J. OPERATOR THEORY
61:1(2009), 191–223

TWO CRITERIA FOR A PATH OF OPERATORS TO HAVE COMMON HYPERCYCLIC VECTORS

KIT C. CHAN and REBECCA SANDERS

Communicated by Kenneth R. Davidson

ABSTRACT. We offer two conditions for a path of bounded linear operators on a Banach space to have a dense G_δ set of common hypercyclic vectors. One of them is an equivalent condition and the other one is a generalization of the hypercyclicity criterion. Using the conditions, we show that between any two hypercyclic unilateral weighted backward shifts, there exists a path of such operators having a dense G_δ set of common hypercyclic vectors. Furthermore, we prove that such a set of vectors exists for a path of scalar multiples of the unweighted shift, reproducing a result of Abakumov and Gordon, and of Costakis and Sambarino. Motivated by our results, we provide an example of a path of unilateral weighted backward shifts that fails to have any common hypercyclic vector. Lastly, we adopt the main results to bilateral weighted shifts.

KEYWORDS: *Hypercyclic operator, hypercyclic vector, unilateral weighted backward shift.*

MSC (2000): Primary 47A16, 47B37; Secondary 46B45.

1. INTRODUCTION

On a separable, infinite dimensional, Banach space X, a bounded linear operator $T : X \to X$ is said to be *hypercyclic* if there exists a vector x in X such that its orbit $\mathrm{Orb}(T, x) = \{T^n x : n \geqslant 0\}$ is dense in X. Such a vector x is called a *hypercyclic vector* for T, and we use $\mathcal{HC}(T)$ to denote the set of all hypercyclic vectors for T. As it turns out, the set $\mathcal{HC}(T)$ is always a dense G_δ subset of X whenever T is hypercyclic; see Kitai Theorem 2.1 of [19]. It easily follows from the Baire Category Theorem that for any countable family $\{T_n : n \geqslant 1\}$ of hypercyclic operators, the set $\bigcap\limits_{n=1}^{\infty} \mathcal{HC}(T_n)$ of vectors that are hypercyclic for each operator T_n is still a dense G_δ set. On the other hand, the above argument fails to apply when we have an uncountable family of hypercyclic operators. This observation motivates us to study the existence of vectors that are hypercyclic for each operator in an uncountable family, with continuity maintained inside the family. To be precise, we need to introduce a few definitions.

Let $B(X)$ denote the algebra of bounded linear operators on the Banach space X, and let I denote an interval of real numbers. A family of operators $\{F_t \in B(X) : t \in I\}$ is said to be *a path of operators* if the map $F : I \to (B(X), \|\cdot\|)$, defined by $F(t) = F_t$, is a continuous map with respect to the usual topology of the real numbers and the operator norm topology on $B(X)$. Furthermore, if the interval $I = [a, b]$, then the path $\{F_t \in B(X) : t \in I\}$ is said to be a *path of operators between F_a and F_b*. For any path, a vector x in X is a *common hypercyclic vector* if x is a hypercyclic vector for every operator in the path; that is, $x \in \bigcap_{t\in I} \mathcal{HC}(F_t)$. For example, León-Saavedra and Müller ([20], Corollary 3) showed that if T is a hypercyclic operator, then the path $\{e^{it}T : t \in [0, 2\pi]\}$ of all rotations of T has a dense G_δ set of common hypercyclic vectors, and indeed they have the exact same set of hypercyclic vectors. Another example concerns unilateral weighted backward shifts $T : \ell^p \to \ell^p$, with $p \geqslant 1$. To explain the details, let $\{e_j : j \geqslant 0\}$ be the canonical basis of ℓ^p; that is, $e_j = (0, 0, \ldots, 0, 1, 0, \ldots)$, where the 1 is in the j-th position. A bounded linear operator $T : \ell^p \to \ell^p$ is said to be a *unilateral weighted backward shift* if there is a positive weight sequence $\{w_j : j \geqslant 1\}$ such that

$$Te_0 = 0, \quad \text{and} \quad Te_j = w_j e_{j-1} \quad \text{when } j \geqslant 1.$$

In the case that all $w_j = 1$, then the operator is simply called *the unilateral backward shift*, and is denoted by B. Since B has norm 1, it cannot be hypercyclic but Rolewicz [21] showed that its multiples $\{tB : t \in (1, \infty)\}$ form a path of hypercyclic unilateral weighted backward shifts, providing the first examples of hypercyclic operators on a Banach space. Then Salas [23] raised the question whether the path has a common hypercyclic vector. Settling this question, Abakumov and Gordon [1] answered in the positive by constructing such vectors. Shortly after, Costakis and Sambarino introduced a sufficient condition ([13], Theorem 12) to show that the path has a dense G_δ set common hypercyclic vectors. Their sufficient condition enabled them to offer another example ([13], Theorem 17) of a dense G_δ set of common hypercyclic vectors for a specific path of unilateral weighted backward shifts whose weights are bounded below by 1. In fact, they used the condition to provide many similar dense G_δ results for some families of translation and differentiation operators defined on the Fréchet space of entire functions. A different sufficient condition for the existence of a dense G_δ set of common hypercyclic vectors was obtained by Bayart and Matheron [7], who showed applications in situations where the condition of Costakis and Sambarino does not apply. In addition, they ([7], Theorem 4.4) also provided such a sufficient condition for a path of unilateral weighted backward shifts whose weights $w_n(t)$ is a nondecreasing function of the path parameter t.

On the other hand, the work of Abakumov and Gordon [1] motivated Bayart [4] to provide a sufficient condition for the path of scalar multiples of a single operator to have a dense G_δ set of common hypercyclic vectors, and apply that to the composition operators on spaces of analytic functions. This topic of research was also explored by Bayart and Grivaux [6].

Common hypercyclic vectors for an uncountable family of operators are also studied in other interesting settings, such as Cesaro hypercyclic operators by Costakis [12], and semigroups of operators by Conejero, Müller, and Peris [10]. Besides common hypercyclic vectors, this "common" phenomenon can be carried over to a hypercyclic subspace; that is, an infinite dimensional closed subspace consisting entirely, except for the zero vector, of hypercyclic vectors. Along this line, Aron, Bès, León, and Peris [3], and Bayart [5] have provided sufficient conditions for a family of hypercyclic operators to have a common hypercyclic subspace.

In the present paper, we study only common hypercyclic vectors for a path of hypercyclic operators. We first obtain results for general hypercyclic operators on a Banach space, and then we specialize our study on the shift operators.

In Section 2, we focus on universality, which is a generalization of hypercyclicity. To provide a definition, let $(T_n)_{n=1}^\infty$ be a sequence of bounded linear operators on a separable, infinite dimensional, Banach space X. The sequence $(T_n)_{n=1}^\infty$ is *universal* if there exists a vector x in X such that the set $\{T_n x : n \geqslant 1\}$ is dense in X. Such a vector x is called a *universal vector* for $(T_n)_{n=1}^\infty$. If the set of all universal vectors, denoted by $\mathcal{U}(T_n)$, is dense in X, then $(T_n)_{n=1}^\infty$ is said to be *densely universal*. In the case that the sequence satisfies the condition $T_n = T^n$, for a single bounded linear operator T on X, the sequence $(T_n)_{n=1}^\infty$ is densely universal if and only if T is hypercyclic. For more details about universality, one may refer to the survey article of Große-Erdmann [17].

The objective of Section 2 is to study *common universal vectors* for a family $\{(F_{t,n})_{n=1}^\infty : t \in I\}$ of bounded linear operators on X with parameter t in an interval I; that is, by definition, those vectors x such that the set $\{F_{t,n}x : n \geqslant 1\}$ is dense for every t in I. Assuming that for each given integer $n \geqslant 1$, the map $t \mapsto F_{t,n}$ defines a path of operators, we provide a necessary and sufficient condition for the family $\{(F_{t,n})_{n=1}^\infty : t \in I\}$ to have a dense G_δ set of common universal vectors $\bigcap_{t\in[a,b]} \mathcal{U}(F_{t,n})$. The condition reduces to the well-known condition of Große-Erdmann [16] and Godefroy and Shapiro ([15], Theorem 1.2) for a sequence of operators $(T_n)_{n=1}^\infty$, when the interval I is taken to be a singleton set. Based on our necessary and sufficient condition we obtain another sufficient condition, which reduces to the most relaxed form of the hypercyclicity criterion when I is singleton and $F_{t,n} = T^n$ for some bounded linear operator T. Hence both conditions that we obtain are natural generalizations of existing criteria for universality and hypercyclicity. The two conditions are different from the sufficient condition obtained by Costakis and Sambarino ([13], Theorem 12), in the sense that their condition requires a comparison of the growth rate of certain quantities with a convergent series, and hence directly addressing the concerns of Bayart and Grivaux ([6], page 292).

In Section 3, we use our necessary and sufficient condition to reproduce the above mentioned result that the path $\{tB : t \in (1, \infty)\}$ has a dense G_δ set of common hypercyclic vectors, when B is the unilateral backward shift. After that, we focus on proving that between any two given hypercyclic unilateral weighted backward shifts, there exists a path of such operators having a dense G_δ set of common hypercyclic vectors, using the sufficient condition in Section 2. The difficulties in the proof arise from the fact that the given shift operators may have infinitely many weights less than 1. As it turns out, this causes a lot of problems in ensuring the continuity of a path. To handle the situation, we need to go through a few technical lemmas in the section.

In Section 4, we provide an example of a path of hypercyclic unilateral weighted backward shifts that fails to have a common hypercyclic vector, justifying our existence result in Section 3. Furthermore, the example also shows that between any two hypercyclic unilateral weighted backward shifts, there exists a path of such operators without a common hypercyclic vector, in contrast with our main result in Section 3. The example also justifies the two conditions found in Section 2, because Salas [22] showed that every hypercyclic unilateral weighted backward shift must satisfy the hypercyclicity criterion. Consequently, our example provides a path of operators each of which satisfies the hypercyclicity criterion, but they altogether do not have a common hypercyclic vector. In that sense, it is natural to generalize in Section 2 the hypercyclicity criterion to the setting of a path.

Lastly in Section 5, we conclude the paper by outlining how to make our results work for bilateral weighted shifts.

2. COMMON UNIVERSAL VECTORS

In this section, we consider a separable, infinite dimensional Banach space X and study a family of bounded linear operators $(F_{t,n})_{n=1}^{\infty}$ in $B(X)$ parametrized by a real variable t in $[a, b]$. Assume that for each integer $n \geqslant 1$, the map $t \mapsto F_{t,n}$ is continuous with respect to the operator norm topology of $B(X)$; that is, it defines a path of operators. We first obtain two general conditions that guarantee the existence of a dense G_δ set of common universal vectors for the family $(F_{t,n})_{n=1}^{\infty}$. These two conditions are generalizations of existing criteria for universality and hypercyclicity. The first one is based on a result of Große-Erdmann [16] and Godefroy and Shapiro ([15], Theorem 1.2): *The sequence $(T_n)_{n=1}^{\infty}$ has a dense G_δ set of universal vectors if and only if for each pair of nonempty open sets U_1, U_2, there exists an integer $n \geqslant 1$ such that*

$$T_n(U_1) \cap U_2 \neq \emptyset. \tag{2.1}$$

When the family of operators becomes uncountable, the Baire Category Theorem fails to provide common universal vectors. Nevertheless, we can still provide the following generalization for a continuous family.

THEOREM 2.1. *For each $t \in [a,b]$, let $(F_{t,n})_{n=1}^{\infty}$ be a sequence of nonzero bounded linear operators on a separable infinite dimensional Banach space X. Suppose for each integer $n \geqslant 1$, the map $t \mapsto F_{t,n}$ defines a path of operators on $[a,b]$. The set $\bigcap_{t\in[a,b]} \mathcal{U}(F_{t,n})$ of common universal vectors is a dense G_δ set if and only if for each pair of nonempty open sets U_1, U_2, there exist a partition $P = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of $[a,b]$, positive integers $n_1, n_2, \ldots, n_k$, and a nonempty open set V such that $V \subseteq U_1$ and $F_{t,n_i}(V) \subseteq U_2$ whenever $1 \leqslant i \leqslant k$ and $t \in [t_{i-1}, t_i]$.*

REMARKS 2.2. (i) Theorem 2.1 can be applied to a family of operators which is parametrized by a multidimensional parameter u in a compact cube of $\mathbb{R}^n$. All one needs to use is a space filling curve $u = u(t)$ to reparametrize the family by a real parameter t in a compact interval $[a,b]$. An example of such an argument is provided in the introduction of Section 4 below.

(ii) Though Theorem 2.1 is stated for the compact interval $[a,b]$, it holds true for any interval I because we can write I as a countable union of compact intervals. Correspondingly, to each of these intervals, Theorem 2.1 provides a dense G_δ set of common universal vectors. By the Baire Category Theorem, the intersection of all those dense G_δ sets provides the desired result. The same argument applies to Theorem 2.4 below as well.

Proof of Theorem 2.1. We first show the forward implication by assuming the family $\{(F_{t,n})_{n=1}^{\infty} : t \in [a,b]\}$ has a dense G_δ set of common universal vectors. Hence, if U_1 is an open subset of X, there is a common universal vector g in U_1. Then for each open subset U_2 of X and for each $s \in [a,b]$, we can find a positive integer n_s such that $F_{s,n_s} g \in U_2$.

By our hypothesis on the family, we see that for each integer $n \geqslant 1$, the map $(t,h) \mapsto F_{t,n}(h)$ is continuous on $[a,b] \times X$. Thus, for each $s \in [a,b]$, there is a relatively open subinterval I_s of $[a,b]$ with $s \in I_s$ and an open subset V_s of X satisfying $g \in V_s \subseteq U_1$ and $F_{t,n_s}(h) \in U_2$ whenever $t \in I_s$ and $h \in V_s$. Since all these subintervals I_s together cover $[a,b]$, there is a finite subcover $I_{s_1}, I_{s_2}, \ldots, I_{s_k}$ which gives rise to a partition P of $[a,b]$. Lastly, we take integers $n_{s_1}, n_{s_2}, \ldots, n_{s_k}$ and take $V = V_{s_1} \cap V_{s_2} \cap \cdots \cap V_{s_k}$ to complete the argument for the forward implication.

For the backward implication, let $h \in X \setminus \{0\}$ and $\varepsilon > 0$, and let denote the set $A(h;\varepsilon) = \{g \in X : \forall t \in [a,b], \exists m \text{ with } F_{t,m} g \in B(h,\varepsilon)\}$. If $\{h_i : i \geqslant 1\}$ is a countable dense set in $X \setminus \{0\}$, then

$$\bigcap_{i=1}^{\infty}\bigcap_{j=1}^{\infty} A(h_i;2^{-j}) = \{g \in X : \forall i,j, \forall t \in [a,b], \exists m \text{ with } F_{t,m} g \in B(h_i, 2^{-j})\} = \bigcap_{t\in[a,b]} \mathcal{U}(F_{t,n}).$$

By the Baire Category Theorem, it suffices to show the set $A(h;\varepsilon)$ is open and dense in X.

