J. OPERATOR THEORY
61:2(2009), 253–278

NORMALITY VIA LOCAL SPECTRAL RADII

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Communicated by Florian-Horia Vasilescu

ABSTRACT. A new criterion for essential normality of unbounded Hilbert space operators is furnished in terms of local spectral radius. Accordingly, extensive study of operators of certain types related to local spectral radius is conducted. Spectral radii of local restrictions of a normal operator are investigated.

KEYWORDS: *Local spectral radius, normal operator, paranormal operator, normaloid operator, algebraic operator, locally algebraic operator, compact operator.*

MSC (2000): Primary 47B15, 47B20; Secondary 47B07.

INTRODUCTION

In this paper, we give conditions for normality of the closure (read: essential normality) of an unbounded Hilbert space operator. Our approach is based on the notion of local spectral radius (cf. [7], [35], [36], [8], [19], [37]; see also [4], [5] for the case of unbounded operators). We attach to any (unbounded) operator a family of invariant linear spaces of C^∞-vectors determined by uniform boundedness of their local spectral radii. Examining the corresponding restrictions (called local restrictions) of the operator in question enables us to distinguish classes of localoid and locally normaloid operators (cf. Section 4). This is preceded by detailed analysis of local spectral radius, as some of its properties are not preserved in the unbounded case (cf. Sections 1 and 3). In Section 2, we provide necessary and sufficient conditions for an abstract transformation of the half real line to come from spectral radii of local restrictions of a normal operator. The advantage of introducing notions of localoid and locally normaloid operators when compared with other known classes of operators (e.g. restriction-normaloid operators) lies in the fact that our attention is confined to the family of selected invariant subspaces which may reduce to a finite set. We show that paranormal operators as well as those satisfying the Kato-Protter inequality form proper subclasses of locally normaloid operators. In Section 5 we discuss the question of when essential normality of an operator follows from essential normality of all its cyclic parts. Having this established, we provide a criterion for essential normality of locally algebraic operators written in terms of local normaloidity (cf. Theorem 6.7).

1. LOCAL SPECTRAL RADIUS

Throughout what follows, $\mathcal{H}$ stands for a complex Hilbert space. By an operator *in* $\mathcal{H}$ we mean a linear mapping $A\colon \mathcal{D}(A) \to \mathcal{H}$ defined on a linear subspace $\mathcal{D}(A)$ of $\mathcal{H}$, called the *domain* of A. We denote by $\mathcal{N}(A)$, A^* and $\overline{A}$ the kernel, the adjoint and the closure of A respectively. We put $\mathcal{D}^\infty(A) = \bigcap_{n=1}^{\infty} \mathcal{D}(A^n)$. A densely defined operator A in $\mathcal{H}$ is called *normal* if it is closed and $A^*A = AA^*$. An operator A in $\mathcal{H}$ is said to be *essentially normal* if A is closable and $\overline{A}$ is normal.

Given an inner product space $\mathcal{D}$, we denote by $\boldsymbol{B}(\mathcal{D})$ the algebra of all bounded linear mappings from $\mathcal{D}$ into $\mathcal{D}$. If $A \in \boldsymbol{B}(\mathcal{D})$, then $r(A)$ stands for the spectral radius of the unique bounded linear extension of A to the completion of $\mathcal{D}$; in the case of $\mathcal{D} = \{0\}$, we put $r(A) = 0$. By the Gelfand theorem $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$.

Given an operator A in $\mathcal{H}$, we define

$$r(A,f) = \limsup_{n\to\infty} \|A^n f\|^{1/n}, \quad f \in \mathcal{D}^\infty(A),$$
$$\mathcal{H}_A(t) = \{f \in \mathcal{D}^\infty(A)\colon r(A,f) \leqslant t\}, \quad t \in [0,\infty].$$

The quantity $r(A,f) \in [0,\infty]$ is called the *local spectral radius* of A at f. Note that the set of all vectors $f \in \mathcal{H}$ such that $r(A,f) < \infty$ is identical with the set of all bounded vectors for A (cf. [10], [32] and [33] for an up-to-date approach). Evidently $\mathcal{H}_A(s) \subseteq \mathcal{H}_A(t)$ whenever $0 \leqslant s \leqslant t$, and $\mathcal{H}_A(\infty) = \mathcal{D}^\infty(A)$. Let us list some basic properties of local spectral radius (consult the proof of Lemma 1 in [8]).

LEMMA 1.1. *If A is an operator in $\mathcal{H}$, then:*

(i) $r(A, \alpha f + \beta g) \leqslant \max\{r(A,f), r(A,g)\}$ *for all* $\alpha, \beta \in \mathbb{C}$ *and* $f, g \in \mathcal{D}^\infty(A)$;

(ii) $r(A, Af) = r(A,f)$ *for all* $f \in \mathcal{D}^\infty(A)$;

(iii) *for every* $t \in [0,\infty]$, $\mathcal{H}_A(t)$ *is a linear subspace of* $\mathcal{D}^\infty(A)$ *which is invariant for* A;

(iv) *for every* $t \in [0,\infty]$, $\mathcal{H}_A(t) = \mathcal{H}_A(\widetilde{t})$, *where* $\widetilde{t} \stackrel{\mathrm{def}}{=} \sup\{r(A,f)\colon f \in \mathcal{H}_A(t)\}$.

For $t \in [0,\infty]$, we put $A_{[t]} = A|_{\mathcal{H}_A(t)}$. We regard $A_{[t]}$ as an operator in $\mathcal{H}$.

We now show to what extent the equality $r(A,f)^k = r(A^k, f)$, originally proved in [8], can be extended to the case of unbounded operators.

LEMMA 1.2. *If A is an operator in $\mathcal{H}$, f is a vector in $\mathcal{D}^\infty(A)$ and $k \geqslant 1$ is an integer, then:*

(i) $r(A,f)^k = \max\{r(A^k, A^j f)\colon j = 0, 1, \ldots, k-1\}$;

(ii) $r(A,f)^k = \max\{r(A^k, A^j f)\colon j = 0,1\}$ *provided* $A_{[t]}$ *is closed (as an operator in* $\mathcal{H}$*) for* $t = r(A,f)$*;*
(iii) $r(A,f)^k = r(A^k, f)$ *provided* $A_{[t]}$ *is bounded for* $t = r(A,f)$*.*

Proof. (i) Each positive integer n can be written as $n = km + j$ with unique nonnegative integers $m \geqslant 0$ and $j \in \{0, \dots, k-1\}$. This leads to

$$r(A,f)^k = \lim_{l\to\infty} \sup_{n\geqslant l} \|A^n f\|^{k/n} = \lim_{l\to\infty} \max_{j=0,\dots,k-1} \sup_{km+j\geqslant l} (\|(A^k)^m A^j f\|^{1/m})^{1/(1+\frac{j}{km})}$$

$$= \max_{j=0,\dots,k-1} \lim_{l\to\infty} \sup_{km+j\geqslant l} (\|(A^k)^m A^j f\|^{1/m})^{1/(1+\frac{j}{km})} = \max_{j=0,\dots,k-1} r(A^k, A^j f).$$

(ii) Set $\|h\|_A = \sqrt{\|h\|^2 + \|Ah\|^2}$ for $h \in \mathcal{H}_A(t)$. Denote by $\mathcal{X}_t$ the inner product space $(\mathcal{H}_A(t), \|\cdot\|_A)$ and define the linear mapping $B_t\colon \mathcal{X}_t \to \mathcal{X}_t$ by $B_t h = Ah$ for $h \in \mathcal{X}_t$. Since $A_{[t]}$ is a closed operator in $\mathcal{H}$, $\mathcal{X}_t$ is a Hilbert space and the operator B_t is closed. By the closed graph theorem $B_t \in \boldsymbol{B}(\mathcal{X}_t)$. Note that $r(A,f) = r(B_t, f)$ and $r(B_t^k, f) = \max\{r(A^k, f), r(A^k, Af)\}$ ($r(B_t, f)$ and $r(B_t^k, f)$ are calculated with respect to $\|\cdot\|_A$). Applying Lemma 1 (6) of [8] to the bounded operator B_t we get

$$r(A,f)^k = r(B_t, f)^k = r(B_t^k, f) = \max\{r(A^k, f), r(A^k, Af)\}.$$

(iii) Applying Lemma 1 (6) of [8] now to the bounded operator $A_{[t]}$, we get the following which completes the proof:

$$r(A,f)^k = r(A_{[t]}, f)^k = r(A_{[t]}^k, f) = r(A^k, f), \quad k \geqslant 1. \quad ∎$$

Note that the case $r(A,f) = \infty$ is excluded in (ii) and (iii) of Lemma 1.2.

COROLLARY 1.3. *If* A *is an operator in* $\mathcal{H}$ *such that all the operators* $A_{[t]}$, $t \in [0,\infty)$*, are bounded, then* $\mathcal{H}_{A^k}(t) = \mathcal{H}_A(\sqrt[k]{t})$ *for all real* $t \geqslant 0$ *and all integers* $k \geqslant 1$.

COROLLARY 1.4. *If* A *is an operator in* $\mathcal{H}$ *and* $k \geqslant 1$ *is an integer, then*
(i) *for every* $f \in \mathcal{D}^\infty(A)$ *there exists* $j \in \{0, \dots, k-1\}$ *such that*

$$r(A, A^j f)^k = r(A^k, A^j f);$$

(ii) *if* $\mathcal{D}^\infty(A) \neq \{0\}$*, then there exists* $f \in \mathcal{D}^\infty(A) \setminus \{0\}$ *such that* $r(A,f)^k = r(A^k, f)$.

Proof. (i) follows from Lemmata 1.1 and 1.2.
(ii) Take any nonzero vector $f \in \mathcal{D}^\infty(A)$. If $A^k f = 0$, then manifestly $r(A,f)^k = r(A^k, f) = 0$. Otherwise, $A^j f \neq 0$ for all $j = 0, \dots, k-1$, which combined with (i) completes the proof. ∎

EXAMPLE 1.5. We show that the equality $r(A,f)^k = r(A^k, f)$ is no longer true for arbitrary unbounded operators. Let $\{e_n\}_{n=0}^\infty$ be an orthonormal basis of $\mathcal{H}$ and $\mathcal{D}$ be the linear span of $\{e_n\}_{n=0}^\infty$. Given a sequence $\{\lambda_n\}_{n=0}^\infty \subseteq (0,\infty)$, we define the operator A in $\mathcal{H}$ by $\mathcal{D}(A) = \mathcal{D}$ and $Ae_n = \lambda_n e_{n+1}$ for all integers $n \geqslant 0$.

Consider the sequence

$$(\lambda_0, \lambda_1, \lambda_2, \dots) = (u, u^{-1}, u^2, u^{-2}, u^3, u^{-3}, \dots),$$

where $u > 1$ is a fixed real number. It follows from Lemma 1.1 that

$$r(A, e_j) = r(A, A^j e_0) = r(A, e_0) = \sqrt{u}, \quad j \geqslant 1. \tag{1.1}$$

Take $f = \sum_{l=0}^{N} \alpha_l e_l \in \mathcal{D} \setminus \{0\}$ with $\{\alpha_l\}_{l=0}^N \subseteq \mathbb{C}$. By Lemma 1.1 (i) we have

$$r(A, f) \leqslant \max\{r(A, e_j) \colon j = 0, \dots, N\} \overset{(1.1)}{=} \sqrt{u},$$

$$r(A, f) = \limsup_{n\to\infty} \Big(\sum_{l=0}^{N} \|\alpha_l A^n e_l\|^2 \Big)^{1/2n} \geqslant \limsup_{n\to\infty} \|\alpha_p A^n e_p\|^{1/n} = \sqrt{u}, \tag{1.2}$$

where p is chosen so that $\alpha_p \neq 0$. As a consequence, we obtain

$$r(A, f) = \sqrt{u}, \quad f \in \mathcal{D} \setminus \{0\}. \tag{1.3}$$

Arguing as above we see that

$$r(A^2, f) = \begin{cases} u & \text{if there exists an odd integer } p \text{ such that } \alpha_p \neq 0, \\ 1 & \text{otherwise.} \end{cases} \tag{1.4}$$

The equalities (1.3) and (1.4) imply that

$$\mathcal{H}_A(t) = \begin{cases} \{0\} & \text{if } t < \sqrt{u}, \\ \mathcal{D} & \text{if } t \geqslant \sqrt{u}, \end{cases} \quad \text{and} \quad \mathcal{H}_{A^2}(t) = \begin{cases} \{0\} & \text{if } t < 1, \\ * & \text{if } 1 \leqslant t < u, \\ \mathcal{D} & \text{if } t \geqslant u, \end{cases} \tag{1.5}$$

where "$*$" stands for the linear span of $\{e_{2j}\}_{j=0}^\infty$.

