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p-OPERATOR SPACES AND FIGÀ-TALAMANCA–HERZ ALGEBRAS

MATTHEW DAWS

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ABSTRACT. We study a generalisation of operator spaces modelled on L_p spaces, instead of Hilbert spaces, using the notion of p-complete boundedness, as studied by Pisier and Le Merdy. We show that the Figà-Talamanca–Herz algebras $A_p(G)$ become quantised Banach algebras in this framework, and that amenability of these algebras corresponds to amenability of the locally compact group G, extending the result of Ruan about $A(G)$. We also show that various notions of multipliers of $A_p(G)$ (including Herz's generalisation of the Fourier–Stieltjes algebra) naturally fit into this framework.

KEYWORDS: *Operator space, locally compact group, SQ_p-space, Figà-Talamanca–Herz algebra, multiplier algebra, amenability.*

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1. INTRODUCTION

The Fourier algebra, $A(G)$, of a locally compact group G is the collection of coefficient functionals $f : G \to \mathbb{C}$ of the form

$$f(g) = [\lambda(g)(x), y] \quad (g \in G),$$

where $x, y \in L_2(G)$ and λ is the left-regular representation of G on $L_2(G)$. Eymard defined and studied this commutative Banach algebra in [10]. For an abelian group G, the Fourier transform shows that $A(G)$ is nothing but $L_1(\widehat{G})$, where $\widehat{G}$ is the dual group of G. As such, $A(G)$ is amenable as a Banach algebra, and for another abelian group H, we have that $A(G)\widehat{\otimes}A(H) = A(G \times H)$. However, as first noted by Johnson in [19], there exist compact groups G for which $A(G)$ is not amenable. Thus the Banach algebra $A(G)$ does not seem to capture some properties of the group G.

In [30], Ruan showed that when $A(G)$ is considered as an operator space (and hence as a quantised Banach algebra), we have that $A(G)$ is amenable if and only if G is amenable, and that $A(G)\widehat{\otimes}A(H) = A(G \times H)$ for all locally compact groups G and H (here we use the operator space projective tensor product). These results provide some compelling evidence that $A(G)$ is best viewed as an operator space, and not simply as a Banach algebra. We remark that the original question of when $A(G)$ is amenable as a Banach algebra was finally settled in [13], where it is shown that $A(G)$ is amenable if and only if G has a finite-index abelian subgroup.

In [11], Figà-Talamanca introduced a natural generalisation of the Fourier algebra, for abelian and compact groups, by replacing $L_2(G)$ by $L_p(G)$. In [18], Herz extended the definition to arbitrary groups, leading to the commutative Banach algebra $A_p(G)$, now called the Figà-Talamanca–Herz algebras. In many ways these algebras behave like $A(G)$; for example, Leptin's theorem (see Theorem 6 of [17] or Section 10 of [27]) states that G is an amenable group if and only if $A_p(G)$ has a bounded approximate identity.

There have been a number of attempts to give $A_p(G)$ an operator space structure. In [32], Runde used some of Pisier's work on interpolation spaces to define an operator space version of $A_p(G)$, denoted $OA_p(G)$. Unfortunately, while $OA_2(G) = A(G)$ as Banach spaces, the operator space structure can differ; furthermore, $OA_p(G)$ can fail to be equal to $A_p(G)$, even as a Banach space, for $p \neq 2$. In [22], the authors use Lambert's ideas of row and column operator spaces to define an operator space structure on $A_p(G)$ which turns $A_p(G)$ into a bounded (but not contractive) quantised Banach algebra, and in such a way that $A_2(G) = A(G)$ completely isometrically. Furthermore, $A_p(G)$ is amenable in this framework if and only if G is an amenable group.

In this paper, we shall use ideas of Pisier and Le Merdy to define the notion of a *p-operator space* (for $1 < p < \infty$, with a 2-operator space being simply an operator space). We show that the algebras $A_p(G)$ then carry a natural p-operator space structure. We investigate the amenability of $A_p(G)$ in this framework, and also study the p-completely bounded multipliers of $A_p(G)$.

2. BANACH SPACES

In this section we shall gather together some basic results on Banach spaces. Let E be a Banach space, and denote by E' the dual space of E. For $x \in E$ and $\mu \in E'$, we write $\langle \mu, x\rangle$ for $\mu(x)$ (we use angle brackets for bilinear products, and occasionally use square brackets for sesquilinear products). There is a canonical isometry $\kappa_E : E \to E''$ defined by $\langle \kappa_E(x), \mu\rangle = \langle \mu, x\rangle$. When κ_E is an isomorphism, we say that E is *reflexive*.

Let E and F be Banach spaces, and consider the algebraic tensor product $E \otimes F$. We define the *projective tensor norm* $\|\cdot\|_\pi$ on $E \otimes F$ by

$$\|\tau\|_\pi = \inf\Big\{ \sum_k \|x_k\|\|y_k\| : \tau = \sum_k x_k \otimes y_k \Big\} \quad (\tau \in E \otimes F).$$

The completion of $E \otimes F$ with respect to $\|\cdot\|_\pi$ is denoted by $E\widehat{\otimes}F$. It is a simple exercise to show that $(E\widehat{\otimes}F)' = \mathcal{B}(E,F') \cong \mathcal{B}(F,E')$ by the identification

$$\langle T, x\otimes y\rangle = \langle T(x), y\rangle \quad (T\in\mathcal{B}(E,F'), x\in E, y\in F).$$

Here we write $\mathcal{B}(E,F)$ for the Banach space of bounded linear operators from E to F. We write $\mathcal{B}(E)$ for $\mathcal{B}(E,E)$.

Alternatively, we may embed $E\otimes F$ into $\mathcal{B}(E',F)$, which leads to the definition of the *injective tensor norm* $\|\cdot\|_\varepsilon$, and the *injective tensor product* $E\check{\otimes}F$. Then $E'\otimes F$ can be identified with the *finite rank operators* from E to F, denoted by $\mathcal{F}(E,F)$. The closure of $\mathcal{F}(E,F)$ in $\mathcal{B}(E,F)$ is the *approximable operators* from E to F, denoted by $\mathcal{A}(E,F)$. Thus $E'\check{\otimes}F = \mathcal{A}(E,F)$.

There is an obvious norm-decreasing map $J : E'\widehat{\otimes}E \to E'\check{\otimes}E = \mathcal{A}(E)$, whose image is the *nuclear operators*, $\mathcal{N}(E)$. We give $\mathcal{N}(E)$ the quotient norm coming from $\mathcal{N}(E) \cong E'\widehat{\otimes}E/\ker J$. When J is injective, we say that E has the *approximation property*. See [34] or [7] for further details on these ideas.

3. AMENABLE BANACH ALGEBRAS

We shall eventually apply our results to the study of when certain Banach algebras are *amenable* (in various senses). However, we shall also need some ideas from this area as we go along, so we introduce the needed ideas now.

Let $\mathcal{A}$ be a Banach algebra, and let E be an $\mathcal{A}$-bimodule. A linear map $d : \mathcal{A}\to E$ is a *derivation* if $d(ab) = a\cdot d(b) + d(a)\cdot b$ for $a,b\in\mathcal{A}$. We shall assume that all our derivations are bounded. For $x\in E$, define $d_x : \mathcal{A}\to E$ by $d_x(a) = a\cdot x - x\cdot a$, for $a\in\mathcal{A}$. Then d_x is a derivation, called an *inner derivation*. A Banach algebra $\mathcal{A}$ is *amenable* when every derivation from $\mathcal{A}$ to a dual bimodule is inner. See the book [31] for details about amenable Banach algebras, for example.

Johnson showed in [20] that for a locally compact group G, one has that G is amenable if and only if the group algebra $L_1(G)$ is amenable. Recall that a group G is *amenable* when there is a left-invariant mean for $L_\infty(G)$. See [26] or [27] for details about amenable groups. Johnson also provided a useful characterisation of when an algebra is amenable.

DEFINITION 3.1. Let $\mathcal{A}$ be a Banach algebra. A bounded net (d_α) in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is an *approximate diagonal* if

$$\lim_\alpha \|a\cdot d_\alpha - d_\alpha\cdot a\| = 0, \quad \lim_\alpha \|a\Delta_{\mathcal{A}}(d_\alpha) - a\| = 0 \quad (a\in\mathcal{A}).$$

Here $\Delta_{\mathcal{A}} : \mathcal{A}\widehat{\otimes}\mathcal{A}\to\mathcal{A}$ is the linearisation of the product, defined by $\Delta_{\mathcal{A}}(a\otimes b) = ab$.

THEOREM 3.2. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ has an approximate diagonal. When G is an amenable group, we may choose an approximate diagonal for $L_1(G)$ which is bounded by* 1.

Proof. See Chapter 2 of [31] for example. ∎

Let $\mathcal{A}$ be a Banach algebra, and let E be a left $\mathcal{A}$-module. Let $\mathcal{A}^c_E = \{T \in \mathcal{B}(E) : T(a \cdot x) = a \cdot T(x) \ (a \in \mathcal{A}, x \in E)\}$, the commutant of $\mathcal{A}$ in E. Then a projection $\mathcal{Q} : \mathcal{B}(E) \to \mathcal{A}^c_E$ is a *quasi-expectation* when $\mathcal{Q}(TSR) = T\mathcal{Q}(S)R$ for $T, R \in \mathcal{A}^c_E$ and $S \in \mathcal{B}(E)$.

PROPOSITION 3.3. *Let $\mathcal{A}$ be an amenable Banach algebra, and let E be a reflexive left $\mathcal{A}$-module. Then there is a quasi-expectation $\mathcal{Q} : \mathcal{B}(E) \to \mathcal{A}^c_E$.*

Proof. We sketch a proof (see Theorem 4.4.11 of [31] for example). Let (d_α) be an approximate diagonal for $\mathcal{A}$, and let $d_\alpha = \sum_{n=1}^\infty a_n^{(\alpha)} \otimes b_n^{(\alpha)}$ for each α. As E is reflexive, by moving to a subnet if necessary, we may define

$$\langle \mu, \mathcal{Q}(T)(x) \rangle = \lim_\alpha \sum_{n=1}^\infty \langle \mu, a_n^{(\alpha)} \cdot T(b_n^{(\alpha)} \cdot x) \rangle \quad (x \in E, \mu \in E', T \in \mathcal{B}(E)).$$

Then $\mathcal{Q}$ is a linear operator, and $\|\mathcal{Q}\| \leqslant \limsup_\alpha \|d_\alpha\|$. Clearly, if $T \in \mathcal{A}^c_E$, then $\mathcal{Q}(T) = T$. Moreover, as d_α is an approximate diagonal, for $x \in E, \mu \in E', a \in \mathcal{A}$ and $T \in \mathcal{B}(E)$,

$$\langle \mu, \mathcal{Q}(T)(a \cdot x) - a \cdot \mathcal{Q}(T)(x) \rangle = \lim_\alpha \sum_{n=1}^\infty \langle \mu, a_n^{(\alpha)} \cdot T(b_n^{(\alpha)} a \cdot x) - a a_n^{(\alpha)} \cdot T(b_n^{(\alpha)} \cdot x) \rangle = 0,$$

so that $\mathcal{Q}(T) \in \mathcal{A}^c_E$. Thus $\mathcal{Q}$ is a projection onto $\mathcal{A}^c_E$. Similarly, for $T, R \in \mathcal{A}^c_E$ and $S \in \mathcal{B}(E)$, it is easy to check that $\mathcal{Q}(TSR) = T\mathcal{Q}(S)R$. ∎

It is shown in [5] that, in a certain sense, the converse to the above is true. When $\mathcal{A}$ is a von Neumann algebra, we follow [31] and define $\mathcal{A}$ to be *Connes-amenable* using the same definition as for amenability, but insisting that everything is suitably weak*-continuous (this is commonly just referred to as the suitable definition of "amenable" for von Neumann algebras). Then $\mathcal{A}$ is Connes-amenable if and only if there is an *expectation* (that is, a norm-one projection) from $\mathcal{B}(H)$ to $\mathcal{A}$, where H is any Hilbert space such that $\mathcal{A} \subseteq \mathcal{B}(H)$ is a concrete realisation of the von Neumann algebra $\mathcal{A}$. It is well-known (see Chapter III, Theorem 3.4 of [37]) that an expectation is always a quasi-expectation.

4. p-OPERATOR SPACES

Let SQ_p be the collection of subspaces of quotients of L_p spaces, where we identify spaces which are isometrically isomorphic. Let μ be a measure, and E a Banach space. We define a norm on the algebraic tensor product $L_p(\mu) \otimes E$ by embedding $L_p(\mu) \otimes E$ into $L_p(\mu, E)$ in the obvious way. Let the completion be denoted by $L_p(\mu) \otimes_p E$. It is easy to see that $L_p(\mu) \otimes E$ is dense in $L_p(\mu, E)$, so that $L_p(\mu) \otimes_p E = L_p(\mu, E)$ isometrically. An important property of SQ_p spaces is the following. For $E, F \in SQ_p$, we have that for $T \in \mathcal{B}(L_p(\mu))$ and $S \in \mathcal{B}(E, F)$, the operator $T \otimes S$ is bounded as an operator from $L_p(\mu) \otimes_p E$ to $L_p(\mu) \otimes_p F$, with norm $\|T\|\|S\|$. See Section 7 of [7] or the survey paper [8] for further information.

For $n \in \mathbb{N}$, let ℓ_p^n be $\mathbb{C}^n$ with the ℓ_p-norm. Similarly, $\ell_p(I)$ is the usual ℓ_p space over an index set I; we set ℓ_p to be $\ell_p(\mathbb{N})$. Throughout, we shall let p' be the conjugate index to p, so that $p^{-1} + p'^{-1} = 1$.

