

## SCHUR MULTIPLIERS AND DE BRANGES–ROVNYAK SPACES: THE MULTISCALE CASE

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*Communicated by Nikolai K. Nikolski*

**ABSTRACT.** We consider bounded linear operators acting on the  $\ell_2$  space indexed by the nodes of a homogeneous tree. Using the Cuntz relations between the primitive shifts on the tree, we generalize the notion of the single-scale time-varying point evaluation and introduce the corresponding reproducing kernel Hilbert space in which Cauchy's formula holds. These notions are then used in the study of the Schur multipliers and of the associated de Branges–Rovnyak spaces. As an application we obtain realization of Schur multipliers as transfer operators of multiscale input-state-output systems.

**KEYWORDS:** *Non-commutative power series, system realization, homogeneous tree.*

**MSC (2000):** Primary 93B28; Secondary 05C05.

### INTRODUCTION

In this paper we consider bounded operators acting on the Hilbert space

$$(0.1) \quad \ell_2(\mathcal{T}) = \left\{ f : \mathcal{T} \longrightarrow \mathbb{C}; \|f\|_{\ell_2}^2 \stackrel{\text{def}}{=} \sum_{t \in \mathcal{T}} |f(t)|^2 < \infty \right\},$$

where  $\mathcal{T}$  is a homogeneous tree of order  $q \geq 1$ , that is, an acyclic, undirected, connected graph such that every node belongs to exactly  $q + 1$  edges (see [26], [18]). Such operators arise in the theory of multiscale linear systems and multiscale stochastic processes. Here we would like to mention the works [13], [12], [14], where Basseville, Benveniste, Nikoukhah and Willsky have developed a theory of *stationary* multiscale systems and *stationary* multiscale stochastic processes. Connections of their theory with the classical setting when  $q = 1$  and the tree  $\mathcal{T}$  is the tree of integers  $\mathbb{Z}$  (what we shall call the single-scale setting) were explored in [8] and [2]. The special case of *isotropic* processes was considered in [9]; a different approach to isotropic processes uses the theory of Gelfand pairs (see [24], [10]).

In what follows we consider the general multiscale setting, without the assumption of stationarity. Some of the results stated here were announced in [5].

In order to explain our approach, let us recall that in the stationary single-scale setting one considers a function

$$s(z) = s_0 + zs_1 + z^2s_2 + \dots,$$

analytic and contractive in the open unit disk  $\mathbb{D}$ , a Schur function. Then the multiplication by  $s(z)$  is a causal contractive operator acting on the Hardy space  $\mathbf{H}_2$  of the unit disk and the kernel

$$(0.2) \quad K_s(z, w) \stackrel{\text{def}}{=} \frac{1 - s(z)s(w)^*}{1 - z\bar{w}}$$

is positive in  $\mathbb{D}$ . The associated reproducing kernel Hilbert space  $\mathbf{H}(s)$  has the form

$$\mathbf{H}(s) = \sqrt{\mathcal{B}_s} \mathbf{H}_2; \quad \|\sqrt{\mathcal{B}_s}f\|_{\mathbf{H}(s)} = \|(I - \pi)f\|_{\mathbf{H}_2},$$

where  $\mathcal{B}_s = I - M_s M_s^*$  and  $\pi$  is the orthogonal projection in  $\mathbf{H}_2$  onto  $\ker \mathcal{B}_s$ . The space  $\mathbf{H}(s)$  is called the de Branges–Rovnyak space associated with the Schur function  $s$ ; see [16], Appendix of [15] and [25]. It is invariant under the action of the backward shift operator  $R_0$ , defined by

$$(0.3) \quad (R_0 f)(z) \stackrel{\text{def}}{=} \frac{f(z) - f(0)}{z}.$$

Moreover, the formulae

$$(0.4) \quad Af = R_0 f, \quad Bc = R_0(s \cdot c),$$

$$(0.5) \quad Cf = f(0), \quad Dc = s(0) \cdot c,$$

where  $f \in \mathbf{H}(s)$ ,  $c \in \mathbb{C}$ , define a coisometry

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathbf{H}(s) \\ \mathbb{C} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{H}(s) \\ \mathbb{C} \end{pmatrix}.$$

In terms of these operators  $A, B, C, D$  the Schur function  $s(z)$  admits the representation

$$(0.6) \quad s(z) = D + zC(I - zA)^{-1}B$$

and the reproducing kernel  $K_s(z, w)$  can be written as

$$(0.7) \quad K_s(z, w) = C(I - zA)^{-1}(I - wA)^{-*}C^*.$$

Conversely, if  $\mathbf{H}$  is a Hilbert space of functions analytic in the open unit disk such that there exists a coisometry

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathbf{H} \\ \mathbb{C} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{H} \\ \mathbb{C} \end{pmatrix},$$

where  $A$  and  $C$  are as in (0.4), (0.5) (in particular, the space  $\mathbf{H}$  is  $R_0$ -invariant), then the formula (0.6) defines a Schur function  $s$ , for which the kernel  $K_s$  is given by (0.7) and the associated de Branges–Rovnyak space coincides with  $\mathbf{H}$ . For this and more general results in the setting of Pontryagin spaces see Theorem 3.12 p. 85 of [6].

The representation (0.6) implies that if a function

$$u(z) = u_0 + zu_1 + z^2u_2 + \cdots \in \mathbf{H}_2$$

is given, then the Taylor coefficients  $y_0, y_1, y_2, \dots$  of the function

$$y(z) = s(z) \cdot u(z) = y_0 + zy_1 + z^2y_2 + \cdots \in \mathbf{H}_2$$

can be recursively determined as follows:

$$(0.8) \quad \begin{cases} x_0 = 0, \\ x_{n+1} = Ax_n + Bu_n, \\ y_n = Cx_n + Du_n, \end{cases}$$

where  $x_n \in \mathbf{H}(s)$ . This fact has many important numerical applications; in the language of system theory it means that the representation (0.6) is a coisometric realization of the Schur function  $s(z)$  as the transfer function of the input-state-output system (0.8) with the state space  $\mathbf{H}(S)$ .

In the non-stationary single-scale setting, the Hardy space is replaced by the space of upper-triangular Hilbert–Schmidt operators, the Schur functions by upper-triangular contractions, the complex variable by the bilateral shift  $Z$  on  $\ell_2(\mathbb{Z})$  and the constants by diagonal operators (see e.g. [3], [20]). In particular, any upper-triangular bounded operator  $S$  can be written as a power series

$$(0.9) \quad S = S_{[0]} + ZS_{[1]} + Z^2S_{[2]} + \cdots,$$

where  $S_{[j]}$  are bounded diagonal operators. In general, a diagonal operator  $D$  does not commute with the shift  $Z$ . However, they satisfy the commutation relation

$$ZD = D^{(1)}Z,$$

where  $D^{(1)} \stackrel{\text{def}}{=} ZDZ^*$  is also a diagonal operator. This fact can be used to define a point evaluation of an upper-triangular bounded operator at a diagonal “constant”. At the same time  $S$  may be viewed as the input-output operator of a *time-varying* causal linear system. In order to construct a non-stationary analogue of the realization (0.6), it is necessary to consider square-summable sequences of inputs rather than a single input, in other words, the operator of multiplication by  $S$  acting on the space of upper-triangular Hilbert–Schmidt operators.

The multiscale setting considered in this paper can be viewed as the natural multidimensional generalization of the single-scale non-stationary case. Here expansions of the form (0.9) are replaced with non-commutative powers series in  $q$  primitive shifts, which satisfy the Cuntz relations (see [19], [17]). Just as in the single-scale case, the coefficients of these series do not commute with the primitive shifts but satisfy certain commutation relations. Thus the multiscale setting is different from such multidimensional settings as the classical theory of formal non-commutative power series with the coefficients which commute with the indeterminates (see [22] and [11] for recent developments), the Arveson space of

the ball in  $\mathbb{C}^n$  (see [21]) and the quaternionic Arveson space (see [4], [7]). In particular, in the last two cases the de Branges–Rovnyak space associated to a Schur multiplier is Gleason-invariant rather than backward shift-invariant.

The paper is organized as follows. Section 1 is of a review nature. It presents the ordering of the homogeneous tree  $\mathcal{T}$  as introduced by Basseville, Benveniste, Nikoukhah and Willsky and the canonical representation of a bounded linear operator on the Hilbert space  $\ell_2(\mathcal{T})$  as developed in [8]. Section 2 discusses causal operators and, in particular, the algebra of constants. In Section 3 we present the point evaluation of a causal operator at a constant. In Section 4 we study the space of causal Hilbert–Schmidt operators which plays here the role of the Hardy space  $\mathbf{H}_2$  of the unit disk. In particular, we present the analogue of Cauchy’s formula; see Theorem 4.4. Schur multipliers, associated kernels, de Branges–Rovnyak spaces and input-state-output systems are studied in Section 5. In the last section we present the analogue of a Blaschke factor in the present setting.

## 1. POWER SERIES REPRESENTATION OF BOUNDED OPERATORS ON $\ell_2(\mathcal{T})$

We start with the ordering of the homogeneous tree  $\mathcal{T}$  of order  $q \geq 1$ . Note that, as follows from the definition (see Introduction), the tree  $\mathcal{T}$  is infinite. For each node  $t \in \mathcal{T}$  we consider infinite paths, which begin at  $t$ . These are infinite sequences of nodes

$$(t_0 = t, t_1, t_2, \dots),$$

where each pair of consecutive nodes  $t_k, t_{k+1}$  is connected by an edge and each two consecutive edges are distinct:

$$t_{k+1} \neq t_k \neq t_{k+2}, \quad k = 0, 1, 2, \dots$$

Two such paths

$$(1.1) \quad (t_0 = t, t_1, t_2, \dots) \quad \text{and} \quad (s_0 = s, s_1, s_2, \dots),$$

which begin at the nodes  $t$  and  $s$ , respectively, are said to be *equivalent* if they coincide modulo finite number of edges: there exist indices  $m, n$  such that

$$(1.2) \quad t_{m+k} = s_{n+k}, \quad k = 0, 1, 2, \dots$$

The equivalence classes of paths with respect to this relation are called the *boundary points* of the tree  $\mathcal{T}$ .

Let us choose and fix some boundary point of  $\mathcal{T}$ , which will be denoted by  $\infty_{\mathcal{T}}$  (in the single scale case, this is  $-\infty$ ). Since the graph  $\mathcal{T}$  is connected and does not contain cycles, for each  $t \in \mathcal{T}$  there exists a unique representative of the equivalence class  $\infty_{\mathcal{T}}$ , which begins at the node  $t$ .

For a pair of nodes  $t, s$ , let the corresponding representatives of the boundary point  $\infty_{\mathcal{T}}$  be given by (1.1). They coincide modulo finite number of edges, that is, (1.2) holds for some  $m$  and  $n$ . Let us choose the minimal  $m$  and  $n$  for

which (1.2) holds. Then we denote the node  $t_m = s_n$  by  $s \wedge t$  and call the number  $m + n$  the *distance*  $\text{dist}(s, t)$  between the nodes  $s$  and  $t$ .

Using these notations, we define the partial ordering  $\preceq$  and the equivalence relation  $\asymp$  as follows:

$$(1.3) \quad s \preceq t \quad \text{if } \text{dist}(s, s \wedge t) \leq \text{dist}(t, s \wedge t).$$

$$(1.4) \quad s \asymp t \quad \text{if } \text{dist}(s, s \wedge t) = \text{dist}(t, s \wedge t).$$

The equivalence classes with respect to the relation  $\asymp$  are called *horocycles*.

Furthermore, we choose and fix  $q$  mappings

$$\alpha_1, \dots, \alpha_q : \mathcal{T} \longrightarrow \mathcal{T}$$

acting on the right

$$\{t\alpha_1, \dots, t\alpha_q\} = \{s \in \mathcal{T} : t \preceq s, \text{dist}(t, s) = 1\}.$$

The mappings  $\alpha_1, \dots, \alpha_q$  are called *primitive shifts*.

