*(A))$, for then we would have

$$C^*(A) = \pi^{-1}(\pi(C^*(A)) = \pi^{-1}(\pi(C^*(B_1, \ldots, B_n))) = C^*(B_1, \ldots, B_n).$$

Here the last equality holds because $C^*(B_1, \ldots, B_n)$ contains all the compacts. Thus, $(B_1, \ldots, B_n)$ needs to be irreducible and $C(\sigma_e(A)) = C^*(f_1, \ldots, f_n)$, which is equivalent to the tuple $(f_1, \ldots, f_n)$ being one-to-one. ∎

COROLLARY 2.4. *If A is an irreducible essentially normal operator, then $C^*(A)$ is generated by n irreducible subnormal operators if and only if there are continuous functions $f_1, \ldots, f_n : \sigma_e(A) \to \mathbb{C}$ such that $f_1(A), \ldots, f_n(A)$ each have the same spectral picture as an irreducible subnormal operator and $(f_1, \ldots, f_n)$ is a one-to-one n-tuple.*

EXAMPLE 2.5. The examples below introduce three one-to-one pairs that will play a significant role in later results.

(i) If $\partial\mathbb{D}$ is the unit circle, then the radial projection R of $\mathbb{C} \setminus \{0\}$ onto $\partial\mathbb{D}$ is defined by $R(z) = \frac{z}{|z|}$. The arc, $\mathcal{A}$ of the parabola $y = x^2$, $x > 0$, intersects the circle $\partial\mathbb{D}$ at a point w_0 and every point in $\mathbb{C} \setminus \{0\}$ is uniquely expressible as $e^{i\theta} z$ where z is a point on $\mathcal{A}$ and $0 \leqslant \theta < 2\pi$. The parabolic projection P of $\mathbb{C} \setminus \{0\}$ onto $\partial\mathbb{D}$ is defined by $P(e^{i\theta} z) = e^{i\theta} w_0$. With these definitions, the pair (R, P) is a one-to-one pair of continuous functions from $\mathbb{C} \setminus \{0\}$ onto $\partial\mathbb{D}$.

(ii) The exterior radial projection R_E and exterior parabolic projection P_E of $\mathbb{C}$ onto $\partial\mathbb{D}$ are defined exactly as R and P on the exterior of $\mathbb{D}$ and as the identity on $\overline{\mathbb{D}}$ itself. The pair (R_E, P_E) is a one-to-one continuous pair.

(iii) If π_x and π_y denote the orthogonal projections onto the x and y axes respectively, then (π_x, π_y) is a one-to-one pair of continuous functions. More generally, if L_1, L_2 are two lines that intersect in a single point and π_{L_1}, π_{L_2} denote the orthogonal projections onto L_1,L_2, then (π_{L_1}, π_{L_2}) is a one-to-one pair of continuous functions.

THEOREM 2.6. *If A is an irreducible essentially normal operator and for some $\lambda \in \mathbb{C} \setminus \sigma_e(A)$ we have $\mathrm{ind}(A - \lambda I) \neq 0$, then there are two unitarily equivalent irreducible subnormal operators S_1 and S_2 such that $C^*(A) = C^*(S_1, S_2)$.*

Proof. Let Δ be a closed disk of radius r centered at w and contained in a component of $\mathbb{C} \setminus \sigma_e(A)$ where $\mathrm{ind}(A - wI) = a_0$ is nonzero. By Theorem 1.2 the spectral picture of each of $R(\frac{A-wI}{r})$ and $P(\frac{A-wI}{r})$ consists of $\partial\mathbb{D}$ together with the index value a_0 in $\mathbb{D}$. It is well known that there exists an irreducible subnormal operator S with spectral picture $\partial\mathbb{D}$ and index value $-|a_0|$ in the interior of $\mathbb{D}$, see McGuire [13] or Feldman and McGuire [8]. Theorem 1.1 implies that a compact perturbation of $R(\frac{A-wI}{r})$, as well as $P(\frac{A-wI}{r})$, is unitarily equivalent to S or S^*. Theorem 2.3 now implies that $C^*(A)$ is generated by two subnormal operators unitarily equivalent to S. ∎

COROLLARY 2.7. *If A is an irreducible essentially normal operator and either $\mathrm{ind}(A - \lambda I) = 1$ or $\mathrm{ind}(A - \lambda I) = -1$ for some $\lambda \in \mathbb{C} \setminus \sigma_e(A)$, then $C^*(A)$ is generated by two unilateral shifts of multiplicity one.*