An abstract characterisation of SQ_p spaces is the following, which goes back to Kwapien (see Theorem 3.2 of [25] for example). For a square matrix $a = (a_{ij}) \in \mathbb{M}_n$, we let a induce an operator on ℓ_p^n, which leads to the norm

$$\|a\|_{\mathcal{B}(\ell_p^n)} = \sup\Big\{ \Big(\sum_{i=1}^n \Big| \sum_{j=1}^n a_{ij} x_j \Big|^p \Big)^{1/p} : (x_j)_{j=1}^n \subseteq \mathbb{C}, \sum_{j=1}^n |x_j|^p \leqslant 1 \Big\}.$$

We have that $E \in SQ_p$ if and only if, for each n and each $a=(a_{ij}) \in \mathbb{M}_n$, we have that

$$\sup\Big\{ \Big(\sum_{i=1}^n \Big\| \sum_{j=1}^n a_{ij} x_j \Big\|^p \Big)^{1/p} : (x_j)_{j=1}^n \subseteq E, \sum_{j=1}^n \|x_j\|^p \leqslant 1 \Big\} \leqslant \|a\|_{\mathcal{B}(\ell_p^n)}.$$

4.1. p-OPERATOR SPACES. We now introduce some ideas studied in [29], and especially [25], although we introduce some new notation. A *concrete p-operator space* is a closed subspace of $\mathcal{B}(E)$, for some $E \in SQ_p$. Notice that we could equally define this by using $\mathcal{B}(E, F)$ instead, for $E, F \in SQ_p$. This follows, as we can identify $\mathcal{B}(E, F)$ with a closed subspace of $\mathcal{B}(E \oplus_p F)$, where $E \oplus_p F$ is the direct sum of E and F together with the norm $\|e \oplus f\| = (\|e\|^p + \|f\|^p)^{1/p}$ for $e \in E$ and $f \in F$.

For a concrete p-operator space $X \subseteq \mathcal{B}(E)$, for each $n > 0$, we define a norm $\|\cdot\|_n$ on $\mathbb{M}_n(X) = \mathbb{M}_n \otimes X$ by identifying $\mathbb{M}_n(X)$ as a subspace of $\mathcal{B}(\ell_p^n \otimes_p E)$. It is easy to see that the norms $\|\cdot\|_n$ satisfy:

- $\mathcal{D}_\infty$: for $u \in \mathbb{M}_n(X)$ and $v \in \mathbb{M}_m(X)$, we have that $\|u \oplus v\|_{n+m} = \max(\|u\|_n, \|v\|_m)$. Here $u \oplus v \in \mathbb{M}_{n+m}(X)$ has block representation $\left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)$.
- $\mathcal{M}_p$: for $u \in \mathbb{M}_m(X)$, $\alpha \in \mathbb{M}_{n,m}$ and $\beta \in \mathbb{M}_{m,n}$, we have that $\|\alpha u \beta\|_n \leqslant \|\alpha\|\|u\|_m\|\beta\|$. Here $\alpha u \beta$ is the obvious matrix product, and we define $\|\alpha\|$ to be the norm of α as a member of $\mathcal{B}(\ell_p^m, \ell_p^n)$, and similarly for β.

An *abstract p-operator space* is a Banach space X together with a family of norms $\|\cdot\|_n$ defined by $\mathbb{M}_n(X)$ satisfying the above two axioms. When $p = 2$, the above axioms are just Ruan's axioms, and so 2-operator spaces are just operator spaces. Here, and throughout, we refer to [9] for details on operator spaces. Then Theorem 4.1 of [25] shows that an abstract p-operator space X can be isometrically embedded in $\mathcal{B}(E)$ for some $E \in SQ_p$, and in such a way that the canonical norms on $\mathbb{M}_n(X)$ arising from this embedding agree with the given norms. Henceforth, we shall just talk of p-operator spaces. We shall tend to abuse notation, and write $\|\cdot\|$ instead of $\|\cdot\|_n$, where there can be no confusion.

The natural morphisms between p-operator spaces are the *p-completely bounded* maps, as first studied in [29]. A linear map $u : X \to Y$ between p-operator spaces induces a map $(u)_n : \mathbb{M}_n(X) \to \mathbb{M}_n(Y)$ in an obvious way. We say that u is *p-completely bounded* if $\|u\|_{pcb} := \sup_n \|(u)_n\| < \infty$. Similarly, we have the notions of p-completely contractive and p-completely isometric. We write $\mathcal{CB}_p(X,Y)$ for the Banach space of all p-completely bounded maps from X to Y.

Pisier proved a factorisation scheme for p-completely bounded maps. Let $E \in \mathcal{S}Q_p$, let J be some index set, and let ϕ_j be a measure, for each $j \in J$. Let $\mathcal{U}$ be an ultrafilter on J, so that we may form the ultraproduct $\widehat{E} = (L_p(\phi_j, E))_{\mathcal{U}}$. Notice that $\widehat{E} \in SQ_p$ (see [14] for details about ultraproducts of Banach spaces). For each $j \in J$, $\mathcal{B}(E)$ acts naturally on $L_p(\phi_j, E)$, and so we get a canonical homomorphism $\pi : \mathcal{B}(E) \to \mathcal{B}(\widehat{E})$. Now suppose that $X \subseteq \mathcal{B}(E)$ is a p-operator space. Let $N \subseteq M \subseteq E$ and $\widehat{N} \subseteq \widehat{M} \subseteq \widehat{E}$ be closed subspaces such that, for each $x \in X$, $\pi(x)$ maps N into $\widehat{N}$ and M into $\widehat{M}$. Hence, for each $x \in X$, $\pi(x)$ naturally induces a map, denoted $\widehat{\pi}(x)$, from $G = M/N$ to $\widehat{G} = \widehat{M}/\widehat{N}$. Notice that $G, \widehat{G} \in SQ_p$. We call the map $\widehat{\pi}$ a *p-representation* from X to $\mathcal{B}(G, \widehat{G})$.

THEOREM 4.1. *Let $E, F \in SQ_p$, let $X \subseteq \mathcal{B}(E)$ be a p-operator space, and let $u : X \to \mathcal{B}(F)$ be a linear map. Then u is p-completely bounded with $\|u\|_{pcb} \leqslant C$ if and only if there exists a p-representation $\widehat{\pi} : X \to \mathcal{B}(G, \widehat{G})$ and operators $U : F \to G$ and $V : \widehat{G} \to F$ such that*

$$u(x) = V\widehat{\pi}(x)U \quad (x \in X).$$

Proof. This is Theorem 2.1 of [29], although we have followed the presentation of [25]. ∎

As noted by Pisier after the statement of Theorem 2.1 in [29], if $X \subseteq \mathcal{B}(E)$ is a unital closed subalgebra, we may suppose that $M = \widehat{M}$ and $N = \widehat{N}$, so that $G = \widehat{G}$.

As for operator spaces, we define a norm on $\mathbb{M}_n(\mathcal{CB}_p(X,Y))$ by identifying this space with $\mathcal{CB}_p(X, \mathbb{M}_n(Y))$. It is then an easy check to see that these norms satisfy the above axioms, and so Le Merdy's theorem tells us that $\mathcal{CB}_p(X,Y)$ is itself a p-operator space.

For the next result, we give $\mathbb{C}$ the obvious p-operator space structure: that is, $\mathbb{M}_n(\mathbb{C}) = \mathcal{B}(\ell_p^n)$.

LEMMA 4.2. *Let X be a p-operator space, and let $\mu \in X'$, the Banach dual space of X. Then μ is p-completely bounded as a map to $\mathbb{C}$, and $\|\mu\|_{pcb} = \|\mu\|$.*

Proof. We cannot simply follow the usual operator-space proof. In the $p = 2$ case, we have Smith's Lemma available, which tells us that for a map $u : X \to \mathbb{M}_n$, we have that $\|u\|_{cb} = \|(u)_n\|$. An examination of the proof of Lemma 2.2.1 in [9] shows that we cannot hope for an extension to the general p case.

We wish to show that $(\mu)_n : \mathbb{M}_n(X) \to \mathcal{B}(\ell_p^n)$ is bounded, with norm $\|\mu\|$. Let $x = (x_{ij})_{i,j=1}^n \in \mathbb{M}_n(X)$, so that $(\mu)_n(x) = (\langle \mu, x_{ij} \rangle)$. Let $\alpha = (\alpha_i)_{i=1}^n \in \ell_p^n$ and

$\beta = (\beta_j)_{j=1}^n \in \ell_{p'}^n$. Then

$$\langle \beta, (\mu)_n(x)(\alpha) \rangle = \sum_{i,j=1}^n \beta_i \langle \mu, x_{ij} \rangle \alpha_j = \Big\langle \mu, \sum_{i,j=1}^n \beta_i x_{ij} \alpha_j \Big\rangle.$$

We may regard α as a member of $\mathbb{M}_{n,1}$, from which it follows that $\|\alpha\|_{\mathcal{B}(\ell_p^1, \ell_p^n)} = \|\alpha\|_p$, and similarly $\beta \in \mathbb{M}_{1,n}$ with $\|\beta\|_{\mathcal{B}(\ell_p^n, \ell_p^1)} = \|\beta\|_{p'}$. So from axiom $\mathcal{M}_p$ it follows that $\|\beta x \alpha\|_1 \leqslant \|\beta\|_{p'} \|x\|_n \|\alpha\|_p$, and so

$$|\langle \beta, (\mu)_n(x)(\alpha) \rangle| \leqslant \|\mu\| \|\beta\|_{p'} \|x\|_n \|\alpha\|_p.$$

This implies that $\|(\mu)_n(x)\| \leqslant \|\mu\| \|x\|_n$, which in turn implies that $\|(\mu)_n\| \leqslant \|\mu\|$, as required. ∎

As this proof indicates, we shall have significant problems extending many results from operator spaces to p-operator spaces. Indeed, the evidence below suggests that the current definitions might be wrong, in that we are unable to prove simple properties which one would naturally want to hold.

We may hence identify X' with $\mathcal{CB}_p(X, \mathbb{C})$, and from this it follows that X' is also a p-operator space. We may use Le Merdy's Theorem to show that X' admits a representation $X' \subseteq \mathcal{B}(E)$ for some $E \in SQ_p$. In fact, in this special case, we have a more concrete embedding.

THEOREM 4.3. *Let X be a p-operator space. There exists a p-complete isometry $\Phi : X' \to \mathcal{B}(\ell_p(I))$ for some index set I.*

Proof. We follow Proposition 3.2.4 of [9]. For each $n \in \mathbb{N}$, let s_n be the unit sphere of $\mathbb{M}_n(X)$, and let $s = \bigcup_n s_n$. For $x \in s$, let $n(x) \in \mathbb{N}$ be such that $x \in s_{n(x)}$. Then let E be the ℓ_p-direct sum of the spaces $\{\ell_p^{n(x)} : x \in s\}$, so that E is isometric to $\ell_p(I)$ for some index set I. For $\mu \in X'$ and $x \in s$, we have that $x(\mu) := (\mu)_n(x) \in \mathbb{M}_{n(x)} = \mathcal{B}(\ell_p^{n(x)})$, with $\|x(\mu)\| \leqslant \|x\| \|\mu\| = \|\mu\|$. For $a = (a_x)_{x \in s} \in E$ and $\mu \in X'$, we may hence define

$$\Phi(\mu)(a) = (x(\mu)(a_x))_{x \in s},$$

and we see that Φ is norm-decreasing. Indeed, clearly μ attains its norm on s_1, so that Φ is an isometry.

For $\mu \in \mathbb{M}_m(X')$, by definition,

$$\|\mu\| = \sup\{|\langle\langle \mu, x \rangle\rangle| : n \in \mathbb{N}, x \in \mathbb{M}_n(X), \|x\| = 1\} = \sup\{|\langle\langle \mu, x \rangle\rangle| : x \in s\}.$$

Following the notation in [9], for $x = (x_{ij}) \in \mathbb{M}_n(X)$ and $\mu = (\mu_{kl}) \in \mathbb{M}_m(X')$, we let $\langle\langle \mu, x \rangle\rangle = (\langle \mu_{kl}, x_{ij} \rangle)_{(k,i),(l,j)} \in \mathbb{M}_m \otimes \mathbb{M}_n = \mathbb{M}_{m \times n}$. We then see that $(\Phi)_n(\mu) = (\langle\langle \mu, x \rangle\rangle)_{x \in s}$, so that $(\Phi)_n$ is an isometry, and hence Φ is a p-complete isometry as required. ∎

We now come to our first problem. Let X be a Banach space, and recall the isometric map $\kappa = \kappa_X : X \to X''$ defined by $\langle \kappa_X(x), \mu \rangle = \langle \mu, x \rangle$ for $x \in X$ and $\mu \in X'$.

PROPOSITION 4.4. *Let X be a p-operator space. Then κ_X is a p-complete contraction. Furthermore, κ_X is a p-complete isometry if and only if $X \subseteq \mathcal{B}(L_p(\phi))$ p-completely isometrically for some measure ϕ.*

Proof. For $x = (x_{ij}) \in \mathbb{M}_n(X)$, by definition,

$$\begin{aligned}\|(\kappa)_n(x)\|_n &= \sup\{\|\langle\langle(\kappa(x_{ij})), \mu\rangle\rangle\| : m \in \mathbb{N}, \mu \in \mathbb{M}_m(X'), \|\mu\|_m = 1\} \\ &= \sup\{\|\langle\langle \mu, x\rangle\rangle\| : m \in \mathbb{N}, \mu \in \mathbb{M}_m(X'), \|\mu\|_m = 1\} \leqslant \|x\|_n,\end{aligned}$$

so that κ is a p-complete contraction.

Suppose now that κ is a p-complete isometry. From the above theorem, we know that $X'' \subseteq \mathcal{B}(\ell_p(I))$ for some index set I. Thus $X = \kappa(X) \subseteq \mathcal{B}(\ell_p(I))$, as required.

Conversely, suppose that $X \subseteq \mathcal{B}(E)$ for $E = L_p(\phi)$ for some measure ϕ. To show that κ is a p-complete isometry, we need to show that for each $x \in \mathbb{M}_n(X)$ and $\varepsilon > 0$ there exists $m \in \mathbb{N}$ and a p-complete contraction $u \in \mathcal{CB}_p(X, \mathbb{M}_m) = \mathbb{M}_m(X')$ with $\|u(x)\|_m \geqslant \|x\|_n - \varepsilon$. When $p = 2$, we may use Lemma 2.3.4 of [9] and take $n = m$ and $\varepsilon = 0$. However, for other values of p we have to work harder.

Let $x = (x_{ij}) \in \mathbb{M}_n(X) \subseteq \mathcal{B}(\ell_p^n \otimes_p E)$. For $\varepsilon > 0$, there exists $(a_i)_{i=1}^n \subseteq E$ with $\sum_{i=1}^n \|a_i\|^p \leqslant 1$ and $\Big(\sum_{i=1}^n \Big\| \sum_{j=1}^n x_{ij}(a_j)\Big\|^p\Big)^{1/p} \geqslant \|x\|_n - \varepsilon$. Let $b_i = \sum_{j=1}^n x_{ij}(a_j)$ for $1 \leqslant i \leqslant n$. Let $\delta > 0$ to be chosen later. By standard properties of $E = L_p(\phi)$, there exists $m \in \mathbb{N}$ and an isometry $U : \ell_p^m \to E$ such that for each j, there exists $f_j \in \ell_p^m$ with $\|U(f_j) - a_j\| < \delta$. Similarly, there exists a contraction $V : E \to \ell_p^m$ such that $(1-\delta)\|b_i\| \leqslant \|V(b_i)\| \leqslant (1+\delta)\|b_i\|$ for each i.