The induced *left* action of the primitive shifts  $\alpha_1, \dots, \alpha_q$  on the space  $\ell_2(\mathcal{T})$  (see (0.1)) is given by:

$$(1.5) \quad (\alpha_j f)(t) \stackrel{\text{def}}{=} f(t\alpha_j), \quad f \in \ell_2(\mathcal{T}), t \in \mathcal{T}, j = 1, \dots, q.$$

Thus the primitive shifts  $\alpha_1, \dots, \alpha_q$  can be also viewed as bounded linear operators on  $\ell_2(\mathcal{T})$ . The adjoint operators are given by

$$(1.6) \quad (\alpha_j^* f)(t) = \begin{cases} f(s) & \text{if } t \text{ is of the form } t = s\alpha_j, \\ 0 & \text{otherwise.} \end{cases}$$

and satisfy the Cuntz relations:

$$(1.7) \quad \alpha_i \alpha_j^* = \delta_{i,j} \cdot I, \quad \sum_{j=1}^q \alpha_j^* \alpha_j = I.$$

In other words, the operator matrix

$$\alpha \stackrel{\text{def}}{=} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_q \end{pmatrix} : \ell_2(\mathcal{T}) \longrightarrow \ell_2(\mathcal{T})^q$$

is unitary:

$$(1.8) \quad \alpha \alpha^* = I_{\ell_2(\mathcal{T})^q}, \quad \alpha^* \alpha = I_{\ell_2(\mathcal{T})}.$$

Since  $\mathcal{T}$  is a tree, the shifts  $\alpha_j$  form a free semigroup, which we denote by  $\mathcal{F}_q$ . Every element  $w \in \mathcal{F}_q$  acts on the tree  $\mathcal{T}$  on the right:

$$t \mapsto tw,$$

and on the space  $\ell_2(\mathcal{T})$  on the left:

$$f \mapsto wf, \quad (wf)(t) = f(tw).$$

The unit element of  $\mathcal{F}_q$  will be denoted by  $\emptyset$ . For  $w \in \mathcal{F}_q$  we also use the notation:

$$(1.9) \quad |w| \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } w = \emptyset, \\ n & \text{if } w = \alpha_{i_1} \cdots \alpha_{i_n}. \end{cases}$$

DEFINITION 1.1. A pair of elements  $w_1, w_2 \in \mathcal{F}_q$  is said to be *reducible* if  $w_1$  and  $w_2$  can be represented as

$$w_1 = \alpha_i v_1, \quad w_2 = \alpha_i v_2$$

for some  $v_1, v_2 \in \mathcal{F}_q$  and some primitive shift  $\alpha_i$ .

REMARK 1.2. Note that a pair of elements  $w_1, w_2 \in \mathcal{F}_q$  is irreducible if and only if there exists  $t \in \mathcal{T}$  such that

$$(1.10) \quad (tw_1) \wedge (tw_2) = t.$$

In this case (1.10) holds for all  $t \in \mathcal{T}$ .

Let  $\mathbf{X}(\mathcal{T})$  denote the  $C^*$ -algebra of bounded linear operators on  $\ell_2(\mathcal{T})$ . The elements of the semigroup  $\mathcal{F}_q$  appear in the non-commutative power series representations of elements of  $\mathbf{X}(\mathcal{T})$ . The coefficients of these power series are diagonal operators with respect to the standard basis  $\chi_t$  of  $\ell_2(\mathcal{T})$ :

$$\chi_t(s) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } t = s, \\ 0 & \text{otherwise.} \end{cases}$$

The precise result can be formulated as follows:

THEOREM 1.3. *Every operator  $S \in \mathbf{X}(\mathcal{T})$  can be represented in the form*

$$(1.11) \quad S = \sum'_{w_1, w_2 \in \mathcal{F}_q} w_1^* w_2 S_{w_1, w_2},$$

where:

(i)  $S_{w_1, w_2}$  are elements of  $\mathbf{X}(\mathcal{T})$  which are diagonal with respect to the standard basis  $\chi_t$  of  $\ell_2(\mathcal{T})$ .

(ii) The notation

$$\sum'_{w_1, w_2 \in \mathcal{F}_q}$$

means that the summation is taken over all the irreducible pairs  $w_1, w_2 \in \mathcal{F}_q$ .

(iii) Convergence is absolute pointwise: for every  $f \in \ell_2(\mathcal{T})$  and  $t \in \mathcal{T}$  the series

$$\sum'_{w_1, w_2 \in \mathcal{F}_q} (w_1^* w_2 S_{w_1, w_2} f)(t),$$

is absolutely convergent and its sum is equal to  $(Sf)(t)$ .

Moreover, the diagonal coefficients  $S_{w_1, w_2}$  of the series (1.11) are partially determined by

$$(1.12) \quad (S_{w_1, w_2} \chi_r)(r) = (S \chi_r)(s w_1) \quad \text{if } r \in \mathcal{T} \text{ is of the form } r = s w_2.$$

The rest of the diagonal entries of  $S_{w_1, w_2}$ , that is, the values  $(S_{w_1, w_2} \chi_r)(r)$  for  $r \notin \mathcal{T} w_2$ , can be assigned arbitrarily (as long as they are uniformly bounded).

*Proof.* Let  $S \in \mathbf{X}(\mathcal{T})$ ,  $f \in \ell_2(\mathcal{T})$  and  $t \in \mathcal{T}$  be fixed. Then

$$(Sf)(t) = \sum_{r \in \mathcal{T}} (S \chi_r)(t) \cdot f(r),$$

where the series is absolutely convergent because of the Cauchy–Schwarz inequality.

Observe that, in view of Remark 1.2, for each  $r \in \mathcal{T}$  there exists a unique triple  $s \in \mathcal{T}$ ,  $w_1, w_2 \in \mathcal{F}_q$  such that

$$t = s w_1, \quad r = s w_2$$

and the pair  $w_1, w_2$  is irreducible. (In particular,  $s = t \wedge r$ .) Therefore, we have

$$(Sf)(t) = \sum'_{s \in \mathcal{T}, w_1, w_2 \in \mathcal{F}_q: t = s w_1} (S \chi_{s w_2})(s w_1) \cdot f(s w_2).$$

Let now  $S_{w_1, w_2}$  be elements of  $\mathbf{X}(\mathcal{T})$  which are diagonal with respect to the standard basis  $\chi_t$  of  $\ell_2(\mathcal{T})$  and satisfy (1.12). Then one can rewrite the last identity as

$$(Sf)(t) = \sum'_{s \in \mathcal{T}, w_1, w_2 \in \mathcal{F}_q: t = s w_1} (S_{w_1, w_2} f)(s w_2) = \sum'_{s \in \mathcal{T}, w_1, w_2 \in \mathcal{F}_q: t = s w_1} (w_1^* w_2 S_{w_1, w_2} f)(t).$$

But, in view of (1.6), for every  $w \in \mathcal{F}_q$  and  $g \in \ell_2(\mathcal{T})$  we have

$$(1.13) \quad (w^* g)(t) = \begin{cases} g(s) & \text{if } t \text{ is of the form } t = s w, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the identity (1.11) holds in the sense of pointwise absolute convergence.

Furthermore, let  $t, s \in \mathcal{T}$  be fixed and let  $S \in \mathbf{X}(\mathcal{T})$  admit (in the sense of the pointwise absolute convergence) a representation of the form (1.11), where the coefficients  $S_{w_1, w_2} \in \mathbf{X}(\mathcal{T})$  are diagonal with respect to the standard basis  $\chi_t$  of  $\ell_2(\mathcal{T})$ . Then

$$(S \chi_t)(s) = \sum'_{w_1, w_2 \in \mathcal{F}_2} (w_1^* w_2 S_{w_1, w_2} \chi_t)(s),$$

but, in view of (1.13), the terms of the series on the right-hand side satisfy the relations

$$(w_1^* w_2 S_{w_1, w_2} \chi_t)(s) = \begin{cases} (S_{w_1, w_2} \chi_t)(t) & \text{if } t = (t \wedge s) w_2 \text{ and } s = (t \wedge s) w_1, \\ 0 & \text{otherwise,} \end{cases}$$

hence the series contains at most one non-zero term and (1.12) follows. ■

REMARK 1.4. In order to avoid ambiguity, we shall usually normalize the diagonal coefficients  $S_{w_1, w_2}$  of the representation (1.11) for an operator  $S \in \mathbf{X}(\mathcal{T})$  as follows:

$$(1.14) \quad S_{w_1, w_2} \chi_r = 0, \quad \text{if } r \notin \mathcal{T} w_2.$$

Since the diagonal coefficients  $S_{w_1, w_2}$  in the expansion (1.11) do not commute, in general, with the shift operators  $w_1^* w_2$ , it is of interest to study the representations (1.11) in the special cases when  $S$  is of the form  $S = Dw^*$  or  $S = Dw$ , where  $D \in \mathbf{X}(\mathcal{T})$  is a diagonal operator with respect to the standard basis  $\chi_t$  of  $\ell_2(\mathcal{T})$  and  $w \in \mathcal{F}_q$ .

LEMMA 1.5. *Let  $D \in \mathbf{X}(\mathcal{T})$  be a diagonal operator with respect to the standard basis  $\chi_t$  of  $\ell_2(\mathcal{T})$  and let  $w \in \mathcal{F}_q$ . Then*

$$Dw^* = w^*D', \quad Dw = wD'',$$

where  $D', D'' \in \mathbf{X}(\mathcal{T})$  are diagonal operators given by

$$D'\chi_t = (D\chi_{tw})(tw) \cdot \chi_t \quad \forall t \in \mathcal{T},$$

$$D''\chi_t = \begin{cases} (D\chi_s)(s) \cdot \chi_t & \text{if } t \in \mathcal{T} \text{ is of the form } t = sw, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* In view of (1.5), (1.6) we have

$$(1.15) \quad w^*\chi_t = \chi_{tw}, \quad w\chi_t = \begin{cases} \chi_s & \text{if } t \text{ is of the form } t = sw, \\ 0 & \text{otherwise.} \end{cases}$$

Since the operators  $D, D', D''$  are diagonal, the rest of the proof is straightforward. ■

## 2. CAUSAL BOUNDED OPERATORS AND CONSTANTS

DEFINITION 2.1. Let  $S \in \mathbf{X}(\mathcal{T})$ . Then  $S$  is said to be causal if for every node  $s \in \mathcal{T}$  and every element  $f \in \ell_2(\mathcal{T})$  such that

$$f(t) = 0 \quad \forall t \preceq s$$

it holds that

$$(Sf)(t) = 0 \quad \forall t \preceq s.$$

EXAMPLE 2.2. For every  $w \in \mathcal{F}_q$  the adjoint operator  $w^* \in \mathbf{X}(\mathcal{T})$  is causal, as follows from (1.15).

PROPOSITION 2.3. *Let  $S \in \mathbf{X}(\mathcal{T})$  be represented by the pointwise absolutely convergent series*

$$S = \sum_{w_1, w_2 \in \mathcal{F}_q} w_1^* w_2 S_{w_1, w_2},$$

where the operators  $S_{w_1, w_2} \in \mathbf{X}(\mathcal{T})$  are diagonal with respect to the standard basis  $\chi_t$  of  $\ell_2(\mathcal{T})$  and normalized by (1.14).

Then  $S$  is causal if and only if

$$S_{w_1, w_2} = 0 \quad \text{whenever } |w_1| < |w_2|.$$

*Proof.* First let us assume that  $S$  is causal. Let  $w_1, w_2 \in \mathcal{F}_q$  be an irreducible pair such that  $|w_1| < |w_2|$  and let  $s \in \mathcal{T}$ . Then, according to (1.3) and (1.4),

$$sw_1 \preceq sw_2 \quad \text{and} \quad sw_1 \not\asymp sw_2.$$

Therefore, from Definition 2.1 and the formula (1.12) of Theorem 1.3 it follows that

$$S_{w_1, w_2} \chi_{sw_2} = (S \chi_{sw_2})(sw_1) \cdot \chi_{sw_2} = 0.$$

In view of (1.14), we conclude that  $S_{w_1, w_2} = 0$ .

Conversely, assume that  $S_{w_1, w_2} = 0$  whenever  $|w_1| < |w_2|$ . Let  $s, t \in \mathcal{T}$  be such that  $t \preceq s$  and  $t \not\asymp s$ . Then there exists a unique pair of elements  $w_1, w_2 \in \mathcal{F}_q$  such that

$$t = (t \wedge s)w_1, \quad s = (t \wedge s)w_2.$$

By definition of  $t \wedge s$ , this pair  $w_1, w_2$  is irreducible. In view of (1.3) and (1.4),  $|w_1| < |w_2|$ . Hence, according to the formula (1.12) of Theorem 1.3,

$$(S \chi_s)(t) = (S_{w_1, w_2} \chi_s)(s) = 0.$$

Thus  $(S \chi_s)(t) = 0$  for every pair of nodes  $s, t \in \mathcal{T}$  such that  $t \preceq s$  and  $t \not\asymp s$ . In view of Definition 2.1, this means that  $S$  is causal.  $\blacksquare$

We shall denote the algebra of causal operators  $S \in \mathbf{X}(\mathcal{T})$  by  $\mathbf{H}(\mathcal{T})$ . Note that, in view of Definition 2.1, the algebra  $\mathbf{H}(\mathcal{T})$  is closed in  $\mathbf{X}(\mathcal{T})$  in the pointwise sense: if a sequence  $S_1, S_2, \dots$  of elements of  $\mathbf{H}(\mathcal{T})$  and an element  $S \in \mathbf{X}(\mathcal{T})$  are such that for every  $f \in \ell_2(\mathcal{T})$  and  $t \in \mathcal{T}$   $\lim_{n \rightarrow \infty} (S_n f)(t) = (S f)(t)$ , then  $S \in \mathbf{H}(\mathcal{T})$ .