To show $A(h;\varepsilon)$ is open, let $g \in A(h;\varepsilon)$. Then for each $s \in [a,b]$, there exists m_s such that $F_{s,m_s} g \in B(h,\varepsilon)$. Using the same argument as in the forward implication, there exist $a = t_0 < t_1 < \cdots < t_k = b$, $\delta > 0$ and positive integers $n_1, n_2, \ldots, n_k$ such that

$$F_{t,n_i}(B(g,\delta)) \subseteq B(h,\varepsilon) \quad \text{whenever } \ t \in [t_{i-1}, t_i].$$

Hence, $B(g,\delta) \subseteq A(h;\varepsilon)$.

To show that $A(h;\varepsilon)$ is dense, let $f \in X$ and let $\varepsilon' > 0$. By assumption, there exists a partition $P = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of $[a,b]$, integers $n_1, n_2, \ldots, n_k$, and a nonempty open set V such that $V \subseteq B(f,\varepsilon')$ and for $1 \leqslant i \leqslant k$, we have $F_{t,n_i}(V) \subseteq B(h,\varepsilon)$ whenever $t \in [t_{i-1}, t_i]$. Therefore, $V \subseteq B(f,\varepsilon') \cap A(h;\varepsilon)$. ∎

In the proof of the backward implication in Theorem 2.1, we first write the set $\bigcap_{t\in[a,b]} \mathcal{U}(F_{t,n})$ as a G_δ set. This is accomplished by only using the hypothesis that for each integer $n \geqslant 1$, the map $t \mapsto F_{t,n}$ defines a path of operators on $[a,b]$. Using the remaining hypothesis of Theorem 2.1, we prove the G_δ set $\bigcap_{t\in[a,b]} \mathcal{U}(F_{t,n})$ is dense. Keeping this in mind, we have a sufficient condition for the set $\bigcap_{t\in[a,b]} \mathcal{U}(F_{t,n})$ to be G_δ set.

COROLLARY 2.3. *For each $t \in [a,b]$, let $(F_{t,n})_{n=1}^\infty$ be a sequence of nonzero bounded linear operators on a separable infinite dimensional Banach space X. Suppose for each integer $n \geqslant 1$, the map $t \mapsto F_{t,n}$ defines a path of operators on $[a,b]$. Then the set $\bigcap_{t\in[a,b]} \mathcal{U}(F_{t,n})$ of common universal vectors is a G_δ set.*

With condition (2.1) in mind, we may attempt to relax the requirement $F_{t,n_i}(V) \subseteq U_2$ in Theorem 2.1 to the condition $F_{t,n_i}(U_1) \cap U_2 \neq \emptyset$. However, this relaxation does not work. For a counterexample, suppose $\{F_t : t \in [a,b]\}$ is a path of hypercyclic operators with no common hypercyclic vector; see Theorem 4.1 below. Then the family $\{(F_{t,n})_{n=1}^\infty : t \in [a,b]\}$, where $F_{t,n} = F_t^n$, has no common universal vector. Let U_1, U_2 be two nonempty open sets. Since F_s is hypercyclic for each $s \in [a,b]$, there exist a vector $g_s \in U_1$ and an integer $m_s \geqslant 1$ such that $F_{s,m_s} g_s \in U_2$. Repeating the same argument as in the proof for the forward implication of Theorem 2.1, we can find a partition $P = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of $[a,b]$, integers $m_{s_1}, m_{s_2}, \ldots, m_{s_k}$, and vectors $g_{s_1}, g_{s_2}, \ldots, g_{s_k}$ satisfying $F_{t,m_{s_i}} g_{s_i} \in U_2$ whenever $t \in [t_{i-1}, t_i]$, and so $F_{t,m_{s_i}}(U_1) \cap U_2 \neq \emptyset$.

Using Theorem 2.1, we now establish a sufficient condition for an uncountable family $\{(F_{t,n})_{n=1}^\infty : t \in [a,b]\}$ to have a dense G_δ set of common universal vectors. Furthermore, in the case when $a = b$, our sufficient condition reduces to the Universality Criterion for the family $(F_n)_{n=1}^\infty$. This criterion coincides with the Hypercyclicity Criterion when $F_n = T^n$ for a single operator T.

The Hypercyclicity Criterion is a well-known sufficient condition for an operator to be hypercyclic. It was first obtained by Kitai ([19], Theorem 1.4) and rediscovered by Gethner and Shapiro ([14], Theorem 2.2) in a more general setting. The following version of the Hypercyclicity Criterion was given by Bès and Peris ([9], Theorem 2.2): *An operator T on X is hypercyclic if there exist dense sets D_1, D_2, a sequence $(m_k)_{k=1}^{\infty}$ of positive integers, and mapping $S_k : D_1 \to X$ satisfying:*

(i) $S_k x \to 0$ *for all* $x \in D_1$;
(ii) $T^{m_k} y \to 0$ *for all* $y \in D_2$;
(iii) $T^{m_k} S_k x \to x$ *for all* $x \in D_1$.

THEOREM 2.4. *For each $t \in [a,b]$, let $(F_{t,n})_{n=1}^{\infty}$ be a sequence of nonzero bounded linear operators on a separable, infinite dimensional Banach space X. Suppose for each integer $n \geqslant 1$, the map $t \mapsto F_{t,n}$ defines a path of operators on $[a,b]$. Further suppose there is a dense set D_1 such that for each $h \in D_1$ and $\varepsilon > 0$, there exist a $\delta > 0$, a dense set D_2, an increasing sequence $(m_k)_{k=1}^{\infty}$ of positive integers, and a set of maps $\{S_{t,k} : D_1 \to X : t \in [a,b], k \geqslant 1\}$ satisfying:*

(i) *for each $t \in [a,b]$, we have $\|S_{t,k}h\| \to 0$ as $k \to \infty$;*
(ii) *for each $f \in D_2$, we have $\|F_{t,m_k} f\| \to 0$ uniformly on $[a,b]$ as $k \to \infty$;*
(iii) *for each $t' \in [a,b]$ and integer $K \geqslant 1$, there is $k \geqslant K$ such that $\|F_{t,m_k} S_{t',k} h - h\| < \varepsilon$ whenever $|t - t'| < \delta$.*

Then the set $\bigcap_{t\in[a,b]} \mathcal{U}(F_{t,n})$ of common universal vectors is a dense G_δ set.

Proof. Let U_1, U_2 be two nonempty open sets. Choose $h \in D_1$ and $\rho > 0$ such that $B(h,\rho) \subseteq U_2$. With $\varepsilon = \frac{\rho}{3}$, the hypothesis of the theorem guarantees the existence of δ, D_2, $(m_k)_{k=1}^{\infty}$ and $\{S_{t,k}\}$ satisfying conditions (i),(ii) and (iii). Let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a,b]$ with $\max_{1\leqslant i\leqslant n} |t_i - t_{i-1}| < \delta$.

CLAIM. *For each i with $1 \leqslant i \leqslant n$, each nonempty open set V and each $K \geqslant 1$, there exist a nonempty open set $V' \subseteq V$ and $k \geqslant K$ such that $F_{t,m_k}(V') \subseteq U_2$ whenever $t \in [t_{i-1}, t_i]$.*

Proof of Claim. Choose $f \in D_2$ and $r > 0$ such that $B(f,r) \subseteq V$. From conditions (i), (ii) and (iii), there exists $k \geqslant K$ such that $\|S_{t_i,k}h\| < r$, and for all $t \in [t_{i-1}, t_i]$, we have $\|F_{t,m_k} f\| < \varepsilon$ and

$$\|F_{t,m_k} S_{t_i,k} h - h\| < \varepsilon.$$

Let $g = f + S_{t_i,k}h$ and let $V' = V \cap B(g, \frac{\varepsilon}{\gamma}) \subseteq V$ where $\gamma = \sup\{\|F_{t,m_k}\| : t \in [t_{i-1}, t_i]\}$. Observe that $\|g - f\| = \|S_{t_i,k}h\| < r$, and so $g \in B(f,r) \cap B(g, \frac{\varepsilon}{\gamma}) \subseteq V' \neq \emptyset$. Next, observe that if $g' \in V'$ and $t \in [t_{i-1}, t_i]$, then

$$\begin{aligned}\|F_{t,m_k} g' - h\| &\leqslant \|F_{t,m_k}\|\|g' - g\| + \|F_{t,m_k} g - h\| \\ &\leqslant \gamma\frac{\varepsilon}{\gamma} + \|F_{t,m_k} f\| + \|F_{t,m_k} S_{t_i,k} h - h\| < \varepsilon + \varepsilon + \varepsilon = \rho.\end{aligned}$$

Therefore, $F_{t,m_k}(V') \subseteq B(h,\rho) \subseteq U_2$, and this finishes the proof of the claim. ∎

Choose $f' \in D_2$ and let $\varepsilon' > 0$ such that $B(f',\varepsilon') \subseteq U_1$. Consider the open set $V_0 = B(f',\varepsilon')$. From the Claim, there exist a nonempty open set $V_1 \subseteq V_0$ and $k_1 \geqslant 1$ such that $F_{t,m_{k_1}}(V_1) \subseteq U_2$ whenever $t \in [t_0,t_1]$. Again, from the Claim, there exist a nonempty open set $V_2 \subseteq V_1$ and $k_2 > k_1$ such that $F_{t,m_{k_2}}(V_2) \subseteq U_2$ whenever $t \in [t_1,t_2]$. Repeating this process n times yields integers $k_1 < k_2 < \cdots < k_n$, and nonempty open sets $V_n \subseteq V_{n-1} \subseteq \cdots \subseteq V_1 \subseteq V_0$ such that $F_{t,m_{k_i}}(V_i) \subseteq U_2$ whenever $t \in [t_{i-1},t_i]$. Note that $V_n \subseteq V_0 \subseteq U_1$, and if $1 \leqslant i \leqslant n$, then $F_{t,m_{k_i}}(V_n) \subseteq F_{t,m_{k_i}}(V_i) \subseteq U_2$ whenever $t \in [t_{i-1},t_i]$. The result now follows by Theorem 2.1. ∎

If $\{F_t : t \in [a,b]\}$ is a path of hypercyclic operators, by letting $F_{t,n} = F_t^n$, one can use Theorem 2.1 or Theorem 2.4 to show the family $\{F_t : t \in [a,b]\}$ has a dense G_δ set of common hypercyclic vectors.

Bès and Peris ([9], Theorem 2.2) showed that an operator T satisfies the Hypercyclicity Criterion if and only if $T \oplus T$ is hypercyclic. Analogous to their result, we provide the following statement.

COROLLARY 2.5. *If all the operators $F_{t,n}$ satisfy the condition $F_{t,n} = F_t^n$ for a family $\{F_t : t \in [a,b]\}$ and also the hypotheses in Theorem* 2.4, *then $(F_t^n \oplus F_t^n)_{n=1}^\infty$ is universal.*

Proof. By Theorem 3.4 in [8], it suffices to show for any $t \in [a,b]$ and any nonempty open sets U, V, W with $0 \in W$, there exists n for which $F_t^n(U) \cap W \neq \emptyset$ and $F_t^n(W) \cap V \neq \emptyset$. Choose $h \in D_1$ and $\varepsilon > 0$ such that $B(h,\varepsilon) \subseteq V$. By assumption, there exist a $\delta > 0$, a dense set D_2, an increasing sequence $(m_k)_{k=1}^\infty$ of positive integers, and a set of maps $\{S_{u,k} : D_1 \to X : u \in [a,b], k \geqslant 1\}$ satisfying (i), (ii) and (iii) in Theorem 2.4 with those h and ε. Choose $f \in D_2 \cap U$. From (i), (ii) and (iii), there exists k such that

$$S_{t,k}h \in W, \quad F_t^{m_k} f \in W, \quad \text{and} \quad F_t^{m_k} S_{t,k}h \in B(h,\varepsilon) \subseteq V.$$

Hence, $F_t^{m_k}(U) \cap W \neq \emptyset$ and $F_t^{m_k}(W) \cap V \neq \emptyset$. ∎

In the case when $a = b$ and there exists an operator T such that $F_{a,n} = T^n$, the above corollary, along with the result of Bès and Peris ([9], Theorem 2.2), gives the following statement.

COROLLARY 2.6. *The operator $T : X \to X$ satisfies the Hypercyclicity Criterion if and only if there exists a dense set D_1 such that for each $h \in D_1$ and $\varepsilon > 0$, we have a dense set D_2, an increasing sequence $(m_k)_{k=1}^\infty$ of positive integers, and maps $S_k : D_1 \to X$ satisfying:*

(i) $S_k h \to 0$ *as* $k \to \infty$;

(ii) *for each* $f \in D_2$, *we have* $T^{m_k} f \to 0$ *as* $k \to \infty$;

(iii) *for each* $K \geqslant 1$, *there exists* $k \geqslant K$ *with* $\|T^{m_k} S_k h - h\| < \varepsilon$.

The conditions in Corollary 2.6 are weaker than the Hypercyclicity Criterion, in the sense that the dense set D_2, the increasing sequence $(m_k)_{k=1}^{\infty}$ can vary with the h and the ε. Moreover, the corollary does not require $T^{m_k}S_k h \to h$ for all $h \in D_1$. It only states that for the fixed h, we have $T^{m_k}S_k h \in B(h,\varepsilon)$ infinitely often.

3. HYPERCYCLIC UNILATERAL SHIFTS

Applying the results in the previous section, we now examine unilateral weighted backward shifts. The importance of this class of operators lies on the fact that they often serve as a testing ground for different research directions in operator theory. By the same token, they offered the first examples of hypercyclic operators on a Banach space; see Rolewicz [21]. In fact, such a hypercyclic shift operator is completely characterized in terms of its weight sequence by Salas ([22], Theorem 2.8): *A unilateral weighted backward shift is hypercyclic if and only if its weight sequence $\{w_j : j \geqslant 1\}$ satisfies*

$$\sup\Big\{\prod_{j=1}^{n} w_j : n \geqslant 1\Big\} = \infty. \tag{3.1}$$

More recently, Große-Erdmann ([18], Theorem 6) put this condition in a more general setting.

The main theorem in this section is to show that for any two given hypercyclic unilateral weighted backward shifts there exists a path of such shifts with a dense G_δ set of common hypercyclic vectors. Paths of unilateral weighted backward shifts provided some of the early examples of an uncountable family of hypercyclic operators having a dense G_δ set of common hypercyclic vectors. Along this line, the first example was studied by Abakumov and Gordon [1] who took the uncountable family $\{zB : |z| > 1\}$ where B is the unilateral backward shift. In fact, their result is equivalent to showing the existence of common hypercyclic vectors for the path $\{tB : t \in (0,\infty)\}$ because León-Saavedra and Müller ([20], Corollary 3) showed that if T is hypercyclic, then any rotation $\mathrm{e}^{\mathrm{i}t}T$ of T has the same set of hypercyclic vectors as T. Costakis and Sambarino ([13], Theorem 4) studied the path in a totally different way, and they ([13], Theorem 17) offered another example of a specific path F_t of unilateral weighted backward shifts whose weights $w_j^{(t)} \geqslant 1$. Generally speaking, for shifts whose weights may be less than 1, the problem becomes more complicated. Nevertheless, we can still use Theorem 2.4 to handle this situation in the later half of the section. First, we use Theorem 2.1 to prove the next result.