In turn, if $(\lambda_0, \lambda_1, \lambda_2, \dots) = (2^{2^0}, 2^{-2^0}, 2^{2^1}, 2^{-2^1}, 2^{2^2}, 2^{-2^2}, \dots)$, then one can show that equalities (1.3), (1.4) and (1.5) remain valid with ∞ in place of u and $\sqrt{u}$.

We now list some basic properties of the function $t \mapsto r(A_{[t]})$.

LEMMA 1.6. *If A is an operator in $\mathcal{H}$ such that $A_{[u]} \in \boldsymbol{B}(\mathcal{H}_A(u))$ for some $u \in [0, \infty)$, then:*

(i) *$A_{[t]} \in \boldsymbol{B}(\mathcal{H}_A(t))$ for all $t \in [0, u]$;*

(ii) *$\mathcal{H}_A(t) \subseteq \mathcal{H}_A(r(A_{[t]}))$ for all $t \in [0, u]$;*

(iii) *$r(A_{[t]}) \leqslant r(A_{[t']})$ whenever $t \in [0, u]$ and $A_{[t']} \in \boldsymbol{B}(\mathcal{H}_A(t'))$, where $t' \overset{\mathsf{def}}{=} r(A_{[t]})$;*

(iv) *$t \leqslant r(A_{[t]})$ whenever $t \in [0, u]$ and $t = r(A, f)$ for some $f \in \mathcal{D}^\infty(A)$;*

(v) *the function $[0, u] \ni t \mapsto r(A_{[t]}) \in [0, \infty)$ is monotonically increasing;*

(vi) *if $t \in [0, u)$ and $r(A_{[t]}) < r(A_{[t+]}) \overset{\mathsf{def}}{=} \lim_{s\to t+} r(A_{[s]})$, then $t < r(A_{[s]})$ for all $s \in (t, u]$ and consequently $t \leqslant r(A_{[t+]})$;*

(vii) *if* $t \in (0,u]$ *and* $r(A_{[t]}) > r(A_{[t-]}) \stackrel{\text{def}}{=} \lim_{s\to t-} r(A_{[s]})$, *then* $r(A_{[t]}) \geqslant t$.

Proof. (i) Evident.

(ii) This follows from the inequality

$$r(A,f) = r(A_{[t]},f) \leqslant r(A_{[t]}), \quad f \in \mathcal{H}_A(t),\ t \in [0,u]. \tag{1.6}$$

(iii) and (v) Apply (ii) and the Gelfand formula for the spectral radius.

(iv) This is a direct consequence of (1.6).

(vi) Suppose, for contradiction, that (vi) fails to hold. Then there exist $t \in [0,u)$ and $s \in (t,u]$ such that $r(A_{[t]}) < r(A_{[t+]})$ and $r(A_{[s]}) \leqslant t$. By (ii) this yields

$$\mathcal{H}_A(s) \subseteq \mathcal{H}_A(r(A_{[s]})) \subseteq \mathcal{H}_A(t) \subseteq \mathcal{H}_A(s). \tag{1.7}$$

Hence $r(A_{[t]}) = r(A_{[s]})$ and thus by (v), $r(A_{[t]}) = r(A_{[t+]})$, which is a contradiction.

(vii) Since $r(A_{[t]}) > r(A_{[t-]})$, we see that $\mathcal{H}_A(s) \subsetneq \mathcal{H}_A(t)$ for every $s \in (0,t)$. Hence for every $s \in (0,t)$ there exists $f_s \in \mathcal{D}^\infty(A)$ such that $s < r(A,f_s) \leqslant t$. Observing that $r(A,f_s) \leqslant r(A_{[t]})$ and letting s tend to t, we get $t \leqslant r(A_{[t]})$. ∎

2. CHARACTERIZATION OF $t \mapsto r(A_{[t]})$ FOR NORMAL OPERATORS

The function $t \mapsto r(A_{[t]})$ can be explicitly computed for normal operators by means of their spectral measures. For $t \in [0,\infty)$, we put $\Delta_t = \{z \in \mathbb{C}\colon |z| \leqslant t\}$, Δ_t° = the interior of Δ_t and $\Gamma_t = \Delta_t \setminus \Delta_t^{\circ}$. By convention $\sup\varnothing = 0$ and $|Y| = \{|z|\colon z \in Y\}$ for $Y \subseteq \mathbb{C}$. As usual, $\sigma(A)$ stands for the spectrum of an operator A.

LEMMA 2.1. *If E is the spectral measure of a normal operator A in $\mathcal{H}$, then for all $t \in [0,\infty)$,*

(i) $\mathcal{H}_A(t) = E(\Delta_t)\mathcal{H}$;

(ii) $r(A_{[t]}) = \begin{cases} t & \text{if } E(\Gamma_t) \neq 0, \\ \sup|\sigma(A) \cap \Delta_t^{\circ}| & \text{if } E(\Gamma_t) = 0; \end{cases}$

(iii) $r(A_{[t]})$ *equals* $\sup|\sigma(A) \cap \Delta_t^{\circ}|$ *or* $\sup|\sigma(A) \cap \Delta_t|$.

Proof. (i) Consult the proof of Proposition 4 in [33].

(ii) and (iii) Take $t \in [0,\infty)$. By (i), $E_{[t]}(\cdot) \stackrel{\text{def}}{=} E(\cdot)|_{E(\Delta_t)\mathcal{H}}$ is the spectral measure of $A_{[t]}$. Since the closed support of the spectral measure of a normal operator coincides with its spectrum, we get

$$\sigma(A_{[t]}) \subseteq \Delta_t \cap \sigma(A) \quad \text{and} \quad \sigma(A_{[t]}) \cap \Delta_t^{\circ} = \sigma(A) \cap \Delta_t^{\circ}. \tag{2.1}$$

Suppose that $E(\Gamma_t) \neq 0$. The standard compactness argument implies that there exists $z_0 \in \Gamma_t$ such that $E(U \cap \Gamma_t) \neq 0$ and consequently $E(U \cap \Delta_t) \neq 0$ for every open neighborhood U of z_0. This means that z_0 belongs to the closed support of $E_{[t]}$. Hence, by (2.1), $r(A_{[t]}) = t$. As a consequence,

$$r(A_{[t]}) = \sup |\sigma(A) \cap \Delta_t|.$$

Suppose now that $E(\Gamma_t) = 0$. If $\sup |\sigma(A) \cap \Delta_t^{\mathrm{o}}| = t$, then by (2.1), $r(A_{[t]}) = t$ and consequently $r(A_{[t]}) = \sup |\sigma(A) \cap \Delta_t|$. In turn, if $s \overset{\mathrm{def}}{=} \sup |\sigma(A) \cap \Delta_t^{\mathrm{o}}| < t$, then $E(\Delta_t^{\mathrm{o}} \setminus \Delta_s) = 0$. Since moreover $E(\Gamma_t) = 0$, we get $E_{[t]}(\mathbb{C} \setminus \Delta_s) = 0$, which yields $r(A_{[t]}) \leqslant s$. If $s = 0$, then evidently $r(A_{[t]}) = s$. Otherwise $s > 0$ and consequently $\Gamma_s \cap \sigma(A) \neq \varnothing$. This combined with (2.1) and $r(A_{[t]}) \leqslant s$ leads to $r(A_{[t]}) = s$. ∎

We now extract some properties of the function $s \mapsto r(A_{[s]})$, where A is a normal operator. Their abstract versions determine all functions of this type (cf. Theorem 2.4). In what follows, we preserve the notation used in Lemma 1.6.

PROPOSITION 2.2. *If A is a normal operator in $\mathcal{H}$ and $t \in [0, \infty)$, then:*
(i) *$r(A_{[t]})$ equals $r(A_{[t-]})$ or $r(A_{[t+]})$;*
(ii) *if the function $s \mapsto r(A_{[s]})$ is not continuous at t, then*

$$r(A_{[t-]}) < r(A_{[t+]}) = t,$$
$$r(A_{[v]}) = r(A_{[t-]}), \quad v \in [r(A_{[t-]}), t);$$

(iii) *if $r(A_{[s]}) = t$ for all $s \in (t, \infty)$, then $r(A_{[t]}) = t$;*
(iv) *if the function $s \mapsto r(A_{[s]})$ is continuous on an open interval $U \subseteq (0, \infty)$, then there exists $c \in [0, \infty]$ such that $r(A_{[s]}) = \min\{s, c\}$ for all $s \in U$.*

Proof. It follows from Lemma 2.1 (ii) that

$$r(A_{[s]}) \leqslant s, \quad s \in [0, \infty). \tag{2.2}$$

The next step of the proof is to show that for every $x \in [0, \infty)$,

$$\text{if } r(A_{[x]}) < x, \text{ then } r(A_{[v]}) = r(A_{[x]}) \text{ for all } v \in [r(A_{[x]}), x]. \tag{2.3}$$

Indeed, if $r(A_{[x]}) < x$, then by Lemma 2.1 (ii), $u \overset{\mathrm{def}}{=} \sup |\sigma(A) \cap \Delta_x^{\mathrm{o}}| < x$ and $E(\Delta_x \setminus \Delta_u) = 0$. This and again Lemma 2.1 (ii) give $r(A_{[v]}) = u$ for all $v \in (u, x]$. Consequently, $r(A_{[x]}) = u$. Hence it remains to verify that $r(A_{[u]}) = u$. Suppose that, contrary to our claim, $z \overset{\mathrm{def}}{=} r(A_{[u]}) < u$. Then, by Lemma 2.1 (ii), $E(\Delta_u \setminus \Delta_z) = 0$. This and $E(\Delta_x^{\mathrm{o}} \setminus \Delta_u) = 0$ yield $E(\Delta_x^{\mathrm{o}} \setminus \Delta_z) = 0$, which in turn implies that $\sigma(A) \cap (\Delta_x^{\mathrm{o}} \setminus \Delta_z) = \varnothing$. Thus $\Gamma_u \cap \sigma(A) = \varnothing$, which contradicts $\sup |\sigma(A) \cap \Delta_x^{\mathrm{o}}| = u > 0$.

We are now ready to prove all the parts of the conclusion using only (2.2), (2.3) and the monotonicity of $s \mapsto r(A_{[s]})$ (without any recourse to Lemma 2.1). This observation is essential in the context of the proof of Proposition 2.5.

(i) If $r(A_{[t]}) = t$, then by (2.2) and the monotonicity of the function $s \mapsto r(A_{[s]})$ we have $r(A_{[t]}) = r(A_{[t+]})$. Otherwise $r(A_{[t]}) < t$, which, together with (2.3), yields $r(A_{[t]}) = r(A_{[t-]})$.

(ii) By discontinuity of the function $s \mapsto r(A_{[s]})$ at t, $r(A_{[t-]}) < r(A_{[t+]}) \leqslant t$. Let $\{t_n\}_{n=1}^\infty \subseteq (r(A_{[t-]}), t)$ be a strictly increasing sequence tending to t. Take $v \in [r(A_{[t-]}), t)$. Since $r(A_{[t_n]}) \leqslant r(A_{[t-]})$ for all $n \geqslant 1$ and $t_n \nearrow t$, there exists an integer $k \geqslant 1$ such that $v \in [r(A_{[t_n]}), t_n)$ for all $n \geqslant k$. In virtue of (2.3) we see that $r(A_{[v]}) = r(A_{[t_n]})$ for all $n \geqslant k$. Letting n tend to ∞, we get $r(A_{[v]}) = r(A_{[t-]})$.