In order to study the algebra  $\mathbf{H}(\mathcal{T})$  further, we consider its subalgebra

$$\mathcal{C} \stackrel{\text{def}}{=} \{S \in \mathbf{X}(\mathcal{T}) : S, S^* \in \mathbf{H}(\mathcal{T})\}.$$

Elements of  $\mathcal{C}$  play the role of constants in the present setting; we note that

$$(2.1) \quad S \in \mathcal{C} \quad \Leftrightarrow \quad S \chi_t \in \overline{\text{span}}\{\chi_s : s \succ t\} \quad \forall t \in \mathcal{T},$$

where  $\overline{\text{span}}$  denotes closed linear span. Thus the subalgebra  $\mathcal{C}$  is closed in  $\mathbf{X}(\mathcal{T})$  in the pointwise sense.

**REMARK 2.4.** Note that, according to Proposition 2.3 and Theorem 1.3, an element  $S$  of the algebra  $\mathbf{X}(\mathcal{T})$  belongs to the subalgebra  $\mathcal{C}$  if and only if it is of the form

$$S = \sum'_{w_1, w_2 \in \mathcal{F}_q, |w_1| = |w_2|} w_1^* w_2 S_{w_1, w_2},$$

where  $S_{w_1, w_2}$  are diagonal operators with respect to the standard basis  $\chi_t$  of  $\ell_2(\mathcal{T})$ .

**THEOREM 2.5.** *Let  $S \in \mathbf{H}(\mathcal{T})$ . Then  $S$  can be represented as the pointwise absolutely convergent series*

$$(2.2) \quad S = \sum_{w \in \mathcal{F}_q} w^* S_{[w]},$$

where  $S_{[w]} \in \mathcal{C}$  are uniquely determined by

$$(2.3) \quad (S_{[w]}\chi_t)(s) = \begin{cases} (S\chi_t)(sw) & \text{if } t \asymp s, \\ 0 & \text{otherwise,} \end{cases}$$

and satisfy

$$(2.4) \quad \|S_{[w]}\| \leq \left\| \sum_{v \in \mathcal{F}_q, |v|=|w|} v^* S_{[v]} \right\| \leq \|S\|.$$

In the proof of Theorem 2.5 we shall use the following lemma:

LEMMA 2.6. *Let  $T \in \mathbf{X}(\mathcal{T})$  and let*

$$(2.5) \quad T = \sum'_{w_1, w_2 \in \mathcal{F}_q} w_1^* w_2 T_{w_1, w_2}, \quad \text{where } T_{w_1, w_2} \text{ are diagonal,}$$

be the pointwise absolutely convergent expansion of  $T$  as in Theorem 1.3. Then the series

$$T_{[\emptyset]} \stackrel{\text{def}}{=} \sum'_{w_1, w_2 \in \mathcal{F}_q, |w_1|=|w_2|} w_1^* w_2 T_{w_1, w_2}$$

converges pointwise absolutely in  $\mathcal{C}$  and

$$(2.6) \quad \|T_{[\emptyset]}\| \leq \|T\|.$$

*Proof.* First we observe that, since the series (2.5) is pointwise absolutely convergent, the series

$$(T_{[\emptyset]}f)(t) \stackrel{\text{def}}{=} \sum'_{w_1, w_2 \in \mathcal{F}_q, |w_1|=|w_2|} (w_1^* w_2 T_{w_1, w_2} f)(t)$$

converges absolutely for each  $f \in \ell_2(\mathcal{T})$  and  $t \in \mathcal{T}$ .

Now let

$$f = \sum_{i=1}^n f_i \chi_{t_i}, \quad \text{where } f_i \in \mathbb{C},$$

let  $\mathfrak{h}_1, \dots, \mathfrak{h}_k$  be horocycles such that

$$\{t_1, \dots, t_n\} \subset \mathfrak{h}_1 \cup \dots \cup \mathfrak{h}_k$$

and let  $\pi_1, \dots, \pi_k$  denote the corresponding orthogonal projections in  $\ell_2(\mathcal{T})$ :

$$\pi_j = \sum_{t \in \mathfrak{h}_j} \langle \cdot, \chi_t \rangle_{\ell_2(\mathcal{T})} \chi_t.$$

For each  $j$  the relations (1.15) imply

$$(T_{[\emptyset]}\pi_j f)(t) = \begin{cases} (\pi_j T \pi_j f)(t) & \text{if } t \in \mathfrak{h}_j, \\ 0 & \text{otherwise,} \end{cases}$$

hence

$$T_{[\emptyset]}\pi_j f = \pi_j T \pi_j f \in \text{ran}(\pi_j).$$

Since

$$f = \sum_{j=1}^k \pi_j f,$$

we observe that  $T_{[\emptyset]}f \in \ell_2(\mathcal{T})$  and, moreover,

$$\begin{aligned} \|T_{[\emptyset]}f\|_{\ell_2(\mathcal{T})}^2 &= \left\| \sum_{j=1}^k T_{[\emptyset]} \pi_j f \right\|_{\ell_2(\mathcal{T})}^2 = \sum_{j=1}^k \|\pi_j T \pi_j f\|_{\ell_2(\mathcal{T})}^2 \\ &\leq \sum_{j=1}^k \|T\|^2 \|\pi_j f\|_{\ell_2(\mathcal{T})}^2 = \|T\|^2 \|f\|_{\ell_2(\mathcal{T})}^2. \end{aligned}$$

Since  $\text{span}\{\chi_t : t \in \mathcal{T}\}$  is dense in  $\ell_2(\mathcal{T})$ , we conclude that

$$T_{[\emptyset]} \in \mathbf{X}(\mathcal{T}), \quad \|T_{[\emptyset]}\| \leq \|T\|.$$

Finally, in view of Remark 2.4,  $T_{[\emptyset]} \in \mathcal{C}$ . ■

*Proof of Theorem 2.5.* According to Theorem 1.3 and Proposition 2.3,  $S$  admits in the sense of pointwise absolute convergence a representation of the form

$$S = \sum'_{w_1, w_2 \in \mathcal{F}_q, |w_1| \geq |w_2|} w_1^* w_2 S_{w_1, w_2}, \quad \text{where } S_{w_1, w_2} \text{ are diagonal.}$$

Denote

$$S_{[w]} \stackrel{\text{def}}{=} (wS)_{[\emptyset]}, \quad w \in \mathcal{F}_q.$$

Then, as follows from (1.7),  $S_{[w]}$  admits in the sense of pointwise absolute convergence the representation

$$S_{[w]} = \sum'_{w_1, w_2 \in \mathcal{F}_q, |w_1| = |w_2|} w_1^* w_2 S_{w_1, w_2}.$$

Hence  $S$  admits the representation (2.2) and the relations (1.12) in Theorem 1.3 imply (2.3).

Finally, we note that, as follows from Lemma 1.5 and (1.7),

$$S_{[w]} = w \sum_{v \in \mathcal{F}_q, |v|=|w|} v^* S_{[v]} \quad \text{and} \quad (Sw)_{[\emptyset]} = \sum_{v \in \mathcal{F}_q, |v|=|w|} v^* S_{[v]} w.$$

Using the identity  $ww^* = I$  and the inequality (2.6) in Lemma 2.6, we obtain

$$\|S_{[w]}\| \leq \left\| \sum_{v \in \mathcal{F}_q, |v|=|w|} v^* S_{[v]} \right\| = \|(Sw)_{[\emptyset]} w^*\| \leq \|Sw\| \leq \|S\|.$$

Thus the inequality (2.4) holds. ■

REMARK 2.7. Let  $c \in \mathcal{C}$ , let  $S, T \in \mathbf{H}(\mathcal{T})$  and let

$$S = \sum_{w \in \mathcal{F}_q} w^* S_{[w]}, \quad T = \sum_{w \in \mathcal{F}_q} w^* T_{[w]}, \quad \text{where } S_{[w]}, T_{[w]} \in \mathcal{C},$$

be the pointwise absolutely convergent expansions of  $S, T$  as in Theorem 2.5. Then

$$Sc + T = \sum_{w \in \mathcal{F}_q} w^* (S_{[w]}c + T_{[w]}),$$

where convergence is again pointwise absolute.

Note that the coefficients  $S_{[w]} \in \mathcal{C}$  in the expansion (2.2) do not commute, in general, with the shift operators  $w^*$ . The following lemma deals with the special case when  $S$  is of the form  $S = Cw^*$ , where  $C \in \mathcal{C}$ .

LEMMA 2.8. *Let  $w \in \mathcal{F}_q$  and let  $C \in \mathcal{C}$ . Then*

$$(2.7) \quad Cw^* = \sum_{v \in \mathcal{F}_q, |v|=|w|} v^* C_v, \quad \text{where } C_v = vCw^* \in \mathcal{C}.$$

*Proof.* Let  $v \in \mathcal{F}_q$  be such that  $|v| = |w|$  and consider the operator  $C_v = vCw^* \in \mathbf{X}(\mathcal{T})$ . For each  $t \in \mathcal{T}$  the relations (2.1) and (1.15) imply that

$$C_v \chi_t = vC\chi_{tw} \in \overline{\text{span}}\{v\chi_u : u \succ tw\} = \overline{\text{span}}\{\chi_s : sv \succ tw\} \subset \overline{\text{span}}\{\chi_s : s \succ t\}$$

and hence, according to (2.1),  $C_v \in \mathcal{C}$ .

Furthermore, we note that the Cuntz relations (1.7) imply

$$(2.8) \quad \sum_{v \in \mathcal{F}_q, |v|=n} v^* v = I, \quad n = 0, 1, 2, \dots$$

Hence

$$Cw^* = \sum_{v \in \mathcal{F}_q, |v|=|w|} v^* vCw^*$$

and (2.7) follows. ■

### 3. THE POINT EVALUATION OF CAUSAL OPERATORS

In this section we define a point evaluation for the elements of  $\mathbf{H}(\mathcal{T})$  at the "points" from  $\mathcal{C}$ .

We consider the set of  $q$ -tuples

$$(3.1) \quad \mathbb{B}(\mathcal{T}) \stackrel{\text{def}}{=} \left\{ c = (c_1 \ \cdots \ c_q) \in \mathcal{C}^q : \lim_{n \rightarrow \infty} \|(c\alpha)^n\|^{1/n} < 1 \right\},$$

which plays the role of the unit disk in the present setting.

Let  $c \in \mathbb{B}(\mathcal{T})$  and let  $S \in \mathbf{H}(\mathcal{T})$ . Let

$$(3.2) \quad S = \sum_{w \in \mathcal{F}_q} w^* S_{[w]}, \quad \text{where } S_{[w]} \in \mathcal{C},$$

be the pointwise absolutely convergent expansion of  $S$  as in Theorem 2.5. Then, in view of the estimate (2.4), the series

$$(3.3) \quad S^\wedge(c) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (c\alpha)^n \left( \sum_{w \in \mathcal{F}_q, |w|=n} w^* S_{[w]} \right)$$

converges absolutely with respect to the operator norm. It follows from Lemma 2.8 and the Cuntz relations (1.7) that each term of the series (3.3) belongs to the algebra of constants  $\mathcal{C}$ , which is closed in  $\mathbf{X}(\mathcal{T})$  in the pointwise sense. Hence  $S^\wedge(c) \in \mathcal{C}$ .

In this way we associate with each operator  $S \in \mathbf{H}(\mathcal{T})$  a mapping  $c \mapsto S^\wedge(c)$  from  $\mathbb{B}(\mathcal{T})$  to the algebra of constants  $\mathcal{C}$ . We shall refer to this mapping as *the point evaluation of  $S$* . Its main properties are listed in the following lemma.

LEMMA 3.1. (i) Let  $F, G \in \mathbf{H}(\mathcal{T})$ ,  $p \in \mathcal{C}$ ,  $c \in \mathbb{B}(\mathcal{T})$ . Then

$$(3.4) \quad (Fp + G)^\wedge(c) = F^\wedge(c) \cdot p + G^\wedge(c),$$

$$(3.5) \quad (FG)^\wedge(c) = (F^\wedge(c) \cdot G)^\wedge(c).$$

(ii) If  $S \in \mathbf{H}(\mathcal{T})$  and  $S^\wedge(c) = 0$  for every  $c \in \mathbb{B}(\mathcal{T})$ , then  $S = 0$ .