PROPOSITION 3.1. *Let $B : \ell^2 \to \ell^2$ be the unilateral backward shift. The set of all common hypercyclic vectors for the family $\{tB : t \in (1,\infty)\}$ is a dense G_δ set.*

Proof. For each $t \in (1, \infty)$, let $F_t = tB$. Observe that

$$\bigcap_{t \in (1,\infty)} \mathcal{HC}(F_t) = \bigcap_{n \in \mathbb{N}} \bigcap_{t \in [1+\frac{1}{n}, n]} \mathcal{HC}(F_t),$$

and so, by the Baire Category Theorem, it suffices to show the subpath $\{F_t : t \in [a, b]\}$ with $1 < a < b$ has a dense G_δ set of common hypercyclic vectors. To do this, we now use Theorem 2.1. Let U_1, U_2 be two nonempty open sets in ℓ^2. Choose $h_1, h_2 \in \operatorname{span}\{e_j : j \geqslant 0\} \setminus \{0\}$, $\varepsilon_1, \varepsilon_2 > 0$, and an integer $m \geqslant 1$ such that $B(h_1, \varepsilon_1) \subseteq U_1$, $B(h_2, \varepsilon_2) \subseteq U_2$, and $\langle h_1, e_j\rangle = \langle h_2, e_j\rangle = 0$ for all $j > m$. Next, choose $x > 0$ such that

$$0 < 1 - \mathrm{e}^{-x} < \frac{\varepsilon_2}{8\|h_2\|}. \tag{3.2}$$

From the definition of F_t, there exists a large enough integer $N \geqslant 1$ such that

$$F_t^n e_j = 0 \quad \text{for } 0 \leqslant j \leqslant m,\ t \in [a, b], \text{ and } n \geqslant N. \tag{3.3}$$

Further suppose N is chosen so that

$$\sum_{i=1}^{\infty} a^{-iN} < \min\Big\{\frac{\varepsilon_1}{\|h_2\|}, \frac{\varepsilon_2}{4\|h_2\|}\Big\}, \tag{3.4}$$

$$\Big|\mathrm{e}^{-x} - \Big(1 + \frac{x}{n}\Big)^{-n}\Big| < \frac{\varepsilon_2}{8\|h_2\|}, \quad \text{for } n \geqslant N. \tag{3.5}$$

Define $t_0 = a$ and $t_i = t_{i-1}(1 + \frac{x}{iN})$ for all $i \geqslant 1$. Note that $t_i \nearrow \infty$; see, for example page 16 of [11]. Thus, there is an integer $k \geqslant 1$ with $a = t_0 < t_1 < \cdots < t_{k-1} < b \leqslant t_k$. Let $S : \ell^2 \to \ell^2$ be the unilateral unweighted forward shift. That is, $Se_j = e_{j+1}$ for $j \geqslant 0$. Let $g = h_1 + \sum_{i=1}^{k} t_i^{-iN} S^{iN} h_2$.

Consider the partition $P = \{a = t_0 < t_1 < \cdots < t_{k-1} < b\}$ of $[a, b]$ and the open set $V = B(g, \delta) \cap U_1 \subseteq U_1$ where $\delta = \frac{\varepsilon_2}{2\|F_b\|^{kN}}$. To show V is nonempty, observe that

$$\|h_1 - g\| \leqslant \sum_{i=1}^{k} t_i^{-iN} \|S^{iN} h_2\| \leqslant \sum_{i=1}^{k} a^{-iN} \|h_2\| < \varepsilon_1,$$

where the last inequality follows from (3.4). Thus, $g \in B(g, \delta) \cap B(h_1, \varepsilon_1) \subseteq B(g, \delta) \cap U_1 = V$.

By Theorem 2.1, it remains to show $F_t^{iN}(V) \subseteq U_2$ whenever $1 \leqslant i \leqslant k$ and $t \in [t_{i-1}, t_i]$. Let $1 \leqslant i_0 \leqslant k$ and suppose $t \in [t_{i_0 - 1}, t_{i_0}]$. From the definition of g and (3.3), we have

$$\|F_t^{i_0 N} g - h_2\| \leqslant \|t_{i_0}^{-i_0 N} F_t^{i_0 N} S^{i_0 N} h_2 - h_2\| + \sum_{i=i_0+1}^{k} \|t_i^{-iN} F_t^{i_0 N} S^{iN} h_2\|. \tag{3.6}$$

To estimate the sum above, we note that if $i_0 \leqslant i \leqslant k$ and $j \geqslant 0$, then $t_i^{-iN} F_t^{i_0 N} S^{iN} e_j = \frac{t^{i_0 N}}{t_i^{iN}} e_{j+(i-i_0)N}$, and so

$$\sum_{i=i_0+1}^{k} \|t_i^{iN} F_t^{i_0 N} S^{iN} h_2\| = \sum_{i=i_0+1}^{k} \frac{t^{i_0 N}}{t_i^{iN}} \|h_2\| \leqslant \sum_{i=i_0+1}^{k} t_i^{-(i-i_0)N} \|h_2\| \tag{3.7}$$
$$\leqslant \sum_{i=i_0+1}^{k} a^{-(i-i_0)N} \|h_2\|$$
$$< \frac{\varepsilon_2}{4}, \quad \text{by (3.4).}$$

To estimate the middle term in (3.6), we note that $t_{i_0} = t_{i_0-1}(1 + \frac{x}{i_0 N})$ by definition, and we get

$$\|t_{i_0}^{-i_0 N} F_t^{i_0 N} S^{i_0 N} h_2 - h_2\| = \left| \left(\frac{t}{t_{i_0}}\right)^{i_0 N} - 1 \right| \|h_2\| \leqslant \left(1 - \left(\frac{t_{i_0-1}}{t_{i_0}}\right)^{i_0 N}\right) \|h_2\|$$
$$= \left(1 - \left(1 + \frac{x}{i_0 N}\right)^{-i_0 N}\right) \|h_2\| \tag{3.8}$$
$$= \left(1 - \mathrm{e}^{-x} + \mathrm{e}^{-x} - \left(1 + \frac{x}{i_0 N}\right)^{-i_0 N}\right) \|h_2\|$$
$$< \frac{\varepsilon_2}{8} + \frac{\varepsilon_2}{8}, \quad \text{by (3.2) and (3.5)}$$
$$= \frac{\varepsilon_2}{4}.$$

Combining (3.7) and (3.8) with (3.6) yields $\|F_t^{i_0 N} g - h_2\| < \frac{\varepsilon_2}{4} + \frac{\varepsilon_2}{4} = \frac{\varepsilon_2}{2}$. If $g' \in V$, then

$$\|F_t^{i_0 N} g' - h_2\| \leqslant \|F_t\|^{i_0 N} \|g' - g\| + \|F_t^{i_0 N} g - h_2\| < \|F_t\|^{i_0 N} \frac{\varepsilon_2}{2\|F_b\|^{kN}} + \frac{\varepsilon_2}{2}$$
$$\leqslant \frac{\varepsilon_2}{2} + \frac{\varepsilon_2}{2}, \quad \text{because } 1 < \|F_t\| \leqslant \|F_b\| = \varepsilon_2.$$

Hence, $F_t^{i_0 N}(V) \subseteq B(h_2, \varepsilon_2) \subseteq U_2$ and it concludes the whole proof. ∎

Using some ideas in the proof of Proposition 3.1, we now prove that there exists a path of hypercyclic operators with a dense G_δ set of common hypercyclic vectors between two particular unilateral weighted shifts whose weights are greater than 1.

PROPOSITION 3.2. *Let $T_0, T_1 : \ell^2 \to \ell^2$ be two unilateral weighted backward shifts with weight sequences $\{w_j : j \geqslant 1\}$ and $\{v_j : j \geqslant 1\}$, respectively, with $w_j \geqslant 1$ and $v_j \geqslant \max\{w_j, 2\}$. If T_0 is hypercyclic, then there exists a path of unilateral weighted backward shifts between T_0 and T_1 such that the set of common hypercyclic vectors for the whole path is a dense G_δ set.*

Proof. For each $t \in [0,1]$, let $F_t = (1-t)T_0 + tT_1$. Note that F_t is a unilateral weighted backward shift with the weight sequence $\{w_j^{(t)} : j \geqslant 1\} = \{(1-t)w_j + tv_j : j \geqslant 1\}$. Clearly, $t \mapsto F_t$ is a continuous map, and so $\{F_t : t \in [0,1]\}$ is a path of operators between $T_0 = F_0$ and $T_1 = F_1$. Next, observe that

$$\bigcap_{t\in[0,1]} \mathcal{HC}(F_t) = \mathcal{HC}(T_0) \cap \Big[\bigcap_{n\in\mathbb{N}} \bigcap_{t\in[\frac{1}{n},1]} \mathcal{HC}(F_t) \Big].$$

Thus, it suffices to show the family $\{F_t : t \in [r,1]\}$ with $0 < r < 1$ has a dense G_δ set of common hypercyclic vectors.

Let $r \in (0,1)$ and let U_1, U_2 be two nonempty open sets in ℓ^2. Choose $h_1, h_2, \varepsilon_1, \varepsilon_2$, and m as in the proof of Proposition 3.1. Next, choose $x > 0$ and an integer $N \geqslant 1$ satisfying (3.2), (3.3) with $a = 0$ and $b = 1$, and (3.5). Further suppose

$$\sum_{i=1}^{\infty} (1+r)^{-iN} < \min\Big\{ \frac{\varepsilon_1}{\|h_2\|}, \frac{\varepsilon_2}{4\|h_2\|} \Big\}. \tag{3.9}$$

Define $t_0 = r$ and $t_i = t_{i-1}(1 + \frac{x}{iN})$ for $i \geqslant 1$. As in the proof of Proposition 3.1, there is an integer k with $r = t_0 < t_1 < \cdots < t_{k-1} < 1 \leqslant t_k$. For each $t \in [r,1]$, define $S_t : \ell^2 \to \ell^2$ by $S_t e_j = \frac{1}{w_{j+1}^{(t)}} e_{j+1}$. Let $g = h_1 + \sum_{i=1}^{k} S_{t_i}^{iN} h_2$.

If we can show $\|h_1 - g\| < \varepsilon_1$, and for all $t \in [t_{i_0-1}, t_{i_0}]$, we have

$$\sum_{i=i_0+1}^{k} \|F_t^{i_0 N} S_{t_i}^{iN} h_2\| < \frac{\varepsilon_2}{4}, \tag{3.10}$$

$$\|F_t^{i_0 N} S_{t_{i_0}}^{i_0 N} h_2 - h_2\| < \frac{\varepsilon_2}{4}, \tag{3.11}$$

then we can estimate $\|F_t^{i_0 N} g - h_2\|$ with the same argument for the two terms of the right hand side of (3.6). This easily allows one to conclude the whole proof in the same way as Proposition 3.1, and we omit the details.

To estimate $\|h_1 - g\|$, we note that by the hypothesis on the weight sequences,

$$w_j^{(t)} = (1-t)w_j + tv_j \geqslant (1-t) + 2t \geqslant 1 + r, \tag{3.12}$$

for all $t \in [r,1]$ and integers $j \geqslant 1$. If $1 \leqslant i \leqslant k$, then

$$\|S_{t_i}^{iN} h_2\|^2 = \sum_{j=0}^{m} \Big[\prod_{l=1}^{iN} w_{j+l}^{(t_i)} \Big]^{-2} |\langle h_2, e_j \rangle|^2 \leqslant (1+r)^{-2iN} \|h_2\|^2,$$

and so

$$\|h_1 - g\| \leqslant \sum_{i=1}^{k} \|S_{t_i}^{iN} h_2\| \leqslant \sum_{i=1}^{k} (1+r)^{-iN} \|h_2\| < \varepsilon_1,$$

where the last inequality follows from (3.9).

To prove (3.10), let $1 \leqslant i_0 \leqslant k$ and let $t \in [t_{i_0-1}, t_{i_0}]$. For $i_0 \leqslant i \leqslant k$, we have

$$F_t^{i_0N} S_{t_i}^{iN} e_j = \Big[\prod_{l=1}^{(i-i_0)N} w_{j+l}^{(t_i)} \Big]^{-1} \prod_{l=(i-i_0)N+1}^{iN} \frac{w_{j+l}^{(t)}}{w_{j+l}^{(t_i)}} e_{j+(i-i_0)N}. \tag{3.13}$$

By (3.12), (3.13) and because $w_j^{(t)} \leqslant w_j^{(t_i)}$, we have $\|F_t^{i_0N} S_{t_i}^{iN} h_2\| \leqslant (1+r)^{-(i-i_0)N} \|h_2\|$. Combining this inequality with (3.9) yields (3.10). To prove (3.11), we note that $t_{i_0} = t_{i_0-1}(1 + \frac{x}{i_0N})$, and hence

$$1 \geqslant \frac{w_j^{(t)}}{w_j^{(t_{i_0})}} \geqslant \frac{w_j^{(t_{i_0-1})}}{w_j^{(t_{i_0})}} = \frac{w_j + t_{i_0-1}(v_j - w_j)}{w_j + t_{i_0}(v_j - w_j)} = \frac{w_j + t_{i_0}(1 + \frac{x}{i_0N})^{-1}(v_j - w_j)}{w_j + t_{i_0}(v_j - w_j)}$$

$$= 1 - \left[1 - \left(1 + \frac{x}{i_0N}\right)^{-1}\right] \frac{t_{i_0}(v_j - w_j)}{w_j + t_{i_0}(v_j - w_j)} \geqslant \left(1 + \frac{x}{i_0N}\right)^{-1}, \tag{3.14}$$

and so, for all j with $0 \leqslant j \leqslant m$, we have

$$\begin{aligned}
\Big| \prod_{l=1}^{i_0N} \frac{w_{j+l}^{(t)}}{w_{j+l}^{(t_{i_0})}} - 1 \Big| &\leqslant \left(1 - \left(1 + \frac{x}{i_0N}\right)^{-i_0N}\right) = 1 - \mathrm{e}^{-x} + \mathrm{e}^{-x} - \left(1 + \frac{x}{i_0N}\right)^{-i_0N} \\
&< \frac{\varepsilon_2}{8\|h_2\|} + \frac{\varepsilon_2}{8\|h_2\|} \quad \text{by (3.2) and (3.5)} \\
&= \frac{\varepsilon_2}{4\|h_2\|}.
\end{aligned} \tag{3.15}$$

Therefore, using (3.13) with $i = i_0$, we get

$$\begin{aligned}
\|F_t^{i_0N} S_{t_{i_0}}^{i_0N} h_2 - h_2\|^2 &= \sum_{j=0}^{m} \Big| \prod_{l=1}^{i_0N} \frac{w_{j+l}^{(t)}}{w_{j+l}^{(t_{i_0})}} - 1 \Big|^2 |\langle h_2, e_j \rangle|^2 \\
&< \left(\frac{\varepsilon_2}{4\|h_2\|}\right)^2 \|h_2\|^2 \quad \text{by (3.15)} \\
&= \left(\frac{\varepsilon_2}{4}\right)^2. \quad \blacksquare
\end{aligned}$$

As a consequence of Proposition 3.2, there exists a path of hypercyclic operators with a dense G_δ set of common hypercyclic vectors between any two hypercyclic unilateral weighted backward shifts whose weight sequences are bounded below by 1. To find such a path, let T_0 and T_1 be two such shifts with the weight sequences $\{w_j : j \geqslant 1\}$ and $\{w'_j : j \geqslant 1\}$, respectively. Let T be the unilateral weighted backward shift with the weight sequence $\{v_j : j \geqslant 1\}$ given by $v_j = \max\{w_j, w'_j, 2\}$. By Proposition 3.2, there exists a path of hypercyclic operators with a dense G_δ set of common hypercyclic vectors between T_0 and T. There is also such a path between T and T_1. Combining the two paths, along with an application of the Baire Category Theorem, gives us the desired path between T_0 and T_1.