We now show that $r(A_{[t+]}) = t$. In the contrary case, $r(A_{[t+]}) < t$ and consequently there exists real $s > t$ such that $r(A_{[s]}) \in [r(A_{[t+]}), t)$. This leads to $r(A_{[s]}) < t < s$. By (2.3), $r(A_{[v]}) = r(A_{[s]})$ for all $v \in (r(A_{[s]}), t)$. Therefore, we have $r(A_{[t-]}) = r(A_{[s]}) \geqslant r(A_{[t+]})$, which is a contradiction.

(iii) Apply (2.3) to some $x > t$.

(iv) We claim that if $r(A_{[v]}) < v$ for some $v \in U$, then $r(A_{[s]}) = r(A_{[v]})$ for all $s \in (r(A_{[v]}), \infty) \cap U$. By (2.3) we see that $r(A_{[s]}) = r(A_{[v]})$ for all $s \in (r(A_{[v]}), v]$. Suppose that, contrary to our claim, the function $s \mapsto r(A_{[s]})$ is not constant on $(r(A_{[v]}), \infty) \cap U$. Then there exists $w \in U$ such that $v < w$ and $r(A_{[v]}) < r(A_{[w]})$. By the Darboux property we can assume without loss of generality that $r(A_{[w]}) < v$. Hence $v \in (r(A_{[w]}), w)$, which by (2.3) contradicts $r(A_{[v]}) < r(A_{[w]})$.

We now show that if $r(A_{[v]}) = v$ for some $v \in U$, then $r(A_{[s]}) = s$ for all $s \in (0, v] \cap U$. Indeed, supposing that $r(A_{[w]}) \neq w$ for some $w \in (0, v) \cap U$, we infer from (2.2) that $r(A_{[w]}) < w$, which contradicts the previous paragraph.

To complete the proof, note that if $r(A_{[s]}) = s$ for all $s \in U$, then we may take $c = \infty$. Otherwise, there exists $v \in U$ such that $r(A_{[v]}) < v$, and so $c \stackrel{\text{def}}{=} r(A_{[v]})$ fits into (iv). This finishes the proof. ∎

Given $a = \{a_n\}_{n=1}^\infty \subseteq \mathbb{C}$, we denote by M_a the *diagonal* operator in ℓ^2 defined by

$$\mathcal{D}(M_a) = \Big\{\{x_n\}_{n=1}^\infty \in \ell^2 \colon \sum_{n=1}^\infty |a_n x_n|^2 < \infty\Big\} \quad \text{and}$$
$$M_a(x) = \{a_n x_n\}_{n=1}^\infty \quad \text{for } x = \{x_n\}_{n=1}^\infty \in \mathcal{D}(M_a).$$

It is well known that M_a is normal and $\sigma(M_a)$ is the closure of the set $\{a_n \colon n \geqslant 1\}$.

COROLLARY 2.3. *If $a = \{a_n\}_{n=1}^\infty \subseteq \mathbb{C}$, then*

$$r((M_a)_{[t]}) = \sup\{|a_n| \colon n \geqslant 1,\ |a_n| \leqslant t\}, \quad t \in [0, \infty). \tag{2.4}$$

Proof. The spectral measure E of M_a is given by $E(\sigma) = M_{a_\sigma}$, where $a_\sigma = \{a_{\sigma,n}\}_{n=1}^\infty$ is the sequence given by $a_{\sigma,n} = 1$ if $a_n \in \sigma$ and $a_{\sigma,n} = 0$ otherwise. Thus $E(\sigma) \neq 0$ if and only if $\sigma \cap \{a_n \colon n \geqslant 1\} \neq \varnothing$. Applying Lemma 2.1 (ii) and the equality $\sup |\sigma(A) \cap \Delta_t^\circ| = \sup\{|a_n| \colon n \geqslant 1,\ |a_n| < t\}$ completes the proof. ∎

We now show that monotonically increasing transformations of $[0, \infty)$ satisfying abstract versions of conditions (i)–(iv) of Proposition 2.2 come from normal operators.

THEOREM 2.4. *If $\varphi\colon [0,\infty) \to [0,\infty)$ is a monotonically increasing function satisfying the following four conditions:*

(i) *for every $t \in [0,\infty)$, $\varphi(t)$ equals $\varphi(t-)$ or $\varphi(t+)$;*

(ii) *for every $t \in (0,\infty)$, if the function φ is not continuous at t, then $\varphi(t-) < \varphi(t+) = t$ and $\varphi(v) = \varphi(t-)$ for all $v \in [\varphi(t-), t)$;*

(iii) *for every $t \in [0,\infty)$, if $\varphi(s) = t$ for all $s \in (t,\infty)$, then $\varphi(t) = t$;*

(iv) *if the function φ is continuous on an open interval $U \subseteq (0,\infty)$, then there exists $c \in [0,\infty]$ such that $\varphi(s) = \min\{s,c\}$ for all $s \in U$;*

then there exists a positive selfadjoint operator A in ℓ^2 such that $\varphi(s) = r(A_{[s]})$ for all $s \in [0,\infty)$.

Proof. In what follows, K' stands for the collection of all accumulation points of a set $K \subseteq \mathbb{R}$. Define the following four sets:

$$\begin{aligned} K_- &= \{t \in [0,\infty)\colon \varphi(t) = \varphi(t-) < \varphi(t+)\},\\ K_+ &= \{t \in [0,\infty)\colon \varphi(t-) < \varphi(t+) = \varphi(t)\},\\ K_{\mathrm{c}} &= \{t \in [0,\infty)\colon \varphi(t-) = \varphi(t) = \varphi(t+)\},\\ K_{\mathrm{r}} &= \{t \in [0,\infty)\colon \varphi^{-1}(\{t\}) \text{ has a nonempty interior}\}. \end{aligned}$$

It follows from (i) that $[0,\infty) = K_- \cup K_+ \cup K_{\mathrm{c}}$ with pairwise disjoint terms. Clearly, $K_{\mathrm{r}} \subseteq \varphi([0,\infty))$ and, by (ii), $K_+ \subseteq \varphi([0,\infty))$. It is also easily seen that the sets K_-, K_+ and K_{r} are at most countable. This implies that there exists a sequence $a = \{a_n\}_{n=1}^\infty \subseteq \varphi([0,\infty))$ dense in $\overline{\varphi([0,\infty))}$ such that $K_+ \cup K_{\mathrm{r}} \subseteq \{a_n\colon n \geqslant 1\}$. Let A be the diagonal operator M_a. Since A is positive and selfadjoint, it remains to show that $\varphi(s) = r(A_{[s]})$ for all $s \in [0,\infty)$. We split the proof into several steps. The monotonicity assumption will be mentioned explicitly only in more subtle cases.

Step 1. $\varphi(s) \leqslant s$ for all $s \in [0,\infty)$ (an analogue of (2.2)).

This can be done by analyzing three possible cases: $s \in K_- \cup K_+$, $s \in K_{\mathrm{c}} \cap (K_- \cup K_+)'$ and $s \in K_{\mathrm{c}} \setminus (K_- \cup K_+)'$.

Step 2. If $t \in K_{\mathrm{r}}$, then either $\varphi^{-1}(\{t\}) = [t,\infty)$ or there exists $u \in (t,\infty) \cap (K_- \cup K_+)$ such that $[t,u) \subseteq \varphi^{-1}(\{t\}) \subseteq [t,u]$.

Taking any nonempty open interval $(\alpha,\beta) \subseteq \varphi^{-1}(\{t\})$, we infer from Step 1 and (ii) that $\alpha \in [t,\infty)$ and $(t,\beta) \subseteq K_{\mathrm{c}}$, which enables us to define

$$u = \sup\{s \in (t,\infty)\colon (t,s) \subseteq K_{\mathrm{c}}\}.$$

Clearly, $u \geqslant \beta$ and $(t,u) \subseteq K_{\mathrm{c}}$. If $u = \infty$, then (iv) and (iii) give $\varphi^{-1}(\{t\}) = [t,\infty)$. Assume that $u < \infty$. It follows from (iv) that $(t,u) \subseteq \varphi^{-1}(\{t\})$. If $u \in K_- \cup K_+$, then applying (ii) we get the other part of the conclusion of Step 2. We are left with verifying that the remaining possibility $u \in K_{\mathrm{c}}$ can never happen. Indeed, the case $u \in K_{\mathrm{c}} \setminus (K_- \cup K_+)'$ immediately contradicts the definition of u. In turn, if $u \in K_{\mathrm{c}} \cap (K_- \cup K_+)'$, then there exists $\{u_n\}_{n=1}^\infty \subseteq (u,\infty) \cap (K_- \cup K_+)$ such that

$u = \lim_{n\to\infty} u_n$. Hence by (ii) and monotonicity of φ we must have $\varphi(s) > u$ for all $s > u$, which means that $\varphi(u+) \geqslant u > t = \varphi(u-)$, a contradiction.

Let us fix a real $t \geqslant 0$. If $t = 0$, then Step 1 and (2.2) imply that $\varphi(0) = r(A_{[0]}) = 0$. In the rest of the proof we assume that $t > 0$.

Step 3. If $t \in K_+$, then $\varphi(t) = r(A_{[t]}) = t$.

This can be deduced from (2.4), $K_+ \subseteq \{a_n : n \geqslant 1\}$ and (ii).

Step 4. If $t \in K_-$, then $\varphi(t) = r(A_{[t]})$.

Using (ii) and Step 2, we can show that $\varphi(s) > \varphi(t+) = t$ for all $s \in (t, \infty)$, which yields $(\varphi(t), t] \cap \varphi([0,\infty)) = \varnothing$. This and $\{a_n\}_{n=1}^{\infty} \subseteq \varphi([0,\infty))$ lead to $(\varphi(t), t] \cap \{a_n : n \geqslant 1\} = \varnothing$. Applying (2.4), we get $r(A_{[t]}) \leqslant \varphi(t)$. However $\varphi(t) \in K_{\mathrm{r}} \subseteq \{a_n : n \geqslant 1\}$, which together with (2.4) and $\varphi(t) < t$ gives $r(A_{[t]}) \geqslant \varphi(t)$.

Step 5. If $\alpha \in [0, t)$ and $\varphi(s) = s$ for every $s \in (\alpha, t)$, then $\varphi(t) = r(A_{[t]}) = t$.

Indeed, by $\overline{\{a_n : n \geqslant 1\}} = \overline{\varphi([0,\infty))}$, there exists $\{k_n\}_{n=1}^{\infty} \subseteq \{1, 2, \ldots\}$ such that $\lim_{n\to\infty} a_{k_n} = t$ and $a_{k_n} < t$ for all integers $n \geqslant 1$. By virtue of (2.2) and (2.4), we have $r(A_{[t]}) = t$. In turn, Step 1 and monotonicity of φ implies $\varphi(t) = t$, as desired.

Step 6. If $t \in K_{\mathrm{c}} \cap (K_- \cup K_+)'$, then $\varphi(t) = r(A_{[t]}) = t$.

Suppose first that φ is continuous on an interval $(t - \varepsilon, t)$ for some $\varepsilon > 0$. We claim that $\varphi(t) = t$. Indeed, since there exists a sequence $\{u_n\}_{n=1}^{\infty} \subseteq (t, \infty) \cap (K_- \cup K_+)$ converging to t, we deduce from (ii) that $\varphi(u_n+) = u_n$ tends to t as $n \to \infty$. This and $t \in K_{\mathrm{c}}$ lead to $\varphi(t) = t$. Next, in view of (iv), we get $\varphi(s) = s$ for all $s \in (t - \varepsilon, t)$, and so Step 5 gives the conclusion. In the other case there exists a strictly increasing sequence $\{t_n\}_{n=1}^{\infty} \subseteq K_- \cup K_+$ tending to t. Condition (ii) accompanied with Steps 3 and 4 implies that

$$t_n = \varphi(t_n+) \leqslant \varphi(t_{n+1}) = r(A_{[t_{n+1}]}) \leqslant r(A_{[t]}), \quad n \geqslant 1.$$

Letting n tend to ∞, and applying $t \in K_{\mathrm{c}}$ and (2.2) proves Step 6.