Define $u : X \to \mathcal{B}(\ell_p^m)$ by $u(x) = VxU$ for $x \in X$. A simple calculation shows that u is a p-complete contraction, as $\|U\|\|V\| \leqslant 1$. Then

$$\|(u)_n(x)\|\Big(\sum_{j=1}^n \|f_j\|^p\Big)^{1/p} \geqslant \Big(\sum_{i=1}^n \Big\|\sum_{j=1}^n V x_{ij} U(f_j)\Big\|^p\Big)^{1/p} \geqslant \|x\|_n - 2\varepsilon,$$

if $\delta > 0$ is sufficiently small. Similarly, if $\delta > 0$ is sufficiently small, then $\sum_{j=1}^n \|U(f_j)\|^p \leqslant \sum_{j=1}^n \|a_j\|^p + \varepsilon \leqslant 1 + \varepsilon$. Hence $\|(u)_n(x)\|_m$ can be chosen to be arbitrarily close to $\|x\|_n$, as required. ∎

The following was communicated to us by Christian Le Merdy. Suppose that $X \subseteq \mathcal{B}(L_p(\phi))$ for some measure ϕ, and that X is finite dimensional with $\mathbb{M}_{n,1}(X) = \ell_p^n(X)$ for each n. Pick $\varepsilon > 0$, and let $(x_1, \ldots, x_n)$ be an ε-dense subset of the unit sphere of X (which exists as X is finite dimensional). Then

$$\Big(\sum_{k=1}^{n}\|x_k\|^p\Big)^{1/p}=\|(x_k)\|_{\mathbb{M}_{n,1}(X)}:=\sup\Big\{\Big(\sum_{k=1}^{n}\|x_k(w)\|^p\Big)^{1/p}:w\in L_p(\phi),\|w\|\leqslant 1\Big\}.$$

Hence there exists $w_\varepsilon \in L_p(\phi)$ with $\|w_\varepsilon\| = 1$ and $\|x_k(w_\varepsilon)\| \geqslant \|x_k\| - \varepsilon = 1-\varepsilon$ for each k. Define $T_\varepsilon : X \to L_p(\phi)$ by $T(x) = x(w_\varepsilon)$ for $x \in X$. For $x \in X$ with $\|x\| = 1$, let $\|x - x_k\| < \varepsilon$, so that

$$1 = \|x\| \geqslant \|T_\varepsilon(x)\| = \|x(w_\varepsilon)\| > \|x_k(w_\varepsilon)\| - \varepsilon \geqslant 1 - 2\varepsilon.$$

By homogeneity, $(1-2\varepsilon)\|x\| \leqslant \|T_\varepsilon(x)\| \leqslant \|x\|$ for each $x \in X$. A simple ultrapower argument then shows that we may construct an isometry $X \to L_p(\psi)$ for some measure ψ (recall that an ultrapower of $L_p(\phi)$ is equal to $L_p(\psi)$ for some ψ).

Now let $E \subseteq \ell_p^m$ be some subspace. We give ℓ_p^m the p-operator space structure given by the identification $\ell_p^m = \mathcal{B}(\mathbb{C}, \ell_p^m)$, and then make E a subspace. Then $\mathbb{M}_{n,1}(\ell_p^m) \subseteq \mathcal{B}(\mathbb{C}, \ell_p^m \otimes_p \ell_p^n)$, so that $\mathbb{M}_{n,1}(\ell_p^m) = \ell_p^n(\ell_p^m)$, and similarly for E. In particular,

$$\mathbb{M}_{n,1}(\ell_p^m/E) = \mathbb{M}_{n,1}(\ell_p^m)/\mathbb{M}_{n,1}(E) = \ell_p^n(\ell_p^m)/\ell_p^n(E) = \ell_p^n(\ell_p^m/E).$$

So, if $\ell_p^m/E \subseteq \mathcal{B}(L_p(\phi))$ for some measure ϕ, then $\ell_p^m/E \subseteq L_p(\psi)$ for some measure ψ. However, for suitable chosen E, this is nonsense. In particular, there exist p-operator spaces X (which may be finite-dimensional) such that κ_X is not a p-complete isometry.

LEMMA 4.5. *Let X and Y be p-operator spaces, and let $u \in \mathcal{CB}_p(X,Y)$. Then $u' \in \mathcal{CB}_p(Y',X')$ and $\|u'\|_{pcb} \leqslant \|u\|_{pcb}$.*

Proof. This follows as for operator spaces, see Proposition 3.2.2 of [9]. We cannot conclude that $\|u'\|_{pcb} = \|u\|_{pcb}$ because of the problems we encountered above. ∎

Combining Theorem 4.3 and Proposition 4.4, we see that for every p-operator space X, we have that $\kappa_{X'} : X' \to X'''$ is a p-complete isometry. Actually, there is a much easier way to see this result. A simple calculation shows that $\kappa_X'\kappa_{X'} = I_{X'}$, and as the identity map if a p-complete isometry, so also must $\kappa_{X'}$ be, as by the lemma, κ_X' is a p-complete contraction.

Let X and Y be p-operator spaces, and let $u \in \mathcal{CB}_p(X,Y)$. Then u is a *p-complete quotient map* if, for each n, $(u)_n$ takes the open unit ball of $\mathbb{M}_n(X)$ onto the open unit ball of $\mathbb{M}_n(Y)$.

LEMMA 4.6. *Let X and Y be p-operator spaces, and let $u : X \to Y$ be a p-complete quotient map. Then $u' : Y' \to X'$ is a p-complete isometry.*

Proof. Let $\mu \in \mathbb{M}_n(Y')$ and $\varepsilon > 0$, so that for some m, there exists $y \in \mathbb{M}_m(Y)$ with $\|y\|_m < 1$ and $|\langle\langle \mu, y\rangle\rangle| \geqslant \|\mu\|_n - \varepsilon$. By assumption, we can find $x \in \mathbb{M}_m(X)$ with $\|x\|_m < 1$ and $u(x) = y$, and so

$$\|(u')_n(\mu)\|_n \geqslant \|\langle\langle \mu, u(x)\rangle\rangle\| \geqslant \|\mu\|_n - \varepsilon,$$

which, as $\varepsilon > 0$ was arbitrary, shows that $\|(u')_n(\mu)\|_n = \|\mu\|_n$, as required. ∎

The lack of a suitable Hahn–Banach theorem for p-operator spaces (when $p = 2$ we have the Arveson–Wittstock theorem Theorem 4.1.5 of [9]) means that we cannot show the converse to the above.

We define subspaces of p-operator spaces in the obvious way. Given a p-operator space X and a closed subspace $Y \subseteq X$, we define a norm on $\mathbb{M}_n(X/Y)$ by identifying this space with $\mathbb{M}_n(X)/\mathbb{M}_n(Y)$. Then, as for operator spaces (see Proposition 3.11 of [9]) it is easy to check that X/Y becomes a p-operator space, and that the quotient map $\pi : X \to X/Y$ is a p-complete quotient map. The above lemma then tells us that $\pi' : (X/Y)' \to X'$ is a p-complete isometry. A simple calculation shows that the image of π' is

$$Y^\perp := \{\mu \in X' : \langle \mu, y\rangle = 0 \ (y \in Y)\},$$

so that we may identify $(X/Y)'$ with $Y^\perp$ p-completely isometrically. Again, we have no such identification of Y' with a suitable quotient of X'.

4.2. TENSOR PRODUCTS. We define the *p-operator space projective tensor norm* on the tensor product of two p-operator space X and Y to be

$$\|\tau\|_\wedge = \inf\{\|\alpha\|\|u\|\|v\|\|\beta\| : \tau = \alpha(u \otimes v)\beta\} \quad (\tau \in \mathbb{M}_n(X \otimes Y)).$$

Here we let $u \in \mathbb{M}_r(X)$ and $v \in \mathbb{M}_s(Y)$, so that $u \otimes v \in \mathbb{M}_{r\times s}(X \otimes Y)$ in a natural way, and we take $\alpha \in \mathbb{M}_{n,r\times s}$ and $\beta \in \mathbb{M}_{r\times s,n}$, so that $\alpha(u \otimes v)\beta \in \mathbb{M}_n(X \otimes Y)$ as required. This is exactly the definition for operator spaces, except that as above, we evaluate $\|\alpha\|$ as a member of $\mathcal{B}(\ell^n_p, \ell^{s\times r}_p)$, and similarly $\|\beta\|$. We shall prove below that $\|\cdot\|_\wedge$ gives $X \otimes Y$ an abstract p-operator space structure. Denote by $X\widehat{\otimes}^p Y$ the completion.

PROPOSITION 4.7. *Let X be a vector space, and for each n, let $\|\cdot\|_n : \mathbb{M}_n(X) \to [0, \infty)$ be a map such that:*

$\mathcal{D}'_\infty$ *: for $u \in \mathbb{M}_n(X)$ and $v \in \mathbb{M}_m(X)$, we have that $\|u \oplus v\|_{n+m} \leqslant \max(\|u\|_n, \|v\|_m)$;*
$\mathcal{M}_p$ *: for $u \in \mathbb{M}_m(X)$, $\alpha \in \mathbb{M}_{n,m}$ and $\beta \in \mathbb{M}_{m,n}$, we have that $\|\alpha u \beta\|_n \leqslant \|\alpha\|\|u\|_m\|\beta\|$.*

Then each $\|\cdot\|_n$ is a norm, and the completion of X becomes an abstract p-operator space.

Proof. This follows exactly as for operator spaces, ([9], Proposition 2.3.6). ∎

PROPOSITION 4.8. *Let X and Y be p-operator spaces. Then $\|\cdot\|_\wedge$ induces a p-operator space structure on $X \otimes Y$. Furthermore, $\|\cdot\|_\wedge$ is the largest such p-operator space norm with the additional property that $\|u \otimes v\| \leqslant \|u\|_r\|v\|_s$ for $u \in \mathbb{M}_r(X)$ and $v \in \mathbb{M}_s(Y)$.*

Proof. This follows as for operator space (see Theorem 7.1.1 of [9]) with minor alterations. In [9], the authors use the C^*-identity, in the $p = 2$ case, to estimate the norm of a matrix $\alpha \in \mathbb{M}_{r,s} = \mathcal{B}(\ell_p^s, \ell_p^r)$ of the block form

$$\alpha = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 \end{pmatrix}.$$

However, we get the entirely elementary estimate that $\|\alpha\| \leqslant \max(\|\alpha_1\|, \|\alpha_2\|)$, which is all that is required. ∎

Let X, Y and Z be p-operator spaces, and let $\psi : X \times Y \to Z$ be a bilinear map. We define bilinear maps

$$(\psi)_{r,s;t,u} : \mathbb{M}_{r,s}(X) \times \mathbb{M}_{t,u}(Y) \to \mathbb{M}_{r\times t,s\times u}(Z); \ \ (x, y) \mapsto (\psi(x_{i,j}, y_{k,l})).$$

Then we let $(\psi)_{r;s} = (\psi)_{r,r;s,s}$, and define

$$\|\psi\|_{pcb} = \sup\{\|(\psi)_{r;s}\| : r, s \in \mathbb{N}\}.$$

This leads to the definition of $\mathcal{CB}_p(X \times Y, Z)$, which can be turned into a p-operator space in the same way as for $\mathcal{CB}_p$.

PROPOSITION 4.9. *Let X, Y and Z be operator spaces. Then we have natural completely isometric identifications*

$$\mathcal{CB}_p(X\widehat{\otimes}^p Y, Z) = \mathcal{CB}_p(X \times Y, Z) = \mathcal{CB}_p(X, \mathcal{CB}_p(Y, Z)).$$

Proof. This follows as for operator spaces, see Proposition 7.1.2 of [9]. ∎

We hence see that, for example, $(X\widehat{\otimes}^p Y)' = \mathcal{CB}_p(X, Y')$. As for operator spaces (see Chapter 7 of [9]), we can now easily show that $X\widehat{\otimes}^p Y = Y\widehat{\otimes}^p X$ naturally, and that the operator $\widehat{\otimes}^p$ is associative. Furthermore, if $u_i : X_i \to Y_i$ are complete contractions for $i = 1, 2$, then $u_1 \otimes u_2$ extends to a complete contraction $X_1\widehat{\otimes}^p X_2 \to Y_1\widehat{\otimes}^p Y_2$.

PROPOSITION 4.10. *Let X, Y, X_1 and Y_1 be p-operator spaces, and let $u : X \to X_1$ and $v : Y \to Y_1$ be p-complete quotient maps. Then $u \otimes v : X\widehat{\otimes}^p Y \to X_1\widehat{\otimes}^p Y_1$ is also a p-complete quotient map. Furthermore,* $\ker(u \otimes v)$ *is the closure of the space*

$$(\ker u) \otimes Y + X \otimes (\ker v) \subseteq X\widehat{\otimes}^p Y.$$

Proof. A careful examination of the proof for operator spaces ([9], Proposition 7.1.7) shows that the proof is equally valid for p-operator spaces. ∎

5. ALGEBRAS

In this section, we shall study weak*-closed subalgebras of $\mathcal{B}(E)$ for an SQ_p space E. The starting point is to look at $\mathcal{B}(E)$ itself, and in particular, its predual $E'\widehat{\otimes}E$.

Let ϕ be a measure, and consider the space $\mathcal{N}(L_p(\phi))$ of nuclear operators on $L_p(\phi)$, so that $\mathcal{N}(L_p(\phi))' = \mathcal{B}(L_p(\phi))$ as explained above. Thus $\mathcal{N}(L_p(\phi))$ carries a natural p-operator space structure by duality.