*Proof.* (i) In view of Remark 2.7, the identity (3.4) follows immediately from the definition (3.3) of the point evaluation. Therefore, it suffices to establish the identity (3.5) for  $F, G$  of the form

$$F = w_1^*, \quad G = w_2^*, \quad \text{where } w_1, w_2 \in \mathcal{F}_q.$$

But Lemma 2.8 implies that

$$\begin{aligned} (w_1^* w_2^*)^\wedge(c) &= (c\alpha)^{(|w_1|+|w_2|)} w_1^* w_2^* = (c\alpha)^{|w_2|} (w_1^*)^\wedge(c) w_2^* \\ &= \sum_{|w|=|w_2|} (c\alpha)^{|w|} w^* (w(w_1^*)^\wedge(c) w_2^*) = \sum_{|w|=|w_2|} (w^*)^\wedge(c) (w(w_1^*)^\wedge(c) w_2^*) \\ &= \left( \sum_{|w|=|w_2|} w^* w (w_1^*)^\wedge(c) w_2^* \right)^\wedge(c) = ((w_1^*)^\wedge(c) w_2^*)^\wedge(c). \end{aligned}$$

(ii) We prove that  $S_{[w]} = 0$  for each  $w \in \mathcal{F}_q$ . We use induction on  $|w|$ . First,

$$S_{[\emptyset]} = S^\wedge(0) = 0.$$

Next, assume that

$$S_{[w]} = 0 \quad \forall w : |w| \leq n$$

and let

$$w_{n+1} = \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{n+1}}.$$

We shall prove that  $S_{[w_{n+1}]} = 0$ .

Denote

$$w_k = \alpha_{i_1} \cdots \alpha_{i_k}, \quad 1 \leq k \leq n+1, \quad w_0 = \emptyset.$$

Fix  $t_0 \in \mathcal{T}$  and consider the diagonal operators  $c_j \in \mathcal{C}$ ,  $1 \leq j \leq q$ , defined as follows:

$$c_j \chi_t = \begin{cases} \chi_{t_0 w_k} & \text{if } j = i_{k+1} \text{ and } t = t_0 w_k, \ 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

We set

$$c = (c_1 \quad \cdots \quad c_q)$$

and observe that, as follows from (1.5),

$$(c\alpha)\chi_t = \begin{cases} \chi_{t_0 w_{k+1}} & \text{if } t = t_0 w_{k+1}, \ 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$(c\alpha)^{n+1} \chi_t = \begin{cases} \chi_{t_0} & \text{if } t = t_0 w_{n+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(c\alpha)^{n+2} = 0.$$

In particular,  $c \in \mathbb{B}(\mathcal{T})$ .

Furthermore, in view of (1.15), for  $w \in \mathcal{F}_q$  such that  $|w| = n+1$  we have

$$(c\alpha)^{n+1} w^* \chi_t = (c\alpha)^{n+1} \chi_{tw} = \begin{cases} \chi_{t_0} & \text{if } t = t_0 \text{ and } w = w_{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus by the induction assumption

$$0 = S^\wedge(c) = (c\alpha)^{n+1} w_{n+1}^* S_{[w_{n+1}]},$$

which implies (see (1.15))

$$0 = S_{[w_{n+1}]}^* w_{n+1} (\alpha^* c^*)^{n+1} \chi_{t_0} = S_{[w_{n+1}]}^* w_{n+1} \chi_{t_0 w_{n+1}} = S_{[w_{n+1}]}^* \chi_{t_0}.$$

Since  $t_0 \in \mathcal{T}$  was chosen arbitrarily,  $S_{[w_{n+1}]} = 0$ . ■

#### 4. THE SPACE OF CAUSAL HILBERT-SCHMIDT OPERATORS

In this section we consider the following spaces of Hilbert–Schmidt operators:

$$\mathbf{H}_2(\mathcal{T}) \stackrel{\text{def}}{=} \{F \in \mathbf{H}(\mathcal{T}); \|F\|_2^2 \stackrel{\text{def}}{=} \text{trace}(F^*F) < \infty\},$$

$$\mathcal{C}_2 \stackrel{\text{def}}{=} \mathcal{C} \cap \mathbf{H}_2(\mathcal{T}).$$

As we shall see from Propositions 4.1 and 4.2 below, the space  $\mathbf{H}_2(\mathcal{T})$  of causal Hilbert–Schmidt operators plays the role of the Hardy space of the unit disk in the present setting: elements of the algebra  $\mathbf{H}(\mathcal{T})$  act on the space  $\mathbf{H}_2(\mathcal{T})$  by multiplication. The space  $\mathcal{C}_2$  is the space of constants which appear in the power series expansions (see Theorem 2.5) of the elements of  $\mathbf{H}_2(\mathcal{T})$ .

PROPOSITION 4.1. (i) *The space  $\mathbf{H}_2(\mathcal{T})$  is a Hilbert space contractively included in  $\mathbf{H}(\mathcal{T})$ :*

$$\forall F \in \mathbf{H}_2(\mathcal{T}) \quad \langle F, F \rangle_2 \leq \|F\|^2.$$

(ii)  $\mathcal{C}_2$  *is a closed subspace of the Hilbert space  $\mathbf{H}_2(\mathcal{T})$ .*

(iii) *For every  $F \in \mathbf{H}_2(\mathcal{T})$  and  $S \in \mathbf{H}(\mathcal{T})$  the operators  $SF$  and  $FS$  belong to  $\mathbf{H}_2(\mathcal{T})$  and it holds that*

$$(4.1) \quad \max(\|SF\|_2, \|FS\|_2) \leq \|S\| \|F\|_2.$$

Moreover, the multiplication operators  $M_S, \widehat{M}_S$  defined on  $\mathbf{H}_2(\mathcal{T})$  by

$$(4.2) \quad M_S F \stackrel{\text{def}}{=} SF, \quad \widehat{M}_S F \stackrel{\text{def}}{=} FS, \quad F \in \mathbf{H}_2(\mathcal{T}),$$

satisfy

$$\|M_S\| = \|\widehat{M}_S\| = \|S\|.$$

*Proof.* It is well known (see, for instance, [23]) that the space of Hilbert–Schmidt operators on a given separable Hilbert space is a Hilbert space. In particular, the space

$$\mathbf{X}_2(\mathcal{T}) \stackrel{\text{def}}{=} \{F \in \mathbf{X}(\mathcal{T}); \|F\|_2^2 \stackrel{\text{def}}{=} \text{trace}(F^*F) < \infty\}$$

is a Hilbert space. For every  $F \in \mathbf{X}_2(\mathcal{T})$  and  $f \in \ell_2(\mathcal{T})$  it holds that

$$\begin{aligned} \|Ff\|_{\ell_2(\mathcal{T})}^2 &= \sum_{t \in \mathcal{T}} |Ff(t)|^2 = \sum_{t \in \mathcal{T}} \left| \sum_{s \in \mathcal{T}} F\chi_s(t) \cdot f(s) \right|^2 \\ &\leq \sum_{t, s \in \mathcal{T}} |F\chi_s(t)|^2 \|f\|_{\ell_2(\mathcal{T})}^2 = \|F\|_2^2 \|f\|_{\ell_2(\mathcal{T})}^2, \end{aligned}$$

hence the space  $\mathbf{X}_2(\mathcal{T})$  is contractively included in  $\mathbf{X}(\mathcal{T})$ .

The space  $\mathbf{H}_2(\mathcal{T})$  is the intersection

$$(4.3) \quad \mathbf{H}_2(\mathcal{T}) = \mathbf{X}_2(\mathcal{T}) \cap \mathbf{H}(\mathcal{T}).$$

Since the algebra  $\mathbf{H}(\mathcal{T})$  is closed in  $\mathbf{X}(\mathcal{T})$  in the pointwise sense, it is also closed with respect to the operator norm. It follows that  $\mathbf{H}_2(\mathcal{T})$  is a closed (with respect to the Hilbert–Schmidt norm  $\|\cdot\|_2$ ) subspace of the Hilbert space  $\mathbf{X}_2(\mathcal{T})$ , which proves the statement (i).

The proof of the statement (ii) is analogous: it uses the fact that the algebra of constants  $\mathcal{C}$  is closed in  $\mathbf{H}(\mathcal{T})$  in the pointwise sense and hence also with respect to the operator norm.

In order to prove the statement (iii), we observe that for every  $S \in \mathbf{X}(\mathcal{T})$  and  $F \in \mathbf{X}_2(\mathcal{T})$  it holds that

$$\begin{aligned} \|SF\|_2^2 &= \sum_{t \in \mathcal{T}} \|SF\chi_t\|_{\ell_2(\mathcal{T})}^2 \leq \|S\|^2 \sum_{t \in \mathcal{T}} \|F\chi_t\|_{\ell_2(\mathcal{T})}^2 = \|S\|^2 \|F\|_2^2; \\ \|FS\|_2^2 &= \|S^*F^*\|_2^2 \leq \|S^*\|^2 \|F^*\|_2^2 = \|S\|^2 \|F\|_2^2. \end{aligned}$$

In particular, taking into account (4.3) and the fact that  $\mathbf{H}(\mathcal{T})$  is an algebra, we may conclude that

$$\forall S \in \mathbf{H}(\mathcal{T}), \forall F \in \mathbf{H}_2(\mathcal{T}) \quad SF, FS \in \mathbf{H}_2(\mathcal{T})$$

and (4.1) holds.

Let  $f \in \text{span}\{\chi_t\}$  and choose  $t_0 \in \mathcal{T}$  such that  $f(t) = 0 \forall t \preceq t_0$ . Consider the operator  $F$  defined by

$$(4.4) \quad Fu = u(t_0) \cdot f, \quad u \in \ell_2(\mathcal{T}).$$

Then

$$\|F\|_2^2 = \sum_{t \in \mathcal{T}} \|F\chi_t\|_{\ell_2(\mathcal{T})}^2 = \|f\|_{\ell_2(\mathcal{T})}^2,$$

hence  $F \in \mathbf{X}_2(\mathcal{T})$ . According to Definition (2.1), the operator  $F$  is causal, hence  $F \in \mathbf{H}_2(\mathcal{T})$ .

Furthermore, let  $S \in \mathbf{H}(\mathcal{T})$ . Then

$$\|SF\|_2^2 = \sum_{t \in \mathcal{T}} \|SF\chi_t\|_{\ell_2(\mathcal{T})}^2 = \|Sf\|_{\ell_2(\mathcal{T})}^2.$$

Since  $\text{span}\{\chi_t\}$  is dense in  $\ell_2(\mathcal{T})$ , we conclude that the left multiplication operator  $M_S$  satisfies  $\|M_S\| \geq \|S\|$ . On the other hand, (4.1) implies  $\|M_S\| \leq \|S\|$ , hence  $\|M_S\| = \|S\|$ .

The proof of the equality  $\|\widehat{M}_S\| = \|S\|$  for the right multiplication operator  $\widehat{M}_S$  is analogous. ■

PROPOSITION 4.2. *Let  $F \in \mathbf{H}(\mathcal{T})$  and let*

$$(4.5) \quad F = \sum_{w \in \mathcal{F}_q} w^* F_{[w]},$$

where  $F_{[w]} \in \mathcal{C}$ , be the pointwise absolutely convergent expansion for  $F$ , as in Theorem 2.5. Then  $F \in \mathbf{H}_2(\mathcal{T})$  if and only if

$$(4.6) \quad \forall w \in \mathcal{F}_q \quad F_{[w]} \in \mathcal{C}_2 \quad \text{and} \quad \sum_{w \in \mathcal{F}_q} \|F_{[w]}\|_2^2 < \infty.$$

In this case the expansion (4.5) converges in the  $\mathbf{H}_2(\mathcal{T})$ -norm and

$$(4.7) \quad \|F\|_2^2 = \sum_{w \in \mathcal{F}_q} \|F_{[w]}\|_2^2.$$

*Proof.* ( $\Rightarrow$ ) Assume that  $F \in \mathbf{H}_2(\mathcal{T})$ . Then, as follows from the relations (2.3) in Theorem 2.5 and the statement (iii) of Proposition 4.1, we have

$$\begin{aligned} \|F\|_2^2 &= \|w\| \|F\|_2^2 \geq \|wF\|_2^2 = \sum_{t,s \in \mathcal{T}} |(wF\chi_t)(s)|^2 \\ &\geq \sum_{\substack{t,s \in \mathcal{T} \\ t \succ s}} |(wF\chi_t)(s)|^2 = \sum_{t,s \in \mathcal{T}} |(F_{[w]}\chi_t)(s)|^2 = \|F_{[w]}\|_2^2, \end{aligned}$$

hence  $F_{[w]} \in \mathcal{C}_2$ .