Proof. If the index of $A - \lambda I$ is ± 1 for λ in a disk of radius r and center w, then $R(\frac{A-wI}{r})$ and $P(\frac{A-wI}{r})$ are unitarily equivalent to a compact perturbation of the unilateral shift of multiplicity one by Theorem 1.1. ∎

COROLLARY 2.8. *If A is an irreducible essentially normal operator and there are two components Ω_1 and Ω_2 of $\mathbb{C} \setminus \sigma_e(A)$ with associated Fredholm indices α_1 and α_2 such that $|\alpha_1 - \alpha_2| = 1$, then $C^*(A)$ is generated by two unilateral shifts of multiplicity one.*

Proof. Let Δ be a closed disk with center w contained in Ω_1 and let ψ be the transformation $\psi(z) = \frac{1}{z-w}$. Note that $\sigma_e(\psi(A))$ is a homeomorphic image of the essential spectrum of A and, by Proposition 1.3, $\psi(A)$ has index value ± 1 in a component of $\mathbb{C} \setminus \sigma_e(\psi(A))$. We may assume $\psi(A)$ is irreducible as $\psi(A)$ can be any operator in $\pi^{-1}(f(\pi(A)))$ and an irreducible compact perturbation of $\psi(A)$ can always be chosen. By Corollary 2.7, $C^*(A) = C^*(\psi(A))$ is generated by two unilateral shifts of multiplicity one. ∎

The following corollary is an immediate consequence of the preceding corollary.

COROLLARY 2.9. *If A is an irreducible essentially normal operator and there is a homeomorphism ψ of $\sigma_e(A)$ such that $\psi(A)$ has index value ± 1 at some point in $\mathbb{C} \setminus \psi(\sigma_e(A))$, then $C^*(A)$ is generated by two unilateral shifts of multiplicity one.*

The following three examples apply the above corollaries in three different instances of varying complexity and indicate how much easier it is to have two subnormal (or hyponormal) generators rather than one. In all three examples the corollary implies $C^*(A)$ is generated by two unilateral shifts of multiplicity one.

EXAMPLE 2.10 (Non-zero index, no single subnormal generator, a single hyponormal generator). Let K consist of the boundary of the unit square together with the vertical line segment from $(\frac{1}{2}, 0)$ to $(\frac{1}{2}, 1)$ and the horizontal line segment from $(0, \frac{1}{2})$ to $(1, \frac{1}{2})$. If A is an irreducible essentially normal operator with $\sigma_{\mathrm{e}}(A) = K$ and $\mathrm{ind}(A - \lambda I) = 0$ for $\lambda \in \mathbb{C} \setminus K \cap \{z : \Re e\{z\} < \frac{1}{2}\}$ and $\mathrm{ind}(A - \lambda I) = 1$ for $\lambda \in \mathbb{C} \setminus K \cap \{z : \Re e\{z\} > \frac{1}{2}\}$, then $C^*(A)$ does not have a single subnormal generator, although it does have a single hyponormal generator (see Feldman and McGuire [10], page 496). However, Corollary 2.8 implies that $C^*(A)$ is generated by two unilateral shifts of multiplicity one.

EXAMPLE 2.11 (Non-zero index, no single hyponormal generator). Let $K_1 = \{z \in \mathbb{C} : 1 \leqslant |z| \leqslant 2\}$, let $K = K_1 \cup \{1 - \frac{1}{n}\}_{n=1}^{\infty} \cup \{2 + \frac{1}{n}\}_{n=1}^{\infty}$. If A is an irreducible essentially normal operator with $\sigma_{\mathrm{e}}(A) = K$ and $\mathrm{ind}(A - \lambda I) = -1$ for $\lambda \in \left(\mathbb{D} \setminus \{1 - \frac{1}{n}\}_{n=1}^{\infty}\right)$, then $C^*(A)$ does not have a hyponormal generator (see Feldman and McGuire [10], page 493). However, Corollary 2.8 implies that $C^*(A)$ is generated by two unilateral shifts of multiplicity one.