One may wonder why the path given in the proof of Proposition 3.2 does not work for an arbitrary pair of hypercyclic unilateral weighted backward shifts. The answer is rather simple. It is just because the set of all such shifts is not convex. To show this, it suffices to find two hypercyclic unilateral weighted backward shifts T_0 and T_1 such that $\frac{1}{2}T_0 + \frac{1}{2}T_1$ is not hypercyclic. For that we first set $n_1 = 1, w_1 = \frac{3}{2}$, and $v_1 = \frac{1}{2}$. Inductively, we define a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers and two sequences $\{w_j : j \geqslant 1\}$, $\{v_j : j \geqslant 1\}$ of weights in the following manner. When k is even, we choose a positive integer $n_k > n_{k-1}$ such that

$$\left(\frac{3}{2}\right)^{n_k} \prod_{j=1}^{n_1+\cdots+n_{k-1}} v_j > k,$$

and let $w_{n_1+\cdots+n_{k-1}+1} = \cdots = w_{n_1+\cdots+n_k} = \frac{1}{2}$ and $v_{n_1+\cdots+n_{k-1}+1} = \cdots = v_{n_1+\cdots+n_k} = \frac{3}{2}$. When k is odd, we choose a positive integer $n_k > n_{k-1}$ such that

$$\left(\frac{3}{2}\right)^{n_k} \prod_{j=1}^{n_1+\cdots+n_{k-1}} w_j > k,$$

and let $w_{n_1+\cdots+n_{k-1}+1} = \cdots = w_{n_1+\cdots+n_k} = \frac{3}{2}$ and $v_{n_1+\cdots+n_{k-1}+1} = \cdots = v_{n_1+\cdots+n_k} = \frac{1}{2}$. Let T_0 and T_1 be the unilateral weighted backward shifts with the weight sequences $\{w_j : j \geqslant 1\}$ and $\{v_j : j \geqslant 1\}$, respectively. It is clear from the definition that $\sup\left\{ \prod_{j=0}^{n} w_j : n \geqslant 1\right\} = \infty$ and $\sup\left\{ \prod_{j=0}^{n} v_j : n \geqslant 1\right\} = \infty$. Hence, by Salas' condition (3.1), the unilateral weighted backward shifts T_0, T_1 are hypercyclic. However, for any integer $j \geqslant 1$, we have $\frac{1}{2}T_0 e_j + \frac{1}{2}T_1 e_j = \frac{1}{2}(w_j + v_j)e_{j-1} = e_{j-1}$. Therefore, $\frac{1}{2}T_0 + \frac{1}{2}T_1$ is not hypercyclic.

In general, for two shifts with weight sequences not bounded below by 1, the above convexity discussion makes it clear that the argument for Proposition 3.2 does not work. Nevertheless, we are able to find a more sophisticated path using a totally different argument which involves grouping weights of a particular path into blocks and considering their geometric means. We first study how to increase and decrease the weights in a block to keep its geometric mean a constant.

LEMMA 3.3. *Suppose $0 < v_1 \leqslant \cdots \leqslant v_m < G < w_1 \leqslant \cdots \leqslant w_n$, where $G = \sqrt[m+n]{v_1 v_2 \cdots v_m w_1 w_2 \cdots w_n}$ is the geometric mean. Define $p(s) = (w_1 - s)(w_2 - s)\cdots(w_n - s)$ and $q(t) = (v_1 + t)(v_2 + t)\cdots(v_n + t)$. For the variables $s \in [0, w_1 - G]$ and $t \in [0, G - v_m]$ satisfying the relation $p(s)q(t) = G^{m+n}$, there exists at most one value of t at which $\frac{\mathrm{d}s}{\mathrm{d}t} = 1$.*

Proof. Clearly the equation $p(s)q(t) = G^{m+n}$ defines s as a strictly increasing function of t under the condition that $s \in [0, w_1 - G]$ and $t \in [0, G - v_m]$. Differentiating with respect to t, we have $p'(s)q(t)\frac{\mathrm{d}s}{\mathrm{d}t} + p(s)q'(t) = 0$. It follows that $\frac{\mathrm{d}s}{\mathrm{d}t} = 1$ if and only if $\frac{p'(s)}{p(s)} = -\frac{q'(t)}{q(t)}$. That is,

$$\sum_{j=1}^{n} \frac{1}{w_j - s} = \sum_{i=1}^{m} \frac{1}{v_i + t},$$

where the right-hand side decreases as t increases, and the left-hand side increases as s increases. Since s is a strictly increasing function of t, there exists at most one value of t so that the left-hand side equals the right-hand side. ∎

The above lemma enables us to construct a function ψ to increase and decrease a finite number of weights to their geometric mean. The function ψ eventually helps us define a path between two weight sequences that is continuous with respect to the supremum norm of the sequences.

LEMMA 3.4. *For $0 < a_1, a_2, \ldots, a_n \leqslant c$ and $G = (a_1 a_2 \cdots a_n)^{1/n}$, there exists a function $\psi : [0, c] \to \mathbb{R}^n$ with the following properties:*

(i) $\psi(0) = (a_1, a_2, \ldots, a_n)$.

(ii) $\psi(c) = (G, G, \ldots, G)$.

(iii) *If t is in $[0, c]$ and $\psi(t) = (b_1, b_2, \ldots, b_n)$, then $b_1 b_2 \cdots b_n = G^n$. Moreover, $a_i \leqslant b_i \leqslant G \leqslant c$ if $a_i \leqslant G$, and $G \leqslant b_i \leqslant a_i \leqslant c$ if $a_i \geqslant G$.*

(iv) $\|\psi(t_1) - \psi(t_2)\|_\infty \leqslant |t_1 - t_2|$, *whenever $t_1, t_2 \in [0, c]$.*

Proof. For notational simplicity, we first rename, reindex, and reorder the given positive numbers $a_1, a_2, \ldots, a_n$ as $v_1, \ldots, v_m, w_1, \ldots, w_n$ such that

$$0 < v_1 \leqslant \cdots \leqslant v_m \leqslant G \leqslant w_1 \leqslant \cdots \leqslant w_n \leqslant c$$

where $G = \sqrt[n+m]{v_1 \cdots v_m w_1 \cdots w_n}$. With this notational change, property (i) becomes $\psi(0) = (v_1, \ldots, v_m, w_1, \ldots, w_n)$. The other properties can be restated easily.

We first provide an outline. We can assume that there is at least one $v_i \neq G$, for otherwise $v_i = w_j = G$ for all i, j, and the lemma is trivial. Hence there is at least one $w_j \neq G$. Furthermore, if $v_m = v_{m-1} = \cdots = v_{m-k} = G$, then we set the m-th, $\ldots$, $(m-k)$-th coordinates of $\psi(t)$ to be the constant value G for all $t \in [0, c]$. The same process applies to the situation when $w_1 = \cdots = w_k = G$. It follows from the above observation that we can assume $v_m \neq G$ and $w_1 \neq G$, and find a procedure to define ψ on a subinterval $[0, d]$ of $[0, c]$ so that at the right-hand end point d of the subinterval, either the m-th coordinate v_m of $\psi(0)$ is increased to G or the $(m+1)$-st coordinate w_1 is decreased to G, or both at the same time. That increased or decreased coordinate is then kept constant G by ψ in the interval $[d, c]$. Then we repeat the procedure to increase at least one v_i to G, or decrease at least one w_j to G, or both on a subinterval whose left-hand end point is d. After applying the procedure at most $m + n$ times, then all coordinates of ψ become G. We must make sure that the lengths of subintervals generated in the procedures do not add up to c so that properties (ii) and (iii) hold and that ψ is continuous and satisfies property (iv).

We now assume $v_m \neq G$ and $w_1 \neq G$, and find the procedure in the above outline. Define $p(s) = (w_1 - s)(w_2 - s) \cdots (w_n - s)$ and $q(t) = (v_1 + t)(v_2 + t) \cdots (v_m + t)$. Clearly the equation $p(s)q(t) = G^{m+n}$ defines s as a strictly increasing function of t under the condition that $s \in [0, w_1 - G]$ and $t \in [0, G - v_m]$. Let $s = f(t)$. As t increases from 0, either t reaches the value $G - v_m$ first, or $s = f(t)$ reaches the value $w_1 - G$ first. In other words, either $f(G - v_m) \leqslant w_1 - G$ or there exists t_0 with $0 < t_0 < G - v_m$ and $f(t_0) = w_1 - G$.

We discuss only the case that $f(G - v_m) \leqslant w_1 - G$. The same idea works for the other case and the details are omitted. By Lemma 3.3, there exists at most one point α in $[0, G - v_m]$ such that $f'(\alpha) = 1$. Owing to this, along with the fact that the derivative f' is continuous, we proceed in four cases.

Case 1. Such α does not exist and $f'(t) < 1$ for all t in $[0, G - v_m]$. Define $\psi : [0, G - v_m] \to \mathbb{R}^{m+n}$ by

$$\psi(t) = (v_1 + t, \ldots, v_m + t, w_1 - f(t), \ldots, w_n - f(t)).$$

Since $f'(t) < 1$ for all t in $[0, G - v_m]$, we use the mean value theorem to see that if $t_1, t_2 \in [0, G - v_m]$ then $|f(t_2) - f(t_1)| \leqslant |t_2 - t_1|$, and hence $\|\psi(t_2) - \psi(t_1)\|_\infty = |t_2 - t_1|$, satisfying property (iv). Note that $\psi(G - v_m) = (v_1 + (G - v_m), \ldots, v_{m-1} + (G - v_m), G, w_1 - f(G - v_m), \ldots, w_n - f(G - v_m))$. Since $f(G - v_m) \leqslant w_1 - G$, we have $w_j - f(G - v_m) \geqslant w_1 - f(G - v_m) \geqslant G$. Observe also that $v_i + (G - v_m) = G - (v_m - v_i) \leqslant G$. We come to the conclusion that Lemma 3.3 applies to these coordinates, $v_i + (G - v_m)$ and $w_j - f(G - v_m)$, whose values are not exactly G.

Case 2. Such α does not exist and $f'(t) > 1$ for all t in $[0, G - v_m]$. Define $\psi : [0, f(G - v_m)] \to \mathbb{R}^{m+n}$ by

$$\psi(s) = (v_1 + f^{-1}(s), \ldots, v_m + f^{-1}(s), w_1 - s, \ldots, w_n - s).$$

Here the inverse function f^{-1} exists and is differentiable because the definition $p(s)q(t) = G^{m+n}$ also defines t as a function of s in their appropriate ranges. Applying the chain rule of differentiation to the equation $f(f^{-1}(s)) = s$, we obtain that $(f^{-1})'(s) \leqslant 1$ whenever s is in $[0, f(G - v_m)]$. As in Case 1, we apply the mean value to conclude that $\|\psi(s_1) - \psi(s_2)\|_\infty = |s_1 - s_2|$, whenever $s_1, s_2 \in [0, f(G - v_m)]$, satisfying property (iv). Note that $\psi(f(G - v_m)) = (v_1 + G - v_m, \ldots, v_{m-1} + G - v_m, G, w_1 - f(G - v_m), \ldots, w_n - f(G - v_m))$, which leads us to the conclusion of Case 1.

Case 3. Such α exists and $f'(t) \leqslant 1$ whenever $t \in [0, \alpha]$, and $f'(t) \geqslant 1$ whenever $t \in [\alpha, G - v_m]$. Define $\psi : [0, f(G - v_m) - f(\alpha) + \alpha] \to \mathbb{R}^{m+n}$ by $\psi(t) = (v_1 + t, \ldots, v_m + t, w_1 - f(t), \ldots, w_n - f(t))$ whenever $t \in [0, \alpha]$, and furthermore

$$\begin{aligned}\psi(s) = (&v_1 + f^{-1}(s + f(\alpha) - \alpha), \ldots, v_m + f^{-1}(s + f(\alpha) - \alpha),\\ &w_1 - (s + f(\alpha) - \alpha), \ldots, w_n - (s + f(\alpha) - \alpha)),\end{aligned}$$

whenever $s \in [\alpha, f(G - v_m) - f(\alpha) + \alpha]$. This definition clearly provides a continuous function ψ. Since $f'(t) \leqslant 1$ on $[0, \alpha]$, we use the mean value theorem to see that $\|\psi(t_1) - \psi(t_2)\|_\infty = |t_1 - t_2|$, whenever $t_1, t_2 \in [0, \alpha]$. In addition, the fact that $f^{-1}(f(t)) = t$ gives the inequality $(f^{-1})'(f(t)) \leqslant 1$ for all $t \in [\alpha, G - v_m]$. It follows that if we put $u(s) = f^{-1}(s + f(\alpha) - \alpha)$, then $\frac{\mathrm{d}u}{\mathrm{d}s} \leqslant 1$ whenever $s \in [\alpha, f(G - v_m) - f(\alpha) + \alpha]$. Hence if $s_1, s_2 \in [\alpha, f(G - v_m) - f(\alpha) + \alpha]$, then by the mean value theorem, $|f^{-1}(s_1 + f(\alpha) - \alpha) - f^{-1}(s_2 + f(\alpha) - \alpha)| \leqslant |s_1 - s_2|$ and hence, $\|\psi(s_1) - \psi(s_2)\|_\infty = |s_1 - s_2|$. Furthermore, if $t \in [0, \alpha]$, and $s \in [\alpha, f(G - v_m) - f(\alpha) + \alpha]$, then $\|\psi(s) - \psi(t)\|_\infty \leqslant \|\psi(s) - \psi(\alpha)\|_\infty + \|\psi(\alpha) - \psi(t)\|_\infty = |s - \alpha| + |\alpha - t| = |s - t|$, satisfying property (iv). At the right-hand end point $f(G - v_m) - f(\alpha) + \alpha$ of the domain of ψ, we have that $\psi(f(G - v_m) - f(\alpha) + \alpha) = (v_1 + (G - v_m), \ldots, v_{m-1} + (G - v_m), G, w_1 - f(G - v_m), \ldots, w_n - f(G - v_m))$, which leads us to the conclusion of Case 1 again.