Furthermore, since  $F$  is causal,

$$(4.8) \quad \begin{aligned} \|F\|_2^2 &= \sum_{s,t \in \mathcal{T}, s \leq t} |(F\chi_s)(t)|^2 = \sum_{s,t \in \mathcal{T}, s \geq t} \sum_{w \in \mathcal{F}_q} |(F\chi_s)(tw)|^2 \\ &= \sum_{s,t \in \mathcal{T}, s \geq t} \sum_{w \in \mathcal{F}_q} |(F_{[w]}\chi_s)(t)|^2 = \sum_{w \in \mathcal{F}_q} \|F_{[w]}\|_2^2. \end{aligned}$$

Thus (4.7) holds.

As a consequence, we obtain the convergence of the expansion (4.5) in the following sense: if we order somehow the countable set  $\mathcal{F}_q$ , say

$$\mathcal{F}_q = \{w_j\}_{j=0}^\infty,$$

then, according to (4.6) and the statement (iii) of Proposition 4.1, for each  $n = 0, 1, 2, \dots$  the finite sum  $\sum_{j=0}^n w_j^* F_{[w_j]}$  belongs to  $\mathbf{H}_2(\mathcal{T})$  and it holds that

$$\left\| F - \sum_{j=0}^n w_j^* F_{[w_j]} \right\|_2^2 = \sum_{j=n+1}^\infty \|F_{[w_j]}\|_2^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

( $\Leftarrow$ ) Assume that  $F \in \mathbf{H}(\mathcal{T})$  and that the coefficients  $F_{[w]}$  of the expansion (4.5) satisfy the conditions (4.6). Then it suffices to reverse the computation (4.8) in order to show that  $F \in \mathbf{H}_2(\mathcal{T})$ . ■

Our next goal is to demonstrate that the space  $\mathbf{H}_2(\mathcal{T})$  has a reproducing kernel structure with respect to the point evaluation defined in the previous section (see (3.3)).

LEMMA 4.3. *Let  $F \in \mathbf{H}_2(\mathcal{T})$  have the expansion (4.5), where  $F_{[w]} \in \mathcal{C}_2$ , and let  $c \in \mathbb{B}(\mathcal{T})$ . Then the following series converges absolutely in  $\mathcal{C}_2$ :*

$$(4.9) \quad F^\wedge(c) = \sum_{n=0}^\infty (c\alpha)^n \left( \sum_{w \in \mathcal{F}_q, |w|=n} w^* F_{[w]} \right).$$

*Proof.* It follows from the identity (4.7) in Proposition 4.2 that

$$\left\| \sum_{w \in \mathcal{F}_q, |w|=n} w^* F_{[w]} \right\|_2 \leq \|F\|_2, \quad n = 0, 1, 2, \dots$$

Hence, in view of the definition (3.1) of  $\mathbb{B}(\mathcal{T})$  and the estimate (4.1) in Proposition 4.1, the series (4.9) converges absolutely in  $\mathbf{H}_2(\mathcal{T})$ . Since each term of the series belongs to  $\mathcal{C}$ , the desired conclusion follows. ■

THEOREM 4.4. *Let  $c \in \mathbb{B}(\mathcal{T})$ . Then the operator  $I - \alpha^* c^*$  is invertible in  $\mathbf{H}(\mathcal{T})$  and its inverse*

$$(4.10) \quad K_\wedge^c \stackrel{\text{def}}{=} (I - \alpha^* c^*)^{-1}$$

satisfies

$$(4.11) \quad \langle F^\wedge(c), k \rangle_2 = \langle F, K_\wedge^c k \rangle_2, \quad \forall F \in \mathbf{H}_2(\mathcal{T}), \forall k \in \mathcal{C}_2.$$

*Proof.* In view of the definition (3.1) of  $\mathbb{B}(\mathcal{T})$ ,  $K_\wedge^c$  is the sum of the absolutely convergent series

$$(4.12) \quad K_\wedge^c = \sum_{n=0}^{\infty} (\alpha^* c^*)^n.$$

Since each term of the series belongs to  $\mathbf{H}(\mathcal{T})$ , so does  $K_\wedge^c$ . Let us now choose and fix an element  $k \in \mathcal{C}_2$  and an element  $F \in \mathbf{H}_2(\mathcal{T})$  with the expansion (4.5), where  $F_{[w]} \in \mathcal{C}_2$ . Then, according to the statement (iii) of Proposition 4.1,

$$K_\wedge^c k = \sum_{n=0}^{\infty} (\alpha^* c^*)^n k \in \mathbf{H}_2(\mathcal{T}),$$

where convergence is absolute with respect to the  $\mathbf{H}_2(\mathcal{T})$ -norm. Since, in view of the identity (2.8),

$$(4.13) \quad (\alpha^* c^*)^n = \sum_{w \in \mathcal{F}_q, |w|=n} w^* w (\alpha^* c^*)^n = \sum_{w \in \mathcal{F}_q, |w|=n} w^* (w^*)^\wedge (c^*)^n,$$

Proposition 4.2 and Lemma 4.3 imply that

$$\begin{aligned} \langle F, K_\wedge^c k \rangle_2 &= \sum_{n=0}^{\infty} \langle F, (\alpha^* c^*)^n k \rangle_2 = \sum_{n=0}^{\infty} \left\langle \sum_{w \in \mathcal{F}_q, |w|=n} w^* F_{[w]}, (\alpha^* c^*)^n k \right\rangle_2 \\ &= \sum_{n=0}^{\infty} \left\langle (c\alpha)^n \sum_{w \in \mathcal{F}_q, |w|=n} w^* F_{[w]}, k \right\rangle_2 = \langle F^\wedge(c), k \rangle_2. \quad \blacksquare \end{aligned}$$

REMARK 4.5. Note that Theorem 4.4 and the statement (ii) of Lemma 3.1 imply that

$$\overline{\text{span}}\{K_\wedge^c k : c \in \mathbb{B}(\mathcal{T}), k \in \mathcal{C}_2\} = \mathbf{H}_2(\mathcal{T}).$$

We close this section with the description of the counterparts of the backward shift operator  $R_0$  in the stationary single-scale setting (see the formula (0.3) in Introduction).

PROPOSITION 4.6. *Let the operators  $A_j : \mathbf{H}_2(\mathcal{T}) \rightarrow \mathbf{H}_2(\mathcal{T})$ ,  $j = 1, \dots, q$ , be defined by*

$$(4.14) \quad A_j F \stackrel{\text{def}}{=} (F - F^\wedge(0))\alpha_j, \quad F \in \mathbf{H}_2(\mathcal{T}), \quad j = 1, \dots, q.$$

Then:

(i) *The operator  $A_j$  is the adjoint of the right multiplication operator  $\widehat{M}_{\alpha_j^*}$ :*

$$A_j = \widehat{M}_{\alpha_j^*}^*, \quad j = 1, \dots, q.$$

(ii) *For  $i, j = 1, \dots, q$  the following relations hold:*

$$A_j \widehat{M}_{\alpha_i^*} = \widehat{M}_{\alpha_i^*} A_j, \quad \widehat{M}_{\alpha_i^*} A_j = \begin{cases} (I - C^*C) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$(4.15) \quad CF \stackrel{\text{def}}{=} F^\wedge(0).$$

(iii) Let  $F \in \mathbf{H}_2(\mathcal{T})$  with the expansion (4.5), where  $F_{[w]} \in \mathcal{C}_2$ , be given. Then for every pair  $w, v \in \mathcal{F}_q$  such that  $|v| = |w|$  it holds that

$$F_{[w]} = w \cdot (CA^v F) \cdot v^*,$$

where

$$(4.16) \quad A^{\alpha_{i_1} \cdots \alpha_{i_k}} \stackrel{\text{def}}{=} A_{i_k} \cdots A_{i_1}, \quad A^\emptyset \stackrel{\text{def}}{=} I.$$

*Proof.* Consider  $F \in \mathbf{H}_2(\mathcal{T})$  with the expansion (4.5), where  $F_{[w]} \in \mathcal{C}_2$ . Then, in view of Proposition 4.2,

$$A_j F = (F - F_{[\emptyset]})\alpha_j = \sum_{w \in \mathcal{F}_q, |w| \geq 1} w^* F_{[w]} \alpha_j \in \mathbf{H}_2(\mathcal{T})$$

and for every  $G \in \mathbf{H}_2(\mathcal{T})$  it holds that

$$\begin{aligned} \langle F, G\alpha_j^* \rangle_2 &= \langle F - F_{[\emptyset]}, G\alpha_j^* \rangle_2 = \text{trace}(\alpha_j G^*(F - F_{[\emptyset]})) \\ &= \text{trace}(G^*(F - F_{[\emptyset]})\alpha_j) = \langle A_j F, G \rangle_2. \end{aligned}$$

This proves the statement (i) of the proposition. The statement (ii) follows immediately from the Cuntz relations (1.7).

In order to prove the statement (iii), let us fix a sequence of indices  $i_1, i_2, \dots$ , such that  $1 \leq i_n \leq q$ . Then

$$\begin{aligned} F &= F_{[\emptyset]} + (F - F_{[\emptyset]}) = CF + (A_{i_1} F)\alpha_{i_1}^* \\ &= CF + (CA_{i_1} F)\alpha_{i_1}^* + (A_{i_2} A_{i_1} F)\alpha_{i_2}^* \alpha_{i_1}^* = \cdots \\ &= \sum_{n=0}^m (CA^{v_n} F)v_n^* + (A^{v_{m+1}} F)v_{m+1}^*, \quad m = 0, 1, 2, \dots \end{aligned}$$

where  $v_0 = \emptyset$ ,  $v_n = \alpha_{i_1} \cdots \alpha_{i_n}$ . Now it follows from the identity (2.3) in Theorem 2.5 that

$$\sum_{|w|=n} w^* F_{[w]} d = (CA^{v_n} F)v_n^* \quad \forall n$$

and hence

$$F_{[w]} = w(CA^{v_{|w|}} F)v_{|w|}^* \quad \forall w.$$

Since the sequence of indices  $i_1, i_2, i_3, \dots$  was chosen arbitrarily, this completes the proof. ■

## 5. SCHUR MULTIPLIERS AND DE BRANGES–ROVNYAK SPACES

DEFINITION 5.1. Let  $S \in \mathbf{H}(\mathcal{T})$  be such that  $\|S\| \leq 1$ . Then  $S$  is said to be a *Schur multiplier*.

THEOREM 5.2. Let a mapping  $s : \mathbb{B}(\mathcal{T}) \mapsto \mathcal{C}$  be given. Then there exists a Schur multiplier  $S \in \mathbf{H}(\mathcal{T})$  such that

$$s(c) = S^\wedge(c) \quad \forall c \in \mathbb{B}(\mathcal{T})$$

if and only if the kernel  $K_s : \mathbb{B}(\mathcal{T}) \times \mathbb{B}(\mathcal{T}) \rightarrow \mathcal{C}$  defined by

$$(5.1) \quad K_s(c, d) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (c\alpha)^n (I - s(c)s(d)^*)(d\alpha)^{n*}, \quad c, d \in \mathbb{B}(\mathcal{T}),$$

is positive: for any  $m \geq 0$ ,  $c_0, \dots, c_m \in \mathbb{B}(\mathcal{T})$ ,  $k_0, \dots, k_m \in \mathcal{C}_2$ , it holds that

$$\sum_{i,j=0}^m \langle K_s(c_i, c_j)k_j, k_i \rangle_2 \geq 0.$$

In this case  $K_s(c, d) = (K_S^d)^\wedge(c)$ , where

$$(5.2) \quad K_S^d \stackrel{\text{def}}{=} (I - SS^\wedge(d)^*)K_\wedge^d, \quad d \in \mathbb{B}(\mathcal{T}).$$

In the proof of Theorem 5.2 we shall use the following lemma:

LEMMA 5.3. (i) Let  $T : \mathbf{H}_2(\mathcal{T}) \rightarrow \mathbf{H}_2(\mathcal{T})$  be a bounded linear operator. Then  $T$  is of the form  $T = M_S$  for some  $S \in \mathbf{H}(\mathcal{T})$  if and only if

$$\forall Q \in \mathbf{H}(\mathcal{T}) \quad T\widehat{M}_Q = \widehat{M}_QT.$$

(ii) Let  $T : \mathcal{C}_2 \rightarrow \mathcal{C}_2$  be a bounded linear operator. Then  $T$  is of the form  $T = M_c$  for some  $c \in \mathcal{C}$  if and only if

$$\forall d \in \mathcal{C} \quad T\widehat{M}_d = \widehat{M}_dT.$$

*Proof.* We shall prove only the statement (i) of the proposition; the proof of the statement (ii) is completely analogous.