EXAMPLE 2.12. Let $K = C_1 \cup C_2 \cup C_3$ where C_1, C_2, and C_3 are the circles of radius one centered at -3, 0, and 3 respectively. Assume A is an irreducible essentially normal operator with $\sigma_{\mathrm{e}}(A) = K$ and index values 3, 5, and 9 in the respective interiors of C_1, C_2, and C_3. If ψ is a homeomorphism of K which acts as the identity on C_2, maps C_1 to the circle of radius $\frac{1}{2}$ centered at the origin in an orientation preserving manner, and which maps C_3 to the circle of radius $\frac{1}{4}$ centered at the origin in a manner that reverses the orientation, then an application of Theorem 1.2 will show that $\psi(A) - \lambda I$ has index -1 for $|\lambda| < \frac{1}{4}$. In this example $C^*(A)$ has a single subnormal generator (see McGuire [12] or Feldman and McGuire [10]), but Corollary 2.9 implies that $C^*(A)$ can also be viewed as being generated by two unilateral shifts of multiplicity one.

THEOREM 2.13. *If A is an irreducible essentially normal operator and the interior of $\sigma_{\mathrm{e}}(A)$ is non-empty, then there are two irreducible subnormal operators S_1 and S_2 that are unitarily equivalent to the dual of the Bergman shift and such that $C^*(A) = C^*(S_1, S_2)$.*

Proof. If $\mathrm{int}[\sigma_{\mathrm{e}}(A)] \neq \emptyset$, then let Δ be a closed disk of radius r and center w contained in $\mathrm{int}[\sigma_{\mathrm{e}}(A)]$ and consider $R_{\mathrm{E}}(\frac{A-wI}{r})$ and $P_{\mathrm{E}}(\frac{A-wI}{r})$ where R_{E} and P_{E} are the exterior radial and parabolic projections. Both $R_{\mathrm{E}}(\frac{A-wI}{r})$ and $P_{\mathrm{E}}(\frac{A-wI}{r})$ have the same spectral picture as the dual of the Bergman shift. Since the dual of the Bergman shift is an irreducible essentially normal subnormal operator, Theorem 1.1 and Theorem 2.3 together imply that $C^*(A)$ has two subnormal generators unitarily equivalent to the dual of the Bergman shift. ∎

3. OPERATORS WITH THIN ESSENTIAL SPECTRA AND INDEX ZERO

The results above indicate that if A is an irreducible essentially normal operator such that either the index of $A - \lambda I$ is nonzero at some point or the essential spectrum of A has non-empty interior, then at most two subnormal operators are required to generate $C^*(A)$. We next turn our intention to the situation where the essential spectrum of A has no interior and the index of $A - \lambda I$ is always zero for $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{e}}(A)$. Our first result of this kind shows that if the essential spectrum of A is an interval (or Jordan arc), then $C^*(A)$ is generated by two subnormal operators each of which is unitarily equivalent to the dual of the Bergman shift. This requires the construction of a one-to-one pair of space filling curves from the interval $[0,1]$ onto the unit square. The curves are Peano curves and their construction closely follows the construction in Sagan [16], page 31. In what follows the notation $t = \underset{b}{.}t_1t_2t_3\cdots$ will be used to denote the base b decimal expansion of a number $t \in [0,1]$.

DEFINITION 3.1. For each $t = \underset{3}{.}t_1t_2t_3\cdots$ in the interval $[0,1]$, let

$$f(t) = \begin{pmatrix} \underset{3}{.}t_1(k^{t_2}t_3)(k^{t_2+t_4}t_5)\cdots \\ \underset{3}{.}(k^{t_1}t_2)(k^{t_1+t_3}t_4)\cdots \end{pmatrix}$$

where $k(t) = 2 - t$ and k^v denotes the vth iterate of k. Similarly for each $t = \underset{5}{.}t_1t_2t_3\cdots$ in the interval $[0,1]$, let

$$g(t) = \begin{pmatrix} \underset{5}{.}t_1(u^{t_2}t_3)(u^{t_2+t_4}t_5)\cdots \\ \underset{5}{.}(u^{t_1}t_2)(u^{t_1+t_3}t_4)\cdots \end{pmatrix}$$

where $u(t) = 4 - t$ and u^v denotes the vth iterate of u.