LEMMA 5.1. *With notation as above, $\mathcal{B}(L_p(\phi)) = \mathcal{N}(L_p(\phi))'$ p-completely isometrically.*

Proof. To ease notation, write $\mathcal{N} = \mathcal{N}(L_p(\phi))$ and $\mathcal{B} = \mathcal{B}(L_p(\phi))$. By definition, for $\tau \in \mathbb{M}_n(\mathcal{N})$, we have that

$$\|\tau\|_n = \sup\{\|\langle\langle \tau, T\rangle\rangle\| : m \in \mathbb{N}, T \in \mathbb{M}_m(\mathcal{B}), \|T\|_m \leqslant 1\}.$$

Here we have identified $\mathbb{M}_n(\mathcal{N})$ with a subspace of $\mathbb{M}_n(\mathcal{B}') = \mathcal{CB}_p(\mathcal{B}, \mathbb{M}_n)$, and it is easy to see that this subspace coincides with the space $\mathcal{CB}_p^\sigma(\mathcal{B}, \mathbb{M}_n)$ of weak*-continuous p-completely bounded maps from $\mathcal{B}$ to $\mathbb{M}_n$.

For $T \in \mathbb{M}_n(\mathcal{B})$, let $\|T\|_{\mathcal{N}'}$ be the norm of T considered as a member of $\mathbb{M}_n(\mathcal{N}') = \mathcal{CB}_p(\mathcal{N}, \mathbb{M}_n)$, so that

$$\|T\|_{\mathcal{N}'} = \sup\{\|\langle\langle \tau, T\rangle\rangle\| : m \in \mathbb{N}, \tau \in \mathbb{M}_m(\mathcal{N}), \|\tau\| \leqslant 1\} \leqslant \|T\|.$$

To show the converse, for $\varepsilon > 0$, we wish to find $\tau \in \mathbb{M}_m(\mathcal{N}) = \mathcal{CB}_p^\sigma(\mathcal{B}, \mathbb{M}_n)$ with $|\langle\langle \tau, T\rangle\rangle| \geqslant \|T\| - \varepsilon$.

By Proposition 4.4, we know that there exists $\tau \in \mathcal{CB}_p(\mathcal{B}, \mathbb{M}_m)$ with this property. Following that proof, we see that τ is defined to be $\tau(T) = VTU$ for $T \in \mathcal{B}$, for suitable $U : \ell_p^m \to L_p(\phi)$ and $V : L_p(\phi) \to \ell_p^m$. A simple calculation shows that such a map is actually in $\mathcal{CB}_p^\sigma(\mathcal{B}, \mathbb{M}_n)$, which completes the proof. ∎

It will be useful to have a more concrete description of the norm on $\mathcal{N}(L_p(\phi))$. For ease of notation, let $\mathcal{N} = \mathcal{N}(L_p(\phi))$ and $\mathcal{B} = \mathcal{B}(L_p(\phi))$. Let $n \in \mathbb{N}$ and $\tau \in \mathbb{M}_n(\mathcal{N})$. Then, as above, $\|\tau\|_n = \sup\{\|\langle\langle T, \tau\rangle\rangle\| : T \in \mathbb{M}_m(\mathcal{B}), \|T\|_m \leqslant 1\}$. For $T \in \mathbb{M}_m(\mathcal{B})$, we have that

$$\|\langle\langle T, \tau\rangle\rangle\| = \sup\Big\{\Big|\sum_{i=1}^n\sum_{k=1}^m\sum_{j=1}^n\sum_{l=1}^m \beta_{ki}\langle T_{kl}, \tau_{ij}\rangle\alpha_{lj}\Big| : \sum_{l,j}|\alpha_{lj}|^p \leqslant 1, \sum_{k,i}|\beta_{ki}|^{p'} \leqslant 1\Big\}.$$

Suppose that $\tau_{ij} = \sum_{r=1}^\infty \mu_r^{(ij)} \otimes x_r^{(ij)} \in L_{p'}(\phi)\widehat{\otimes}L_p(\phi)$ for each i, j. Treat $T = (T_{kl}) \in \mathbb{M}_m(\mathcal{B})$ as an operator on $\ell_p^m \otimes_p L_p(\phi)$, given by $T(\delta_l \otimes x) = \sum_{k=1}^m \delta_k \otimes T_{kl}(x)$. Then

$$\begin{aligned}\|\langle\langle T, \tau\rangle\rangle\| &= \sup\Big\{\Big|\sum_{i,j,k,l}\sum_{r=1}^\infty \beta_{ki}\langle \mu_r^{(ij)}, T_{kl}(x_r^{(ij)})\rangle\alpha_{lj}\Big| : \sum_{l,j}|\alpha_{lj}|^p \leqslant 1, \sum_{k,i}|\beta_{ki}|^{p'} \leqslant 1\Big\}\\ &= \sup\Big\{\Big|\sum_{i,j,k,l}\sum_{r=1}^\infty \langle \beta_{ki}\delta_k^* \otimes \mu_r^{(ij)}, T(\alpha_{lj}\delta_l \otimes x_r^{(ij)})\rangle\Big| : \sum_{l,j}|\alpha_{lj}|^p \leqslant 1, \sum_{k,i}|\beta_{ki}|^{p'} \leqslant 1\Big\}\\ &= \sup\Big\{\Big|\sum_{i,j}\sum_{r=1}^\infty \langle \eta_i \otimes \mu_r^{(ij)}, T(\gamma_j \otimes x_r^{(ij)})\rangle\Big| : \sum_j\|\gamma_j\|^p \leqslant 1, \sum_i\|\eta_i\|^{p'} \leqslant 1\Big\},\end{aligned}$$

where we have $(\eta_i) \subseteq \ell_{p'}^m$ and $(\gamma_j) \subseteq \ell_p^m$. Thus, by the usual duality between $\mathcal{N}(\ell_p^m \otimes_p L_p(\phi))$ and $\mathcal{B}(\ell_p^m \otimes_p L_p(\phi))$, we see that

$$(5.1)\quad \|\tau\|_n = \sup\Big\{\Big\|\sum_{r=1}^{\infty}\sum_{i,j=1}^{n}(\eta_i\otimes\mu_r^{(ij)})\otimes(\gamma_j\otimes x_r^{(ij)})\Big\|_\pi : \sum_i\|\eta_i\|^{p'}\leqslant 1, \sum_j\|\gamma_j\|^p\leqslant 1\Big\},$$

where now m is also free to vary.

Let $\mathcal{N}_n^p = \mathcal{N}(\ell_p^n)$, so by the lemma, $(\mathcal{N}_n^p)' = \mathcal{B}(\ell_p^n) = \mathbb{M}_n$. For a p-operator space X, we hence have that

$$(\mathcal{N}_n^p\widehat{\otimes}^p X)' = \mathcal{CB}_p(X, (\mathcal{N}_n^p)') = \mathcal{CB}_p(X, \mathbb{M}_n) = \mathbb{M}_n(X').$$

In particular, $(\mathcal{N}_n^p\widehat{\otimes}^p\mathcal{N}_m^p)' = \mathbb{M}_n((\mathcal{N}_m^p)') = \mathbb{M}_n(\mathbb{M}_m) = \mathbb{M}_{n\times m}$, and so, as everything is finite-dimensional,

$$\mathcal{N}_n^p\widehat{\otimes}^p\mathcal{N}_m^p = \mathcal{N}_{n\times m}^p,$$

isometrically.

PROPOSITION 5.2. *We have a natural isometric identification*

$$\mathcal{N}(\ell_p)\widehat{\otimes}^p\mathcal{N}(\ell_p) = \mathcal{N}(\ell_p\otimes_p\ell_p).$$

Proof. We follow the proof of Proposition 7.2.1 in [9]. For $n\in\mathbb{N}$, let $\iota_n : \ell_p^n \to \ell_p$ be the inclusion onto the first n co-ordinates, and let $p_n : \ell_p \to \ell_p^n$ be the natural projection. Thus the maps

$$j_n : \mathcal{N}(\ell_p^n) \to \mathcal{N}(\ell_p);\ \tau\mapsto \iota_n\tau p_n,\quad P_n : \mathcal{N}(\ell_p)\to\mathcal{N}(\ell_p^n);\ \sigma\mapsto p_n\sigma\iota_n,$$

are, respectively, a complete isometry and a complete quotient map such that $P_n j_n$ is the identity. Thus j_nP_n is a completely contractive projection of $\mathcal{N}(\ell_p)$ onto $\mathcal{N}_n^p$.

For n, m, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{N}_p^n\widehat{\otimes}^p\mathcal{N}_p^m & \longrightarrow & \mathcal{N}(\ell_p^n\otimes_p\ell_p^m) \\ \big\downarrow{\scriptstyle j_n\otimes j_m} & & \big\downarrow \\ \mathcal{N}(\ell_p)\widehat{\otimes}^p\mathcal{N}(\ell_p) & \longrightarrow & \mathcal{N}(\ell_p\otimes_p\ell_p)\,. \end{array}$$

As above, we know that the top row is a isometry. From the previous paragraph, we know that $j_n\otimes j_m$ is a complete isometry, and similarly, the right column is a complete isometry. The union of the spaces $\mathcal{N}_p^n\otimes\mathcal{N}_p^m$ is norm dense in $\mathcal{N}(\ell_p)\widehat{\otimes}^p\mathcal{N}(\ell_p)$, and the union of the spaces $\mathcal{N}(\ell_p^n\otimes_p\ell_p^m)$ is norm dense in $\mathcal{N}(\ell_p\otimes_p\ell_p)$. Hence, as all the maps are coherent, we conclude that the bottom row must also be a isometry, as required. ∎

PROPOSITION 5.3. *Let ϕ and λ be measures. We have a natural isometric identification*

$$\mathcal{N}(L_p(\phi))\widehat{\otimes}^p\mathcal{N}(L_p(\lambda)) = \mathcal{N}(L_p(\phi\times\lambda)).$$

Proof. Spaces of the form $L_p(\mu)$ admit a net of subspaces (E_i) whose union is dense, and such that each E_i is 1-complemented, and isometric to ℓ_p^n for some n. Hence we may directly adapt the above proof. ∎

Suppose that such a net of subspaces (E_i) exists for some $E \in SQ_p$. Then it is easily seen that E is a $\mathcal{L}^g_{p,1}$ space, as defined in Section 23 of [7]. By Theorem 23.2 of [7], E is thus isometric to a 1-complemented subspace of some L_p space, and is thus isometric to an L_p space (see [39]). Hence the above proposition is the best we can do, at least using this method of proof.

We wish to further study the norm on $\mathbb{M}_n(\mathcal{N}(E))$, for $E \in SQ_p$. Suppose that E has the approximation property (eventually, we shall have to assume that $E = L_p(\phi)$ anyway) so that $\mathcal{K}(E)' = \mathcal{N}(E)$. Define $T_n(\mathcal{K}(E))$ to be the vector space $\mathbb{M}_n(\mathcal{K}(E))$ together with the norm defined by, for $K = (k_{ij})_{i,j=1}^n$,

$$\|K\|_{T_n(\mathcal{K}(E))} = \inf\Big\{ \|T\|_m \Big(\sum_{i,k} |\alpha_{ik}|^p \Big)^{1/p} \Big(\sum_{i,k} |\beta_{ik}|^{p'} \Big)^{1/p'} \Big\},$$

where we take the infimum over $m \in \mathbb{N}$ and $T \in \mathbb{M}_m(\mathcal{K}(E))$ such that for each i,j, $k_{ij} = \sum_{k,l=1}^m \beta_{ki} T_{kl} \alpha_{lj}$. We define a bilinear mapping $\mathbb{M}_n(\mathcal{N}(E)) \times T_n(\mathcal{K}(E)) \to \mathbb{C}$ by

$$\langle \tau, K \rangle = \sum_{i,j=1}^n \langle \tau_{ij}, K_{ij} \rangle \quad (\tau = (\tau_{i,j}) \in \mathbb{M}_n(\mathcal{N}(E)), K = (k_{ij}) \in T_n(\mathcal{K}(E))).$$

By formula (5.1) it is immediate that $|\langle \tau, K \rangle| \leqslant \|\tau\|_n \|K\|_{T_n(\mathcal{K}(E))}$.

Let $\Gamma \in T_n(\mathcal{K}(E))'$, and for each i,j, define $\tau_{ij} \in \mathcal{N}(E)$ by $\langle \tau_{ij}, k \rangle = \langle \Gamma, \delta_{ij} \otimes k \rangle$ for $k \in \mathcal{K}(E)$. Here $\delta_{ij} \otimes k \in T_n(\mathcal{K}(E))$ is the matrix with k in the (i,j) entry, and 0 elsewhere. Then $\|\delta_{ij} \otimes k\|_{T_n(\mathcal{K}(E))} \leqslant \|k\|$, so that τ_{ij} is well-defined, and $\|\tau_{ij}\| \leqslant \|\Gamma\|$. Let $\tau = (\tau_{ij}) \in \mathbb{M}_n(\mathcal{N}(E))$. Let $\tau_{ij} = \sum_r \mu_r^{(ij)} \otimes x_r^{(ij)}$ for each i,j. Then let $T \in \mathbb{M}_m(\mathcal{K}(E))$, so that

$$\|\langle\langle T, \tau \rangle\rangle\| = \sup\Big\{ \Big| \sum_{i,j,k,l} \sum_{r=1}^\infty \beta_{ki} \langle \mu_r^{(ij)}, T_{kl}(x_r^{(ij)}) \rangle \alpha_{lj} \Big| : \sum_{l,j} |\alpha_{lj}|^p \leqslant 1, \sum_{k,i} |\beta_{ki}|^{p'} \leqslant 1 \Big\} = |\langle \tau, K \rangle|,$$

where $K = (k_{ij}) \in T_n(\mathcal{K}(E))$ is defined by $k_{ij} = \sum_{k,l=1}^m \beta_{ki} T_{kl} \alpha_{lj}$. By definition, $\|K\|_{T_n(\mathcal{K}(E))} \leqslant \|T\|$, so by the definition of $\mathbb{M}_n(\mathcal{N}(E))$, we conclude that $T_n(\mathcal{K}(E))' = \mathbb{M}_n(\mathcal{N}(E))$ isometrically. Here we move from taking a supremum over $\mathbb{M}_m(\mathcal{B}(E))$ to $\mathbb{M}_m(\mathcal{K}(E))$, which we may do by approximation, as E has the (metric) approximation property.

Define $T_n(\mathcal{B}(E))$ in a similar way to the definition of $T_n(\mathcal{K}(E))$. Given $T = (T_{ij}) \in \mathbb{M}_n(\mathcal{B}(E))$ and $\tau = (\tau_{ij}) \in \mathbb{M}_n(\mathcal{N}(E))$, so that we see that $|\langle T, \tau \rangle| \leqslant \|T\|_n \|\tau\|_n$ immediately. Proceeding as above, we may at least identify $\mathbb{M}_n(\mathcal{N}(E))'$ with $T_n(\mathcal{B}(E))$ as vector spaces.