Case 4. Such α exists and $f'(t) \geqslant 1$ for all $t \in [0, \alpha]$ and $f'(t) \leqslant 1$ for all $t \in [\alpha, G - v_m]$. Define $\psi : [0, (G - v_m) + f(\alpha) - \alpha] \to \mathbb{R}^{m+n}$ by $\psi(s) = (v_1 + f^{-1}(s), \ldots, v_m + f^{-1}(s), w_1 + s, \ldots, w_n + s)$ whenever $s \in [0, f(\alpha)]$, and furthermore

$$\begin{aligned}\psi(t) = (&v_1 + t + \alpha - f(\alpha), \ldots, v_m + t + \alpha - f(\alpha),\\ &w_1 - f(t + \alpha - f(\alpha)), \ldots, w_n - f(t + \alpha - f(\alpha)),\end{aligned}$$

whenever $t \in [f(\alpha), (G - v_m) + f(\alpha) - \alpha]$. This definition clearly provides a continuous function ψ. Since $f^{-1}(f(t)) = t$, and $f'(t) \geqslant 1$ whenever $t \in [0, \alpha]$, we have $(f^{-1})'(f(t)) \leqslant 1$. Thus, if $t \in [0, \alpha]$, then

$$\frac{\mathrm{d}f^{-1}(s)}{\mathrm{d}s} = \frac{\mathrm{d}t}{\mathrm{d}s} \leqslant 1.$$

It follows that if $s_1, s_2 \in [0, f(\alpha)]$, then $|f^{-1}(s_1) - f^{-1}(s_2)| \leqslant |s_1 - s_2|$ and hence $\|\psi(s_1) - \psi(s_2)\|_\infty = |s_1 - s_2|$. Note that $f'(t + \alpha - f(\alpha)) \leqslant 1$ whenever $t \in [f(\alpha), (G - v_m) + f(\alpha) - \alpha]$. Thus, if $t_1, t_2 \in [f(\alpha), (G - v_m) + f(\alpha) - \alpha]$, then $|f(t_1 + \alpha - f(\alpha)) - f(t_2 + \alpha - f(\alpha))| \leqslant |t_1 - t_2|$, and so $\|\psi(t_1) - \psi(t_2)\|_\infty = |t_1 - t_2|$. One can then check that if $s \in [0, f(\alpha)]$ and $t \in [f(\alpha), (G - v_m) + f(\alpha) - \alpha]$, then $\|\psi(t) - \psi(s)\|_\infty \leqslant |t - s|$, showing that property (iv) is satisfied. At the right-hand end point of its domain, $\psi((G - v_m) + f(\alpha) - \alpha) = (v_1 + (G - v_m), \ldots, v_{m-1} + (G - v_m), G, w_1 - f(G - v_m), \ldots, w_n - f(G - v_m))$. We come to the conclusion of Case 1.

Observe that in all four cases, the weights below G increase towards G, and the weights above G decrease towards G. The above procedure is given for the case $f(G - v_m) \leqslant w_1 - G$. The procedure proceeds in a similar fashion for the other case that there exists t_0 with $0 < t_0 < G - v_m$ and $f(t_0) = w_1 - G$. In either case, we obtain a continuous function ψ on a subinterval $I' = [0, x']$ of $[0, c]$. Let a' = length of subinterval of I' on which $f'(t) \leqslant 1$, and b' = length of subinterval of I' on which $(f^{-1})'(s) \leqslant 1$. Clearly $a' + b'$ = length of I', from the procedure. Furthermore, at the right-hand end point x' of the interval I', if we write $\psi(x') = (v'_1, \ldots, v'_m, w'_1, \ldots, w'_n)$, then either $v'_m = G$ or $w'_1 = G$. Moreover, $v'_i \geqslant v_i + a'$ and $w'_j \leqslant w_j - b'$ for all i, j, and hence in particular, $v'_1 \geqslant v_1 + a'$ and $w'_n \leqslant w_n - b'$.

Since G is the geometric mean, we see that $v'_1 = G$ if and only if $w'_n = G$. If $v'_1 \neq G$, then we repeat the above procedure to continue the definition of ψ on an interval $I'' = [x', x'']$ whose left-hand end point x' coincides with the right-hand end point of the interval I'. At the right-hand end point x'', we have $\psi(x'') = (v''_1, \ldots, v''_m, w''_1, \ldots, w''_n)$ with at least two values of v''_{m-1}, v''_m, w''_1, w''_2 equal to G. Let a'' = length of subinterval of I'' on which $f'(t) < 1$, and let b'' = length of subinterval of I'' on which $(f^{-1})'(s) < 1$. Clearly $a'' + b''$ = length of I'', and $v''_1 \geqslant v_1 + a''$ and $w''_n \leqslant w_n - b''$

We apply the procedure again if $v''_1 \neq G$. After a finite number, say k with k at most $m + n$, of applications of the procedure, we have ψ defined on an interval $I^{(k)}$ whose length is

$$\begin{aligned}(a' + b') + (a'' + b'') + \cdots + (a^{(k)} + b^{(k)}) &= (a' + \cdots + a^{(k)}) + (b' + \cdots + b^{(k)}) \\ &\leqslant (G - v_1) + (w_n - G) = w_n - v_1 < c.\end{aligned}$$

At the right-hand end point $x^{(k)}$ of the interval $I^{(k)}$, we have that $\psi(x^{(k)}) = (G, G, \ldots, G)$. Thus, we can continue to define ψ on the interval $[x^{(k)}, c]$ by setting it equal to the constant vector $(G, G, \ldots, G)$. This completes the proof of the lemma. ∎

Lemma 3.4 enables us to define a path for a finite number of weights that can keep the value of their product constant. All those paths combined give a path for the entire weight sequence with continuity guaranteed by property (iv) of the Lemma 3.4. With this idea, we are now ready to generalize the result in Proposition 3.2 and produce the main theorem of this section.

THEOREM 3.5. *Let $T_0, T_1 : \ell^2 \to \ell^2$ be two hypercyclic, unilateral weighted backward shifts. Then there exists a path of unilateral weighted backward shifts between T_0 and T_1 such that the set of common hypercyclic vectors for the whole path is a dense G_δ set.*

Proof. As we have remarked after the proof of Proposition 3.2, the theorem holds for two hypercyclic unilateral weighted backward shifts whose weight sequences are bounded below by 1. To prove the theorem, it suffices to find a path of unilateral weighted backward shifts with a dense G_δ set of common hypercyclic vectors between T_0 and a hypercyclic unilateral weighted shift whose weight sequence is bounded below by 1. Then the same argument produces such a path for T_1, and an application of the Baire Category Theorem finishes the whole proof.

Let $\{w_j : j \geqslant 1\}$ be the weight sequence for T_0 and let $c = \|T_0\| = \sup\{w_j : j \geqslant 1\} > 1$. Before we construct the desired path, we first need to prove a claim about the weight sequence $\{w_j : j \geqslant 1\}$.

CLAIM 1. *There exists a sequence $(n_k)_{k=0}^{\infty}$ of nonnegative integers such that:*
(i) $n_0 = 0$ *and* $n_k > n_{k-1} + 2k$ *for each* $k \geqslant 1$*;*
(ii) *for each* $k \geqslant 1$*, we have* $\prod_{i=n_{k-1}+1}^{n_k} w_i > c^{n_{k-1}+4k}$*;*
(iii) *for each* $k \geqslant 1$ *and* $0 \leqslant j \leqslant k-1$*, we have* $w_{n_k-j} > c^{-j}$*.*

Proof of Claim 1. Set $n_0 = 0$. Since T_0 is a hypercyclic unilateral weighted backward shift, by Salas' Condition (3.1), the set

$$A_1 = \Big\{ n \in \mathbb{N} : n \geqslant 2 \text{ and } \prod_{i=1}^{n} w_i > c^4 \Big\}$$

is a nonempty subset of the natural numbers $\mathbb{N}$. Set $n_1 = \min A_1$. Inductively define $(n_k)_{k=0}^{\infty}$ by letting

$$A_k = \Big\{ n \in \mathbb{N} : n \geqslant n_{k-1} + 2k \text{ and } \prod_{i=n_{k-1}+1}^{n} w_i > c^{n_{k-1}+4k} \Big\} \neq \varnothing,$$

and setting $n_k = \min A_k$. Clearly, the sequence $(n_k)_{k=0}^{\infty}$ satisfies (i) and (ii). To show (iii), let $k \geqslant 1$. Since $c = \sup\{w_j : j \geqslant 1\}$ and $\prod_{i=n_{k-1}+1}^{n_k} w_i > c^{n_{k-1}+4k}$, it follows that $n_k - n_{k-1} > n_{k-1} + 4k$. Thus, $n_k - j > n_{k-1} + 2k$ whenever $0 \leqslant j \leqslant k$. Furthermore, since $n_k = \min A_k$, we get

$$w_{n_{k-1}+1} w_{n_{k-1}+2} \cdots w_{n_k-j} \leqslant c^{n_{k-1}+4k} \quad \text{whenever } 0 \leqslant j \leqslant k.$$

Let $0 \leqslant j \leqslant k-1$. If we had $w_{n_k-j} w_{n_k-j+1} \cdots w_{n_k} \leqslant 1$, then it would follow that

$$c^{n_{k-1}+4k} < w_{n_{k-1}+1} w_{n_{k-1}+2} \cdots w_{n_k} \leqslant w_{n_{k-1}+1} \cdots w_{n_k-(j+1)} \leqslant c^{n_{k-1}+4k},$$

which would be a contradiction. Hence, we must have

$$1 < w_{n_k-j} w_{n_k-j+1} \cdots w_{n_k} \leqslant c^j w_{n_k-j}.$$

Thus, $w_{n_k-j} > c^{-j}$ and this concludes the proof of Claim 1. ∎

For each $k \geqslant 1$, let $\varphi_k : [0, c] \to \mathbb{R}^n$ be a mapping satisfying the conditions in Lemma 3.4 with $n = n_k - n_{k-1}$, $\alpha_1 = w_{n_{k-1}+1}, \alpha_2 = w_{n_{k-1}+2}, \ldots, \alpha_n = w_{n_k}$ and $c = \sup\{w_j : j \geqslant 1\}$. For each $k \geqslant 1$, each i with $1 \leqslant i \leqslant n_k - n_{k-1}$ and $t \in [0, c]$, let $w_{n_{k-1}+i}^{(t)}$ be the i-th entry in the vector $\varphi_k(t)$. Thus, if we put the terms $w_i^{(t)}$ together as a vector, then it follows from conditions (i) and (ii) in Lemma 3.4 and from condition (ii) in Claim 1 that

$$(w_1^{(0)}, w_2^{(0)}, w_3^{(0)}, \ldots) = (w_1, w_2, w_3, \ldots), \tag{3.16}$$

$$(w_1^{(c)}, w_2^{(c)}, \ldots) = (\underbrace{G_1, \ldots, G_1}_{n_1 - n_0 \text{copies}}, \underbrace{G_2, \ldots, G_2}_{n_2 - n_1 \text{copies}}, \underbrace{G_3, \ldots G_3}_{n_3 - n_2 \text{copies}}, \ldots), \tag{3.17}$$

where each $G_k = \Big(\prod_{i=n_{k-1}+1}^{n_k} w_i \Big)^{1/(n_k - n_{k-1})} > c^{(n_{k-1}+4k)/(n_k-n_{k-1})} > 1$. From condition (iii) in Lemma 3.4, we get

$$\prod_{i=1}^{n_k} w_i^{(t)} = \prod_{i=1}^{n_k} w_i > c^{\sum_{i=1}^{k} n_{i-1}+4i} \quad \text{for all } t \in [0,c]. \tag{3.18}$$

Moreover, if $k \geqslant 1$ and $1 \leqslant i \leqslant n_k - n_{k-1}$, then we see that whenever $w_{n_{k-1}+i} \geqslant G_k$,

$$1 \leqslant G_k \leqslant w^{(t)}_{n_{k-1}+i} \leqslant w_{n_{k-1}+i} \leqslant c, \tag{3.19}$$

and also whenever $w_{n_{k-1}+i} \leqslant G_k$,

$$w_{n_{k-1}+i} \leqslant w^{(t)}_{n_{k-1}+i} \leqslant G_k \leqslant c. \tag{3.20}$$

Let $\beta_j = \inf\{w_j^{(t)} : t \in [0,c]\}$. From (3.19) and (3.20), we get $\beta_j > 0$. Moreover, with Claim 1, for $k \geqslant 1$ and $0 \leqslant j \leqslant k-1$, we get

$$c^{-j} < \beta_{n_k - j}. \tag{3.21}$$

From condition (iv) in Lemma 3.4, for each $k \geqslant 1$ and $1 \leqslant i \leqslant n_k - n_{k-1}$, we have

$$|w^{(t)}_{n_{k-1}+i} - w^{(t')}_{n_{k-1}+i}| \leqslant \|\varphi_k(t) - \varphi_k(t')\|_\infty \leqslant |t - t'|. \tag{3.22}$$

For each $t \in [0,c]$, let F_t be the unilateral weighted backward shift with the weight sequence $\{w_j^{(t)} : j \geqslant 1\}$. From (3.16) and (3.17), we get $F_0 = T_0$ and F_c is a unilateral weighted backward shift whose weight sequence is bounded below by 1. The map $t \mapsto F_t$ is continuous by (3.22). We now use Theorem 2.4 to show the path $\{F_t : t \in [0,c]\}$ of unilateral weighted backward shifts between $F_0 = T_0$ and F_c has a dense G_δ set of common hypercyclic vectors.