( $\Rightarrow$ ). The "only if" direction is clear: for every  $S, Q \in \mathbf{H}(\mathcal{T})$  and  $F \in \mathbf{H}_2(\mathcal{T})$  it holds that

$$M_S\widehat{M}_QF = SFQ = \widehat{M}_QM_SF.$$

( $\Leftarrow$ ). Assume that an operator  $T$ , which commutes with every  $\widehat{M}_Q$ , is given. For each  $t \in \mathcal{T}$  let us consider the projection  $\pi_t \in \mathcal{C}_2$ ,

$$\pi_t u \stackrel{\text{def}}{=} u(t)\chi_t, \quad u \in \ell_2(\mathcal{T}),$$

and define an operator  $S$  on  $\text{span}\{\chi_t\}$  by

$$S\chi_t = (T\pi_t)\chi_t.$$

Let  $f \in \text{span}\{\chi_t\}$  and choose  $t_0 \in \mathcal{T}$  such that  $f(t) = 0 \forall t \preceq t_0$ . Consider  $F \in \mathbf{H}_2(\mathcal{T})$  defined by

$$Fu = u(t_0) \cdot f, \quad u \in \ell_2(\mathcal{T}),$$

as in the proof of Proposition 4.1 (see (4.4)). Then

$$F = \sum_{t \in \mathcal{T}} f(t) \cdot F_t^{t_0}, \quad \text{where } F_t^{t_0} u = u(t_0) \chi_t, \quad u \in \ell_2(\mathcal{T}).$$

Note that  $F_t^{t_0} = \pi_t F_t^{t_0} = F_t^{t_0} \pi_{t_0} \in \mathbf{H}_2(\mathcal{T})$ , hence for every  $u \in \ell_2(\mathcal{T})$  we have

$$\begin{aligned} (TF)u &= u(t_0) \sum_{t \in \mathcal{T}} f(t) (TF_t^{t_0}) \chi_{t_0} = u(t_0) \sum_{t \in \mathcal{T}} f(t) (T\pi_t) F_t^{t_0} \chi_{t_0} \\ &= u(t_0) \sum_{t \in \mathcal{T}} f(t) (T\pi_t) \chi_t = u(t_0) S f = S F u. \end{aligned}$$

It follows that

$$\|Sf\|_{\ell_2(\mathcal{T})} \leq \|TF\| \leq \|TF\|_2 \leq \|T\| \cdot \|F\|_2 = \|T\| \cdot \|f\|_{\ell_2(\mathcal{T})}$$

and  $(Sf)(t) = 0 \forall t \preceq t_0$ . Since

$$\overline{\text{span}}\{\chi_t : t \in \mathcal{T}\} = \ell_2(\mathcal{T}) \quad \text{and} \quad \overline{\text{span}}\{F_t^s : t, s \in \mathcal{T}, s \preceq t\} = \mathbf{H}_2(\mathcal{T}),$$

we conclude that

$$S \in \mathbf{H}(\mathcal{T}) \quad \text{and} \quad T = M_S. \quad \blacksquare$$

*Proof of Theorem 5.2.* Let us assume first that  $s(c) = S^\wedge(c)$ , where  $S$  is a Schur multiplier. According to Theorem 4.4, for  $k \in \mathbf{C}_2$  and  $F \in \mathbf{H}_2(\mathcal{T})$  we have

$$\langle K_\wedge^d k, SF \rangle_2 = \langle K_\wedge^d k, S^\wedge(d)F \rangle_2 = \langle S^\wedge(d)^* K_\wedge^d k, F \rangle_2,$$

hence

$$(5.3) \quad M_S^*(K_\wedge^d k) = S^\wedge(d)^* K_\wedge^d k,$$

$$(5.4) \quad K_S^d k = (I - M_S M_S^*)(K_\wedge^d k).$$

Furthermore, it follows from the identities (4.12) and (4.13), the definition (3.3) of the point evaluation and the statement (i) in Lemma 3.1 that

$$\begin{aligned} (K_S^d)^\wedge(c) &= (K_\wedge^d)^\wedge(c) - (s(c)s(d)^* K_\wedge^d)^\wedge(c) \\ &= \sum_{n=0}^{\infty} (c\alpha)^n (I - s(c)s(d)^*) (\alpha^* d^*)^n = K_s(c, d). \end{aligned}$$

Now, given  $c_0, \dots, c_m \in \mathbb{B}(\mathcal{T})$ ,  $k_0, \dots, k_m \in \mathbf{C}_2$ , we observe that

$$\sum_{i,j=0}^m \langle K_s(c_i, c_j) k_j, k_i \rangle_2 = \left\langle (I - M_S M_S^*) \left( \sum_{j=0}^m K_\wedge^{c_j} k_j \right), \sum_{i=0}^m K_\wedge^{c_i} k_i \right\rangle_2 \geq 0,$$

because the operator  $I - M_S M_S^*$  is positive. Thus the kernel  $K_s(c, d)$  is positive.

Conversely, assume that the kernel  $K_s(c, d)$  is positive and define on

$$\text{span}\{K_\wedge^d k : d \in \mathbb{B}(\mathcal{T}), k \in \mathbf{C}_2\}$$

an operator  $T$  by

$$T(K_\wedge^d k) = s(d)^* K_\wedge^d k.$$

Then  $T$  is a well-defined contraction, because

$$\left\| T \sum_{j=0}^m K_{\wedge}^{c_j} k_j \right\|_2^2 = \left\| \sum_{j=0}^m K_{\wedge}^{c_j} k_j \right\|_2^2 - \sum_{i,j=0}^m \langle K_s(c_i, c_j) k_j, k_i \rangle_2 \leq \left\| \sum_{j=0}^m K_{\wedge}^{c_j} k_j \right\|_2^2.$$

Hence, in view of Remark 4.5,  $T$  can be extended as a contraction on  $\mathbf{H}_2(\mathcal{T})$ .

The adjoint operator has the property

$$(5.5) \quad (T^*F)^\wedge(c) = (s(c)F)^\wedge(c) \quad \forall F \in \mathbf{H}_2(\mathcal{T}), \forall c \in \mathbb{B}(\mathcal{T}).$$

In view of the statement (i) in Lemma 3.1, for every  $Q \in \mathbf{H}(\mathcal{T})$ ,  $F \in \mathbf{H}_2(\mathcal{T})$ ,  $c \in \mathbb{B}(\mathcal{T})$  we have

$$\begin{aligned} (T^* \widehat{M}_Q F)^\wedge(c) &= (s(c)FQ)^\wedge(c) = ((s(c)F)^\wedge(c)Q)^\wedge(c) \\ &= ((T^*F)^\wedge(c)Q)^\wedge(c) = (\widehat{M}_Q T^*F)^\wedge(c), \end{aligned}$$

which, according to the statement (ii) of the same Lemma 3.1, implies

$$T^* \widehat{M}_Q = \widehat{M}_Q T^* \quad \forall Q \in \mathbf{H}(\mathcal{T}).$$

Now it follows from Lemma 5.3 that there exists a Schur multiplier  $S$  such that  $T^* = M_S$ . In view of (5.5) and Lemma 3.1, this Schur multiplier  $S$  satisfies

$$S^\wedge(c) = s(c) \quad \forall c \in \mathbb{B}(\mathcal{T}). \quad \blacksquare$$

We recognize in the kernel (5.1) an analogue of the kernel (0.2), mentioned in Introduction. As in the single-scale case, we consider the associated de Branges–Rovnyak space defined below.

DEFINITION 5.4. Let  $S \in \mathbf{H}(\mathcal{T})$  be a Schur multiplier, let

$$\mathcal{B}_S \stackrel{\text{def}}{=} I - M_S M_S^*$$

and let  $\pi_S$  denote the orthogonal projection in  $\mathbf{H}_2(\mathcal{T})$  onto  $\ker \mathcal{B}_S$ . The Hilbert space  $\mathbf{H}(S)$ , defined by

$$\mathbf{H}(S) = \sqrt{\mathcal{B}_S} \mathbf{H}_2(\mathcal{T}); \quad \|\sqrt{\mathcal{B}_S} F\|_{\mathbf{H}(S)} = \|(I - \pi_S)F\|_{\mathbf{H}_2(\mathcal{T})},$$

is said to be the *de Branges–Rovnyak space* associated with  $S$ .

In the sequel we shall use the following terminology:

DEFINITION 5.5. Let  $\mathbf{H}$  be a Hilbert space of elements of  $\mathbf{H}_2(\mathcal{T})$ . The space  $\mathbf{H}$  is said to be *right  $\mathcal{C}$ -invariant* if for every  $F, G \in \mathbf{H}$  and  $c \in \mathcal{C}$  it holds that

$$Fc \in \mathbf{H}, \quad \|Fc\|_{\mathbf{H}} \leq \|F\|_{\mathbf{H}} \|c\|, \quad \langle Fc, G \rangle_{\mathbf{H}} = \langle F, Gc^* \rangle_{\mathbf{H}}.$$

PROPOSITION 5.6. Let  $S \in \mathbf{H}(\mathcal{T})$  be a Schur multiplier. Then the de Branges–Rovnyak space  $\mathbf{H}(S)$ , associated with  $S$ , is a right  $\mathcal{C}$ -invariant Hilbert space.

Furthermore, for every  $F \in \mathbf{H}(S)$ ,  $k \in \mathcal{C}_2$ ,  $c \in \mathbb{B}(\mathcal{T})$  it holds that

$$(5.6) \quad K_S^c k \in \mathbf{H}(S) \quad \text{and} \quad \langle F, K_S^c k \rangle_{\mathbf{H}(S)} = \langle F^\wedge(c), k \rangle_2,$$

where  $K_S^c$  is as in (5.2). In particular,

$$(5.7) \quad \mathbf{H}(S) = \overline{\text{span}}\{K_S^c k : k \in \mathcal{C}_2, c \in \mathbb{B}(\mathcal{T})\}.$$

*Proof.* Let  $c \in \mathcal{C}$ . Since the adjoint of the right multiplication operator  $\widehat{M}_c$  in  $\mathbf{H}_2(\mathcal{T})$  is given by  $\widehat{M}_c^* = \widehat{M}_{c^*}$ , Lemma 5.3 implies that the operators  $\widehat{M}_c$  and  $\mathcal{B}_S$  commute. Hence, in view of Definitions 5.4, 5.5 and the statement (iii) of Proposition 4.1, the space  $\mathbf{H}(S)$  is right  $\mathcal{C}$ -invariant.

Furthermore, as follows from (5.4) and Definition 5.4, for every  $k \in \mathcal{C}_2$   $K_S^c k \in \mathbf{H}(S)$ . Moreover, for every  $F \in \mathbf{H}(S)$  we have

$$\langle F, K_S^c k \rangle_{\mathbf{H}(S)} = \langle F, K_\wedge^c k \rangle_2.$$

Thus the identity (4.11) in Theorem 4.4 implies (5.6) and the statement (ii) in Lemma 3.1 implies (5.7). ■

**THEOREM 5.7.** *Let  $S \in \mathbf{H}(\mathcal{T})$  be a Schur multiplier and let  $\mathbf{H}(S)$  be the associated de Branges–Rovnyak space. Set*

$$(5.8) \quad \begin{aligned} A_j F &= (F - F^\wedge(0))\alpha_j, & B_j d &= (S - S^\wedge(0))d\alpha_j, \\ C F &= F^\wedge(0), & D d &= S^\wedge(0)d, \end{aligned}$$

where  $1 \leq j \leq q$ ,  $F \in \mathbf{H}(S)$ ,  $d \in \mathcal{C}_2$ . Then the following statements hold true:

(i) *The formulae (5.8) define a bounded linear operator*

$$V_j = \begin{pmatrix} A_j & B_j \\ C & D \end{pmatrix} : \begin{pmatrix} \mathbf{H}(S) \\ \mathcal{C}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{H}(S) \\ \mathcal{C}_2 \end{pmatrix},$$

which satisfies

$$(5.9) \quad V_j V_j^* = \begin{pmatrix} \widehat{M}_{\alpha_j^* \alpha_j} & 0 \\ 0 & I \end{pmatrix}.$$

In particular, the space  $\mathbf{H}(S)$  is  $A_j$ -invariant for  $j = 1, \dots, q$ .

(ii) *The operators  $A_j, B_j, C, D$  satisfy the relations*

$$(5.10) \quad A_\ell F = (A_j F)\alpha_j^* \alpha_\ell, \quad B_\ell d = (B_j d)\alpha_j^* \alpha_\ell,$$

$$(5.11) \quad A_j(Fc) = (A_j F)\alpha_j^* c\alpha_j, \quad B_j(dc) = (B_j d)\alpha_j^* c\alpha_j,$$

$$(5.12) \quad C(Fc) = (CF)c, \quad D(dc) = (Dd)c$$

for every  $F \in \mathbf{H}(S)$ ,  $c \in \mathcal{C}$ ,  $d \in \mathcal{C}_2$ ,  $1 \leq j, \ell \leq q$ .