PROPOSITION 5.4. *Let ϕ be a measure, and let $E = L_p(\phi)$. Then $\mathbb{M}_n(\mathcal{N}(E))' = T_n(\mathcal{B}(E))$ isometrically.*

Proof. Suppose firstly that E is finite-dimensional (that is, $E = \ell_p^N$ for some N). Then $\mathcal{B}(E) = \mathcal{K}(E)$, and as the space $\mathbb{M}_n(\mathcal{N}(E))$ is finite-dimensional, we see that $\mathbb{M}_n(\mathcal{N}(E))' = T_n(\mathcal{B}(E))$. The general case then follows by a finite-dimensional decomposition argument, as used in Proposition 5.2.

Indeed, let $F \subseteq E$ be a 1-complemented finite-dimensional subspace. Thus $\mathcal{N}(\ell_p^m \otimes_p F) \subseteq \mathcal{N}(\ell_p^m \otimes_p E)$ isometrically, for each m. It follows that $\mathbb{M}_n(\mathcal{N}(F)) \subseteq \mathbb{M}_n(\mathcal{N}(E))$ isometrically, and so the natural map $\mathbb{M}_n(\mathcal{N}(E))' \to \mathbb{M}_n(\mathcal{N}(F))'$ is a quotient map. Similarly, we may check that the natural map $T_n(\mathcal{B}(E)) \to T_n(\mathcal{B}(F))$ (induced by the projection of E onto F) is a quotient map. Thus we have the following diagram

$$\begin{tikzcd} \mathbb{M}_n(\mathcal{N}(F))' & \mathbb{M}_n(\mathcal{N}(E))' \arrow[l] \\ T_n(\mathcal{B}(F)) \arrow[u, "\cong"] & T_n(\mathcal{B}(E)). \arrow[l, "\phi_F"'] \arrow[u, "\psi"'] \end{tikzcd}$$

The map on the left is norm-decreasing, while the map on the right is an isometric isomorphism. Let $T \in T_n(\mathcal{B}(E))$, and we may easily check that

$$\|T\|_{T_n(\mathcal{B}(E))} = \sup\{\|\phi_F(T)\|_{T_n(\mathcal{B}(F))} : F \subseteq E\}.$$

The supremum is taken over 1-complemented subspaces of E, of course. A similar equality holds for $\psi(T)$, and it follows that $\|\psi(T)\|_{\mathbb{M}_n(\mathcal{N}(E))'} = \|T\|_{T_n(\mathcal{B}(E))}$, as required. ∎

As before, this method of proof does not readily generalise to spaces other than $L_p(\phi)$.

5.1. GENERAL WEAK*-CLOSED ALGEBRAS. Let $E = L_p(\phi)$ for some measure ϕ, and let $\mathcal{A} \subseteq \mathcal{B}(E)$ be a weak*-closed algebra. The predual of $\mathcal{A}$, denoted $\mathcal{A}_*$, may be identified with the quotient $\mathcal{A}_* = \mathcal{N}(E)/{}^{\perp}\mathcal{A}$, where

$${}^{\perp}\mathcal{A} = \{\tau \in \mathcal{N}(E) : \langle a, \tau \rangle = 0 \ (a \in \mathcal{A})\}.$$

Clearly $\mathcal{A}$ carries a canonical p-operator space structure, and we can use this to induce a p-operator space structure on $\mathcal{A}_*$. We shall call this the *dual structure* on $\mathcal{A}_*$.

PROPOSITION 5.5. *Let $\mathcal{A} \subseteq \mathcal{B}(L_p(\phi))$ be a weak*-closed subalgebra, for some measure ϕ. Give $\mathcal{A}_*$ the dual structure. Then $\mathcal{A}_*' = \mathcal{A}$ p-completely isometrically.*

Proof. This follows in an analogous way to the proof of Lemma 5.1. To be precise, let $T \in \mathbb{M}_n(\mathcal{A})$ and $\varepsilon > 0$. Then there exists $m \in \mathbb{N}$ and maps $U : \ell_p^m \to L_p(\phi)$ and $V : L_p(\phi) \to \ell_p^m$ such that $\|U\| = \|V\| = 1$ and, if $\tau \in \mathcal{CB}_p(\mathcal{A}, \mathbb{M}_m)$ is defined by $\tau(a) = VaU$, then $\|\langle\langle a, \tau \rangle\rangle\| \geqslant (\|a\| - \varepsilon)\|\tau\|$.

Define $\sigma \in \mathbb{M}_m(\mathcal{A}_*) = \mathcal{CB}_p^\sigma(\mathcal{A}, \mathbb{M}_m)$ by setting

$$\sigma_{ij} = \tau_{ij} + {}^\perp\mathcal{A} \in \mathcal{N}(L_p(\phi))/{}^\perp\mathcal{A} = \mathcal{A}_* \quad (1 \leqslant i, j \leqslant m).$$

Then $\langle\langle a, \tau\rangle\rangle = \langle\langle a, \sigma\rangle\rangle$, and we claim that $\|\sigma\| \leqslant \|\tau\|$, which will complete the proof. To show this claim, it suffices to show that as an operator in $\mathcal{CB}_p(\mathcal{A}, \mathbb{M}_m)$, σ is a contraction. This is immediate however, as σ agrees with τ on $\mathcal{A}$. ∎

Notice $\mathcal{A}_*$ is also a quotient of $\mathcal{N}(E)$, and so we could define a p-operator space structure on $\mathcal{A}_*$ by insisting that the quotient map $\pi : \mathcal{N}(E) \to \mathcal{A}_*$ is a p-complete quotient mapping. We shall call this the *quotient structure*. By Lemma 4.6, when $\mathcal{A}_*$ has the quotient structure, the inclusion $\pi' : \mathcal{A} = \mathcal{A}_*' \to \mathcal{N}(E)' = \mathcal{B}(E)$ is a p-complete isometry. Thus $\mathcal{A}$ carries the same p-operator space structure, irrespective of the p-operator space structure put on $\mathcal{A}_*$. We also see that, in general, the quotient norm dominates the dual norm on $\mathbb{M}_n(\mathcal{A}_*)$ for each n. When $p = 2$, we may immediate conclude that the two structures on $\mathcal{A}_*$ coincide, but for other values of p, the lack of a suitable Hahn–Banach result means that we cannot conclude this. We shall later show that this problem seems to have some link with amenability (see Theorem 7.1), a result we prepare for now.

Let $E = L_p(\phi)$ for some measure ϕ. From Proposition 5.4, we know that $\mathbb{M}_n(\mathcal{N}(E))' = T_n(\mathcal{B}(E))$ isometrically. We may regard $(\pi)_n$ as a map from $\mathbb{M}_n(\mathcal{N}(E))$ to $\mathbb{M}_n(\mathcal{A}_*)$, which is defined to be a quotient map when $\mathcal{A}_*$ carries the quotient structure. Thus $(\pi)_n' : \mathbb{M}_n(\mathcal{A}_*)' \to \mathbb{M}_n(\mathcal{N}(E))' = T_n(\mathcal{B}(E))$ is an isometry which maps onto $(\ker(\pi)_n)^\perp$. It is easy to see that $\tau \in \mathbb{M}_n(\mathcal{N}(E))$ lies in $\ker(\pi)_n$ if and only if $\tau_{ij} \in \ker \pi$ for each i, j. Hence it follows that $T \in T_n(\mathcal{B}(E))$ lies in the image of $(\pi)_n'$ if and only if $T_{ij} \in \mathcal{A}$ for each i, j.

From the definition of $T_n(\mathcal{B}(E))$, we see that the quotient structure norm on $\mathbb{M}_n(\mathcal{A}_*)$ may be computed by considering matrices $T = (T_{ij})$ such that $T_{ij} = \sum_{k,l} \beta_{ik} S_{kl} \alpha_{lj} \in \mathcal{A}$ for some $S \in \mathbb{M}_m(\mathcal{B}(E))$ of norm one, and suitable α and β. By definition, the dual structure norm may be computed by exactly the same method, only now we must ensure that $S_{kl} \in \mathcal{A}$ for each k, l, and not only that $T_{ij} \in \mathcal{A}$ for each i, j.

PROPOSITION 5.6. *Let $\mathcal{A}$ and $\mathcal{A}_*$ be as above, and suppose that there is a p-completely contractive projection from $\mathcal{B}(E)$ onto $\mathcal{A}$. Then the two p-operator space structures on $\mathcal{A}_*$ coincide.*

Proof. This is immediate, as given $T = (T_{ij})$ with $T_{ij} = \sum_{k,l} \beta_{ik} S_{kl} \alpha_{lj} \in \mathcal{A}$ for each i, j, then we have that $P(T_{ij}) = \sum_{k,l} \beta_{ik} P(S_{kl}) \alpha_{lj} \in \mathcal{A}$, where $P(S_{kl}) \in \mathcal{A}$ for each k, l. As $\|(P(S_{kl}))\|_n = \|(P)_n(S)\|_n \leqslant \|P\|_{\mathrm{pcb}} \|S\|_n = \|S\|_n$, the claim follows. ∎

6. TENSOR PRODUCTS OF ALGEBRAS

For two von Neumann algebras R and S, there is a natural tensor product of their preduals R_* and S_* such that $R_* \otimes S_*$ is the predual of the von Neumann algebra tensor product $R\overline{\otimes}S$. A key fact about operator spaces ([9], Theorem 7.2.4) is that $R_*\widehat{\otimes}^2 S_*$ agrees with the predual of $R\overline{\otimes}S$. In this section, we shall explore how this result is proved, and shall lay the foundations for analogous proofs, in the $p \neq 2$ case, in some rather special cases.

We shall now study Slice Maps, following the presentation in Section 7.2 of [9]. Let ϕ_1, ϕ_2 be measures, and set $E = L_p(\phi_1)$ and $F = L_p(\phi_2)$. Let $w_1 \in \mathcal{N}(E)$, so that we have a map $w_1 \otimes I : \mathcal{B}(E) \otimes \mathcal{B}(F) \to \mathcal{B}(F)$ given by $(w_1 \otimes I)(T \otimes S) = \langle T, w_1\rangle S$.

LEMMA 6.1. *There exists a weak*-continuous map $R(w_1) : \mathcal{B}(E \otimes_p F) \to \mathcal{B}(F)$ such that $R(w_1)$, when restricted to $\mathcal{B}(E) \otimes \mathcal{B}(F)$, agrees with $w_1 \otimes I$. Furthermore, $R(w_1)$ is p-completely bounded with $\|R(w_1)\|_{pcb} = \|w_1\|$.*

Proof. For $u \in \mathcal{B}(E \otimes_p F)$, define $R(w_1)(u) \in \mathcal{B}(F) = \mathcal{N}(F)'$ by

$$\langle R(w_1)(u), \tau\rangle = \langle u, w_1 \otimes \tau\rangle \quad (\tau \in \mathcal{N}(F)).$$

Then clearly $R(w_1)(u) \in \mathcal{B}(F)$ and $\|R(w_1)(u)\| \leqslant \|u\|\|w_1\|$. Obviously $R(w_1) : \mathcal{B}(E \otimes_p F) \to \mathcal{B}(F)$ is linear, and is thus a bounded operator which clearly extends $w_1 \otimes I$. Furthermore, we may define $r(w_1) : \mathcal{N}(F) \to \mathcal{N}(E \otimes_p F)$ by

$$r(w_1)(\tau) = w_1 \otimes \tau \in \mathcal{N}(E) \otimes \mathcal{N}(F) \subseteq \mathcal{N}(E \otimes_p F) \quad (\tau \in \mathcal{N}(F)),$$

and then we clearly see that $r(w_1)' = R(w_1)$, so that $R(w_1)$ is weak*-continuous.

By Proposition 5.3, $\mathcal{N}(E)\widehat{\otimes}^p\mathcal{N}(F) = \mathcal{N}(E \otimes_p F)$, and so $\mathcal{B}(E \otimes_p F) = \mathcal{CB}_p(\mathcal{N}(E), \mathcal{B}(F))$ p-completely isometrically. Concretely, this second identification is given as follows. For $u \in \mathcal{B}(E \otimes_p F)$, we define $\Lambda(u) \in \mathcal{CB}_p(\mathcal{N}(E), \mathcal{B}(F))$ by

$$\Lambda(u)(w_1) = R(w_1)(u) \quad (w_1 \in \mathcal{N}(E)).$$

Let $U \in \mathbb{M}_n(\mathcal{B}(E \otimes_p F))$ so that $(R(w_1))_n(U) \in \mathbb{M}_n(\mathcal{B}(F))$. Then

$$(R(w_1))_n(U) = (R(w_1)(U_{ij})) = (\Lambda(U_{ij})(w_1)) = (\Lambda)_n(U)(w_1),$$

so that $\|(R(w_1))_n(U)\| = \|(\Lambda)_n(U)(w_1)\| \leqslant \|\Lambda\|_{pcb}\|U\|\|w_1\| = \|U\|\|w_1\|$, and so $\|(R(w_1))_n\| \leqslant \|w_1\|$, implying that $\|R(w_1)\|_{pcb} \leqslant \|w_1\|$. Clearly then $\|R(w_1)\|_{pcb} = \|w_1\|$, as required. ∎

Similarly, we may work "on the left", leading to the definition of $L(w_2) : \mathcal{B}(E \otimes_p F) \to \mathcal{B}(E)$ for $w_2 \in \mathcal{N}(F)$.

Given weak*-closed subalgebras $\mathcal{A} \subseteq \mathcal{B}(E)$ and $\mathcal{B} \subseteq \mathcal{B}(F)$, we define $\mathcal{A}\overline{\otimes}\mathcal{B}$ to be the weak*-closure of $\mathcal{A} \otimes \mathcal{B}$ in $\mathcal{B}(E \otimes_p F) = \mathcal{B}(L_p(\phi_1 \times \phi_2))$. We define the *Fubini product* $\mathcal{A} \otimes_{\mathrm{F}} \mathcal{B}$ to be the subspace

$$\{u \in \mathcal{B}(E \otimes_p F) : R(w_1)(u) \in \mathcal{B}, L(w_2)(u) \in \mathcal{A}\ (w_1 \in \mathcal{N}(E), w_2 \in \mathcal{N}(F))\}.$$

As $R(w_1)$ and $L(w_2)$ are weak*-continuous, we immediately see that $\mathcal{A}\overline{\otimes}\mathcal{B} \subseteq \mathcal{A} \otimes_{\mathrm{F}} \mathcal{B}$.