Note that $f(0) = g(0) = (0,0)$ and that the representation of $t = 1$ in base 3 or base 5 is uniquely given by $\underset{3}{.}222\cdots$ and $\underset{5}{.}444\cdots$ respectively. For our purposes both 0 and 1 will not be viewed as having finite ternary or base five expansions. Every other finite ternary or base five expansion can also be written as an infinite ternary with trailing 2's or an infinite base five decimal with trailing 4's. The curve f above is Peano's space filling curve and the curve g is a mild variation of f. Both f and g are well defined, continuous, and map the interval $[0,1]$ onto the unit square (see Chapter 3 of Sagan [16]). Moreover if

$$f(t) = \begin{pmatrix} {}_{3}^{.}\beta_1\beta_2\beta_3\cdots \\ {}_{3}^{.}\gamma_1\gamma_2\gamma_3\cdots \end{pmatrix},$$

then one can recursively solve for $t_{2n-1} = k^{t_2+t_4+\cdots+t_{2n-2}}\beta_n$ and $t_{2n} = k^{t_1+t_34+\cdots+t_{2n-1}}$ γ_n. Similarly if

$$g(t) = \begin{pmatrix} {}_{5}^{.}\beta_1\beta_2\beta_3\cdots \\ {}_{5}^{.}\gamma_1\gamma_2\gamma_3\cdots \end{pmatrix},$$

then one can recursively solve for $t_{2n-1} = u^{t_2+t_4+\cdots+t_{2n-2}}\beta_n$ and $t_{2n} = u^{t_1+t_34+\cdots+t_{2n-1}}$ γ_n.

Neither f or g is injective. However if (β, γ) is a point in the unit square with a unique ternary expansion, then $f^{-1}(\beta, \gamma)$ is uniquely determined. Similarly if (β, γ) is a point in the unit square with a unique base five expansion, then $g^{-1}(\beta, \gamma)$ is uniquely determined. If one of β or γ has a unique expansion and the other has a finite expansion, then at most two preimages exist for each map. If both β and γ have finite expansions, then possibly four preimages exist.

PROPOSITION 3.2. *If $f(t) = (\beta, \gamma)$ where exactly one of β or γ has a finite ternary expansion, then there are two possibilities for t, denoted by t_a and t_b, that satisfy $t_a + t_b$ is a number of the form $B + \frac{A}{3^s}$ where $0 < A < 3^s$ and either $B = 0$ or $B = 1$.*

Proof. Assume β has a finite ternary expansion and $\gamma = {}_{3}^{.}\gamma_1\gamma_2\gamma_3\cdots$ has a unique infinite expansion. First note that if $f(t_a) = {}_{3}^{.}\beta_1\beta_2\beta_3\cdots\beta_s000\cdots$ where $\beta_s \neq 0$ and $t_a = {}_{3}^{.}a_1a_2a_3\cdots$, then $a_1 = \beta_1$, $a_2 = (k^{a_1}\gamma_1)$, $a_3 = (k^{a_2}\beta_2)$, $a_4 = (k^{a_1+a_3}\gamma_2)$, $a_5 = k^{a_2+a_4}\beta_3, \cdots$. Next note that if we write β with trailing 2's and seek $t_b = {}_{3}^{.}b_1b_2b_3\cdots$ where $f(t_b) = {}_{3}^{.}\beta_1\beta_2\beta_3\cdots(\beta_s - 1)222\cdots$, then $b_j = a_j$ for $j = 1, \ldots, 2s-2$. Since $\beta_s \neq 0$, β_s is either 1 or 2. If $\beta_s = 1$, then $a_{2s-1} = 1$, $\beta_s - 1 = 0$ and b_{2s-1} is either 0 or 2. If $\beta_s = 2$, then $\beta_s - 1 = 1$, $b_{2s-1} = 1$, and a_{2s-1} is either 0 or 2. Thus $a_{2s-1} + b_{2s-1}$ is either 1 or 3. Hence the $2s-1$ digit in the ternary expansion of $t_a + t_b$ is either 0 or 1. This insures that $t_a + t_b$ is not equal to 1. Additionally note that a_{2s-1} and b_{2s-1} have opposite parity, meaning that one is even and the other is odd.

Now let $\tau = a_1 + a_3 + \cdots + a_{2s-3}$ and $\sigma = a_2 + a_4 + \cdots + a_{2s-2}$. Since $a_{2s} = k^{\tau+a_{2s-1}}\gamma_s$, $b_{2s} = k^{\tau+b_{2s-1}}\gamma_s$, and a_{2s-1} has opposite parity to b_{2s-1}, we see that $b_{2s} = ka_{2s} = 2 - a_{2s}$ or $a_{2s} + b_{2s} = 2$. Since $a_{2s+1} = k^{\sigma+a_{2s}}0$, a_{2s+1} is even. Similarly for $j \geqslant s$, $a_{2j+1} = k^{\sigma+a_{2s}+\cdots+a_{2j}}0$ is even. However, $b_{2s+1} = k^{\sigma+b_{2s}}2 = ka_{2s+1}$ implies that $b_{2s+1} = 2$ when $a_{2s} = 0$ and $b_{2s+1} = 0$ when $a_{2s} = 2$. Proceding in the same fashion, one obtains that $a_j + b_j = 2$ for all $j \geqslant 2s$. Since $\sum\limits_{j=2s}^{\infty} \frac{2}{3^j} = \frac{1}{3^{2s-1}}$, the result follows.