Let $D_1 = \operatorname{span}\{e_j : j \geqslant 0\} \setminus \{0\}$. Let $h \in D_1$ and let $\varepsilon > 0$. Since $h \in D_1$, there exists $N \geqslant 2$ such that

$$h = \sum_{j=1}^{N} \langle h, e_j \rangle e_j.$$

Let $m_k = 0$ if $1 \leqslant k \leqslant N$, and let $m_k = n_k - N$ if $k \geqslant N+1$. Set

$$\delta = \frac{\varepsilon}{N c^{N^2} M \|h\|} \quad \text{where } M = \max\Big\{1, \frac{1}{\beta_1}, \frac{1}{\beta_1\beta_2}, \dots, \frac{1}{\beta_1 \cdots \beta_N}\Big\}.$$

Let $D_2 = D_1$. Lastly, for each $t \in [0, c]$, define $S_{t,k} \equiv 0$ if $1 \leqslant k \leqslant N$. Otherwise, define $S_{t,k} : D_1 \to \ell^2$ by

$$S_{t,k}e_j = \begin{cases} \dfrac{\prod_{i=1}^{N} w_{m_k+i}^{(t)}}{\prod_{i=1}^{n_k} w_i} e_{m_k} & \text{if } j = 0, \\ \dfrac{\prod_{i=1}^{j} w_i^{(t)} \prod_{i=j+1}^{N} w_{m_k+i}^{(t)}}{\prod_{i=1}^{n_k} w_i} e_{m_k+j} & \text{if } 1 \leqslant j \leqslant N-1, \\ \dfrac{\prod_{i=1}^{N} w_i^{(t)}}{\prod_{i=1}^{n_k} w_i} e_{m_k+N} & \text{if } j = N, \\ 0, & \text{otherwise.} \end{cases}$$

To prove condition (i) in Theorem 2.4, observe that by (3.18),(3.19), and (3.20), for any j with $0 \leqslant j \leqslant N$, we have $|\langle S_{t,k}e_j, e_{j+m_k}\rangle| \leqslant c^m c^{-\sum_{i=1}^{k}(n_{i-1}+4i)}$, and so

$$\|S_{t,k}h\| \leqslant c^N c^{-\sum_{i=1}^{k}(n_{i-1}+4i)} \|h\| \to 0 \quad \text{as } k \to \infty.$$

Since F_t is a unilateral weighted backward shift, for each $f \in D_2$, $F_t^n f = 0$ for all $t \in [0, c]$ and for all sufficiently large n. This proves condition (ii) of Theorem 2.4. To establish its condition (iii), we first need another claim:

CLAIM 2. *For each $k \geqslant N+1$ and $1 \leqslant l \leqslant N$, we have*

$$\Big|\prod_{i=1}^{l} \frac{w_i^{(t')}}{w_i^{(t)}} - 1\Big| \leqslant MNc^{N^2}|t-t'|, \quad \textit{and} \quad \Big|\prod_{i=0}^{l-1} \frac{w_{n_k-i}^{(t')}}{w_{n_k-i}^{(t)}} - 1\Big| \leqslant MNc^{N^2}|t-t'|.$$

Furthermore, if $1 \leqslant j \leqslant N-1$, then

$$\Big|\prod_{i=1}^{j} \frac{w_i^{(t')}}{w_i^{(t)}} \prod_{i=0}^{N-j-1} \frac{w_{n_k-i}^{(t')}}{w_{n_k-i}^{(t)}} - 1\Big| \leqslant MNc^{N^2}|t-t'|.$$

Proof of Claim 2. Using equation (3.22) and the triangle inequality, one easily sees that

$$|w_{i_1}^{(t')} w_{i_2}^{(t')} \cdots w_{i_n}^{(t')} - w_{i_1}^{(t)} w_{i_2}^{(t)} \cdots w_{i_n}^{(t)}| \leqslant nc^{n-1}|t-t'| \tag{3.23}$$

for any $i_1, i_2, \dots, i_n \in \mathbb{N}$. From (3.23) and the definitions of β_i and M, it follows that if $1 \leqslant l \leqslant N$, then

$$\begin{aligned} \Big|\prod_{i=1}^{l} \frac{w_i^{(t')}}{w_i^{(t)}} - 1\Big| &= \Big(\prod_{i=1}^{l} w_i^{(t)}\Big)^{-1} \Big|\prod_{i=1}^{l} w_i^{(t')} - \prod_{i=1}^{l} w_i^{(t)}\Big| \\ &\leqslant \Big(\prod_{i=1}^{l} \beta_i\Big)^{-1} lc^{l-1}|t-t'| \leqslant MNc^{N^2}|t-t'|, \end{aligned}$$

and furthermore if $k \geqslant N+1$,

$$\Big|\prod_{i=0}^{l-1} \frac{w_{n_k-i}^{(t')}}{w_{n_k-i}^{(t)}} - 1\Big| = \Big(\prod_{i=0}^{l-1} w_{n_k-i}^{(t)}\Big)^{-1} \Big|\prod_{i=0}^{l-1} w_{n_k-i}^{(t')} - \prod_{i=0}^{l-1} w_{n_k-i}^{(t)}\Big| \leqslant \Big(\prod_{i=0}^{l-1} \beta_{n_k-i}\Big)^{-1} lc^{l-1}|t-t'|$$
$$\leqslant \Big(\prod_{i=0}^{l-1} c^i\Big) lc^{l-1}|t-t'|, \quad \text{by (3.21)}$$
$$\leqslant MNc^{N^2}|t-t'|.$$

Using a similar argument, if $1 \leqslant j \leqslant N-1$, then

$$\Big|\prod_{i=1}^{j} \frac{w_i^{(t')}}{w_i^{(t)}} \prod_{i=0}^{N-j-1} \frac{w_{n_k-i}^{(t')}}{w_{n_k-i}^{(t)}} - 1\Big| \leqslant \frac{\prod_{i=0}^{N-j-1} c^i}{\prod_{i=1}^{j} \beta_i} \Big|\prod_{i=1}^{j} w_i^{(t')} \prod_{i=0}^{N-j-1} w_{n_k-i}^{(t')} - \prod_{i=1}^{j} w_i^{(t)} \prod_{i=0}^{N-j-1} w_{n_k-i}^{(t)}\Big|$$
$$\leqslant Mc^{(N-j)(N-j-1)} Nc^{N-1}|t-t'| \leqslant MNc^{N^2}|t-t'|,$$

which finishes the proof of Claim 2. ∎

Let $k \geqslant N+1$. Observe that $\langle F_t^{m_k} S_{t',k} h - h, e_j\rangle = 0$ whenever $j \geqslant N+1$. Next, note that

$$\langle F_t^{m_k} S_{t',k} h, e_0\rangle = \langle h, e_0\rangle \langle F_t^{m_k} S_{t',k} e_0, e_0\rangle = \langle h, e_0\rangle \frac{\prod_{i=1}^{N} w_{m_k+i}^{(t')}}{\prod_{i=1}^{n_k} w_i} \prod_{i=1}^{m_k} w_i^{(t)} \tag{3.24}$$
$$= \langle h, e_0\rangle \frac{\prod_{i=1}^{N} w_{m_k+i}^{(t')}}{\prod_{i=1}^{n_k} w_i^{(t)}} \prod_{i=1}^{m_k} w_i^{(t)} \quad \text{by (3.18)}$$
$$= \langle h, e_0\rangle \prod_{i=1}^{N} \frac{w_{m_k+i}^{(t')}}{w_{m_k+i}^{(t)}}$$
$$= \langle h, e_0\rangle \prod_{i=0}^{N-1} \frac{w_{n_k-i}^{(t')}}{w_{n_k-i}^{(t)}} \quad \text{because } m_k = n_k - N.$$

and so, by Claim 2, we have

$$|\langle F_t^{m_k} S_{t',k} h - h, e_0\rangle| = |\langle h, e_0\rangle| \Big|\prod_{i=0}^{N-1} \frac{w_{n_k-i}^{(t')}}{w_{n_k-i}^{(t)}} - 1\Big| \leqslant |\langle h, e_0\rangle| MNc^{N^2}|t-t'|.$$

Using a similar argument,

$$\langle F_t^{m_k} S_{t',k} h, e_N\rangle = \langle h, e_N\rangle \frac{\prod_{i=1}^{N} w_i^{(t')}}{\prod_{i=1}^{n_k} w_i} \prod_{i=1}^{m_k} w_{N+i}^{(t)}$$
$$= \langle h, e_N\rangle \frac{\prod_{i=1}^{N} w_i^{(t')}}{\prod_{i=1}^{n_k} w_i^{(t)}} \prod_{i=1}^{m_k} w_{N+i}^{(t)} = \langle h, e_n\rangle \prod_{i=1}^{N} \frac{w_i^{(t')}}{w_i^{(t)}},$$

and for $1 \leqslant j \leqslant N-1$,

$$\begin{aligned}\langle F_t^{m_k} S_{t',k} h, e_j\rangle &= \langle h, e_j\rangle \frac{\prod_{i=1}^{j} w_i^{(t')} \prod_{i=j+1}^{N} w_{m_k+i}^{(t')}}{\prod_{i=1}^{n_k} w_i} \prod_{i=1}^{m_k} w_{j+i}^{(t)} \\ &= \langle h, e_j\rangle \frac{\prod_{i=1}^{j} w_i^{(t')} \prod_{i=j+1}^{N} w_{m_k+i}^{(t')}}{\prod_{i=1}^{n_k} w_i^{(t)}} \prod_{i=1}^{m_k} w_{j+i}^{(t)} = \langle h, e_j\rangle \prod_{i=1}^{j} \frac{w_i^{(t')}}{w_i^{(t')}} \prod_{i=0}^{N-j-1} \frac{w_{n_k-i}^{(t')}}{w_{n_k-i}^{(t)}}.\end{aligned}$$

Thus, by Claim 2,

$$|\langle F_t^{m_k} S_{t',k} h - h, e_N\rangle| = |\langle h, e_N\rangle| \Big| \prod_{i=1}^{N} \frac{w_i^{(t')}}{w_i^{(t)}} - 1 \Big| \leqslant |\langle h, e_N\rangle| MNc^{N^2} |t-t'|,$$

and for $1 \leqslant j \leqslant N-1$,

$$|\langle F_t^{m_k} S_{t',k} h - h, e_j\rangle| = |\langle h, e_j\rangle| \Big| \prod_{i=1}^{j} \frac{w_i^{(t')}}{w_i^{(t)}} \prod_{i=0}^{N-j-1} \frac{w_{n_k-i}^{(t')}}{w_{n_k-i}^{(t)}} - 1 \Big| \leqslant |\langle h, e_j\rangle| MNc^{N^2} |t-t'|.$$

Therefore,

$$\|F_t^{m_k} S_{t',k} h - h\| \leqslant MNc^{N^2} |t-t'| \|h\|.$$

Recall that $\delta = \frac{\varepsilon}{MNc^{N^2}\|h\|}$. If $k \geqslant N+1$ and $t' \in [0, c]$, then

$$\|F_t^{m_k} S_{t',k} h - h\| < \varepsilon \quad \text{whenever } |t-t'| < \delta,$$

which proves condition (iii). ∎

As an immediate consequence of the above theorem, we obtain the following result.

COROLLARY 3.6. *The hypercyclic unilateral weighted backward shifts form a path connected subset of $B(\ell^2)$.*

Since the corollary does not involve common hypercyclic vectors, it can be directly proved without using our previous results. For example, take two hypercyclic unilateral weighted backward shifts T_0 and T_1 with positive weight sequences $\{w_j : j \geqslant 1\}$ and $\{v_j : j \geqslant 1\}$. Then, by symmetry, it suffices for us to find a path of hypercyclic unilateral weighted backward shifts between T_0 and the shift whose weight sequence is $\{\max\{w_j, v_j\} : j \geqslant 1\}$. This can be done by taking the path $\{F_t : t \in [0,1]\}$ where F_t is the unilateral weighted backward shift whose j-th weight is given by $(1-t)w_j + t\max\{w_j, v_j\}$, which is at least w_j. Salas' Condition (3.1) shows that each F_t along the path is hypercyclic.

4. NO COMMON HYPERCYCLIC VECTORS

The results in the previous two sections may lead one to wonder whether every path of hypercyclic operators must have a dense G_δ set of common hypercyclic vectors. At first glance, one may attempt to prove that by taking a countable subset of hypercyclic operators that are dense in the path. Using the fact that every hypercyclic operator has a dense G_δ set of hypercyclic vectors, one may hope to finish the proof by using the Baire Category Theorem and passing some of the hypercyclic vectors to the entire path by continuity. However, this argument does not work, not even for the special case of a path of unilateral weighted backward shifts, as we show in Theorem 4.1 below.

An example of a family of hypercyclic operators having no common hypercyclic vectors was provided by Borichev who took the family consisting of all hypercyclic operators $z_1 B \oplus z_2 B$, where B is the unilateral backward shift, and (z_1, z_2) in the unbounded region $R = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|, |z_2| > 1\}$; see page 495 of [1]. He also showed that if $\Omega \subseteq (1, \infty) \times (1, \infty)$ has positive Lebesgue measure, then the family $\{sB \oplus tB : (s,t) \in \Omega\}$ has no common hypercyclic vector; see Remark 6.3 of [7]. Thus, if we take a space filling curve $F : [0,1] \to [2,3] \times [2,3]$, then we have a path of operators of the form $sB \oplus tB$ having no common hypercyclic vector. This example leads us to wonder whether such a result can be obtained without the direct sum, and in particular with weighted shifts.

THEOREM 4.1. *There exists a path of hypercyclic unilateral weighted backward shifts on ℓ^2 having no common hypercyclic vector.*

Proof. For each $t \in [0, \frac{1}{4}]$, let F_t be a unilateral weighted backward shift whose weight sequence $\{w_j^{(t)} : j \geqslant 1\}$ is defined as follows. First, let $k_0 = m_0 = 1$, and $\alpha_0 = 0$, and inductively set for each $j \geqslant 0$,

$$
\begin{aligned}
k_{j+1} &= (2 + \alpha_j + m_j)^3, \\
m_{j+1} &= (j+1) + (k_0 + k_1 + \cdots + k_j + k_{j+1}), \text{ and} \\
\alpha_{j+1} &= m_{j+1} + (k_0 + k_1 + \cdots + k_j + k_{j+1}).
\end{aligned}
$$

Second, we observe that

$$m_{j+1} - \alpha_j - m_j = 1 + k_{j+1} - \alpha_j = 1 + (2 + \alpha_j + m_j)^3 - \alpha_j \geqslant 3,$$

and so $3 + \alpha_j + m_j \leqslant m_{j+1} \leqslant \alpha_{j+1}$, which enables us to define the weights $w_i^{(t)}$ block by block inductively, depending on the subindices i. For any integer $j \geqslant 0$, the $j+1$-st block B_{j+1} consists of integers i with $1 + \alpha_j \leqslant i \leqslant \alpha_{j+1}$, and the above observation shows that the integer $m_{j+1} \in B_{j+1}$. For the subindices in the block

B_{j+1}, we define

$$\begin{aligned} w_{1+\alpha_j}^{(t)} &= \left(\frac{1}{2}+t\right)^{k_j}, \\ w_{2+\alpha_j}^{(t)} &= \cdots = w_{1+\alpha_j+m_j}^{(t)} = 2, \\ w_{2+\alpha_j+m_j}^{(t)} &= 2^{-(m_j+k_0+k_1+\cdots+k_{j+1})}, \\ w_{3+\alpha_j+m_j}^{(t)} &= \cdots = w_{m_{j+1}}^{(t)} = 1, \text{ and} \\ w_{1+m_{j+1}}^{(t)} &= \cdots = w_{\alpha_{j+1}}^{(t)} = 2. \end{aligned}$$