In general, we can only say a little about $\mathcal{A}\overline{\otimes}\mathcal{B}$. Let $w_1 \in \mathcal{N}(E)$, and consider the map $R(w_1)$ restricted to $\mathcal{A}\overline{\otimes}\mathcal{B}$, which by weak*-continuity maps into $\mathcal{B}$. Suppose that $w_2 \in \mathcal{N}(E)$ is such that $w_1 - w_2 \in {}^\perp\mathcal{A}$. Then, for any $\tau \in \mathcal{N}(F)$, clearly $(w_1 - w_2) \otimes \tau$ annihilates $\mathcal{A} \otimes \mathcal{B}$, and so

$$\langle R(w_1 - w_2)(T), \tau\rangle = \langle T, (w_1 - w_2) \otimes \tau\rangle = 0 \quad (T \in \mathcal{A}\overline{\otimes}\mathcal{B}).$$

Hence R becomes a well-defined map $\mathcal{N}(E)/{}^\perp\mathcal{A} = \mathcal{A}_* \to \mathcal{CB}_p(\mathcal{A}\overline{\otimes}\mathcal{B}, \mathcal{B})$, and similarly for L.

Now define a map $\delta : \mathcal{A}\overline{\otimes}\mathcal{B} \to (\mathcal{A}_*\widehat{\otimes}^p\mathcal{B}_*)' = \mathcal{CB}_p(\mathcal{B}_*, \mathcal{A})$ by

$$\langle \delta(T), \tau \otimes \sigma\rangle = \langle R(\tau)(T), \sigma\rangle = \langle L(\sigma)(T), \tau\rangle \quad (T \in \mathcal{A}\overline{\otimes}\mathcal{B}, \tau \in \mathcal{A}_*, \sigma \in \mathcal{B}_*).$$

Here we identify $(\mathcal{A}_*\widehat{\otimes}^p\mathcal{B}_*)'$ with $\mathcal{CB}_p(\mathcal{B}_*, \mathcal{A})$, instead of $\mathcal{CB}_p(\mathcal{A}_*, \mathcal{B})$, for convenience, as above we have been working mainly with the map R, and not L. The other choice follows by symmetry, of course.

PROPOSITION 6.2. *With notation as above, and giving $\mathcal{A}_*$ and $\mathcal{B}_*$ the dual structures, we have that δ is a p-complete contraction.*

Proof. Let $T \in \mathbb{M}_n(\mathcal{A}\overline{\otimes}\mathcal{B})$, let $\sigma \in \mathbb{M}_m(\mathcal{B}_*)$, and let $a = ((\delta)_n(T))_m(\sigma) \in \mathbb{M}_{n\times m}(\mathcal{A})$. Notice that

$$a_{ik,jl} = \delta(T_{ij})(\sigma_{kl}) = L(\sigma_{kl})(T_{ij}) \quad (1 \leqslant i, j \leqslant n, 1 \leqslant k, l \leqslant m).$$

We shall, for the proof, give $\mathcal{A}_*$ the quotient structure in order to evaluate the norm on $\mathbb{M}_{n\times m}(\mathcal{A})$. Let $\tau \in \mathbb{M}_r(\mathcal{A}_*)$, and let $\varepsilon > 0$. We may find $\widehat{\tau} \in \mathbb{M}_r(\mathcal{N}(E))$ such that $\widehat{\tau}$ maps to τ, and $\|\widehat{\tau}\|_r \leqslant \|\tau\|_r + \varepsilon$. As in the proof of Lemma 6.1, we p-completely isometrically identify $\mathcal{B}(E \otimes_p F)$ with $\mathcal{CB}_p(\mathcal{N}(E), \mathcal{B}(F))$ by the map Λ. Then we have that

$$\begin{aligned}\|\langle\langle a, \tau\rangle\rangle\| &= \|\langle\langle L(\sigma_{kl})(T_{ij}), \tau_{st}\rangle\rangle\| = \|\langle\langle R(\tau_{st})(T_{ij}), \sigma_{kl}\rangle\rangle\| = \|\langle\langle \Lambda(T_{ij})(\widehat{\tau}_{st}), \sigma_{kl}\rangle\rangle\| \\ &\leqslant \|((\Lambda)_n(T))_r(\widehat{\tau})\|_{n\times r}\|\sigma\|_m \leqslant \|((\Lambda)_n(T))\|_n\|\widehat{\tau}\|_r\|\sigma\|_m \\ &\leqslant \|\Lambda\|_{\mathrm{pcb}}\|T\|_n\|\widehat{\tau}\|_r\|\sigma\|_m \leqslant \|T\|_n\|\sigma\|_m(\|\tau\|_r + \varepsilon).\end{aligned}$$

As τ was arbitrary, we see that $\|a\|_{n\times m} \leqslant \|T\|_n\|\sigma\|_m$. As σ was arbitrary, we see that $\|(\delta)_n(T)\|_{\mathrm{pcb}} \leqslant \|T\|_n$. Finally, as T was arbitrary, we conclude that δ is a p-complete contraction, as required. ∎

Now give $\mathcal{A}_*$ and $\mathcal{B}_*$ the quotient structures. Then by Proposition 4.10, the obvious map

$$\pi_* : \mathcal{N}(E)\widehat{\otimes}^p\mathcal{N}(F) \to \mathcal{A}_*\widehat{\otimes}^p\mathcal{B}_*$$

is a p-complete quotient map. Thus

$$\pi := \pi_*' : (\mathcal{A}_*\widehat{\otimes}^p\mathcal{B}_*)' \to \mathcal{B}(E \otimes_p F)$$

is a p-complete isometry.

THEOREM 6.3. *With notation as above, the map π is a weak*-homeomorphic p-completely isometric map with range equal to $\mathcal{A} \otimes_{\mathrm{F}} \mathcal{B}$. Furthermore, π takes $\overline{\mathcal{A} \otimes \mathcal{B}}$, defined to be the weak*-closure of $\mathcal{A} \otimes \mathcal{B}$ in $(\mathcal{A}_* \widehat{\otimes}^p \mathcal{B}_*)'$, onto $\mathcal{A} \overline{\otimes} \mathcal{B}$.*

Proof. This follows as for operator spaces (given properties of $\widehat{\otimes}^p$ which we established in Proposition 4.10), see Proposition 7.2.3 of [9]. ∎

Finally, we study maps on algebras, and links to complete boundedness.

THEOREM 6.4. *Let ϕ_1 and ϕ_2 be measures, and let $E = L_p(\phi_1)$ and $F = L_p(\phi_2)$. Let $\mathcal{A} \subseteq \mathcal{B}(E)$ be a weak*-closed algebra, and let $M \in \mathcal{CB}_p(\mathcal{A})$ be weak*-continuous. For any weak*-closed algebra $\mathcal{B} \subseteq \mathcal{B}(F)$, there exists a weak*-continuous map $\widehat{M} \in \mathcal{B}(\mathcal{A}\overline{\otimes}\mathcal{B})$ such that $\widehat{M}(a \otimes b) = M(a) \otimes b$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, and $\|\widehat{M}\| \leqslant \|M\|_{pcb}$.*

Proof. We may suppose that $\phi_1 = \phi_2$ is the counting measure on $\mathbb{N}$. The general case will follow in the same way as Proposition 5.3 follows from Proposition 5.2. Hence $E = F = \ell_p$.

Let $P_n : \ell_p \to \ell_p^n$ be the projection onto the first n coordinates, and $\iota_n : \ell_p^n \to \ell_p$ be the canonical inclusion map. Define $\alpha_n : \mathcal{B}(\ell_p(\mathbb{N} \times \mathbb{N})) \to \mathcal{B}(\ell_p)\overline{\otimes}\mathcal{B}(\ell_p^n) = \mathbb{M}_n(\mathcal{B}(\ell_p))$ by

$$\alpha_n(T) = (L(P_n'(\delta_i^*) \otimes \iota_n(\delta_j))(T))_{ij} \quad (T \in \mathcal{B}(\ell_p(\mathbb{N} \times \mathbb{N})), 1 \leqslant i, j \leqslant n).$$

Let $x = (x_j)_{j=1}^n \subseteq \ell_p$ and $\mu = (\mu_i)_{i=1}^n \subseteq \ell_{p'}$, and define $y = \sum_{j=1}^n x_j \otimes \iota_n(\delta_j) \in \ell_p \otimes_p \ell_p$ and $\lambda = \sum_{i=1}^n \mu_i \otimes P_n'(\delta_i^*)$. Then

$$\|y\| = \Big\|(I \otimes \iota_n)\Big(\sum_{j=1}^n x_j \otimes \delta_j\Big)\Big\| \leqslant \Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p},$$

and similarly $\|\lambda\|^{p'} \leqslant \sum_i \|\mu_i\|^{p'}$. Then

$$\begin{aligned}
|\langle \mu, \alpha_n(T)(x)\rangle| &= \Big|\sum_{i,j=1}^n \langle \mu_i, L(P_n'(\delta_i^*) \otimes \iota_n(\delta_j))(T)(x_j)\rangle\Big| \\
&= \Big|\sum_{i,j=1}^n \langle T, ((\mu_i \otimes x_j) \otimes P_n'(\delta_i^*) \otimes \iota_n(\delta_j))\rangle\Big| \\
&= \Big|\Big\langle T, \sum_{i,j=1}^n (\mu_i \otimes P_n'(\delta_i^*)) \otimes (x_j \otimes \iota_n(\delta_j))\Big\rangle\Big| \\
&= |\langle T, \lambda \otimes y\rangle| \leqslant \|T\|\Big(\sum_{i=1}^n \|\mu_i\|^{p'}\Big)^{1/p'}\Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p}.
\end{aligned}$$

Thus $\|\alpha(T)\| \leqslant \|T\|$, so that α is a contraction. It is easy to show that α is weak*-continuous. We have defined α in such a way that $\alpha(T \otimes S) = T \otimes P_n S \iota_n$ for $S, T \in \mathcal{B}(\ell_p)$.

In a similar way, we may define a weak*-continuous contraction $\beta : \mathbb{M}_n(\mathcal{B}(\ell_p)) \to \mathcal{B}(\ell_p(\mathbb{N}\times\mathbb{N}))$ such that $\beta(T\otimes S) = T\otimes \iota_n S P_n$ for $T\in\mathcal{B}(\ell_p)$ and $S\in\mathbb{M}_n$.

By weak*-continuity, we see that $\alpha_n(T)\in\mathbb{M}_n(\mathcal{A})$ for $T\in\mathcal{A}\overline{\otimes}\mathcal{B}(\ell_p)$. As $M\in\mathcal{CB}_p(\mathcal{A})$, by definition, we have that $(M\otimes I_n)\alpha_n : \mathcal{A}\overline{\otimes}\mathcal{B}(\ell_p)\to\mathbb{M}_n(\mathcal{A})$ is bounded, with $\|(M\otimes I_n)\alpha_n\| \leqslant \|M\|_{p\mathrm{cb}}$. Thus $\beta_n(M\otimes I_n)\alpha_n : \mathcal{A}\overline{\otimes}\mathcal{B}(\ell_p)\to \mathcal{A}\overline{\otimes}\mathcal{B}(\ell_p)$ is bounded with $\|\beta_n(M\otimes I_n)\alpha_n\|\leqslant\|M\|_{p\mathrm{cb}}$. As α_n,β_n and $M\otimes I_n$ are weak*-continuous, so is $\beta_n(M\otimes I_n)\alpha_n$.

Let (n_α) be a subnet of $\mathbb{N}$ such that the net $\beta_{n_\alpha}(M\otimes I_{n_\alpha})\alpha_{n_\alpha}(T)$ converges in the weak*-topology, for each $T\in\mathcal{A}\overline{\otimes}\mathcal{B}(\ell_p)$, say converging to $M_0(T)\in\mathcal{A}\overline{\otimes}\mathcal{B}(\ell_p)$. Then M_0 is linear and bounded, with $\|M_0\|\leqslant\|M\|_{p\mathrm{cb}}$. Then, for $i,j,k,l\in\mathbb{N}$, $a\in\mathcal{A}$ and $S\in\mathcal{B}(\ell_p)$,

$$\begin{aligned}&\lim_\alpha\langle\delta_i^*\otimes\delta_j^*,\beta_{n_\alpha}(M\otimes I_{n_\alpha})\alpha_{n_\alpha}(a\otimes S)(\delta_k\otimes\delta_l)\rangle\\ &\qquad=\lim_\alpha\langle\delta_i^*,M(a)(\delta_k)\rangle\langle\delta_j^*,\iota_{n_\alpha}P_{n_\alpha}S\iota_{n_\alpha}P_{n_\alpha}(\delta_l)\rangle=\langle\delta_i^*,M(a)(\delta_k)\rangle\langle\delta_j^*,S(\delta_l)\rangle,\end{aligned}$$

as eventually, $\iota_{n_\alpha}P_{n_\alpha}(\delta_l)=\delta_l$ and so forth. Thus

$$M_0(a\otimes S)=\lim_\alpha\beta_{n_\alpha}(M\otimes I_{n_\alpha})\alpha_{n_\alpha}(a\otimes S)=M(a)\otimes S\quad(a\in\mathcal{A},S\in\mathcal{B}(\ell_p)),$$

with the limit taken in the weak*-topology.