The case when β has a unique infinite ternary expansion and γ has a finite ternary expansion is similar. ∎

PROPOSITION 3.3. *If $g(t) = (\beta, \gamma)$ where exactly one of β or γ has a finite base 5 expansion, then there are two possibilities for t, denoted by t_a and t_b, that satisfy $t_a + t_b$ is a number of the form $B + \frac{A}{5^r}$ where $0 < A < 5^r$ and either $B = 0$ or $B = 1$.*

Proof. This proof is very similar to the base three case. ∎

PROPOSITION 3.4. *If $f(t) = (\beta, \gamma)$ where both β and γ have finite ternary expansions, then t is of the form $\frac{A}{3^s}$ where $0 < A < 3^s$ or of the form $\frac{A}{3^s} + \frac{1}{4 \cdot 3^s}$.*

Proof. Assume first that β and γ are written as $\beta = \underset{3}{.}\beta_1\beta_2\beta_3 \cdots \beta_k 000 \cdots$ and $\gamma = \underset{3}{.}\gamma_1\gamma_2\gamma_3 \cdots \gamma_r 000 \cdots$. If $f(t) = \beta$ where $t = \underset{3}{.}t_1 t_2 t_3 \cdots$, then for all sufficiently large j, $t_{2j} = k^{t_1 + t_2 + \cdots + t_{2j-1}} 0$ and $t_{2j+1} = k^{t_2 + t_4 + \cdots + t_{2j}} 0$. Thus for large j, t_{2j} and t_{2j+1} are both constant sequences consisting of either solely 0's or solely 2's. Hence the ternary expansion of t must be either of the form $\underset{3}{.}t_1 t_2 t_3 \cdots t_s 000 \cdots$, $\underset{3}{.}t_1 t_2 t_3 \cdots t_s 222 \cdots$, or $\underset{3}{.}t_1 t_2 t_3 \cdots t_s 0202 \cdots$.

The remaining three cases where

(i) $\beta = \underset{3}{.}\beta_1\beta_2\beta_3 \cdots \beta_k 000 \cdots$ and $\gamma = \underset{3}{.}\gamma_1\gamma_2\gamma_3 \cdots \gamma_r 222 \cdots$,
(ii) $\beta = \underset{3}{.}\beta_1\beta_2\beta_3 \cdots \beta_k 222 \cdots$ and $\gamma = \underset{3}{.}\gamma_1\gamma_2\gamma_3 \cdots \gamma_r 000 \cdots$,
(iii) $\beta = \underset{3}{.}\beta_1\beta_2\beta_3 \cdots \beta_k 222 \cdots$ and $\gamma = \underset{3}{.}\gamma_1\gamma_2\gamma_3 \cdots \gamma_r 222 \cdots$,

are all similar and for large j, t_{2j} and t_{2j+1} are both constant sequences consisting of either solely 0's or solely 2's. Thus, in all cases the ternary expansion of t is of the form

(i) $\underset{3}{.}t_1 t_2 t_3 \cdots t_s 000 \cdots$,
(ii) $\underset{3}{.}t_1 t_2 t_3 \cdots t_s 222 \cdots$, or
(iii) $\underset{3}{.}t_1 t_2 t_3 \cdots t_s 0202 \cdots$.