Step 7. If $t \in K_{\mathrm{c}} \setminus (K_- \cup K_+)'$, then $\varphi(t) = r(A_{[t]})$.

Indeed, there exists a nonempty interval $J = (\alpha, \beta)$ containing t and such that φ is continuous on J. By virtue of (iv), we can reduce the proof of Step 7 to considering the following two cases:

(a) $\varphi(s) = \varphi(t)$ for all $s \in J$,
(b) $\varphi(s) = s$ for all $s \in (\alpha, t)$.

If (a) holds, then $\varphi(t) \in K_{\mathrm{r}}$, which by Step 2, monotonicity of φ, (i) and (ii) lead either to $(\varphi(t), \infty) \cap \varphi([0,\infty)) = \varnothing$ or to $(\varphi(t), u) \cap \varphi([0,\infty)) = \varnothing$, where $u \in [\beta, \infty) \cap (K_- \cup K_+)$. Applying (2.4), Step 1 and the inclusions $K_{\mathrm{r}} \subseteq \{a_n : n \geqslant 1\} \subseteq \varphi([0,\infty))$, we get $\varphi(t) = r(A_{[t]})$. The case (b) can be deduced from Step 5. ∎

In view of the proof of Theorem 2.4 it is clear that in general for a given transformation φ there may exist many positive selfadjoint operators A such that $\varphi(s) = r(A_{[s]})$ for every $s \in [0,\infty)$. It is also worth mentioning that none of the conditions (i)–(iv) can be removed from Theorem 2.4 without affecting its conclusion.

Arguing as in the proof of Theorem 2.4, we can obtain the following result.

PROPOSITION 2.5. *Suppose that $\varphi\colon [0,\infty) \to [0,\infty)$ is an arbitrary function. If φ is monotonically increasing, then φ satisfies the conditions* (i)–(iv) *of Theorem* 2.4 *if and only if the following two conditions hold:*

(i) *for every $x \in [0,\infty)$, $\varphi(x) \leqslant x$;*

(ii) *for every $x \in [0,\infty)$, if $\varphi(x) < x$, then $\varphi(v) = \varphi(x)$ for all $v \in [\varphi(x), x]$.*

If φ has left-hand and right-hand limits at each point of $[0,\infty)$ and satisfies the conditions (i), (ii) *and* (iv) *of Theorem* 2.4, *then φ is monotonically increasing.*

3. EXAMPLES OF DISCONTINUITY OF $t \mapsto r(A_{[t]})$

EXAMPLE 3.1. For normal operators A, the function $t \mapsto r(A_{[t]})$ may have either left-hand or right-hand discontinuity (but never simultaneously, cf. Proposition 2.2). Indeed, if α, β and γ are positive real numbers such that $\alpha < \beta < \gamma$, and $a = \{a_n\}_{n=1}^{\infty}$ is a sequence whose entries form a dense subset of $[0,\alpha] \cup [\beta,\gamma]$, then by (2.4) the function $t \mapsto r((M_a)_{[t]})$ is left-discontinuous at β whenever $a_n = \beta$ for some $n \geqslant 1$, and right-discontinuous at β otherwise.

EXAMPLE 3.2. Let $\mathbb{Z}$ stand for the set of all integers. Denote by $\ell^2(\mathbb{Z})$ the Hilbert space of all square summable two-sided complex sequences. For $u \in [0,1]$, we define the bounded linear operator T_u on $\ell^2(\mathbb{Z})$ by

$$T_u(\ldots, x_{-1}, \boxed{x_0}, x_1, \ldots) = (\ldots, x_{-2}, \boxed{x_{-1}}, ux_0, ux_1, \ldots), \quad \{x_j\}_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}),$$

where the boxed entries occupy the zeroth position. We show that

$$r((T_u)_{[t]}) = \begin{cases} 0 & \text{if } t \in [0,u), \\ 1 & \text{if } t \in [u,\infty). \end{cases} \tag{3.1}$$

Indeed, note first that if $x \in \ell^2(\mathbb{Z})$ is nonzero, then $\|T_u x\| \geqslant u\|x\|$, which, when iterated, leads to $r(T_u, x) \geqslant u$. Hence $\mathcal{H}_{T_u}(t) = \{0\}$ for all $t \in [0,u)$. On the other hand if $t \in [u,\infty)$, then $e_n \in \mathcal{H}_{T_u}(t)$ for every $n \in \mathbb{Z}$, where $\{e_n\}_{n\in\mathbb{Z}}$ is the standard "0–1" orthonormal basis of $\ell^2(\mathbb{Z})$. This and Lemma 1.1 (iii) imply $r((T_u)_{[t]}) = r(T_u)$. However, $\|T_u^k\| = 1$ for all integers $k \geqslant 1$, which gives $r(T_u) = 1$. This justifies (3.1).

Take now $u, v \in \mathbb{R}$ such that $0 \leqslant u < v$. Since $\mathcal{H}_{\alpha A}(t) = \mathcal{H}_A(t/\alpha)$ and consequently $r((\alpha A)_{[t]}) = \alpha\, r(A_{[t/\alpha]})$ for $A \in \boldsymbol{B}(\mathcal{H})$ and $\alpha \in (0,\infty)$, we deduce from (3.1) that the operator $S_{u,v} \overset{\text{def}}{=} vT_{u/v} \in \boldsymbol{B}(\ell^2(\mathbb{Z}))$ has the following property:

$$r((S_{u,v})_{[t]}) = \begin{cases} 0 & \text{if } t \in [0,u), \\ v & \text{if } t \in [u,\infty). \end{cases} \tag{3.2}$$

EXAMPLE 3.3. Take $u \in [0,1)$ and a sequence $\{u_n\}_{n=1}^\infty \subseteq (u,1)$ tending to u. Set $T = \bigoplus_{n=1}^\infty T_{u_n}$, where $T_{u_n} \in \boldsymbol{B}(\ell^2(\mathbb{Z}))$ are as in Example 3.2. The Hilbert space $\mathcal{H}$ on which the bounded operator T acts is the orthogonal sum of $\aleph_0$ copies of $\ell^2(\mathbb{Z})$. We show that

$$r(T_{[t]}) = \begin{cases} 0 & \text{if } t \in [0,u], \\ 1 & \text{if } t \in (u,\infty). \end{cases} \tag{3.3}$$

For this, observe that if $f = \bigoplus_{n=1}^\infty f_n \in \mathcal{H}$ and $f_k \neq 0$ for some $k \geqslant 1$, then

$$r(T,f) \geqslant r(T_{u_k}, f_k) \geqslant u_k > u. \tag{3.4}$$

The first inequality can be proved similarly to (1.2), while the second one is justified in Example 3.2. Condition (3.4) implies that $\mathcal{H}_T(t) = \{0\}$ for all $t \in [0,u]$. Fix now $t \in (u,\infty)$ and choose an integer $k_0 \geqslant 1$ such that $u_{k_0} \leqslant t$. Denote by $\mathcal{X}$ the set of all $f = \bigoplus_{n=1}^\infty f_n \in \mathcal{H}$ such that $f_n = 0$ for each $n \neq k_0$, and $f_{k_0} \in \{e_j \colon j \in \mathbb{Z}\}$, where e_j's are as in Example 3.2. Since $r(T_{u_{k_0}}, e_j) = u_{k_0}$ for all integers j, we see that $\mathcal{X} \subseteq \mathcal{H}_T(t)$. Hence, again by Lemma 1.1 (iii), the linear span $\mathcal{E}$ of $\mathcal{X}$ is contained in $\mathcal{H}_T(t)$. Consequently

$$1 = \|T\| \geqslant r(T_{[t]}) \geqslant r(T_{[t]}|_{\mathcal{E}}) = r(T_{u_{k_0}}) = 1,$$

the last equality being shown in Example 3.2. This completes the proof of (3.3).

Employing (3.3) and the same "scaling" procedure as in Example 3.2, we construct for each $(u,w) \in \mathbb{R}^2$ with $0 \leqslant u < w$ the operator $T \in \boldsymbol{B}(\mathcal{H})$ such that

$$r(T_{[t]}) = \begin{cases} 0 & \text{if } t \in [0,u], \\ w & \text{if } t \in (u,\infty). \end{cases} \tag{3.5}$$

LEMMA 3.4. *Let A_j be an operator with invariant domain acting in a Hilbert space $\mathcal{H}_j$, $j = 1,2,\ldots,n$. If $A = \bigoplus_{j=1}^n A_j$, then:*

(i) $r(A,f) = \max_{j=1,\ldots,n} r(A_j, f_j)$ *for all* $f = \bigoplus_{j=1}^n f_j \in \mathcal{D}(A)$;

(ii) $\mathcal{H}_A(t) = \bigoplus_{j=1}^n \mathcal{H}_{A_j}(t)$ *for all* $t \in [0,\infty)$;

(iii) $r(A_{[t]}) = \max_{j=1,\ldots,n} r((A_j)_{[t]})$ *for all* $t \in [0,\infty)$ *such that* $A_{[t]} \in \boldsymbol{B}(\mathcal{H}_A(t))$.

Proof. Arguing as in (1.2) and using Lemma 1.1 (i) we obtain (i). Condition (ii) follows from (i), while (iii) is a consequence of (ii). ∎

EXAMPLE 3.5. Take $u, v, w \in \mathbb{R}$ such that $0 \leqslant u < v < w$. Let $T \in \boldsymbol{B}(\mathcal{H})$ satisfy (3.5) and $S_{u,v}$ be as in (3.2). If $A = T \oplus S_{u,v}$, then by Lemma 3.4 we have

$$r(A_{[t]}) = \begin{cases} 0 & \text{if } t \in [0, u), \\ v & \text{if } t = u, \\ w & \text{if } t \in (u, \infty). \end{cases} \tag{3.6}$$

This means that the function $s \mapsto r(A_{[s]})$ is both left- and right-discontinuous at u, and $r(A_{[t]}) > t$ for all $t \in [u, w)$. In turn, the operator $A = S_{u,v}$ satisfies (3.6) with $0 \leqslant u < v = w$, while the operator $A = T \oplus uI$ ($I =$ the identity operator on a nonzero Hilbert space) fulfills (3.6) with $0 \leqslant u = v < w$. Finally, note that by Lemma 1.6 (vii) there is no bounded operator A satisfying (3.6) with $0 < v < u$.

4. LOCALOID AND LOCALLY NORMALOID OPERATORS

We now distinguish two new classes of operators. An operator A in $\mathcal{H}$ is said to be *localoid* if $A_{[t]} \in \boldsymbol{B}(\mathcal{H}_A(t))$ and $r(A_{[t]}) \leqslant t$ for every real $t > 0$. This definition remains unchanged if we let t range over $[0, \infty)$. Indeed, Lemma 1.6 leads to $r(A_{[0]}) \leqslant r(A_{[t]}) \leqslant t$ for all real $t > 0$, and so $r(A_{[0]}) = r(A_{[0+]}) = 0$. The following fact is a consequence of Lemma 1.6.

PROPOSITION 4.1. *If A is a localoid operator in $\mathcal{H}$, then:*

(i) $\mathcal{H}_A(t) = \mathcal{H}_A(r(A_{[t]}))$ *for all* $t \in [0, \infty)$;

(ii) $r(A_{[t]}) = r(A_{[t']})$ *with* $t' \stackrel{\text{def}}{=} r(A_{[t]})$ *for all* $t \in [0, \infty)$;

(iii) $r(A_{[t]}) = t$ *whenever* $t \in [0, \infty)$ *and* $t = r(A, f)$ *for some* $f \in \mathcal{D}^\infty(A)$;

(iv) $r(A_{[t-]}) \leqslant r(A_{[t]}) \leqslant r(A_{[t+]}) \leqslant t$ *for all* $t \in [0, \infty)$;

(v) *if* $t \in [0, \infty)$ *and* $r(A_{[t]}) < r(A_{[t+]})$, *then* $r(A_{[t+]}) = t$;

(vi) $r(A_{[t]})$ *equals* $r(A_{[t-]})$ *or* $r(A_{[t+]})$ *for all* $t \in [0, \infty)$.