Since all positive integers i are partitioned into blocks, the definition for all $w_i^{(t)}$ is completed. In general, for any integer $m \geqslant 0$, and any $i \in B_{m+1}$, the weight $w_i^{(t)}$ is a nonconstant function of t if and only if $i = 1 + \alpha_m$, and in that case $2^{-k_m} \leqslant w_{1+\alpha_m}^{(t)} \leqslant 1$. Hence, a little calculation involving the definitions shows that

$$2^{-k_m} \leqslant w_{1+\alpha_m}^{(t)} \cdots w_{\alpha_{m+1}}^{(t)} \leqslant 1, \tag{4.1}$$

which we use to estimate the product $w_1^{(t)} w_2^{(t)} \cdots w_n^{(t)}$ with $1 + \alpha_j \leqslant n \leqslant \alpha_{j+1}$. To begin, we write

$$\prod_{i=1}^{n} w_i^{(t)} = \prod_{i=1}^{\alpha_1} w_i^{(t)} \prod_{i=1+\alpha_1}^{\alpha_2} w_i^{(t)} \cdots \prod_{i=1+\alpha_{j-1}}^{\alpha_j} w_i^{(t)} \prod_{i=1+\alpha_j}^{n} w_i^{(t)}.$$

Hence, by using the definition of the weights, and repeatedly using inequality (4.1) with $m = 0, \ldots, j-1$, we see that if $1 + \alpha_j \leqslant n \leqslant 1 + \alpha_j + m_j$ then

$$2^{-(k_0+\cdots+k_j)+n-1-\alpha_j} \leqslant \prod_{i=1}^{n} w_i^{(t)} \leqslant 2^{n-1-\alpha_j}, \tag{4.2}$$

and in particular, when $n = 1 + \alpha_j + m_j$ we have

$$2^{m_j-(k_0+\cdots+k_j)} \leqslant w_1^{(t)} \cdots w_{1+\alpha_j+m_j}^{(t)} \leqslant 2^{m_j}. \tag{4.3}$$

With this inequality, we continue to estimate the product for higher values of n. If $2 + \alpha_j + m_j \leqslant n \leqslant m_{j+1}$ then we have

$$2^{-2(k_0+\cdots+k_j)-k_{j+1}} \leqslant \prod_{i=1}^{n} w_i^{(t)} \leqslant 2^{-(k_0+\cdots+k_{j+1})}, \tag{4.4}$$

and furthermore if $1 + m_{j+1} \leqslant n \leqslant \alpha_{j+1}$ then we have

$$2^{n-m_{j+1}-k_{j+1}-2(k_0+\cdots+k_j)} \leqslant \prod_{i=1}^{n} w_i^{(t)} \leqslant 2^{n-m_{j+1}-(k_0+\cdots+k_{j+1})}. \tag{4.5}$$

We remark that inequality (4.3) gives

$$\prod_{i=1}^{1+\alpha_j+m_j} w_i^{(t)} \geqslant 2^{-(k_0+\cdots+k_j)+m_j} = 2^j,$$

which goes to ∞, as $j \to \infty$. This shows that each unilateral weighted backward shift F_t with weight sequence $\{w_j^{(t)} : j \geqslant 1\}$ satisfies Salas' Condition (3.1). In other words, the family $\{F_t : t \in [0, \frac{1}{4}]\}$ consists entirely of hypercyclic operators.

To show $\{F_t : t \in [0, \frac{1}{4}]\}$ is a path of hypercyclic operators, we now show the map $t \mapsto F_t$ is continuous on $[0, \frac{1}{4}]$. Note that $w_i^{(t)}$ is a constant unless the subindex i is in the form $1 + \alpha_j$. By the mean value theorem, we see that if $f(x) = w_{1+\alpha_j}^{(x)}$, then for all $s, t \in [0, \frac{1}{4}]$ we have

$$\Big|\frac{f(t) - f(s)}{t - s}\Big| \leqslant \Big|f'\Big(\frac{1}{4}\Big)\Big| = k_j\Big(\frac{3}{4}\Big)^{-1+k_j},$$

which are bounded above by a positive number, say C, that is independent of s, t, and j. Thus, the operator norm $\|F_s - F_t\| \leqslant C|s - t|$.

To finish the whole proof, we must show that our path $\{F_t : t \in [0, \frac{1}{4}]\}$ of hypercyclic operators does not have a common hypercyclic vector. Since every nonzero scalar multiple of a hypercyclic vector is a hypercyclic vector, it suffices to show that every unit vector $h = (a_0, a_1, a_2, \ldots)$ in ℓ^2 is not a common hypercyclic vector. For that, we now investigate the block structure of the weights $w_i^{(t)}$. Recall $B_{j+1} = \{n \in \mathbb{N} : 1 + \alpha_j \leqslant n \leqslant \alpha_{j+1}\}$, and let

$$Q_h = \{n \in \mathbb{N} : \|F_t^n h - 2e_0\| < 1 \text{ for some } t \in [0, \tfrac{1}{4}]\}.$$

CLAIM. *The set $Q_h \cap B_{j+1}$ contains at most $k_j^{1/3}$ integers.*

Proof of Claim. To prove the claim, we first note that h is a unit vector, and so each a_i satisfies $|a_i| \leqslant 1$. Then we observe that if $n = 1 + \alpha_j$ or $2 + \alpha_j + m_j \leqslant n \leqslant \alpha_{j+1}$, then by inequalities (4.2), (4.4), and (4.5) we have

$$\prod_{i=1}^{n} w_i^{(t)} \leqslant 1, \quad \text{and so } |a_n| \prod_{i=1}^{n} w_i^{(t)} \leqslant 1.$$

Hence,

$$\|F_t^n h - 2e_0\| \geqslant |\langle F_t^n h - 2e_0, e_0\rangle| = \Big|a_n \prod_{i=1}^{n} w_i^{(t)} - 2\Big| \geqslant 1,$$

which shows that $n \notin Q_h$. Consequently, if $n \in Q_h \cap B_{j+1}$ then $2 + \alpha_j \leqslant n \leqslant 1 + \alpha_j + m_j$, and there are at most m_j of such integers. Thus if we use N to denote the least integer in $Q_h \cap B_{j+1}$, then the largest such integer is at most $N + m_j - 1$.

We observe that if $n \in Q_h \cap B_{j+1}$ with $n \neq N$, then there exists $s \in [0, \frac{1}{4}]$ such that $\|F_s^n h - 2e_0\| < 1$, from which it follows that $|a_n| \prod\limits_{i=1}^{n} w_i^{(s)} > 1$. This inequality, along with (4.2), implies that

$$|a_n| > \Big(\prod_{i=1}^{n} w_i^{(s)}\Big)^{-1} \geqslant \frac{1}{2^{n-1-\alpha_j}}. \tag{4.6}$$

Since N is the least integer in $Q_h \cap B_{j+1}$, there exists $t \in [0, \frac{1}{4}]$ such that $\|F_t^N h - 2e_0\| < 1$, and hence,

$$1 > \sum_{\ell=N+1}^{\infty} \Big(|a_\ell| \prod_{i=\ell-N+1}^{\ell} w_i^{(t)}\Big)^2 \geqslant \Big(|a_n| \prod_{i=n-N+1}^{n} w_i^{(t)}\Big)^2 = |a_n|^2 \Big(\frac{\prod_{i=1}^{n} w_i^{(t)}}{\prod_{i=1}^{n-N} w_i^{(t)}}\Big)^2.$$

By taking the square root, rearranging terms, and using inequalities (4.2) and (4.6), we have

$$\prod_{i=1}^{n-N} w_i^{(t)} > |a_n| \prod_{i=1}^{n} w_i^{(t)} > \frac{1}{2^{n-1-\alpha_j}} 2^{-(k_0+\cdots+k_j)+n-1-\alpha_j} = 2^{-(k_0+\cdots+k_j)}. \tag{4.7}$$

Since $0 < n - N \leqslant m_j$ and $m_j \in B_j$, we observe that if $2 + \alpha_{j-1} + m_{j-1} \leqslant n - N \leqslant m_j$, then by inequality (4.4),

$$\prod_{i=1}^{n-N} w_i^{(t)} \leqslant 2^{-(k_0+k_1+\cdots+k_j)},$$

contradicting inequality (4.7). Hence $n - N \leqslant 1 + \alpha_{j-1} + m_{j-1}$; that is, $N < n \leqslant N + 1 + \alpha_{j-1} + m_{j-1}$. It follows that there are at most $1 + \alpha_{j-1} + m_{j-1}$ of such integers n. Along with N itself, we conclude that $Q_h \cap B_{j+1}$ contains at most $2 + \alpha_{j-1} + m_{j-1} = k_j^{1/3}$ integers, finishing the proof for our claim. ∎

We now turn our attention to the set $A_{h,n}$ defined by

$$A_{h,n} = \{t \in [0, \tfrac{1}{4}] : \|F_t^n h - 2e_0\| < 1\}.$$

If $n \in Q_h \cap B_{j+1}$, then $A_{h,n} \neq$ the empty set and we let $a = \inf A_{h,n}$ and $b = \sup A_{h,n}$. Hence if $t = a$ or b, then $|a_n w_1^{(t)} w_2^{(t)} \cdots w_n^{(t)} - 2| \leqslant 1$, and so, $1 \leqslant |a_n| w_1^{(t)} w_2^{(t)} \cdots w_n^{(t)} \leqslant 3$. Using the middle expression with $t = a$ and b respectively, we take the ratio to obtain

$$\frac{w_1^{(b)} w_2^{(b)} \cdots w_n^{(b)}}{w_1^{(a)} w_2^{(a)} \cdots w_n^{(a)}} \leqslant 3.$$

Since $w_i^{(t)}$ is a constant whenever $i \neq 1+\alpha_0, 1+\alpha_1, \ldots, 1+\alpha_j$, we obtain that

$$\frac{w_1^{(b)} w_{1+\alpha_1}^{(b)} \cdots w_{1+\alpha_j}^{(b)}}{w_1^{(a)} w_{1+\alpha_1}^{(a)} \cdots w_{1+\alpha_j}^{(a)}} \leqslant 3,$$

or equivalently,

$$\frac{(1+2b)^1(1+2b)^{k_1}\cdots(1+2b)^{k_j}}{(1+2a)^1(1+2a)^{k_1}\cdots(1+2a)^{k_j}} = \Big(\frac{1+2b}{1+2a}\Big)^{1+k_1+\cdots+k_j} \leqslant 3.$$

Putting $\varepsilon_j = (1+k_1+\cdots+k_j)^{-1}$, we rewrite the above inequality as $1+\frac{2(b-a)}{1+2a} \leqslant 3^{\varepsilon_j}$. It follows that

$$b-a \leqslant \frac{1+2a}{2}(3^{\varepsilon_j}-1) < 3^{\varepsilon_j}-1.$$

We continue the proof using the Lebesgue outer measure λ^* on the real line and note that if $n \in Q_h \cap B_{j+1}$ then $\lambda^*(A_{h,n}) \leqslant b-a < 3^{\varepsilon_j}-1$, which, long with the claim, leads to $\lambda^*\Big(\bigcup_{n\in B_{j+1}} A_{h,n}\Big) < k_j^{1/3}(3^{\varepsilon_j}-1)$. Hence,

$$\lambda^*\Big(\bigcup_{j=1}^{\infty}\bigcup_{n\in B_{j+1}} A_{h,n}\Big) < \sum_{j=1}^{\infty} k_j^{1/3}(3^{\varepsilon_j}-1),$$

which turns out to be a convergent series. To show that, we first use the definition of ε_j to see that $k_j^{1/3}\varepsilon_j < k_j^{-2/3}$, and hence

$$\sum_{j=1}^{\infty} k_j^{1/3}(3^{\varepsilon_j}-1) < \sum_{j=1}^{\infty}\frac{1}{k_j^{2/3}}\frac{3^{\varepsilon_j}-1}{\varepsilon_j}.$$

Then the convergence of the series immediately follows from the observation that $k_j \geqslant (2+j)^3$ by its definition, along with the observation that $\lim \varepsilon_j = 0$ and thus $\lim \frac{3^{\varepsilon_j}-1}{\varepsilon_j} = \ln 3$. The convergence implies there is a positive integer M such that

$$\begin{aligned} &\lambda^*\{t \in [0, \tfrac{1}{4}] : \|F_t^n h - 2e_0\| < 1, \text{and } n \geqslant 1+\alpha_M\} \\ &\qquad = \lambda^*\Big(\bigcup_{j=M}^{\infty}\bigcup_{n\in B_{j+1}} A_{h,n}\Big) < \sum_{j=M}^{\infty} k_j^{1/3}(3^{\varepsilon_j}-1) < \frac{1}{4}. \end{aligned} \tag{4.8}$$

To finish the whole proof, we remark that if h were a common hypercyclic vector for the path $\{F_t : t \in [0, \frac{1}{4}]\}$, then for any positive integer K, the vector $F_t^K h$ would be a hypercyclic vector for each operator F_t. This contradicts equality (4.8) with $M = K$. ∎

If $F : [0, \frac{1}{4}] \to B(\ell^2(\mathbb{Z}))$ is the path of hypercyclic, unilateral weighted backward shifts defined in the proof of the previous theorem, then for any two hypercyclic unilateral weighted backward shifts, we can join them to F_0 and $F_{1/4}$ respectively, with two paths of such operators by Theorem 3.5. Combining the above three paths as one path, we have the following corollary in contrast of Theorem 3.5.

COROLLARY 4.2. *Between any two hypercyclic, unilateral weighted backward shifts, there is a path of such operators without any common hypercyclic vector.*

As a trivial consequence of Theorem 4.1, we have the following statement: *All hypercyclic unilateral weighted backward shifts on ℓ^2 do not have a common hypercyclic vector.*

A result of similar nature was found by Aron, Bès, León, and Peris ([3], Exemple 2.2) who exhibited a family of hypercyclic unilateral weighted backward shifts on different orthonormal basis of a Hilbert space such that the family has no common hypercyclic vector, but each operator in the family has a hypercyclic subspace.

Our statement above can be proved without using Theorem 4.1. For example, we can directly show that for any vector

$$f = (a_0, a_1, a_2, \dots)$$

in ℓ^2, there exists a hypercyclic unilateral weighted shift T for which f is not a hypercyclic vector.

Since f is in ℓ^2, we have $a_j \to 0$ as $j \to \infty$. Hence, there exists a sequence of strictly increasing positive integers $\{n_k : k \geqslant 1\}$ such that

$$|a_j| < \frac{1}{k+1} \quad \text{whenever } \; n_k \leqslant j < n_{k+1}.$$

For this sequence $\{n_k\}$, we let $w_1 = \cdots = w_{n_1} = 1$, and if $k \geqslant 1$, we let

$$\overbrace{w_{1+n_k} = w_{2+n_k} = \cdots = w_{n_{k+1}}}^{n_{k+1}-n_k \text{ terms}} = \Big(\frac{k+1}{k}\Big)^{1/(n_{k+1}-n_k)}.$$

It is easy to check that $1 < w_j < 2$, whenever $j \geqslant 1$. It follows that if $T : \ell^2 \to \ell^2$ is the unilateral weighted backward shift defined by

$$Te_j = \begin{cases} w_j e_{j-1} & \text{if } \; j \geqslant 1, \\ 0 & \text{if } \; j = 0, \end{cases}$$

then $\|T\| \leqslant 2$. The operator T is hypercyclic because

$$\begin{aligned} \sup\{w_1 \cdots w_j : j \geqslant 1\} \geqslant \sup\{w_1 \cdots w_{n_{k+1}} : k \geqslant 1\} &= \sup\Big\{1 \cdot \frac{2}{1} \cdot \frac{3}{2} \cdots \frac{k+1}{k} : k \geqslant 1\Big\} \\ &= \sup\{k+1 : k \geqslant 1\} = \infty, \end{aligned}$$

satisfying the Salas' condition (3.1).