In the first two instances, t is of the form $\frac{B}{3^s}$ for some $B < 3^S$. Note the expansion of t cannot consist of all 2's as that would imply $f(t) = (1, 1)$ and we have excluded 1 as having a "finite" ternary expansion. In the last case t is of the form $\frac{B}{3^s} + \sum_{j=s+2}^{\infty} \frac{2}{3^j} = \frac{B}{3^s} + \frac{1}{4 \cdot 3^s}$. ∎

PROPOSITION 3.5. *If $g(t) = (\beta, \gamma)$ where both β and γ have finite base five expansions, then t is of the form $\frac{A}{5^r}$ where $0 < A < 5^r$ or of the form $\frac{A}{5^r} + \frac{1}{6 \cdot 5^r}$.*

Proof. This proof is very similar to the base three case. ∎

PROPOSITION 3.6. *If $f(t_1) = f(t_2)$, then $g(t_1) \neq g(t_2)$.*

Proof. If $f(t_1) = f(t_2) = (\beta, \gamma)$ is such that β and γ both have finite ternary expansions, then neither β or γ has a finite base five expansion. Hence $g(t_1) \neq g(t_2)$.

If $f(t_1) = f(t_2) = (\beta, \gamma)$ and exactly one of β and γ has a finite ternary expansion, then $t_1 + t_2 = \frac{A}{3^s}$ or $t_1 + t_2 = 1 + \frac{A}{3^s}$ for some positive integer s and $0 < A < 3^s$. Thus at most one of the pair (β, γ) can have a finite base five expansion and if $g(t_1) = g(t_2) = (\beta, \gamma)$, then $t_1 + t_2 = \frac{B}{5^r}$ or $t_1 + t_2 = 1 + \frac{B}{5^r}$ for some positive integer r and $0 < B < 5^r$.

Hence there are four cases to check:

(i) $t_1 + t_2 = \frac{A}{3^s} = \frac{B}{5^r}$;
(ii) $t_1 + t_2 = \frac{A}{3^s} = 1 + \frac{B}{5^r}$;
(iii) $t_1 + t_2 = 1 + \frac{A}{3^s} = \frac{B}{5^r}$;
(iv) $t_1 + t_2 = 1 + \frac{A}{3^s} = 1 + \frac{B}{5^r}$.

In each case, clearing the denominators and using the prime factorization theorem establishes that t_1 cannot equal t_2. ∎

THEOREM 3.7. *If A is an irreducible essentially normal operator with essential spectrum equal to a Jordan arc, then $C^*(A) = C^*(S_1, S_2)$ where S_1 and S_2 are unitarily equivalent to the dual of the Bergman shift.*

Proof. Let ϕ be a homeomorphism of the essential spectrum of A onto the interval $[0,1]$. Let ψ be a homeomorphism of the unit square onto the closed unit disk $\overline{\mathbb{D}}$. With f and g as in Definiton 3.1, Proposition 3.6 implies that $(\psi \circ f \circ \phi, \psi \circ g \circ \phi)$ is a one-to-one pair. Since both $f(A)$ and $g(A)$ have essential spectrum $\overline{\mathbb{D}}$, Theorem 1.1 implies that there are compact perturbations of $f(A)$ and $g(A)$ that are unitarily equivalent to the dual of the Bergman shift. The result follows from Theorem 2.3. ∎

Before proceeding with our final result, we require two preliminary lemmas.

LEMMA 3.8. *If $F = (f_1, \dots, f_m)$ is a one-to-one continuous m-tuple from K to L and $G = (g_1, \dots, g_n)$ is a one-to-one n-tuple from L to M, then the mn-tuple $G \circ F = (g_j(f_i))_{i=1,\dots,m, j=1,\dots,n}$ is a one-to-one continuous mn-tuple from K to M.*

Proof. The continuity of $G \circ F$ is clear. Suppose $(G \circ F)(x_1) = (G \circ F)(x_2)$ for some $x_1, x_2 \in K$. Since G is a one-to-one n-tuple, $f_i(x_1) = f_i(x_2)$ for $i = 1, \dots, m$. Since F is a one-to-one m tuple, $x_1 = x_2$. ∎

LEMMA 3.9. *There exist continuous functions ϕ and ψ from the Cantor set $\mathcal{C}$ onto the unit square $[0,1]^2$ such that (ϕ, ψ) is a one-to-one pair.*

Proof. Assume h is a continuous function from $\mathcal{C}$ onto $[0,1]$ and recall that $\mathcal{C}$ is homeomorphic to $\mathcal{C} \times \mathcal{C}$. It is well known that $(h(x), h(y))$ then provides a continuous function from $\mathcal{C} \times \mathcal{C}$ onto $[0,1]^2$. Hence, given such an h, the homeomorphism from $\mathcal{C}$ to $\mathcal{C} \times \mathcal{C}$ followed by $(h(x), h(y))$ provides a continuous mapping of $\mathcal{C}$ onto $[0,1]^2$. We will provide a one-to-one pair of continuous functions from $\mathcal{C}$ onto $[0,1]$ which by the above remarks will produce the desired one-to-one pair from $\mathcal{C}$ onto $[0,1]^2$.