(iii) *Let*

$$(5.13) \quad S = \sum_{w \in \mathcal{F}_q} w^* S_{[w]}, \quad \text{where } S_{[w]} \in \mathcal{C}$$

be the pointwise absolutely convergent expansion of  $S$  as in Theorem 2.5. Then for every  $d \in \mathcal{C}_2$  it holds that

$$(5.14) \quad S_{[w]} d = \begin{cases} Dd & \text{if } w = \emptyset, \\ w(CA^v B_j d)v^* \alpha_j^* & \forall w, v \in \mathcal{F}_q : |w| = |v| + 1, \forall j : 1 \leq j \leq q, \end{cases}$$

where

$$(5.15) \quad A^{\alpha_{i_1} \cdots \alpha_{i_k}} \stackrel{\text{def}}{=} A_{i_k} \cdots A_{i_1}, \quad A^\emptyset \stackrel{\text{def}}{=} I.$$

*Proof.* (i) By the idea of Theorem 2.3 in [4], we define a linear operator

$$\widehat{V}_j : \text{span} \left\{ \begin{pmatrix} K_S^c d \\ e \end{pmatrix} : c \in \mathbb{B}(\mathcal{T}), d, e \in \mathcal{C}_2 \right\} \longrightarrow \mathbf{H}(S) \oplus \mathcal{C}_2$$

by

$$(5.16) \quad \widehat{V}_j \begin{pmatrix} K_S^c d \\ e \end{pmatrix} = \begin{pmatrix} \mathcal{B}_S \\ CM_S^* \end{pmatrix} (K_\wedge^c d \alpha_j^* + e).$$

We claim that the operator  $\widehat{V}_j$  is well-defined and contractive; moreover, for every  $F_1, F_2 \in \text{span} \left\{ \begin{pmatrix} K_S^c d \\ e \end{pmatrix} : c \in \mathbb{B}(\mathcal{T}), d, e \in \mathcal{C}_2 \right\}$

$$(5.17) \quad \langle \widehat{V}_j F_1, \widehat{V}_j F_2 \rangle_{\mathbf{H}(S) \oplus \mathcal{C}_2} = \left\langle \begin{pmatrix} \widehat{M}_{\alpha_j^* \alpha_j} & 0 \\ 0 & I \end{pmatrix} F_1, F_2 \right\rangle_{\mathbf{H}(S) \oplus \mathcal{C}_2}.$$

Indeed, denote for the moment by  $C^*$  the adjoint of  $C$  in  $\mathbf{H}_2(\mathcal{T})$  (that is, the injection of  $\mathcal{C}_2$  into  $\mathbf{H}_2(\mathcal{T})$ ). Then, in view of Definition 5.4 and the statements (i) and (ii) in Proposition 4.6, we obtain

$$\begin{aligned} & \left\langle \widehat{V}_j \begin{pmatrix} K_S^{c_1} d_1 \\ e_1 \end{pmatrix}, \widehat{V}_j \begin{pmatrix} K_S^{c_2} d_2 \\ e_2 \end{pmatrix} \right\rangle_{\mathcal{H}(S) \oplus \mathcal{C}_2} \\ &= \left\langle \begin{pmatrix} \mathcal{B}_S \\ CM_S^* \end{pmatrix} (K_\wedge^{c_1} d_1 \alpha_j^* + e_1), \begin{pmatrix} \mathcal{B}_S \\ CM_S^* \end{pmatrix} (K_\wedge^{c_2} d_2 \alpha_j^* + e_2) \right\rangle_{\mathcal{H}(S) \oplus \mathcal{C}_2} \\ &= \langle (\mathcal{B}_S + M_S C^* C M_S^*) (K_\wedge^{c_1} d_1 \alpha_j^* + e_1), K_\wedge^{c_2} d_2 \alpha_j^* + e_2 \rangle_2 \\ &= \langle (I - M_S \widehat{M}_{\alpha_j^*} A_j M_S^*) (K_\wedge^{c_1} d_1 \alpha_j^* + e_1), K_\wedge^{c_2} d_2 \alpha_j^* + e_2 \rangle_2 \\ &= \langle (I - \widehat{M}_{\alpha_j^*} M_S M_S^* A_j) (K_\wedge^{c_1} d_1 \alpha_j^* + e_1), K_\wedge^{c_2} d_2 \alpha_j^* + e_2 \rangle_2 \\ &= \langle A_j \widehat{M}_{\alpha_j^*} K_\wedge^{c_1} d_1, K_\wedge^{c_2} d_2 \rangle_2 + \langle e_1, e_2 \rangle_2 - \langle A_j \widehat{M}_{\alpha_j^*} M_S M_S^* A_j \widehat{M}_{\alpha_j^*} K_\wedge^{c_1} d_1, K_\wedge^{c_2} d_2 \rangle_2 \\ &= \langle \widehat{M}_{\alpha_j^* \alpha_j} \mathcal{B}_S K_\wedge^{c_1} d_1, K_\wedge^{c_2} d_2 \rangle_2 + \langle e_1, e_2 \rangle_2 \\ &= \langle \widehat{M}_{\alpha_j^* \alpha_j} K_S^{c_1} d_1, K_S^{c_2} d_2 \rangle_{\mathbf{H}(S)} + \langle e_1, e_2 \rangle_2. \end{aligned}$$

Thus (5.17) holds. Since, according to Proposition 5.6, the space  $\mathbf{H}(S)$  is right  $\mathcal{C}$ -invariant and  $\|\alpha_j^* \alpha_j\| = 1$ , we conclude that  $\widehat{V}_j$  is well-defined and contractive. Since the span of  $K_S^c k$  is dense in  $\mathbf{H}(S)$  (see (5.7)),  $\widehat{V}_j$  can be extended as a contraction on  $\mathcal{H}(S) \oplus \mathcal{C}_2$ , which satisfies

$$\widehat{V}_j^* \widehat{V}_j = \begin{pmatrix} \widehat{M}_{\alpha_j^* \alpha_j} & 0 \\ 0 & I \end{pmatrix}.$$

In order to complete the proof of statement (i), it suffices to observe that  $V_j = \widehat{V}_j^*$ .

(ii) The identities (5.10)–(5.12) follow immediately from (5.8) and the Cuntz relations (1.7).

(iii) The proof parallels the proof of the statement (iii) in Proposition 4.6. Let us fix a sequence of indices  $j = i_0, i_1, i_2, i_3, \dots, 1 \leq i_n \leq q$ . Then for  $d \in \mathcal{C}_2$  we have

$$\begin{aligned} Sd &= Dd + (S - S_{[\emptyset]})d = Dd + (B_j d)\alpha_j^* = Dd + (CB_j d)\alpha_j^* + (A_{i_1} B_j d)\alpha_{i_1}^* \alpha_j^* = \dots \\ &= Dd + \sum_{n=0}^m (CA^{\mu_n} B_j d)\mu_n^* \alpha_j^* + (A^{\mu_{m+1}} B_j d)\mu_{m+1}^* \alpha_j^*, \quad m = 0, 1, 2, \dots \end{aligned}$$

where  $\mu_0 = \emptyset$ ,  $\mu_n = \alpha_{i_1} \cdots \alpha_{i_n}$ . Now it follows from the identity (2.3) in Theorem 2.5, applied to  $Sd$ , that

$$\sum_{|w|=n+1} w^* S_{[w]} d = (CA^{\mu_n} B_j d)\mu_n^* \alpha_j^* \quad \forall n \geq 0$$

and hence

$$S_{[w]} d = w(CA^{|\mu|w-1} B_j d)\mu_{|w|}^*, \quad |w| \geq 1.$$

Since the sequence of indices  $i_0, i_1, i_2, i_3, \dots$  was chosen arbitrarily, we obtain (5.14). ■

A result which is converse to Theorem 5.7 can be formulated as follows:

**THEOREM 5.8.** *Let  $\mathbf{H}$  be a right  $\mathcal{C}$ -invariant Hilbert space included in  $\mathbf{H}_2(\mathcal{T})$ . Assume that for some  $j, 1 \leq j \leq q$ , there exists a bounded linear operator*

$$V_j = \begin{pmatrix} A_j & B_j \\ C & D \end{pmatrix} : \begin{pmatrix} \mathbf{H} \\ \mathcal{C}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{H} \\ \mathcal{C}_2 \end{pmatrix},$$

for which the relations (5.11), (5.12) and (5.9) hold true. Then:

(i) The series

$$(5.18) \quad S = \sum_{w \in \mathcal{F}_q} w^* S_{[w]},$$

where the coefficients  $S_{[w]} \in \mathcal{C}$  are determined by

$$(5.19) \quad \forall d \in \mathcal{C}_2 \quad S_{[w]} d = \begin{cases} Dd & \text{if } w = \emptyset, \\ w(CA_j^{n-1} B_j d)\alpha_j^{n*} & \text{if } |w| = n \geq 1, \end{cases}$$

defines (in the sense of pointwise absolute convergence) a Schur multiplier  $S \in \mathbf{H}(\mathcal{T})$ .

(ii) The series

$$(5.20) \quad E_c F = \sum_{n=0}^{\infty} (c\alpha)^n \cdot (CA_j^n F) \cdot \alpha_j^{n*}, \quad F \in \mathbf{H}, c \in \mathbb{B}(\mathcal{T})$$

converges absolutely in  $\mathcal{C}_2$  and defines a bounded linear operator  $E_c$  from  $\mathbf{H}$  to  $\mathcal{C}_2$ . This operator  $E_c$  and the kernel  $K_S^c$  defined in (5.2) satisfy

$$(5.21) \quad (K_S^d)^\wedge(c)k = E_c E_d^* k \quad \forall c, d \in \mathbb{B}(\mathcal{T}), k \in \mathcal{C}_2.$$

(iii) If the operators  $A_j$  and  $C$  are as in (5.8), then  $\mathbf{H}$  is the de Branges–Rovnyak space associated with the Schur multiplier  $S$ :

$$\mathbf{H} = \mathbf{H}(S).$$

*Proof.* Let  $c \in \mathbb{B}(\mathcal{T})$ . Then, as follows from (5.9) and (3.1), the series (5.20) is absolutely convergent in the  $\mathcal{C}_2$ -norm and defines a bounded linear operator  $E_c$  from  $\mathbf{H}$  to  $\mathcal{C}_2$ .

Let us consider the linear operator  $s(c) : \mathcal{C}_2 \rightarrow \mathcal{C}_2$ , defined by

$$(5.22) \quad s(c)k \stackrel{\text{def}}{=} Dk + \sum_{n=1}^{\infty} (c\alpha)^n \cdot (CA_j^{n-1}B_jk) \cdot \alpha_j^{n*}, \quad k \in \mathcal{C}_2.$$

Here, in view of (5.9) and (3.1), the series is absolutely convergent in the  $\mathcal{C}_2$ -norm and the operator  $s(c)$  is bounded. Moreover, as follows from (5.11) and (5.12), the operator  $s(c)$  commutes with  $\widehat{M}_d$  for every  $d \in \mathcal{C}$ . Hence, according to the statement (ii) of Lemma 5.3,  $s(c) \in \mathcal{C}$ .

Next we observe that

$$\left( (M_{c\alpha} \widehat{M}_{\alpha_j^*}) E_c \quad I \right) V_j = (E_c \quad s(c)).$$

Note that, as follows from (5.11) and (5.12), the self-adjoint operator  $\widehat{M}_{\alpha_j^* \alpha_j}$  commutes with  $E_d$  for every  $d \in \mathcal{C}$ . From (5.9) we obtain

$$\begin{aligned} & (E_c \quad s(c)) (E_d \quad s(d))^* \\ &= \left( (M_{c\alpha} \widehat{M}_{\alpha_j^*}) E_c \quad I \right) V_j V_j^* \left( (M_{d\alpha} \widehat{M}_{\alpha_j^*}) E_d \quad I \right)^* \\ &= (M_{c\alpha} \widehat{M}_{\alpha_j^*}) E_c \widehat{M}_{\alpha_j^* \alpha_j} E_d^* (M_{\alpha^* d^*} \widehat{M}_{\alpha_j}) + I = (M_{c\alpha} \widehat{M}_{\alpha_j^*}) E_c E_d^* (M_{\alpha^* d^*} \widehat{M}_{\alpha_j}) + I, \end{aligned}$$

hence

$$I - s(c)s(d)^* = E_c E_d^* - (M_{c\alpha} \widehat{M}_{\alpha_j^*}) E_c E_d^* (M_{\alpha^* d^*} \widehat{M}_{\alpha_j}).$$

Therefore, for every  $k \in \mathcal{C}_2$  the kernel  $K_s(c, d)$ , which appears in the equation (5.1) of Theorem 5.2, satisfies

$$\begin{aligned} & K_s(c, d)k \\ &= \sum_{n=0}^{\infty} (c\alpha)^n (I - s(c)s(d)^*) (\alpha^* d^*)^n k \\ &= \sum_{n=0}^{\infty} (c\alpha)^n ((I - s(c)s(d)^*) ((\alpha^* d^*)^n k \alpha_j^n)) \alpha_j^{n*} \\ &= \sum_{n=0}^{\infty} (c\alpha)^n (E_c E_d^* (\alpha^* d^*)^n k \alpha_j^n) \alpha_j^{n*} - \sum_{n=0}^{\infty} (c\alpha)^{n+1} (E_c E_d^* (\alpha^* d^*)^{n+1} k \alpha_j^{n+1}) \alpha_j^{(n+1)*} = E_c E_d^* k. \end{aligned}$$

In particular, the kernel  $K_s(c, d)$  is positive. Now, according to Theorem 5.2, there exists a Schur multiplier  $S \in \mathbf{H}(\mathcal{T})$  such that

$$s(c) = S^\wedge(c) \quad \forall c \in \mathbb{B}(\mathcal{T})$$

and the identity (5.2) implies (5.21).