Let $\mathcal{A}_*=\mathcal{N}(\ell_p)/{}^\perp\mathcal{A}$ be the predual of $\mathcal{A}$, and let $m\in\mathcal{B}(\mathcal{A}_*)$ be such that $m'=M$. Let $\theta:\mathcal{A}_*\otimes\mathcal{N}(\ell_p)\to(\mathcal{A}\overline{\otimes}\mathcal{B}(\ell_p))_*=\mathcal{N}(\ell_p(\mathbb{N}\times\mathbb{N}))/{}^\perp(\mathcal{A}\otimes\mathcal{B}(\ell_p))$ be the canonical map given by

$$\langle a\otimes S,\theta(\tau\otimes\sigma)\rangle=\langle a,\tau\rangle\langle S,\sigma\rangle\quad(a\in\mathcal{A},S\in\mathcal{B}(\ell_p),\tau\in\mathcal{A}_*,\sigma\in\mathcal{N}(\ell_p)).$$

Then θ is injective, and we claim that θ has dense range. If not, then there exists a non-zero $T\in\mathcal{A}\overline{\otimes}\mathcal{B}(\ell_p)$ such that $\langle T,\theta(\tau\otimes\sigma)\rangle=0$ for $\tau\in\mathcal{A}_*$ and $\sigma\in\mathcal{N}(\ell_p)$. There hence exists $x\in\ell_p\otimes_p\ell_p$ and $\mu\in\ell_{p'}\otimes_{p'}\ell_{p'}$ with $\langle\mu,T(x)\rangle\neq0$. By approximation, we may suppose that $x=\sum_{n=1}^N x_n\otimes y_n$ and $\mu=\sum_{m=1}^M\mu_m\otimes\lambda_m$. Then

$$0\neq\sum_{n,m}\langle\mu_m\otimes\lambda_m,T(x_n\otimes y_n)\rangle=\sum_{n,m}\langle T,\theta((\mu_m\otimes x_n+{}^\perp\mathcal{A})\otimes(\lambda_m\otimes y_n))\rangle,$$

a contradiction. For $a\in\mathcal{A},S\in\mathcal{B}(\ell_p),\tau\in\mathcal{A}_*$ and $\sigma\in\mathcal{N}(\ell_p)$, we have that

$$\langle M_0(a\otimes S),\theta(\tau\otimes\sigma)\rangle=\langle a,m(\tau)\rangle\langle S,\sigma\rangle=\langle a\otimes S,\theta(m(\tau)\otimes\sigma)\rangle.$$

We hence see that $m\otimes I$ extends continuously to a bounded map on $(\mathcal{A}\overline{\otimes}\mathcal{B}(\ell_p))_*$, and so by weak*-density, M_0 is weak*-continuous.

Finally, for $a\in\mathcal{A}$ and $b\in\mathcal{B}$, we have that $a\otimes b\in\mathcal{A}\overline{\otimes}\mathcal{B}\subseteq\mathcal{A}\overline{\otimes}\mathcal{B}(\ell_p)$, and $M_0(a\otimes b)=M(a)\otimes b$. As M_0 is weak*-continuous, we hence see that $M_0(\mathcal{A}\overline{\otimes}\mathcal{B})\subseteq\mathcal{A}\overline{\otimes}\mathcal{B}$, and so we may set $\widehat{M}$ to be M_0 restricted to $\mathcal{A}\overline{\otimes}\mathcal{B}$, completing the proof. ∎

7. FIGÀ-TALAMANCA–HERZ ALGEBRAS

We shall briefly introduce the Figà-Talamanca–Herz algebras, following the notation of [18] (which means that, compared to some authors, we swap the indexes p and p').

Let G be a locally compact group, and let $\lambda_p : G \to \mathcal{B}(L_p(G))$ be the *left regular representation*, defined by

$$\lambda_p(s)(f)(t) = f(s^{-1}t) \quad (s, t \in G, f \in L_p(G)).$$

We shall also need to use the *right regular representation*, which is defined by

$$\rho_p(s)(f)(t) = f(ts)\Delta_G(s)^{1/p} \quad (s, t \in G, f \in L_p(G)),$$

where Δ_G is the modular function of G. See Section 8 for further details about group representations. Let $C(G)$ be the space of continuous functions from G to $\mathbb{C}$, let $C_{00}(G) \subseteq C(G)$ be the subspace of functions with compact support, and let $C_0(G)$ be its closure. We then define a map $\Lambda_p : L_{p'}(G)\widehat{\otimes}L_p(G) \to C_0(G)$ by

$$\Lambda_p(g \otimes f)(s) = \langle g, \lambda_p(s)(f)\rangle \quad (s \in G, f \in L_p(G), g \in L_{p'}(G)).$$

That Λ_p maps into $C(G)$ follows as λ_p is continuous; that Λ_p maps into $C_0(G)$ follows as $C_{00}(G)$ is dense in $L_p(G)$ and $L_{p'}(G)$. Then $A_p(G)$ is defined to be the *coimage* of Λ_p. That is, we identify the image of Λ_p with the Banach space $L_{p'}(G)\widehat{\otimes}L_p(G)/\ker\Lambda_p$, the latter defining the norm on $A_p(G)$. As shown in [18], $A_p(G)$ becomes a Banach algebra under pointwise operations. When $p = 2$, $A_2(G)$ agrees with the Fourier Algebra $A(G)$, as studied in [10].

By standard Banach space results, we see that the dual of $A_p(G)$ may be identified with the space

$$PM_p(G) = \{T \in \mathcal{B}(L_p(G)) : \langle T, \tau\rangle = 0 \ (\tau \in \ker\Lambda_p)\}.$$

Notice that $\lambda_p(G) = \{\lambda_p(s) : s \in G\} \subseteq PM_p(G)$, and that the weak*-closure of $\lambda_p(G)$ is equal to $PM_p(G)$. It is then easy to show that $PM_p(G)$ is a subalgebra of $\mathcal{B}(L_p(G))$ (see, for example, Section 10 of [27]). When $p = 2$, we have that $PM_2(G) = VN(G)$, the group von Neumann algebra of G. The duality between $A_p(G)$ and $PM_p(G)$ is

$$\langle T, \Lambda_p(g \otimes f)\rangle = \langle g, T(f)\rangle \quad (T \in PM_p(G), g \in L_{p'}(G), f \in L_p(G)).$$

As $PM_p(G) \subseteq \mathcal{B}(L_p(G))$, we see that $PM_p(G)$ carries a natural p-operator space structure. As in Section 5.1, we may hence induce the dual p-operator space structure on $A_p(G)$. Alternatively, we may induce the quotient structure on $A_p(G)$, by defining the map $\Lambda_p : \mathcal{N}(L_p(G)) \to A_p(G)$ to be a p-complete quotient map.

When G is amenable, the algebra $PM_p(G)$ is easier to handle. In particular, we have Theorem 5 of [17], which shows that when G is amenable, we have that $PM_p(G) = \mathrm{CONV}_p(G) := \{T \in \mathcal{B}(E) : T\rho_p(s) = \rho_p(s)T \ (s \in G)\}$

THEOREM 7.1. *Let G be an amenable locally compact group. Then the two natural p-operator space structures on $A_p(G)$, as defined above, agree.*

Proof. Let $E = L_p(G)$. By Proposition 5.6, it suffices to prove that there is a p-completely contractive projection from $\mathcal{B}(E)$ onto $PM_p(G)$. We shall now show that when G is amenable, such a projection exists.

We may regard E as a left $L_1(G)$-module for the ρ_p action, so that $L_1(G)^c_E = \mathrm{CONV}_p(G)$, in the notation of Section 3. Combining Theorem 3.2 and Proposition 3.3 yields that there is a contractive projection $\mathcal{Q} : \mathcal{B}(E) \to \mathrm{CONV}_p(G) = PM_p(G)$.

However, we need to show that $\mathcal{Q}$ is actually p-completely contractive. Let (d_α) be an approximate diagonal of bound one for $L_1(G)$, and let $d_\alpha = \sum_{n=1}^{\infty} a_n^{(\alpha)} \otimes b_n^{(\alpha)} \in L_1(G)\widehat{\otimes}L_1(G)$ for each α. Let $T \in \mathbb{M}_n(\mathcal{B}(E))$, let $(x_j)_{j=1}^n \subseteq E$, and let $(\mu_i)_{i=1}^n \subseteq E'$, so that

$$\begin{aligned}
\Big|\sum_{i,j=1}^n \langle \mu_i, \mathcal{Q}(T_{ij})(x_j)\rangle\Big| &= \lim_\alpha \Big| \sum_{i,j=1}^n \sum_k \langle \mu_i, \rho_p(a_k^{(\alpha)}) T_{ij} \rho_p(b_k^{(\alpha)})(x_j)\rangle \Big| \\
&\leqslant \lim_\alpha \sum_k \Big| \sum_{i,j=1}^n \langle \rho_p(a_k^{(\alpha)})'(\mu_i), T_{ij}\rho_p(b_k^{(\alpha)})(x_j)\rangle \Big| \\
&\leqslant \lim_\alpha \sum_k \|T\|_n \Big(\sum_{i=1}^n \|\rho_p(a_k^{(\alpha)})'(\mu_i)\|^{p'}\Big)^{1/p'} \Big(\sum_{j=1}^n \|\rho_p(b_k^{(\alpha)})(x_j)\|^p\Big)^{1/p} \\
&\leqslant \|T\|_n \lim_\alpha \sum_k \|\rho_p(a_k^{(\alpha)})\| \|\rho_p(b_k^{(\alpha)})\| \Big(\sum_{i=1}^n \|\mu_i\|^{p'}\Big)^{1/p'} \Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p} \\
&\leqslant \|T\|_n \Big(\sum_{i=1}^n \|\mu_i\|^{p'}\Big)^{1/p'} \Big(\sum_{j=1}^n \|x_j\|^p\Big)^{1/p}.
\end{aligned}$$

Thus $\|\mathcal{Q}\|_n \leqslant 1$, and so $\|\mathcal{Q}\|_{pcb} = 1$, as required. ∎

In Section 1.31 of [26], the class of groups G such that $PF_2(G)$ is an amenable Banach algebra is discussed: it is somewhat larger than the class of amenable groups. When $PF_2(G)$ is amenable, by weak*-density, we see that $VN(G) = PM_2(G)$ is Connes-amenable, and this is enough to ensure a projection $\mathcal{B}(L_2(G))$ to $PM_2(G)$ (actually, such a projection is automatically completely positive, and hence completely contractive, see Chapter XV, Corollary 1.3 of [38]). For example, page 84 of [26] shows that $VN(SL(2,\mathbb{R}))$ is Connes-amenable, while $SL(2,\mathbb{R})$ is not amenable. Of course, in the $p = 2$ case the above theorem is not necessary. In the $p \neq 2$ case, we are not aware of a systematic investigation of when $PM_p(G)$, for $p \neq 2$, is Connes-amenable (see Theorem 4.4.13 of [31] for some partial results). Furthermore, even if we have a projection $\mathcal{B}(L_p(G)) \to PM_p(G)$, it is unclear that this projection is necessarily p-completely contractive. It seems possible that the above proof could hence be extended to some non-amenable groups.

However, the existence of a projection onto $PM_p(G)$ is very far from being necessary, so it also seems possible that another method of proof could extend the above result to a much larger class of groups (or even maybe all groups).

We know that p-operator spaces are much easier to work with when they embed into an L_p space. Henceforth, we shall assume that $A_p(G)$ carries the dual structure. We shall resort to the above theorem when it is necessary to use the quotient structure (which is in many ways the more natural structure).

Our next task is to show that $A_p(G)$ is an algebra is the category of p-operator spaces. This is equivalent to saying that the algebra product defines a bounded (indeed, contractive) map $\Delta : A_p(G)\widehat{\otimes}^p A_p(G) \to A_p(G)$. Suppose that $\Delta' : PM_p(G) \to (A_p(G)\widehat{\otimes} A_p(G))' = \mathcal{CB}_p(A_p(G), A_p(G)') = \mathcal{CB}_p(A_p(G), PM_p(G))$ is a p-complete contraction. Then so is Δ'', and hence also $\Delta''\kappa_{A_p(G)\widehat{\otimes}^p A_p(G)} = \kappa_{A_p(G)}\Delta$. As $\kappa_{A_p(G)}$ is a p-complete isometry, we conclude that Δ is a p-complete contraction.

Define $PM_p(G)\overline{\otimes}PM_p(G) \subseteq \mathcal{B}(L_p(G\times G))$, as in Section 6.

PROPOSITION 7.2. *Let G and H be locally compact groups. Then $PM_p(G)\overline{\otimes} PM_p(H) = PM_p(G\times H)$.*

Proof. By definition, $PM_p(G)\overline{\otimes}PM_p(H)$ is the weak*-closure of $PM_p(G)\otimes PM_p(H)$ in $\mathcal{B}(L_p(G)\otimes_p L_p(H)) = \mathcal{B}(L_p(G\times H))$. For this proof, let $\lambda_p^G : G \to \mathcal{B}(L_p(G))$ be the left-regular representation, and define λ_p^H and $\lambda_p^{G\times H}$ similarly. Then it is simple to verify that

$$\lambda_p^G(s)\otimes\lambda_p^H(t) = \lambda_p^{G\times H}(s,t) \quad (s\in G, t\in H).$$

Hence we see immediately that $PM_p(G\times H) \subseteq PM_p(G)\overline{\otimes}PM_p(H)$, as $PM_p(G\times H)$ is the weak*-closure of the span of the image of $\lambda_p^{G\times H}$.

Conversely, we shall show that $\lambda_p^G(G)\otimes PM_p(H) \subseteq PM_p(G\times H)$, and by symmetry that $PM_p(G)\otimes\lambda_p^H(H) \subseteq PM_p(G\times H)$. Thus, for $S\in PM_p(G)$ and $T\in PM_p(H)$, we have that

$$S\otimes T = (S\otimes\lambda_p^H(e_H))(\lambda_p^G(e_G)\otimes T) \in PM_p(G\times H),$$

where e_G, e_H, is the unit of G, respectively H. As $PM_p(G\times H)$ is weak*-closed, we conclude that $PM_p(G)\overline{\otimes}PM_p(H) \subseteq PM_p(G\times H)$, completing the proof.

To show that $\lambda_p^G(G)\otimes PM_p(H) \subseteq PM_p(G\times H)$ we shall show that $\ker\Lambda_p^{G\times H} \subseteq {}^{\perp}(\lambda_p^G(G)\otimes PM_p(H))$. Let $\tau = \sum_{n=1}^{\infty}\mu_n\otimes x_n \in \ker\Lambda_p^{G\times H} \subseteq L_{p'}(G\times H)\otimes L_p(G\times H)$. We regard $L_p(G\times H)$ as $L_p(G, L_p(H))$, and so we regard each x_n as a function from G to $L_p(H)$. Similarly $L_{p'}(G\times H) = L_{p'}(G, L_{p'}(H))$. Fix $u\in G$, so that

$$0 = \langle\lambda_p^G(u)\otimes\lambda_p^H(v), \tau\rangle = \sum_{n=1}^{\infty}\int_G \langle\mu_n(s), \lambda_p^H(v)(x_n(u^{-1}s))\rangle \,\mathrm{d}s \quad (v\in H).$$

For each n, define $y_n \in L_p(G, L_p(H))$ by $y_n(s) = x_n(u^{-1}s)$ for $s \in G$. Thus

$$0 = \sum_{n=1}^{\infty} \langle \mu_n, (I \otimes \lambda_p^H(v))(y_n) \rangle \quad (v \in H).$$

By using Herz's ideas in Lemma 0 of [18], this implies that

$$0 = \sum_{n=1}^{\infty} \langle \mu_n, (I \otimes T)(y_n) \rangle = \langle \lambda_p^G(u) \otimes T, \tau \rangle \quad (T \in PM_p(H)).$$

As $u \in G$ was arbitrary, the proof is complete. ∎

Define $W : L_p(G \times G) \to L_p(G)$ by

$$(Wf)(s,t) = f(s,st) \quad (f \in L_p(G \times G), s,t \in G),$$

so that W is an invertible isometry. Define

$$\Gamma : PM_p(G) \to PM_p(G)\overline{\otimes}PM_p(G); \quad T \mapsto W^{-1}(T \otimes I)W \quad (T \in PM_p(G)).$$

Let $f \in L_p(G \times G)$ and $s \in G$. Then

$$\begin{aligned}(\Gamma(\lambda_p(s))(f))(r,t) &= (W^{-1}(\lambda_p(s) \otimes I)W(f))(r,t) = ((\lambda_p(s) \otimes I)W(f))(r, r^{-1}t) \\ &= (Wf)(s^{-1}r, r^{-1}t) = f(s^{-1}r, s^{-1}t),\end{aligned}$$

for $r,t \in G$. Thus $\Gamma(\lambda_p(s)) = \lambda_p(s) \otimes \lambda_p(s)$.