View $\mathcal{C}$ as the set of all numbers in $[0,1]$ whose ternary decimal expansion consists of only the digits 0 and 2 and let ϕ be the usual Cantor function which maps $\sum_{k=1}^{\infty} b_k \frac{1}{3^k}$ to $\sum_{k=1}^{\infty} \frac{b_k}{2} \frac{1}{2^k}$. The function ϕ identifies consecutive endpoints in $\mathcal{C}$, but otherwise is one-to-one.

To define ψ, first note that $\mathcal{C}$ consists of numbers whose base nine decimal expansions contain only the digits 0, 2, 6, and 8. For each $t = \underset{9}{.} t_1 t_2 t_3 \cdots$ in $\mathcal{C}$, we define $\psi(t)$ to be the base four decimal $\underset{4}{.}\kappa(t_1)\kappa(t_2)\kappa(t_3)\cdots$ where $\kappa(0) = 2$, $\kappa(2) = 0$, $\kappa(6) = 3$, $\kappa(8) = 1$. It is straightforward to verify that ψ is a continuous function from $\mathcal{C}$ onto $[0,1]$. The consecutive endpoints $\frac{1}{3} = \underset{9}{.}2888\cdots$ and $\frac{2}{3} = \underset{9}{.}6000\cdots$ of $\mathcal{C}$ satisfy $\psi(\frac{1}{3}) = \underset{4}{.}0111\cdots = \frac{1}{2} \neq \underset{4}{.}3222\cdots = \psi_1(\frac{2}{3})$. Similarly all consecutive endpoints of $\mathcal{C}$ that are identified by ϕ are separated by ψ. Hence the pair (ϕ, ψ) is one-to-one. ∎

LEMMA 3.10. *If K is a compact subset of $\mathbb{R}$ that has no interior and C is any Cantor set in $\mathbb{R}$, then K is homeomorphic to a subset of C. Furthermore if $a \in K$ and $b \in C$, then we may choose the homeomorphism to map a to b.*

Proof. First construct a Cantor set X in $\mathbb{R}$ that contains K. Since any two Cantor sets are homeomorphic, let $h_1 : X \to C$ be a homeomorphism. Also since Cantor sets are homogeneous, there is a homeomorphism $h_2 : C \to C$ such that $h_2(h_1(a)) = b$. Then $h_2 \circ h_1 : K \to C$ is the desired embedding. ∎

THEOREM 3.11. *If A is an irreducible essentially normal operator with an uncountable spectrum, then at most eight irreducible subnormal operators are required in order to generate $C^*(A)$.*

Proof. By our previous results, at most two subnormal operators are required to generate $C^*(A)$ if the essential spectrum of A has non-empty interior. Hence we may assume that $K = \sigma_e(A)$ is an uncountable set with no interior. Since K is uncountable either $\pi_x(K)$ or $\pi_y(K)$ is uncountable. Assuming $\pi_x(K)$ is uncountable, either $\pi_y(K)$ is uncountable or there exists a value y_0 such that $\pi_y^{-1}(y_0)$ is uncountable. In that case the line L given by $y = x$ is such that $\pi_L(K)$ is uncountable. By part (iii) of Example 2.5 $(\pi_x(K), \pi_L(K))$ is a one-to-one continuous pair. By a further homeomorphism of the plane, we may assume that L is the y coordinate axis and that $\pi_x(K)$ and $\pi_y(K)$ are both uncountable subsets respectively of the x and y axes. By part (iii) of Example 2.5, $Q = (\pi_x, \pi_y)$ is a one-to-one pair.

We will treat separately the cases where $\pi_x(K)$ or $\pi_y(K)$ have interior. If, for example, $\pi_x(K)$ has interior, then we will produce a one-to-one pair $F = (f_1, f_2)$

that maps $\pi_x(K)$ onto $[0,1]$. Letting G be the one-to-one pair of Definition 3.1 and Proposition 3.6, Lemma 3.8 implies that the composition $G \circ F \circ Q$ will be a one-to-one eight tuple from K onto the unit square. If $\pi(K)$ has no interior, then we need only produce a one-to-one pair $F = (f_1, f_2)$ that maps $\pi_x(K)$ onto a Cantor set and then use Lemma 3.9 to obtain a one-to-one pair $G = (g_1, g_2)$ that maps the Cantor set onto the unit square. As before the composition $G \circ F \circ Q$ will be a one-to-one eight tuple from K onto the unit square.