COROLLARY 4.2. *If A is a localoid operator in $\mathcal{H}$ and s, t are real numbers such that $0 \leqslant t \leqslant s$, then the following conditions are equivalent:*

(i) $\mathcal{H}_A(t) = \mathcal{H}_A(s)$;

(ii) $\mathcal{H}_A(r(A_{[t]})) = \mathcal{H}_A(r(A_{[s]}))$;

(iii) $r(A_{[t]}) = r(A_{[s]})$;

(iv) $r(A_{[s]}) \leqslant t$.

Proof. Apply Proposition 4.1 (i), the localoidity of A, and (1.7). ∎

Closed localoid operators can be characterized in a topological manner.

LEMMA 4.3. *Let A be a closed operator in $\mathcal{H}$. Then the following conditions are equivalent:*

(i) *A is localoid;*

(ii) *for every $t \in (0,\infty)$, $\mathcal{H}_A(t)$ is a closed subspace of $\mathcal{H}$;*

(iii) *for every $t \in (0,\infty)$, $A_{[t]} \in \boldsymbol{B}(\mathcal{H}_A(t))$ and $r(A_{[t]}) = r(A, f_t)$ for some $f_t \in \mathcal{H}_A(t)$.*

Moreover, if (i) *holds, then* (ii) *and* (iii) *are also valid for $t = 0$.*

Proof. (i)⇒(ii) Suppose that $t \in (0,\infty)$ and $f \in \overline{\mathcal{H}_A(t)}$. Since $\overline{A_{[t]}} \subseteq A$ and $\overline{A_{[t]}} \in \boldsymbol{B}(\overline{\mathcal{H}_A(t)})$, we obtain $\overline{A_{[t]}} = A|_{\overline{\mathcal{H}_A(t)}}$ and thus $\overline{\mathcal{H}_A(t)} \subseteq \mathcal{D}^\infty(A)$. This yields $r(A,f) = r(\overline{A_{[t]}}, f) \leqslant r(\overline{A_{[t]}}) \leqslant t$, which means that $f \in \mathcal{H}_A(t)$. Hence $\mathcal{H}_A(t) = \overline{\mathcal{H}_A(t)}$. Since $\mathcal{H}_A(0) = \bigcap_{s>0} \mathcal{H}_A(s)$, we see that $\mathcal{H}_A(0)$ is closed.

(ii)⇒(i) By the closed graph theorem $A_{[t]} \in \boldsymbol{B}(\mathcal{H}_A(t))$ for every $t \in (0,\infty)$. Owing to Lemma 2 of [8] (see also Theorem 1 of [19]), we have

$$r(A_{[t]}) = \max_{f \in \mathcal{H}_A(t)} r(A_{[t]}, f) = \max_{f \in \mathcal{H}_A(t)} r(A,f) \leqslant t, \quad t \in (0,\infty).$$

(i)⇒(iii) In view of implication (i)⇒(ii), $\mathcal{H}_A(t)$ is a closed subspace of $\mathcal{H}$, and so (iii) is a direct consequence of Lemma 2 in [8] applied to $A_{[t]}$ ($t \in [0,\infty)$).

(iii)⇒(i) As $f_t \in \mathcal{H}_A(t)$, we get $r(A_{[t]}) = r(A, f_t) \leqslant t$ for all real $t > 0$. ∎

An operator A in $\mathcal{H}$ is called a *locally normaloid* operator if $A_{[t]} \in \boldsymbol{B}(\mathcal{H}_A(t))$ and $\|A_{[t]}\| \leqslant t$ for all real $t > 0$. We may again admit $t = 0$ without harming the definition. It is clear that every locally normaloid operator is localoid. The reverse implication is no longer true (consider a nonzero quasinilpotent $A \in \boldsymbol{B}(\mathcal{H})$).

Locally normaloid operators can be characterized as follows.

LEMMA 4.4. *If A is an operator in $\mathcal{H}$, then the following conditions are equivalent:*

(i) *A is locally normaloid;*

(ii) *$\|Af\| \leqslant r(A,f)\|f\|$ for all $f \in \mathcal{D}^\infty(A)$;*

(iii) *the sequence $\{\|(tA)^n f\|\}_{n=0}^\infty$ is monotonically decreasing for all $f \in \mathcal{H}_{tA}(1)$ and $t \in (0,\infty)$;*

(iv) *$\sup_{n\geqslant 1} \|(tA)^n f\| \leqslant \|f\|$ for all $f \in \mathcal{H}_{tA}(1)$ and $t \in (0,\infty)$.*

Proof. (i)⇒(ii) Fix $f \in \mathcal{D}^\infty(A)$. Suppose that $t \stackrel{\text{def}}{=} r(A,f) < \infty$. Then $f \in \mathcal{H}_A(t)$, and consequently by (i) we have $\|Af\| = \|A_{[t]}f\| \leqslant t\|f\| = r(A,f)\|f\|$. The case $r(A,f) = \infty$ is obvious.

(ii)⇒(iii) Fix $f \in \mathcal{D}^\infty(A)$ and $t \in (0,\infty)$ such that $r(tA,f) \leqslant 1$. It follows from Lemma 1.1 (ii) that $r(A, A^n f) = r(A,f) \leqslant \frac{1}{t}$ for all $n \geqslant 0$. Thus by (ii) we have $\|A^{n+1}f\| = \|A(A^n f)\| \leqslant \frac{1}{t}\|A^n f\|$ for all $n \geqslant 0$, which gives $\|(tA)^{n+1}f\| \leqslant \|(tA)^n f\|$ for all $n \geqslant 0$.

(iii)⇒(iv) Evident.

(iv)⇒(i) If $t \in (0,\infty)$ and $f \in \mathcal{H}_A(t)$, then $r(\frac{1}{t}A, f) \leqslant 1$ and hence, by (iv), $\|\frac{1}{t}Af\| \leqslant \|f\|$. This completes the proof. ∎

We now show that under some extra assumption localoidity may imply local normaloidity. Following [1], we say that an operator A in $\mathcal{H}$ is *restriction-normaloid* if for every linear subspace $\mathcal{E}$ of $\mathcal{D}(A)$ such that $A|_{\mathcal{E}} \in \boldsymbol{B}(\mathcal{E})$, the equality $\|A|_{\mathcal{E}}\| = r(A|_{\mathcal{E}})$ holds (equivalently: $\|A|_{\mathcal{E}}\| = \sup\{|\langle Af, f\rangle| \colon f \in \mathcal{E}, \|f\| = 1\}$; cf. Section 2.5.4, Theorem 2 of [11]). Note that for $A \in \boldsymbol{B}(\mathcal{H})$, our definition agrees with the one in [1].

THEOREM 4.5. *Let A be an operator in $\mathcal{H}$. Then the following conditions are equivalent:*

(i) *A is locally normaloid;*

(ii) *A is localoid and restriction-normaloid.*

Moreover, if A is closed, then (i) *is equivalent to:*

(iii) *$\mathcal{H}_A(t)$ is a closed subspace of $\mathcal{H}$ and (note that by the closed graph theorem, $A_{[t]} \in \boldsymbol{B}(\mathcal{H}_A(t))$) $\|A_{[t]}\| = r(A_{[t]})$ for all $t \in (0,\infty)$.*

Proof. (i)⇒(ii) Evidently A is localoid. Let $\mathcal{E}$ be a linear subspace of $\mathcal{D}(A)$ such that $A|_{\mathcal{E}} \in \boldsymbol{B}(\mathcal{E})$. It follows from the implication (i)⇒(ii) of Lemma 4.4 that

$$\|A|_{\mathcal{E}}f\| = \|Af\| \leqslant r(A,f)\|f\| = r(A|_{\mathcal{E}}, f)\|f\| \leqslant r(A|_{\mathcal{E}})\|f\|, \quad f \in \mathcal{E},$$

which leads to $\|A|_{\mathcal{E}}\| \leqslant r(A|_{\mathcal{E}})$. Hence $\|A|_{\mathcal{E}}\| = r(A|_{\mathcal{E}})$.

(ii)⇒(i) By our assumption we have $\|A_{[t]}\| = r(A_{[t]}) \leqslant t$ for all $t \in (0,\infty)$.

(i)⇔(iii) This can be deduced from Lemma 4.3 and implication (i)⇒(ii). ∎

Note that the assumption on restriction-normaloidity cannot be dropped in Theorem 4.5 (ii) (consider a nonzero quasinilpotent operator). It is an open question whether there exist restriction-normaloid operators which are not locally normaloid.

The following simple fact is stated without proof.

LEMMA 4.6. *If A is a localoid (respectively locally normaloid) operator in $\mathcal{H}$, then:*

(i) *A^k is localoid (respectively locally normaloid) for every integer $k \geqslant 1$;*

(ii) *$A|_{\mathcal{E}}$ is localoid (respectively locally normaloid) for every linear subspace $\mathcal{E}$ of $\mathcal{D}(A)$ such that $A(\mathcal{E}) \subseteq \mathcal{E}$.*

We show by example that both localoidity and local normaloidity are not preserved when taking adjoints (see also Example 4.10).

EXAMPLE 4.7. Let A be the bounded weighted shift on $\mathcal{H}$ given by $Ae_n = \lambda_n e_{n+1}$ for all $n \geqslant 0$, where $\{\lambda_n\}_{n=0}^\infty$ is a sequence of positive real numbers and $\{e_n\}_{n=0}^\infty$ is an orthonormal basis of $\mathcal{H}$. We show that A^* is localoid if and only if $r(A) = 0$. The "if" part is plain due to $r(A) = r(A^*)$. For the "only if" part, note that $A^{*(n+1)}e_n = 0$ for all $n \geqslant 0$, which implies that $e_n \in \mathcal{H}_{A^*}(0)$ for all $n \geqslant 0$. By Lemma 1.1, $\mathcal{H}_{A^*}(0)$ is dense in $\mathcal{H}$ and consequently by Lemma 4.3, $\mathcal{H}_{A^*}(0) = \mathcal{H}$.

This yields $r(A) = r(A^*) = r((A^*)_{[0]}) \leqslant 0$, which completes the proof. Both possibilities $r(A) > 0$ and $r(A) = 0$ can occur in the context of weighted shifts (cf. [27]).

Following [34], we indicate two subclasses of locally normaloid operators.

PROPOSITION 4.8. *Every operator A in $\mathcal{H}$ satisfying the following inequality is locally normaloid:*

$$(4.1) \qquad (-1)^j \,\mathfrak{Re}\langle A^j f, f\rangle \geqslant 0, \quad f \in \mathcal{D}^\infty(A),\ j = 1, 2, \ldots.$$

Proof. By the Protter inequality (cf. Theorem 1 of [24]), we have

$$\|Af\| \leqslant b_n \|f\|^{n/(n+1)} \|A^{n+1} f\|^{1/(n+1)}, \quad f \in \mathcal{D}^\infty(A),\ n = 1, 2, \ldots,$$

where $b_n^2 = (n+1)n^{-n/(n+1)}$. Letting n tend to ∞ gives $\|Af\| \leqslant r(A,f)\|f\|$ for all $f \in \mathcal{D}^\infty(A)$. This implies that A is locally normaloid (cf. Lemma 4.4). ∎

Recall that an operator A in $\mathcal{H}$ is said to be *paranormal* if $\|Ah\|^2 \leqslant \|A^2 h\| \|h\|$ for all $h \in \mathcal{D}(A^2)$ (cf. [16], [22], [34], [11]). The following fact is a direct consequence of Proposition 1 in [34] and Lemma 8 of [32].