The vector f is not a hypercyclic vector for T because if $n_k < n \leqslant n_{k+1}$, then

$$|\langle T^n f, e_0\rangle| = w_1 \cdots w_n |a_n| \leqslant w_1 \cdots w_{n_{k+1}} \cdot \frac{1}{k+1} = 1 \cdot \frac{2}{1} \cdot \frac{3}{2} \cdots \frac{k+1}{k} \cdot \frac{1}{k+1} = 1,$$

and so the orbit $\mathrm{orb}(T, f) = \{f, Tf, T^2 f, \dots\}$ is not dense in ℓ^2, completing the proof for our statement.

5. HYPERCYCLIC BILATERAL SHIFTS

After examining paths of unilateral weighted backward shifts in Sections 3 and 4, the next natural step is to study paths of bilateral weighted shifts. Let $\{e_j : j \in \mathbb{Z}\}$ be the canonical orthonormal basis of $\ell^2(\mathbb{Z})$. That is, e_j is the bilateral sequence $(\dots, 0, 1, 0, \dots)$ where the 1 is in the j-th position. A bounded linear operator $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is said to be a *bilateral weighted (backward) shift* if there exists a positive weight sequence $\{w_j : j \in \mathbb{Z}\}$ such that $Te_j = w_j e_{j-1}$ for all $j \in \mathbb{Z}$. Salas' Theorem 2.1 of [22] characterized the hypercyclicity of a bilateral weighted forward shift, defined by $Te_j = w_j e_{j+1}$, completely in terms of its weight sequence. Since the forward and backward shifts are unitarily equivalent in the bilateral case, Salas' result can be restated as follows: *A bilateral weighted shift with the weight sequence $\{w_j : j \in \mathbb{Z}\}$ is hypercyclic if and only if for each $\varepsilon > 0$ and $q \geqslant 1$, there exists an integer $n \geqslant 1$ such that*

$$\prod_{i=1}^{n} w_{j+i} > \frac{1}{\varepsilon} \quad \textit{and} \quad \prod_{i=0}^{n-1} w_{j-i} < \varepsilon \quad \textit{for } |j| \leqslant q. \tag{5.1}$$

As in the unilateral case, we have the following result about the existence of paths between bilateral weighted shifts.

THEOREM 5.1. *Let $T_0, T_1 : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be two hypercyclic, bilateral weighted shifts. Then there exists a path of bilateral weighted shifts between T_0 and T_1 such that the set of common hypercyclic vectors for the whole path is a dense G_δ set.*

Due to the similarity in the shifting pattern of unilateral weighted backward shifts and bilateral weighted shifts, the proof of Theorem 5.1 involves techniques similar to those in Section 3, and so we outline the steps and leave the details to the reader.

To begin, let $\{w_j : j \in \mathbb{Z}\}$ and $\{w'_j : j \in \mathbb{Z}\}$ be the weight sequences for the hypercyclic bilateral weighted shifts T_0 and T_1 respectively. First suppose that $w_j = w'_j < \frac{1}{2}$ for all $j \leqslant 0$ and $w_j, w'_j \geqslant 1$ for all $j \geqslant 1$. That is, T_0 and T_1 have the same nonpositive indexed weights and they are all less than $\frac{1}{2}$ while the positive indexed weights of T_0 and T_1 are bounded below by 1. Using the techniques in the proof of Proposition 3.1 and Proposition 3.2 on the positive indexed weights, we construct a path of bilateral weighted shifts between T_0 and T_1 with a dense G_δ set of common hypercyclic vectors.

For the case when $\{w_j : j \in \mathbb{Z}\}$ and $\{w'_j : j \in \mathbb{Z}\}$ are two arbitrary weight sequences for the hypercyclic bilateral weighted shifts T_0 and T_1, it suffices to show that there is a path of hypercyclic bilateral weighted shifts with a dense G_δ set of common hypercyclic vectors between T_0 and a bilateral shift whose weight sequence $\{a_j : j \in \mathbb{Z}\}$ satisfies $a_j = \min\{w_j, w'_j, \frac{1}{2}\}$ for all $j \leqslant 0$ and $a_j \geqslant 1$ for all $j \geqslant 1$. The desired path is comprised of two paths. For the first path, we lower the nonpositive indexed weights of T_0 to $a_j = \min\{w_j, w'_j, \frac{1}{2}\}$ while keeping the positive indexed weights of T_0 fixed. To be precise, for each $t \in [0,1]$, define $F_t = (1-t)T_0 + tA$ where A is the bilateral weighted shift whose weight sequence $\{a'_j : j \in \mathbb{Z}\}$ is given by $a'_j = \min\{w_j, w'_j, \frac{1}{2}\}$ for $j \leqslant 0$ and $a'_j = w_j$ for $j \geqslant 1$. To show the path $\{F_t : t \in [0,1]\}$ between T_0 and A has a dense G_δ set of common hypercyclic vectors, we use Theorem 2.4. By Salas' Condition (5.1), there exists an increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers satisfying

$$\prod_{i=1}^{n_k} \frac{1}{w_{j+i}} < \frac{1}{k} \quad \text{and} \quad \prod_{i=0}^{n_k-1} w_{j-i} < \frac{1}{k} \quad \text{whenever } |j| \leqslant k. \tag{5.2}$$

Let $D_1 = \operatorname{span}\{e_j : j \in \mathbb{Z}\}$. If $h \in D_1$ and $\varepsilon > 0$, let $D_2 = D_1$ and define $S_t : D_1 \to \ell^2(\mathbb{Z})$ by $S_t e_j = [(1-t)w_{j+1} + ta'_{j+1}]^{-1} e_{j+1}$ for $j \leqslant -1$ and $S_t e_j = w_{j+1}^{-1} e_{j+1}$ for $j \geqslant 0$. For each $t \in [0,1]$, we get $S_t^{n_k} h \to 0$ as $k \to \infty$ by (5.2). Since we are only lowering the nonnegative indexed weights, for any $f \in D_2$, we get $F_t^{n_k} f \to 0$ uniformly on $[0,1]$ as $k \to \infty$. Furthermore, we can find a $\delta > 0$ such that $\|F_t^{n_k} S_{t'}^{n_k} h - h\| < \varepsilon$ whenever $|t - t'| < \delta$.

For the second path, we change the positive indexed weights of A to values greater than 1 while keeping the nonpositive indexed weights of A fixed. To create this path, we apply techniques almost identical to those in the proof of Theorem 3.5 to increase or decrease the positive indexed weights of A in blocks while keeping the geometric mean in each block constant. By choosing the blocks of weights in the same fashion as in the proof of Theorem 3.5, we create a path of bilateral weighted shifts with a dense G_δ set of common hypercyclic vectors between A and a bilateral weighted shift whose weight sequence $\{a_j : j \in \mathbb{Z}\}$ satisfies $a_j = a'_j = \min\{w_j, w'_j, \frac{1}{2}\}$ for all $j \leqslant 0$ and $a_j \geqslant 1$ for all $j \geqslant 1$.

An immediate consequence of the Theorem 5.1 is the path connectedness of the hypercyclic bilateral weighted shifts.

COROLLARY 5.2. *The hypercyclic bilateral weighted shifts form a path connected subset of* $B(\ell^2(\mathbb{Z}))$.

Even though the hypercyclic bilateral shifts are path connected in $B(\ell^2(\mathbb{Z}))$, as in the unilateral situation, the set of all such shifts do not form a convex set. To prove this, let $\{w_j : j \geqslant 1\}$ and $\{v_j : j \geqslant 1\}$ be the two weight sequences of the hypercyclic unilateral weighted backward shifts given in Section 3 satisfying $\frac{1}{2}(w_j + v_j) = 1$ for all $j \geqslant 1$. Define $T_0, T_1 : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by $T_0 e_j = w_j e_{j-1}$ and $T_1 e_j = v_j e_{j-1}$ if $j \geqslant 1$ and $T_0 e_j = T_1 e_j = \frac{1}{2} e_{j-1}$ if $j \leqslant 0$. By Salas' Conditions (3.1) and (5.1), we get T_0 and T_1 are hypercyclic. However, $\frac{1}{2} T_0 e_j + \frac{1}{2} T_1 e_j = \frac{1}{2}(w_j + v_j) e_{j-1} = e_{j-1}$ for all $j \geqslant 1$, and so $\frac{1}{2} T_0 + \frac{1}{2} T_1$ is not hypercyclic.

In Section 4, we proved that there exists paths of hypercyclic unilateral weighted backward shifts without a common hypercyclic vector; see Theorem 4.1. We have the same result for bilateral weighted shifts.

THEOREM 5.3. *There exists a path of hypercyclic bilateral weighted shifts on $\ell^2(\mathbb{Z})$ having no common hypercyclic vector.*

Proof. To construct such a path, first let $\{F_t : t \in [0, \frac{1}{4}]\}$ be the path of hypercyclic unilateral weighted backward shifts given in the proof of Theorem 4.1 having no common hypercyclic vector. Let $\{w_j^{(t)} : j \geqslant 1\}$ be the weight sequence of F_t. For each $t \in [0, \frac{1}{4}]$, define $G_t : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ by $G_t e_j = w_j^{(t)} e_{j-1}$ for $j \geqslant 1$ and $G_t e_j = \frac{1}{2} e_{j-1}$ for $j \leqslant 0$. Since $t \mapsto F_t$ is continuous, the map $t \mapsto G_t$ is also continuous. By Salas' Condition (5.1), we get G_t is hypercyclic for each $t \in [0, \frac{1}{4}]$, and so $\{G_t : t \in [0, \frac{1}{4}]\}$ is a path of hypercyclic bilateral shifts. Lastly, one can easily see from the definitions of F_t and G_t that if $(\dots, f_{-1}, f_0, f_1, \dots) \in \mathcal{HC}(G_t)$, then $(f_0, f_1, \dots) \in \mathcal{HC}(F_t)$. From this observation, we get that the path $\bigcap\limits_{t\in[0,1/4]} \mathcal{HC}(G_t) = \emptyset$ because $\bigcap\limits_{t\in[0,1/4]} \mathcal{HC}(F_t) = \emptyset$. ∎

The previous theorem implies that between any two hypercyclic, bilateral weighted shifts, there is a path of such operators with no common hypercyclic vector, similar to the ideas in Corollary 4.2.

To conclude the paper, we remark that the results in Sections 3, 4, and 5 hold for shift operators defined on ℓ^p with $1 \leqslant p < \infty$, though we state and prove them only for ℓ^2. One easily adapts the above techniques and arguments for those spaces, with the appropriate adjustments.

Acknowledgements. The authors wish to thank Juan Bès for many inspiring conversations related to this research, and also the referee for many valuable suggestions.

REFERENCES

[1] E. ABAKUMOV, J. GORDON, Common hypercyclic vectors for multiplies of backward shift, *J. Funct. Anal.* **200**(2003), 494–504.

[2] S.I. ANSARI, Hypercyclic and cyclic vectors, *J. Funct. Anal.* **128**(1995), 374–383.

[3] R. ARON, J. BÈS, F. LEÓN, A. PERIS, Operators with common hypercyclic subspaces, *J. Operator Theory* **54**(2005), 251–260.

[4] F. BAYART, Common hypercyclic vectors for composition operators, *J. Operator Theory* **52**(2004), 353–370.

[5] F. BAYART, Common hypercyclic subspaces, *Integral Equations Operator Theory* **53**(2005), 467–476.

[6] F. BAYART, S. GRIVAUX, Hypercyclicity and unimodular point spectrum, *J. Funct. Anal.* **226**(2005), 281–300.

[7] F. BAYART, E. MATHERON, How to get common universal vectors, *Indiana Univ. Math. J.* **56**(2007), 553–580.

[8] L. BERNAL-GONZÁLEZ, K.-G. GROSSE-ERDMANN, The Hypercyclic Criterion for sequences of operators, *Studia Math.* **157**(2003), 17–32.

[9] J. BÈS, A. PERIS, Hereditarily hypercyclic operators, *J. Funct. Anal.* **167**(1999), 94–112.

[10] J.A. CONEJERO, V. MÜLLER, A. PERIS, Hypercyclic behaviour of operators in a hypercyclic C_0-semigroup, *J. Funct. Anal.* **244**(2007), 342–348.

[11] J.B. CONWAY, *A Course in Functional Analysis, Second Edition*, Springer-Verlag, New York 1990.

[12] G. COSTAKIS, Common Cesaro hypercyclic vectors, preprint, 2006.

[13] G. COSTAKIS, M. SAMBARINO, Genericity of wild holomorphic functions and common hypercyclic vectors, *Adv. Math.* **182**(2004), 278–306.

[14] R.M. GETHNER, J.H. SHAPIRO, Universal vectors for operators on spaces of holomorphic functions, *Proc. Amer. Math. Soc.* **100**(1987), 281–288.

[15] G. GODEFROY, J.H. SHAPIRO, Operators with dense, invariant cyclic vector manifolds, *J. Funct. Anal.* **98**(1991), 229–269.

[16] K.-G. GROSSE-ERDMANN, Holomorphe Monster und universelle Funktionen, *Mitt. Math. Sem. Giessen* **176**(1987).

[17] K.-G. GROSSE-ERDMANN, Universal families and hypercyclic operators, *Bull. Amer. Math. Soc.* **36**(1999), 345–381.

[18] K.-G. GROSSE-ERDMANN, Hypercyclic and chaotic weighted shifts, *Studia Math.* **139**(2000), 47–68.

[19] C. KITAI, Invariant closed sets for linear operators, Ph.D. Dissertation, Univ. of Toronto, Toronto 1982.

[20] F. LEÓN–SAAVEDRA, V. MÜLLER, Rotations of hypercyclic and supercyclic operators, *Integral Equations Operator Theory* **50**(2004), 385–391.

[21] S. ROLEWICZ, On orbits of elements, *Studia Math.* **32**(1969), 17–22.

[22] H. SALAS, Hypercyclic weighted shifts, *Trans. Amer. Math. Soc.* **347**(1995), 993–1004.

[23] H. SALAS, Supercyclicity and weighted shifts, *Studia Math.* **135**(1999), 55–74.

KIT C. CHAN, DEPT. OF MATHEMATICS AND STATISTICS, BOWLING GREEN STATE UNIVERSITY, BOWLING GREEN, 43403, USA
E-mail address: kchan@bgnet.bgsu.edu

REBECCA SANDERS, DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, MARQUETTE UNIVERSITY, MILWAUKEE, 53201, USA
E-mail address: rsanders@mscs.mu.edu

Received July 19, 2006.