If $\pi_x(K)$ has interior, then without loss of generality we may assume that $\pi_x(K)$ contains the interval $[-1,1]$. It is easy to see that $F = (f_1, f_2)$ is a one-to-one continuous pair from $\pi_x(K)$ onto $[0,1]$ if

$$f_1(x) = \begin{cases} 0 & \text{if } x < -1, \\ \frac{x+1}{2} & \text{if } x \in [-1,1], \\ 1 & \text{if } x > 1, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} \frac{1+\frac{1}{x}}{2} & \text{if } x \notin [-1,1], \\ \frac{1+x}{2} & \text{if } x \in [-1,1]. \end{cases}$$

For the case where $\pi_x(K)$ has no interior, first write $\pi_x(K) = C \cup \Delta$ where C is a set homeomorphic to the Cantor set and Δ is a countable set. (To see how to do this see the proof of Lemma 5.1 on page 23 of Oxtoby [14].) Note $\pi_x(K)$ is contained in $[-N, N]$ for some N. Consider $\pi_x(K) \setminus C$ which consists of a countable union of disjoint open intervals $\{I_j = (a_J, b_J)\}_{j=1}^{\infty}$ where each a_j and b_j belong to C. For each j, find a point x_j in $I_j \setminus \Delta$ and consider $[x_j, b_j]$. We now define two maps $g_1, g_2 : C \cup \Delta \to C$. Both maps are the identity on C.

The first map g_1 will map $\Delta \cap [x_j, b_j]$ to the endpoint b_j and maps $\Delta \cap [a_j, x_j]$ to the endpoint a_j. Since $\pi_x(K)$ is closed, x_j is not a limit point of Δ and hence g_1 is well defined and continuous.

For the second map we will define g_2 on $\Delta \cap [x_j, b_j]$ as the homeomorphism guaranteed by Lemma 3.10 that maps $\Delta \cap [x_j, b_j]$ into C and satisfies $h(b_j) = b_j$. Similarly, g_2 is defined on $\Delta \cap [a_j, x_j]$ as a homeomorphism that maps $\Delta \cap [a_j, x_j]$ into C and satisfies $g_2(a_j) = a_j$. In this manner g_2 becomes a continuous function mapping $\Delta \cup C$ into C.

Finally, if h is a homeomorphism from C to the canonical Cantor set $\mathcal{C}$, then the pair $G = (h \circ g_1, h \circ g_2)$ is a continuous one-to-one pair of $\pi_x(K)$ onto $\mathcal{C}$. By our remarks above this completes the proof. ∎

COROLLARY 3.12. *If A is an irreducible essentially normal operator, then $C^*(A)$ has a finite number of irreducible subnormal generators if and only if the essential spectrum of A is uncountable.*

Proof. If $\sigma_e(A)$ is uncountable, then Theorem 3.11 says that there are eight irreducible subnormal generators. For the converse, if $\sigma_e(A)$ is countable, then for any continuous function f on $\sigma_e(A)$, $f(\sigma_e(A)$ would also be countable and thus $f(A)$ could not have the same spectral picture as an irreducible subnormal operator. Thus by Corollary 2.4, $C^*(A)$ cannot be generated by a finite number of irreducible subnormal operators. ∎

Theorem 3.11 asserts that *at most eight* irreducible subnormal operators are needed to generate $C^*(A)$ for any irreducible essentially normal operator A with uncountable essential spectrum. Moreover, the subnormal operators are either irreducible subnormal operators with spectrum the closed unit disk and finite index in the open unit disk or they are unitarily equivalent to the dual of the Bergman shift. In most instances at most two irreducible subnormal generators are required. It is unknown whether the number eight is sharp.

4. QUESTIONS

(1) If A is an irreducible essentially normal operator with an uncountable essential spectrum, then do there always exist two irreducible subnormal operators S_1 and S_2 such that $C^*(A) = C^*(S_1, S_2)$?

(2) If the answer to (1) is no, then what is the fewest number of irreducible subnormal operators $S_1, \dots, S_k$ necessary in order that

$$C^*(A) = C^*(S_1, \dots, S_k)?$$

Note that by Theorem 3.11, $k \leqslant 8$.

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