PROPOSITION 4.9. *If A is a paranormal operator in $\mathcal{H}$, then A is locally normaloid and for all $f \in \mathcal{D}^\infty(A)$, the sequence $\{\|A^n f\|^{1/n}\}_{n=1}^\infty$ is convergent in $[0, \infty]$.*

EXAMPLE 4.10. We give an example of a bounded locally normaloid operator, which is not paranormal and which fails to fulfill (4.1). Assume that $\mathcal{H}$ is a separable infinite dimensional Hilbert space with an orthonormal basis $\{e_n\}_{n=0}^\infty$. Let $\{\varepsilon_n\}_{n=0}^\infty$ be a sequence of nonnegative real numbers which converges to 0. Then evidently

$$(4.2) \qquad \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^n \varepsilon_j = 0.$$

Let $A \in \boldsymbol{B}(\mathcal{H})$ be the weighted shift given by $Ae_n = \mathrm{e}^{-\varepsilon_n} e_{n+1}$ for all $n \geqslant 0$. Since $\sup_{n\geqslant 0} \mathrm{e}^{-\varepsilon_n} = 1$, we see that $\|A\| = 1$. It follows from (4.2) that

$$(4.3) \qquad r(A, e_j) = \limsup_{n\to\infty} \mathrm{e}^{-\frac{\varepsilon_j + \cdots + \varepsilon_{j+n-1}}{n}} = 1, \quad j \geqslant 0.$$

We claim that $r(A,f) = 1$ for all $f \in \mathcal{H} \setminus \{0\}$. Indeed, the inequality "$\leqslant$" follows from $r(A,f) \leqslant r(A) \leqslant \|A\| = 1$. For the opposite inequality, write $f = \sum_{j=0}^\infty \alpha_j e_j$ with nonzero $\{\alpha_j\}_{j=0}^\infty \in \ell^2$. Let p be such that $\alpha_p \neq 0$. Computing as in (1.2) and applying (4.3) we get $r(A,f) \geqslant r(A, \alpha_p e_p) = 1$, which proves our claim. By contractivity of A this implies that $\|Af\| \leqslant \|f\| = r(A,f)\|f\|$ for all $f \in \mathcal{H}$. Hence, by Lemma 4.4, A is locally normaloid.

To see that A does not satisfy (4.1), take $f = \sum_{k=0}^{\infty} \frac{1}{2^k} e_k$ and compute

$$(-1)^j \,\mathfrak{Re}\langle A^j f, f\rangle = \Big(-\frac{1}{2}\Big)^j \sum_{k=0}^{\infty} \mathrm{e}^{-(\varepsilon_k + \cdots + \varepsilon_{k+j-1})} \Big(\frac{1}{2}\Big)^{2k}, \quad j \geqslant 1.$$

To disprove the paranormality of A we have to assume that the sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ is not monotonically decreasing. Then A can not be paranormal because otherwise

$$\mathrm{e}^{-2\varepsilon_n} = \|Ae_n\|^2 \leqslant \|A^2 e_n\| = \mathrm{e}^{-\varepsilon_n}\,\mathrm{e}^{-\varepsilon_{n+1}}, \quad n \geqslant 0,$$

which is a contradiction.

5. NORMALITY FROM CYCLIC NORMALITY

In this section we investigate the question of essential normality of an operator whose all cyclic (or multicyclic) parts are essentially normal. This question was settled by Nussbaum in the case of symmetric operators (cf. Theorem 1 of [20]). We begin with an auxiliary result.

PROPOSITION 5.1. *Let A be a closed operator in $\mathcal{H}$ and let $\mathcal{X}$ be the set of all vectors $h \in \mathcal{H}$ for which there exists a closed linear subspace $\mathcal{K}$ of $\mathcal{H}$ reducing A to a normal operator and containing h. If $\mathcal{X}$ is total in $\mathcal{H}$, then A is normal.*

Proof. Note first that A is densely defined. One can deduce from Theorem of [30] (e.g. making use of its "moreover" part) that there exists the greatest closed linear subspace of $\mathcal{H}$ which reduces A to a normal operator; denote it by $\mathcal{H}_{\mathrm{n}}$. It is now clear that $\mathcal{X} \subseteq \mathcal{H}_{\mathrm{n}}$, which yields $\mathcal{H}_{\mathrm{n}} = \mathcal{H}$. The proof is complete. ∎

Recall that $e \in \mathcal{D}^{\infty}(A)$ is an *analytic vector* of an operator A in $\mathcal{H}$ if there exists a constant $r > 0$ such that $\sum_{n=0}^{\infty} \|A^n e\| \frac{r^n}{n!} < \infty$. A densely defined operator A in $\mathcal{H}$ is said to be *subnormal* if there exists a complex Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ (an isometric embedding) and a normal operator N in $\mathcal{K}$ such that $\mathcal{D}(A) \subseteq \mathcal{D}(N)$ and $Ah = Nh$ for all $h \in \mathcal{D}(A)$. A densely defined operator A in $\mathcal{H}$ is called *hyponormal* if $\mathcal{D}(A) \subseteq \mathcal{D}(A^*)$ and $\|A^* h\| \leqslant \|Ah\|$ for all $h \in \mathcal{D}(A)$.

THEOREM 5.2. *If A is a densely defined operator in $\mathcal{H}$ with invariant domain, then each of the following three conditions implies the essential normality of A:*

(i) *$\mathcal{D}(A)$ consists of analytic vectors of A and for every $e \in \mathcal{D}(A)$, the operator $A|_{\mathcal{D}_e}$ is essentially normal in $\mathcal{H}_e \overset{\mathsf{def}}{=} \overline{\mathcal{D}_e}$, where $\mathcal{D}_e \overset{\mathsf{def}}{=} \operatorname{lin}\{A^n e \colon n \geqslant 0\}$;*

(ii) *A is hyponormal and the set of all vectors $e \in \mathcal{D}(A)$ for which the operator $A|_{\mathcal{D}_e}$ is essentially normal in $\mathcal{H}_e$ is total in $\mathcal{H}$;*

(iii) *for every finite subset $\mathcal{E}$ of $\mathcal{D}(A)$, the operator $A|_{\mathcal{D}_{\mathcal{E}}}$ is essentially normal in $\mathcal{H}_{\mathcal{E}} \overset{\mathsf{def}}{=} \overline{\mathcal{D}_{\mathcal{E}}}$, where $\mathcal{D}_{\mathcal{E}} \overset{\mathsf{def}}{=} \operatorname{lin}\{A^n e \colon n \geqslant 0,\, e \in \mathcal{E}\}$.*

Proof. (i) If $e \in \mathcal{D}(A)$, then $\|A^n e\|^2 = \int_0^\infty t^n \langle E_e(\mathrm{d}t)e, e\rangle$ for all integers $n \geqslant 0$, where E_e is the spectral measure of $(A|_{\mathcal{D}_e})^*(A|_{\mathcal{D}_e})^-$. By Theorem 7 of [29], the operator A is subnormal. This implies that A is closable. Since, as easily checked, Corollary 1 of [30] remains valid for hyponormal operators, and each subnormal operator is hyponormal, we see that $\mathcal{H}_e$ reduces $\overline{A}$ to a normal operator for every $e \in \mathcal{D}(A)$. This means that the operator $\overline{A}$ satisfies all the assumptions of Proposition 5.1. Hence $\overline{A}$ is normal, as desired.

(ii) Argue as in the proof of (i).

(iii) Since A is subnormal if and only if $A|_{\mathcal{D}_\mathcal{E}}$ is subnormal for every finite subset $\mathcal{E}$ of $\mathcal{D}(A)$ (cf. Theorem 3 of [29]), we deduce from our assumptions that A is subnormal. Applying Corollary 1 of [30] and Proposition 5.1 as in the proof of (i) completes the proof of Theorem 5.2. ∎

6. ALL THIS FOR LOCALLY ALGEBRAIC AND COMPACT OPERATORS

Below, $\mathbb{K}[X]$ stands for the ring of all polynomials in indeterminate X with coefficients in a field $\mathbb{K}$. An operator A in $\mathcal{H}$ is called *algebraic* if there exists a nonzero polynomial $p \in \mathbb{C}[X]$ such that $p(A)f = 0$ for all $f \in \mathcal{D}(p(A)) \overset{\text{def}}{=} \mathcal{D}(A^{\deg p})$. Recall that there are algebraic operators with invariant domains which are not closable (cf. Example 3.2 of [21]). For the sake of self-containedness, we include the proof of the following basic fact about algebraic operators.

LEMMA 6.1. *Let $p_1, \ldots, p_m \in \mathbb{K}[X]$ be a collection of pairwise relatively prime polynomials, $\mathcal{X}$ be a linear space over $\mathbb{K}$ and $A : \mathcal{X} \to \mathcal{X}$ be a linear mapping such that $p_1(A)\cdots p_m(A) = 0$. Then:*

(i) *all the spaces $\mathcal{N}(p_j(A))$, $j = 1, \ldots, m$, are invariant for A;*

(ii) *$\mathcal{X} = \mathcal{N}(p_1(A)) \dotplus \cdots \dotplus \mathcal{N}(p_m(A))$, where $\dotplus$ denotes the direct sum;*

(iii) *$\mathcal{N}(p_{i_1}(A)\cdots p_{i_s}(A)) = \mathcal{N}(p_{i_1}(A)) \dotplus \cdots \dotplus \mathcal{N}(p_{i_s}(A))$ for all finite sequences of integers $1 \leqslant i_1 < \cdots < i_s \leqslant m$;*

(iv) *$\mathcal{N}(p_j(A)) \neq \{0\}$ for every $j \in \{1, \ldots, m\}$ such that $\prod_{k\neq j} p_k(A) \neq 0$.*

Proof. The invariance of $\mathcal{N}(p_j(A))$ under A is readily checked. The proof of (ii) is by induction on m. The case $m = 1$ is clear. Suppose (ii) is valid for a fixed integer $m \geqslant 1$. Assume that $p_1, \ldots, p_{m+1}$ are pairwise relatively prime and $p_1(A)\cdots p_{m+1}(A) = 0$. Set $q = p_1 \cdots p_m$. By relative primeness of q and p_{m+1} there exist polynomials $u, v \in \mathbb{K}[X]$ such that $uq + vp_{m+1} = 1$ (combine Corollary III.6.4, Theorem III.3.9 and Theorem III.3.11 of [12]). Then

$$x = q(A)u(A)x + p_{m+1}(A)v(A)x, \quad x \in \mathcal{X}. \tag{6.1}$$

As $q(A)p_{m+1}(A) = 0$, we see that

(6.2) $$p_{m+1}(A)v(A)x \in \mathcal{N}(q(A)) \quad \text{and} \quad q(A)u(A)x \in \mathcal{N}(p_{m+1}(A)) \quad \text{for all } x \in \mathcal{X}.$$

Define $\mathcal{Y} = \mathcal{N}(q(A))$ and $\mathcal{Z} = \mathcal{N}(p_{m+1}(A))$. Then (6.1) and (6.2) imply that

(6.3) $$\mathcal{X} = \mathcal{Y} \dotplus \mathcal{Z}.$$

Since $\mathcal{Y}$ is an invariant subspace for the mapping A, $p_1(A|_{\mathcal{Y}}) \cdots p_m(A|_{\mathcal{Y}}) = 0$ and $\mathcal{N}(p_j(A|_{\mathcal{Y}})) = \mathcal{N}(p_j(A))$ for all $j = 1, \ldots, m$, we infer from the induction hypothesis that $\mathcal{Y} = \mathcal{N}(p_1(A)) \dotplus \cdots \dotplus \mathcal{N}(p_m(A))$. By (6.3) this completes the induction argument.

Applying (ii) to the restriction of A to $\mathcal{V} \stackrel{\text{def}}{=} \mathcal{N}(p_{i_1}(A) \cdots p_{i_s}(A))$, we get $\mathcal{V} = \mathcal{N}(p_{i_1}(A|_{\mathcal{V}})) \dotplus \cdots \dotplus \mathcal{N}(p_{i_s}(A|_{\mathcal{V}}))$. Since $\mathcal{N}(p_{i_k}(A|_{\mathcal{V}})) = \mathcal{N}(p_{i_k}(A))$ for all $k = 1, \ldots, s$, the proof of (iii) is finished.