Furthermore, in view of (5.22) and the statement (ii) of Lemma 3.1, we obtain the formulae (5.19) for the coefficients  $S_{[w]}$  of the pointwise absolutely convergent expansion (5.18) of  $S$ . This completes the proof of the statements (i) and (ii) of the theorem.

In order to prove the statement (iii), we note that if the operators  $A_j$  and  $C$  are as in (5.8), then, according to the statement (iii) of Proposition 4.6 and the definition (3.3) of the point evaluation,

$$E_c F = F^\wedge(c) \quad \forall F \in \mathbf{H}, c \in \mathbb{B}(\mathcal{T}).$$

Therefore, the identity (5.21) implies

$$\begin{aligned} K_5^\zeta k &= E_c^* k \in \mathbf{H} \quad \forall c \in \mathbb{B}(\mathcal{T}), k \in \mathcal{C}_2; \\ \langle F, K_5^\zeta k \rangle_{\mathbf{H}} &= \langle F^\wedge(c), k \rangle_2, \quad \forall F \in \mathbf{H}, c \in \mathbb{B}(\mathcal{T}), k \in \mathcal{C}_2. \end{aligned}$$

In view of Proposition 5.6 and the statement (ii) of Lemma 3.1,

$$\mathbf{H} = \overline{\text{span}}\{K_5^\zeta k : k \in \mathcal{C}_2, c \in \mathbb{B}(\mathcal{T})\} = \mathbf{H}(S) \quad \text{and} \quad \|\cdot\|_{\mathbf{H}} = \|\cdot\|_{\mathbf{H}(S)}. \quad \blacksquare$$

The formulae (5.14) in Theorem 5.7 play the role of the backward shift realization (0.6) in the present setting: they allow to represent a given Schur multiplier  $S$  as the transfer operator of a multiscale input-state-output system, as described in the following theorem.

**THEOREM 5.9.** *Let  $S \in \mathbf{H}(\mathcal{T})$  be a Schur multiplier and let  $\mathbf{H}(S)$  be the associated de Branges–Rovnyak space. Let the operators*

$$V_j = \begin{pmatrix} A_j & B_j \\ C & D \end{pmatrix} : \begin{pmatrix} \mathbf{H}(S) \\ \mathcal{C}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbf{H}(S) \\ \mathcal{C}_2 \end{pmatrix}, \quad 1 \leq j \leq q,$$

be defined by (5.8) as in Theorem 5.7. Let  $U \in \mathbf{H}_2(\mathcal{T})$ ,

$$U = \sum_{w \in \mathcal{F}_q} w^* U_{[w]}, \quad \text{where } U_{[w]} \in \mathcal{C}_2,$$

and let  $Y = SU \in \mathbf{H}_2(\mathcal{T})$ . Then the coefficients  $Y_{[w]} \in \mathcal{C}_2$  of the expansion

$$Y = \sum_{w \in \mathcal{F}_q} w^* Y_{[w]}$$

satisfy the recurrent relations

$$(5.23) \quad \begin{cases} X_\emptyset = 0, \\ X_{w\alpha_j} = A_j X_w + B_j U_w, \\ Y_{[w]} = w(CX_v + DU_v)v^* \quad \forall v \in \mathcal{F}_q : |v| = |w|, \end{cases}$$

where

$$U_w \stackrel{\text{def}}{=} \sum_{v: |v|=|w|} v^* U_{[v]} w.$$

*Proof.* The case  $w = \emptyset$  is trivial, so let us assume  $n = |w| \geq 1$ . Let  $v = \alpha_{i_1} \cdots \alpha_{i_n}$  and denote

$$v_k = \alpha_{i_1} \cdots \alpha_{i_k}, \quad v_{\bar{k}} = \alpha_{i_{k+1}} \cdots \alpha_{i_n}, \quad v_0 = v_{\bar{n}} = \emptyset.$$

As follows from (5.14),

$$\begin{aligned} \sum_{u \in \mathcal{F}_q, |u|=n} u^* Y_{[u]} &= \sum_{\mu, \nu \in \mathcal{F}_q, |\mu|+|\nu|=n} \mu^* S_{[\mu]} \nu^* U_{[\nu]} \\ &= \sum_{k=0}^{n-1} \sum_{u \in \mathcal{F}_q, |u|=k} C A^{v_{\bar{k+1}}} B_{i_{k+1}} (u^* U_{[u]} v_k) v^* + \sum_{u \in \mathcal{F}_q, |u|=n} D (u^* U_{[u]} v) v^* \\ &= \left( C \sum_{k=0}^{n-1} A^{v_{\bar{k+1}}} B_{i_{k+1}} U_{v_k} + D U_v \right) v^*, \end{aligned}$$

hence, according to (1.7),

$$Y_{[w]} = w \left( C \sum_{0 \leq k \leq n-1} A^{v_{\bar{k+1}}} B_{i_{k+1}} U_{v_k} + D U_v \right) v^*.$$

Denote

$$X_v = \sum_{0 \leq k \leq |v|-1} A^{v_{\bar{k+1}}} B_{i_{k+1}} U_{v_k}, \quad v \in \mathcal{F}_q, |v| \geq 1.$$

Then

$$X_{v\alpha_j} = \sum_{0 \leq k \leq |v|-1} A_j A^{v_{\bar{k+1}}} B_{i_{k+1}} U_{v_k} + B_j U_v = A_j X_v + B_j U_v.$$

In the case  $v = \alpha_j$  we have

$$X_{\alpha_j} = B_j U_{\emptyset} = A_j X_{\emptyset} + B_j U_{\emptyset}.$$

Thus the relations (5.23) hold. ■

## 6. THE BLASCHKE FACTORS

In this section we present an important example of Schur multiplier, which plays a role in interpolation. We follow the ideas of [3], p. 86–90. Let  $c \in \mathbb{B}(\mathcal{T})$  and consider the operator

$$R_c \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} (c\alpha)^n (\alpha^* c^*)^n = (K_{\wedge}^c)^{\wedge}(c) \in \mathcal{C}.$$

Then  $R_c > 0$  and, moreover,

$$(6.1) \quad R_c = I + c(\alpha R_c \alpha^*) c^*.$$

Hence  $R_c > I$ . Next we define

$$L_c \stackrel{\text{def}}{=} \alpha(R_c - R_c \alpha^* c^* R_c^{-1} c \alpha R_c) \alpha^* \in \mathcal{C}^{q \times q}.$$

PROPOSITION 6.1. *The following holds:*

$$(6.2) \quad L_c > 0,$$

$$(6.3) \quad L_c^{-1} = c^* c + \alpha R_c^{-1} \alpha^* = I + c^* c - \alpha c L_c c^* \alpha^*,$$

$$(6.4) \quad c L_c = R_c^{-1} c \alpha R_c \alpha^*.$$

*Proof.* Taking into account (6.1),

$$\begin{aligned} L_c(c^* c + \alpha R_c^{-1} \alpha^*) &= \alpha R_c \alpha^* c^* c + I - \alpha R_c \alpha^* c^* R_c^{-1} c \alpha R_c \alpha^* c^* c - \alpha R_c \alpha^* c^* R_c^{-1} c \\ &= \alpha R_c \alpha^* c^* c + I - \alpha R_c \alpha^* c^* R_c^{-1} (R_c - I) c - \alpha R_c \alpha^* c^* R_c^{-1} c = I. \end{aligned}$$

Since  $c^* c + \alpha R_c^{-1} \alpha^*$  is positive definite, this means

$$L_c = (c^* c + \alpha R_c^{-1} \alpha^*)^{-1} > 0.$$

The rest of the identities follow from (6.1) analogously. ■

Note that

$$(L_c c^* \alpha^*)^n = (\alpha R_c \alpha^* c^* R_c^{-1} \alpha^*)^n = \alpha R_c \alpha^* c^* (\alpha^* c^*)^{n-1} R_c^{-1} \alpha^*,$$

hence  $\alpha L_c c^* \alpha^* \in \mathbb{B}(\mathcal{T})$ .

DEFINITION 6.2. Let  $c \in \mathbb{B}(\mathcal{T})$ . The operator

$$B_c = (\alpha^* - c)(1 - L_c c^* \alpha^*)^{-1} \sqrt{L_c} \in \mathbf{H}(\mathcal{T})$$

is called the *Blaschke factor*, corresponding to the constant  $c$ .

PROPOSITION 6.3. *The operator  $B_c$  is unitary. In particular, the multiplication operator  $\mathcal{M}_{B_c} : \mathbf{H}_2(\mathcal{T})^q \rightarrow \mathbf{H}_2(\mathcal{T})$  is an isometry.*

*Proof.* We have

$$(I - \alpha c L_c) L_c^{-1} (I - L_c c^* \alpha^*) = (\alpha - c^*)(\alpha^* - c),$$

hence  $B_c^* B_c = I$  and  $M_{B_c}$  is an isometry. Furthermore,

$$(\alpha^* - c) = \alpha^* (I - \alpha c)$$

has a bounded inverse (not causal), hence  $B_c$  is also invertible and unitary. ■

THEOREM 6.4. *Let  $F \in \mathbf{H}_2(\mathcal{T})$ ,  $c \in \mathbb{B}(\mathcal{T})$ . Then  $F^\wedge(c) = 0$  if and only if  $F$  is of the form*

$$F = B_c \cdot G,$$

where  $G \in \mathbf{H}_2(\mathcal{T})^q$ ,  $\|G\|_2 = \|F\|_2$ .

*Proof.* First, let us assume that  $F = B_c \cdot G$ , where  $G \in \mathbf{H}_2(\mathcal{T})^q$ . Then  $\|F\|_2 = \|G\|_2$  by Proposition 6.3. Furthermore, as follows from Lemma 3.1,  $B_c^\wedge(c) = 0$  and  $F^\wedge(c) = (B_c G)^\wedge(c) = 0$ .

Conversely, assume that  $F$  has the expansion  $F = \sum_{w \in \mathcal{F}_q} w^* F_{[w]}$  and that  $F^\wedge(c) = 0$ . Then  $F$  is represented by the series

$$F = F - F^\wedge(c) = \sum_{w \in \mathcal{F}_q} (w^* - (w^*)^\wedge(c)) F_{[w]}.$$

Denote  $G' \stackrel{\text{def}}{=} (I - \alpha c)^{-1} \alpha F$ . Then  $F = (\alpha^* - c) G'$ ,  $G' \in \mathbf{X}_2(\mathcal{T})^q$  and, moreover,

$$\begin{aligned} G' &= \sum_{w \in \mathcal{F}_q} (I - \alpha c)^{-1} \alpha (w^* - (w^*)^\wedge(c)) F_{[w]} \\ &= \sum_{w \in \mathcal{F}_q, |w| \geq 1} \alpha (I + c\alpha + (c\alpha)^2 + \dots + (c\alpha)^{|w|-1}) w^* F_{[w]}. \end{aligned}$$

Since each term of the last series is causal,  $G' \in \mathbf{H}_2(\mathcal{T})^q$ . It remains to define

$$G \stackrel{\text{def}}{=} \sqrt{L_c^{-1} (I - L_c c^* \alpha^*)} G'$$

to complete the proof. ■

*Acknowledgements.* D. Alpay thanks the Earl Katz family for endowing the chair which supports his research and NWO, the Netherlands Organization for Scientific Research (grant B 61-524). The research of A. Dijkstra was in part supported by the Center for Advanced Studies in Mathematics (CASM) of the Department of Mathematics, Ben-Gurion University.

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Received June 4, 2006.