If $\prod_{k \neq j} p_k(A) \neq 0$ and $\mathcal{N}(p_j(A)) = \{0\}$ for some $j \in \{1, \ldots, m\}$, then $p_j(A)$ is injective, which contradicts $p_j(A) \prod_{k \neq j} p_k(A) = 0$. Hence (iv) is proved. ∎

Let A be an algebraic operator in $\mathcal{H}$ such that $\mathcal{D}^\infty(A) \neq \{0\}$. Since $A_{[\infty]}$ is also algebraic, we find a unique monic polynomial p of minimal degree (call it *minimal*) such that $p(A_{[\infty]}) = 0$. By the fundamental theorem of algebra, $p(z) = (z - z_1)^{n_1} \cdots (z - z_m)^{n_m}$ with unique integers $n_1, \ldots, n_m \geqslant 1$ and complex numbers $z_1, \ldots, z_m$ such that $z_j \neq z_k$ for all $j \neq k$. Owing to Lemma 6.1, we have

(6.4) $$\begin{gathered} \mathcal{D}^\infty(A) = \mathcal{N}((A_{[\infty]} - z_1)^{n_1}) \dotplus \cdots \dotplus \mathcal{N}((A_{[\infty]} - z_m)^{n_m}), \\ \mathcal{N}((A_{[\infty]} - z_j)^{n_j}) \neq \{0\}, \quad j = 1, \ldots, m. \end{gathered}$$

Thus every $f \in \mathcal{D}^\infty(A)$ can be uniquely decomposed as $f = f_1 + \cdots + f_m$ with $f_j \in \mathcal{N}((A_{[\infty]} - z_j)^{n_j})$. We set $\mathfrak{n}_f(A) = \{j \in \{1, \ldots, m\} : f_j \neq 0\}$. If no confusion can arise, we write $\mathfrak{n}_f$ instead of $\mathfrak{n}_f(A)$.

PROPOSITION 6.2. *Let A be an algebraic operator in $\mathcal{H}$. Then for every nonzero vector $f \in \mathcal{D}^\infty(A)$ the sequence $\{\|A^n f\|^{1/n}\}_{n=1}^\infty$ is convergent to* $\max\{|z_j| : j \in \mathfrak{n}_f\}$, *where $z_1, \ldots, z_m$ and $\mathfrak{n}_f$ are as above. Moreover, if $t \geqslant$* $\max\{|z_j| : 1 \leqslant j \leqslant m\}$, *then $\mathcal{H}_A(t) = \mathcal{D}^\infty(A)$.*

Proof. We start with an auxiliary fact.

SUBLEMMA 6.3. *Assume that $T\colon \mathcal{D} \to \mathcal{D}$ is a linear mapping on an inner product space $\mathcal{D}$, $f \in \mathcal{D} \setminus \{0\}$, $z \in \mathbb{C} \setminus \{0\}$ and $k \geqslant 1$ is an integer. If $(T - z)^k f = 0$, then there exists an integer $N \geqslant 0$ such that the sequence $\{\frac{1}{z^n n^N} T^n f\}_{n=1}^\infty$ is convergent to an element of $\mathcal{D} \setminus \{0\}$.*

Indeed, there exists an integer $N \geqslant 0$ such that $(T-z)^N f \neq 0$ and $(T-z)^j f = 0$ for all $j \geqslant N+1$. The case $N = 0$ is easily seen to be true. If $N \geqslant 1$, then

$$\begin{aligned}\frac{1}{z^n n^N} T^n f &= \frac{1}{z^n n^N}((T-z)+z)^n f = \frac{1}{n^N}\sum_{j=0}^{N}\binom{n}{j}\Big(\frac{T-z}{z}\Big)^j f \\ &= \frac{1}{n^N}\binom{n}{N}\Big(\frac{T-z}{z}\Big)^N f + \frac{1}{n}\sum_{j=0}^{N-1}\frac{1}{n^{N-1}}\binom{n}{j}\Big(\frac{T-z}{z}\Big)^j f, \quad n \geqslant N.\end{aligned}$$

Since $\lim_{n\to\infty}\frac{1}{n^N}\binom{n}{N} = \frac{1}{N!}$ and each sequence $\{\frac{1}{n^{N-1}}\binom{n}{j}\}_{n=N}^{\infty}$, $j = 0,\dots,N-1$, is bounded, we get $\lim_{n\to\infty}\frac{1}{z^n n^N} T^n f = \frac{1}{N!}(\frac{T-z}{z})^N f \in \mathcal{D}\setminus\{0\}$.

We now turn to the proof of Proposition 6.2. There is no loss of generality in assuming that $A = A_{[\infty]}$. The spaces $\mathcal{X}_j \stackrel{\mathsf{def}}{=} \mathcal{N}((A-z_j)^{n_j})$, $j = 1,\dots,m$, are invariant for A and, by (6.4), $\mathcal{D}(A) = \mathcal{X}_1 \dotplus \cdots \dotplus \mathcal{X}_m$. Take a nonzero $f \in \mathcal{D}(A)$. Then $f = f_1 + \cdots + f_m$ with the unique vectors $f_j \in \mathcal{X}_j$. Define

$$\mathfrak{n}_f^* = \{j \in \mathfrak{n}_f\colon |z_j| > 0\}.$$

We will only handle the nontrivial case of $\mathfrak{n}_f^* \neq \varnothing$. Owing to the sublemma, there exist nonnegative integers $\{N_j\}_{j\in\mathfrak{n}_f^*}$ such that

$$g_j \stackrel{\mathsf{def}}{=} \lim_{n\to\infty}\frac{1}{z_j^n n^{N_j}} A^n f_j \in \mathcal{X}_j\setminus\{0\}, \quad j \in \mathfrak{n}_f^*. \tag{6.5}$$

Set $r = \max\{|z_j|\colon j \in \mathfrak{n}_f^*\}$ and $N = \max\{N_j\colon j \in \mathfrak{n}_f^*,\ |z_j| = r\}$. Remark that $r > 0$. We divide $\mathfrak{n}_f^*$ into two disjoint sets

$$\begin{aligned}J_1 &= \{j \in \mathfrak{n}_f^*\colon |z_j| = r,\ N_j = N\},\\ J_2 &= \{j \in \mathfrak{n}_f^*\colon |z_j| = r,\ N_j < N\} \cup \{j \in \mathfrak{n}_f^*\colon |z_j| < r\}.\end{aligned}$$

This enables us to write (with special care for $j \in \mathfrak{n}_f \setminus \mathfrak{n}_f^*$)

$$\frac{1}{r^n n^N} A^n f = \sum_{j\in J_1}\frac{1}{r^n n^N} A^n f_j + \sum_{j\in J_2}\frac{1}{r^n n^N} A^n f_j, \quad n \geqslant \varkappa, \tag{6.6}$$

where $\varkappa \stackrel{\mathsf{def}}{=} \max\{n_1,\dots,n_m\}$. Note that

$$\frac{1}{r^n n^N} A^n f_j = \frac{1}{n^{N-N_j}}\Big(\frac{z_j}{r}\Big)^n \frac{1}{z_j^n n^{N_j}} A^n f_j, \quad j \in \mathfrak{n}_f^*,\ n \geqslant \varkappa. \tag{6.7}$$

If $j \in J_2$, then $\lim_{n\to\infty}\frac{1}{n^{N-N_j}}(\frac{z_j}{r})^n = 0$, which together with (6.5) shows that

$$\text{the sequence } \Big\{\sum_{j\in J_2}\frac{1}{r^n n^N} A^n f_j\Big\}_{n=1}^{\infty} \text{ tends to 0 as } n \to \infty. \tag{6.8}$$

Fix an arbitrary strictly increasing sequence $\{l_n\}_{n=1}^{\infty}$ of positive integers. Passing to a subsequence if necessary, we can assume that for each $j \in J_1$ the sequence

$\{(\frac{z_j}{r})^{l_n}\}_{n=1}^{\infty}$ is convergent to a complex number α_j of absolute value 1. Hence calling upon (6.6), (6.7), (6.8) and (6.5) we see that

$$\lim_{n\to\infty} \frac{1}{r^{l_n} l_n^N} A^{l_n} f = \sum_{j\in J_1} \alpha_j g_j.$$

Since $g_j \in \mathcal{X}_j \setminus \{0\}$ and $\alpha_j \neq 0$ for all $j \in J_1$, and $\mathcal{D}(A) = \mathcal{X}_1 \dotplus \cdots \dotplus \mathcal{X}_m$, the above limit is nonzero. This implies that $\lim\limits_{n\to\infty} \|A^{l_n} f\|^{1/l_n} = r$. Summarizing, we have proved that every subsequence of the sequence $\{\|A^n f\|^{1/n}\}_{n=1}^{\infty}$ admits a subsequence which is convergent to r. Hence $\{\|A^n f\|^{1/n}\}_{n=1}^{\infty}$ is convergent to r. This immediately implies the "moreover" part of the conclusion. ∎

In the case of algebraic operators localoidity reduces to the boundedness of $A_{[\infty]}$.

PROPOSITION 6.4. *If A is an algebraic operator in $\mathcal{H}$, then the following conditions are equivalent:*

(i) *A is localoid;*
(ii) *$A_{[t]} \in \boldsymbol{B}(\mathcal{H}_A(t))$ for every $t \in (0,\infty)$;*
(iii) *$A_{[\infty]}$ is bounded.*

In particular, every algebraic operator $A \in \boldsymbol{B}(\mathcal{H})$ is localoid.

Proof. (i)⇒(ii) Evident.

(ii)⇒(iii) Apply Proposition 6.2.

(iii)⇒(i) Clearly, there is no loss of generality in assuming that $A = A_{[\infty]}$. Next, since $\overline{A}\colon \overline{\mathcal{D}(A)} \to \overline{\mathcal{D}(A)}$ is a bounded algebraic operator, Lemma 4.6 enables us to reduce the proof to the case $A \in \boldsymbol{B}(\mathcal{H})$. Preserving the notation from the proof of Proposition 6.2, we define the set $\mathfrak{m}_t = \{j \in \{1,\ldots,m\}\colon |z_j| \leqslant t\}$ for $t \in (0,\infty)$. It follows from Proposition 6.2 that

$$\mathcal{H}_A(t) = \begin{cases} \dotplus_{j\in\mathfrak{m}_t} \mathcal{X}_j & \text{if } \mathfrak{m}_t \neq \varnothing, \\ \{0\} & \text{if } \mathfrak{m}_t = \varnothing, \end{cases} \qquad t \in (0,\infty).$$

This and part (iii) of Lemma 6.1 imply that $\mathcal{H}_A(t)$ is a closed subspace of $\mathcal{H}$ for every $t \in (0,\infty)$. Hence, by Lemma 4.3, A is localoid. ∎

Bounded algebraic operators on $\mathcal{H}$ may not be locally normaloid (cf. Theorem 6.5). However, there are unbounded closed densely defined nilpotents and idempotents with invariant domains (cf. [21]) which are evidently not localoid.

We now formulate a criterion for essential normality of algebraic operators.

THEOREM 6.5. *Let A be a densely defined algebraic operator in $\mathcal{H}$ with invariant domain. Then the following conditions are equivalent:*

(i) *A is closable and $\overline{A}$ is a bounded normal operator on $\mathcal{H}$;*
(ii) *A is paranormal;*

(iii) *A is locally normaloid.*

Proof. We need only consider the case $\mathcal{H} \neq \{0\}$.

(i)$\Rightarrow$(ii) This is a well known fact (cf. Proposition 3 of [34]).

(ii)$\Rightarrow$(iii) Apply Proposition 4.9.

(iii)$\Rightarrow$(i) One way of proving this implication is to mimic the proof of Proposition 6.1 in [28] and to use Lemma 4.4 and the discussion preceding Proposition 6.2. The other possibility is to reduce our case to bounded algebraic operators satisfying condition (iii) of Lemma 4.4, which would enable us to apply directly Proposition 6.1 of [28]. Hence, in view of Lemma 4.4, it suffices to show that $\overline{A}$ is a bounded algebraic locally normaloid operator. We preserve the notation from the proof of Proposition 6.2. By Proposition 6.4, A is closable and $\overline{A} \in \boldsymbol{B}(\mathcal{H})$. It is clear that $\overline{A}$ is algebraic and the minimal polynomials of A and $\overline{A}$ coincide. It follows from Lemma 6.1 that

$$\begin{aligned} &\mathcal{D}(A) = \mathcal{N}((A-z_1)^{n_1}) \dotplus \cdots \dotplus \mathcal{N}((A-z_m)^{n_m}),\\ &\mathcal{H} = \mathcal{N}((\overline{A}-z_1)^{n_1}) \dotplus \cdots \dotplus \mathcal{N}((\overline{A}-z_m)^{n_m}), \qquad (6.9)\\ &\mathcal{N}((A-z_j)^{n_j}) \subseteq \mathcal{N}((\overline{A}-z_j)^{n_j}), \quad j = 1, \ldots, m. \end{aligned}$$

Fix real $t > 0$. By Proposition 6.4 and Lemma 4.3, the space $\mathcal{H}_{\overline{A}}(t)$ is closed. Hence $\overline{\mathcal{H}_A(t)} \subseteq \mathcal{H}_{\overline{A}}(t)$. We now justify the converse inclusion. Take a nonzero vector $f = \sum_{j\in \mathfrak{n}_f(\overline{A})} f_j \in \mathcal{H}_{\overline{A}}(t)$ with the unique nonzero vectors $f_j \in \mathcal{N}((\overline{A}-z_j)^{n_j})$. Let us abbreviate $\mathfrak{n}_f(\overline{A})$ to σ. We show that the space $\mathcal{U} \overset{\text{def}}{=} \dotplus_{j\in\sigma} \mathcal{N}((\overline{A}-z_j)^{n_j})$ is a range of a projection $P \in \boldsymbol{B}(\mathcal{H})$ (a priori not orthogonal) such that

$$P(\mathcal{D}(A)) \subseteq \dotplus_{j\in\sigma} \mathcal{N}((A-z_j)^{n_j}).$$

If $\sigma = \{1, \ldots, m\}$, then $P =$ the identity operator is the only possible choice. Otherwise, the existence of such a projection follows from the equality $\mathcal{H} = \mathcal{U} \dotplus \mathcal{V}$, where $\mathcal{V} \overset{\text{def}}{=} \dotplus_{j\notin\sigma} \mathcal{N}((\overline{A}-z_j)^{n_j})$, the closedness of the spaces $\mathcal{U}$ and $\mathcal{V}$ (use part (iii) of Lemma 6.1) and (6.9). Since $\mathcal{D}(A)$ is dense in $\mathcal{H}$, there exists a sequence $\{g_n\}_{n=1}^\infty \subseteq \mathcal{D}(A)$ which converges to f. Then the sequence $\{Pg_n\}_{n=1}^\infty \subseteq \dotplus_{j\in\sigma} \mathcal{N}((A-z_j)^{n_j})$ converges to f. According to Proposition 6.2, $r(A, Pg_n) \leqslant r(\overline{A}, f)$ for all $n \geqslant 1$. This shows that $\mathcal{H}_{\overline{A}}(t) \subseteq \overline{\mathcal{H}_A(t)}$. Thus we have proved that $\mathcal{H}_{\overline{A}}(t) = \overline{\mathcal{H}_A(t)}$ for all real $t > 0$. As A is a bounded locally normaloid operator, we get $\|(\overline{A})_{[t]}\| = \|\overline{A_{[t]}}\| = \|A_{[t]}\| \leqslant t$ for all real $t > 0$, which shows that $\overline{A}$ is locally normaloid. ∎

Regarding Theorem 6.5 (ii), we point out that the boundedness of algebraic paranormal operators can also be deduced from Theorem 1 of [31]. The essential part of the proof of Theorem 6.5 consists in transferring the problem from the unbounded operator case to the bounded one and then to apply Proposition 6.1 of [28] (which in fact coincides with Corollary 6.6 below). Though the question of characterizing normality found numerous solutions in literature (e.g. [13], [1], [2], [3], [23], [25], [18]), we have not been able to come across any result directly implying the following.

COROLLARY 6.6. *Every algebraic restriction-normaloid operator* $A \in \boldsymbol{B}(\mathcal{H})$ *is normal.*

Proof. By Proposition 6.4, the operator A is localoid. This and Theorem 4.5 imply that A is locally normaloid. Applying Theorem 6.5 completes the proof. ∎

Corollary 6.6 is no longer true if we drop the algebraicity assumption even though the spectrum were finite. Indeed, according to Theorem 2 of [26], there exists a non-normal contraction with the prescribed finite spectrum contained in the unit circle, and as such is restriction-normaloid (cf. Theorem 1 of [14]). If we do not insist that the spectrum is finite, then there are obvious examples of such operators, e.g. nonunitary isometries.

An operator A in $\mathcal{H}$ with invariant domain is said to be *locally algebraic* if for every $e \in \mathcal{D}(A)$ there exists a nonzero polynomial $p_e \in \mathbb{C}[X]$ such that $p_e(A)e = 0$. According to Lemma 14 of [17] every locally algebraic bounded operator on a Banach space is automatically algebraic. This statement is no longer true for unbounded operators, e.g. every unbounded diagonal operator, when considered on "finite" vectors, is locally algebraic but not algebraic. Below we state the locally algebraic version of Theorem 6.5. Proposition 6.2 may also be adapted to this context.

THEOREM 6.7. *Let* A *be a densely defined locally algebraic operator in* $\mathcal{H}$ *with invariant domain. Then the following conditions are equivalent:*

(i) *A is essentially normal;*
(ii) *A is paranormal;*
(iii) *A is locally normaloid.*

Proof. (i)⇒(ii)⇒(iii) Mimic the appropriate parts of the proof of Theorem 6.5.

(iii)⇒(i) We preserve the notation introduced in Theorem 5.2. Take $e \in \mathcal{D}(A)$. Then $A|_{\mathcal{D}_e}$ is a densely defined algebraic operator in $\mathcal{H}_e$ with invariant domain. Since by Lemma 4.6 the operator $A|_{\mathcal{D}_e}$ is locally normaloid, we infer from Theorem 6.5 that $A|_{\mathcal{D}_e}$ is bounded and essentially normal in $\mathcal{H}_e$. This implies that e is an analytic vector of A. Applying Theorem 5.2 (i) completes the proof. ∎

The alternative proof of Theorem 6.7 consists in employing Theorem 5.2 (iii).

The following result is a counterpart of Proposition 6.2 for compact operators. Let $A \in \boldsymbol{B}(\mathcal{H})$ be a compact operator. Given an isolated point $z \in \mathbb{C}$ of the spectrum $\sigma(A)$ of A, we denote by $P_z \in \boldsymbol{B}(\mathcal{H})$ the Riesz projection attached to

$\{z\}$ (cf. Chapter VII.3 of [9]). In turn, $Q_t \in \boldsymbol{B}(\mathcal{H})$ stands for the Riesz projection associated with the closed set $\sigma(A) \cap \{z \in \mathbb{C}\colon |z| \leqslant t\}$, where $t \in (0,\infty)$. Define

$$\mathfrak{k}_f(A) = \{z \in \mathbb{C}\colon z \text{ is an isolated point of } \sigma(A) \text{ and } P_z f \neq 0\}, \quad f \in \mathcal{H}.$$

Recall that (below I stands for the identity operator on $\mathcal{H}$)

$$Q_t = I - \sum_{z\in\sigma(A),\,|z|>t} P_z, \quad t \in (0,\infty). \tag{6.10}$$

PROPOSITION 6.8. *Let $A \in \boldsymbol{B}(\mathcal{H})$ be a compact operator. Then A is localoid and for every vector $f \in \mathcal{H}$, the sequence $\{\|A^n f\|^{1/n}\}_{n=1}^\infty$ is convergent and*

$$\lim_{n\to\infty} \|A^n f\|^{1/n} = \begin{cases} \max\{|z|\colon z \in \mathfrak{k}_f(A)\} & \text{if } \mathfrak{k}_f(A) \neq \varnothing, \\ 0 & \text{otherwise.} \end{cases} \tag{6.11}$$

Moreover, the following equality holds:

$$\mathcal{H}_A(t) = Q_t(\mathcal{H}), \quad t \in (0,\infty). \tag{6.12}$$

Proof. Take $f \in \mathcal{H} \setminus \{0\}$. Consider three cases.

(i) $\mathfrak{k}_f(A) \neq \varnothing$ and $\varkappa \stackrel{\text{def}}{=} \max\{|z|\colon z \in \mathfrak{k}_f(A)\} > 0$. Then, by (6.10), $f \in Q_\varkappa(\mathcal{H})$. Since $\sigma(A|_{Q_\varkappa(\mathcal{H})}) = \{z \in \sigma(A)\colon |z| \leqslant \varkappa\}$ and $r(A|_{Q_\varkappa(\mathcal{H})}) = \varkappa$, we can apply Proposition 1 of [8] to the operator $A|_{Q_\varkappa(\mathcal{H})}$, which yields $\lim_{n\to\infty}\|A^n f\|^{1/n}=\varkappa$.

(ii) $\mathfrak{k}_f(A) \neq \varnothing$ and $\varkappa = 0$. Since now 0 is an isolated point of $\sigma(A)$ and $P_0 = I - \sum_{z\in\sigma(A)\setminus\{0\}} P_z$, we get $f \in P_0(\mathcal{H})$. Hence

$$r(A,f) = r(A|_{P_0(\mathcal{H})}, f) \leqslant r(A|_{P_0(\mathcal{H})}) = 0.$$

(iii) $\mathfrak{k}_f(A) = \varnothing$. Observe first that 0 is an accumulation point of $\sigma(A)$. For otherwise $\sigma(A)$ is finite and $P_z f = 0$ for all $z \in \sigma(A)$, which implies $f = 0$, a contradiction. Take $w \in \sigma(A) \setminus \{0\}$. Since by (6.10), $f \in Q_{|w|}(\mathcal{H})$, we get

$$r(A,f) = r(A|_{Q_{|w|}(\mathcal{H})}, f) \leqslant r(A|_{Q_{|w|}(\mathcal{H})}) = |w|.$$

Letting w tend to zero completes the proof of (6.11).

Noticing that $\mathcal{H}_A(t) = Q_t(\mathcal{H}) = \mathcal{H}$ for all positive $t \in [r(A),\infty)$, we see that (6.12) has to be verified only for $t \in (0, r(A))$. This can be done with the help of (6.11) via (6.10) and the fact that the ranges of Riesz projections $\{P_z\colon z \in \sigma(A),\, |z| > t\}$ are linearly independent. Condition (6.12) and Lemma 4.3 imply that A is localoid. ∎

In view of Theorem 4.5 and Proposition 6.4, within the class of bounded algebraic operators there is no difference between the notions of a restriction-normaloid operator and a locally normaloid operator. In turn, by Proposition 6.8, we see that the same observation remains valid for the class of compact operators. Hence, owing to Theorem 1 of [15] and Proposition 4.9, we obtain the following corollary.

COROLLARY 6.9. *A compact operator $A \in \boldsymbol{B}(\mathcal{H})$ is normal if and only if it is locally normaloid.*

Acknowledgements. The first and the third authors were supported by the KBN grant 2 P03A 037 024. The second author was supported by a grant (R14-2003-006-01000-0) from the Korea Research Foundation. A substantial part of this paper was written while the first and the third authors visited Kyungpook National University during the fall of 2005; they wish to thank the faculty and administration for their warm hospitality.

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Received August 17, 2006; revised April 5, 2007.