Recall the definition of the map $\delta : PM_p(G)\overline{\otimes}PM_p(G) \to (A_p(G)\widehat{\otimes}^p A_p(G))'$, which is a p-complete contraction by Proposition 6.2. For $a, b \in A_p(G)$ and $s \in G$, we have that

$$\langle \delta\Gamma(\lambda_p(s)), a \otimes b \rangle = \langle \lambda_p(s) \otimes \lambda_p(s), a \otimes b \rangle = a(s)b(s) = (ab)(s) = \langle \lambda_p(s), \Delta(a \otimes b) \rangle.$$

Thus $\Delta' = \delta\Gamma$. In particular, as Γ is clearly a p-complete contraction, so is Δ', as required.

THEOREM 7.3. *Let G and H be amenable locally compact groups. Then $A_p(G)\widehat{\otimes}^p A_p(H) = A_p(G \times H)$ isometrically.*

Proof. This proof is an adaptation of Theorem 7.2.4 in [9]. By Theorem 7.1, we have that the two p-operator space structures agree on $A_p(G)$ and $A_p(H)$. By Theorem 6.3, the map $\pi_* : \mathcal{N}(L_p(G))\widehat{\otimes}^p \mathcal{N}(L_p(H)) \to A_p(G)\widehat{\otimes}^p A_p(H)$ is a p-complete quotient map, so that $\pi = \pi_*' : (A_p(G)\widehat{\otimes}^p A_p(H))' \to \mathcal{B}(L_p(G \times H))$ is a p-complete isometry onto its range, which is $PM_p(G) \otimes_{\mathrm{F}} PM_p(H)$.

For $w \in \mathcal{N}(L_p(G))$, recall the definition of $R(w)$ from Section 6. Let $T \in PM_p(G) \otimes_{\mathrm{F}} PM_p(H) \subseteq \mathcal{B}(L_p(G) \otimes_p L_p(H))$, so by definition $R(w)(T) \in PM_p(H) = \mathrm{CONV}_p(H)$ for each $w \in \mathcal{N}(L_p(G))$. Thus, for $s \in H$, $R(w)(T)\rho_p(s) = \rho_p(s)R(w)(T)$. By weak*-continuity, this implies that

$$R(w)(T(I \otimes \rho_p(s))) = R(w)((I \otimes \rho_p(s))T).$$

As w is arbitrary, this is that $T(I \otimes \rho_p(s)) = (I \otimes \rho_p(s))(T)$ for each $s \in H$. By symmetry, we also see that $(\rho_p(t) \otimes I)T = T(\rho_p(t) \otimes I)$ for $t \in G$. Consequently

T commutes with $\rho_p((t,s))$ for $(t,s) \in G \times H$, that is, $T \in \mathrm{CONV}_p(G \times H) = PM_p(G \times H)$, as $G \times H$ is amenable.

Thus $PM_p(G) \otimes_{\mathrm{F}} PM_p(H) = PM_p(G) \overline{\otimes} PM_p(H) = PM_p(G \times H)$. As π is a homeomorphism, we conclude that $(A_p(G) \widehat{\otimes}^p A_p(H))' = A_p(G \times H)' = PM_p(G \times H)$ p-completely isometrically. As the quotient and dual structures agree on $A_p(G \times H)$, and $\pi = \pi'_*$ is weak*-continuous, this implies that $A_p(G) \widehat{\otimes}^p A_p(H) = A_p(G \times H)$, as required. ∎

In the above proof we use the fact that when G is an amenable group, we have that $PM_p(G) = \mathrm{CONV}_p(G)$. As communicated to us by Professor Figà-Talamanca, in [3], Cowling shows that $PM_p(G) = \mathrm{CONV}_p(G)$ for $G = SL(2, \mathbb{R})$ and $G = \mathbb{F}_2$. Actually, the proof for $\mathbb{F}_2$ is not correct, but can be corrected using results of Haagerup, as done in Theorem 4.9, Chapter 8 of [12]. It is apparently unknown if $PM_p(G) = \mathrm{CONV}_p(G)$ for all groups G. We conclude that the main sticking point in this section is Theorem 7.1.

Finally, we shall show that $A_p(G)$ is amenable in the category of p-operator spaces if and only if G is an amenable group. By "amenable in the category of p-operator spaces", we mean that every p-completely bounded derivation from $A_p(G)$ to a p-completely contractive dual $A_p(G)$-bimodule is inner. The equivalence of this to $A_p(G)$ having an approximate diagonal in $A_p(G) \widehat{\otimes}^p A_p(G)$ follows from exactly the same argument as used for amenability of Banach algebras (compare with Section of [30]). We shall make heavy use of the already established result in the $p = 2$ case, which is Theorem 3.6 of [30].

THEOREM 7.4. *Let G be a locally compact group. Then $A_p(G)$ is p-operator space amenable if and only if G is an amenable group.*

Proof. Suppose that $A_p(G)$ is p-operator space amenable. Then, in particular, $A_p(G)$ has a bounded approximate identity, and so by Leptin's Theorem (compare Theorem 6 of [17]) G is amenable. We remark that the proof, for a Banach algebra $\mathcal{A}$, that $\mathcal{A}$ amenable implies that $\mathcal{A}$ has a bounded approximate identity easily transfers to the category of p-operator spaces (see, for example, Proposition 1.19 of [20]).

Conversely, suppose that G is an amenable group. Then $A_p(G) \widehat{\otimes}^p A_p(G) = A_p(G \times G)$. As $G \times G$ is amenable, by Theorem C of [18], identification of functions gives a norm-decreasing homomorphism $A_2(G \times G) \to A_p(G \times G)$ which has dense range. By Ruan's Theorem, $A_2(G \times G) = A_2(G) \widehat{\otimes}^2 A_2(G)$ contains a bounded approximate diagonal, and hence so does $A_p(G \times G)$. Thus $A_p(G)$ is p-operator space amenable. ∎

7.1. FURTHER HOMOLOGICAL PROPERTIES. Amenability fits into the study of Hochschild cohomology of Banach algebras, and there are further (co)homological properties of Banach algebras which are widely studied. See Chapter 4 of [31] for an introduction to these ideas. As for amenability, when

$A(G)$ is considered as an operator space, homological properties of $A(G)$ depend upon the group G in the same (or dual) way to the way that properties of $L_1(G)$ depend upon G.

In [40], Wood considers *biprojectivity*, and shows that $A(G)$ is biprojective (with the operator space structure) if and only if G is discrete. Conversely, Helemskii (see [15]) showed that $L_1(G)$ is biprojective (as a Banach alegbra) if and only if G is compact (and we view discreteness and compactness as being dual properties, as in the abelian case).

First, some terminology. Let $\mathcal{A}$ be a Banach algebra, let E and F be $\mathcal{A}$-bimodules, and let $\theta \in \mathcal{B}(E,F)$. We say that θ is an *module homomorphism* if $\theta(a \cdot x \cdot b) = a \cdot \theta(x) \cdot b$ for $a, b \in \mathcal{A}$ and $x \in E$. We say that θ is *admissible* if there exists $\phi \in \mathcal{B}(F,E)$ with $\theta\phi\theta = \theta$. We say that an $\mathcal{A}$-bimodule E is *biprojective* when, given $\mathcal{A}$-bimodules F and G, a surjective, admissible module map $\phi : F \to G$ and a module map $\theta : E \to G$, there exists a module map $\psi : E \to F$ with $\phi\psi = \theta$.

In [40], Wood first adapts these ideas to the category of operator spaces. Subject to some technicalities (as usual, to do with duality) it seems rather likely that this carries over easily to the p-operator space situtation. Wood next proves that the multiplication map $A(G)\widehat{\otimes}^2 A(G) \to A(G)$ is surjective. This uses a number of results, including that $A(G)\widehat{\otimes}^2 A(G) = A(G \times G)$ for all groups G, which we have not been able to generalise to the $A_p(G)$ case. Furthermore, this fact is again used in the proof of the main theorem, ([40], Theorem 4.5).

A Banach algebra $\mathcal{A}$ is *weakly-amenable* when every bounded derivation to $\mathcal{A}'$ is inner. When $\mathcal{A}$ is commutative, this is equivalent to the (more natural) condition that every derivation into a *symmetric* $\mathcal{A}$-bimodule E is zero. Here an $\mathcal{A}$-bimodule E is *symmetric* if $a \cdot x = x \cdot a$ for each $a \in \mathcal{A}$ and $x \in E$. It is easy to translate these conditions into the category of operator spaces, and in [36] Spronk shows that $A(G)$ is always weakly-amenable in the category of operator spaces.

Again, we can translate these ideas over to p-operator spaces, but, again, we find that we need properties of the projective tensor norm which we have not been able to establish in full generality (it is, of course, pointless to restrict to amenable groups G, as then $A_p(G)$ is amenable, and so trivially weakly-amenable). Furthermore, Spronk uses simple facts about representations on Hilbert spaces which seem unlikely to hold for SQ_p spaces, as we lack things like orthogonal projections.

In [35], Samei develops the theory of algebras he called *hyper-Tauberian*, and uses this theory to give a simple and elegant proof that $A(G)$ is weakly-amenable, as an operator space. Indeed, Samei's argument easily extends to the $A_p(G)$ algebras, when given the operator space structure constructed in [22]. This operator space structure suffers from the same issue we have, in that $A_p(G)\widehat{\otimes}^2 A_p(G)$ need not, seemmingly, be anything useful, when G is not amenable. Samei sidesteps this issue by first working with $A(G)$ and then transfering the result to $A_p(G)$, see Theorem 28 of [35] for details.

We can immediately adapt Samei's definition of what it means to be hyper-Tauberian to the p-operator space setting, and show that a hyper-Tauberian algebra is weakly-amenable. It remains to show that $A_p(G)$ is indeed hyper-Tauberian as a p-operator space. Unfortunately, we again hit a problem here, as we cannot lift results from $A(G)$ to $A_p(G)$ (as $A(G)$ is not a p-operator space!) and a direct argument, at least following Samei, would again require us to know what $A_p(G)\widehat{\otimes}^p A_p(G)$ is. It at least seems possible that a new direct argument could work for $A_p(G)$ in the p-operator space setting, but we have not been able to make progress in this direction.

8. MULTIPLIERS

In this section we shall study multipliers of Figà-Talamanca–Herz Algebras. Much of the hard work is already in the literature, but often without direct connections being drawn. We try to collect together these results in a unified setting here.

It shall be helpful to sketch some results on group representations. Let G be a locally compact group, and let E be a *reflexive* Banach space. We shall define a *group representation of* G *on* E to be a group homomorphism $\pi : G \to \mathcal{B}(E)$ such that $\pi(s)$ is an isometry for each $s \in G$, and for each $x \in E$ and $\mu \in E'$, the map $G \to \mathbb{C}; s \mapsto \langle \mu, \pi(s)(x)\rangle$ is continuous. Then π extends to a norm-decreasing homomorphism $\pi : L_1(G) \to \mathcal{B}(E)$ by integration.

We shall now sketch the converse to this, which is folklore. Let $\pi : L_1(G) \to \mathcal{B}(E)$ be a norm-decreasing homomorphism. As is standard (see Theorem 3.3.23 of [4] for example) $L_1(G)$ contains an approximate identity (e_α) of bound 1. For $s \in G$ and $f \in L_1(G)$, define $s \cdot f \in L_1(G)$ by $(s \cdot f)(t) = f(s^{-1}t)$ for $t \in G$. We may define a map $\sigma : G \to \mathcal{B}(E)$ by

$$\langle \mu, \sigma(s)(x)\rangle = \lim_\alpha \langle \mu, (s \cdot e_\alpha)(x)\rangle \quad (x \in E, \mu \in E').$$

Then there exists a subspace F of E such that, by restriction, σ becomes a group representation $\sigma : G \to \mathcal{B}(F)$. In fact, there is a contractive projection $P : E \to F$ such that $P\pi(f)P = \pi(f)$ for $f \in L_1(G)$, so that the action of π on the kernel of P is trivial, and so we loose nothing by restricting to F. Applying the previous paragraph to σ yields the homomorphism π, restricted to F. By the Cohen Factorisation Theorem, we have that $F = \{\pi(f)(x) : x \in E, f \in L_1(G)\}$.

Now define a map $\Pi : E'\widehat{\otimes}E \to C(G)$ by

$$\Pi(\mu \otimes x)(s) = \langle \mu, \pi(s)(x)\rangle \quad (\mu \otimes x \in E'\widehat{\otimes}E, s \in G).$$

Here $C(G)$ is the space of continuous functions on G; that Π maps into $C(G)$ follows by the continuity assumption on π. We let $A(\pi)$ be the co-image of Π:

that is, $A(\pi)$ is the image of Π in $C(G)$, but with the norm induced by identifying $A(\pi)$ with the quotient $E'\widehat{\otimes}E/\ker\Pi$. As explained by Herz in [18], the obvious definition of equivalent group representations is a rather strong condition, while $A(\pi)$ gives a more interesting notion of equivalence (for example, $A(\pi)$ is one-dimensional if and only if π is trivial).

Recall the left-regular representation $\lambda_p : G \to \mathcal{B}(L_p(G))$. Then $A_p(G) = A(\lambda_p)$. Let $\pi : G \to \mathcal{B}(E)$ be some group representation, and let $I_E : G \to \mathcal{B}(E)$ be the trivial representation on E. Herz shows that λ_p has the useful property that $A(\lambda_p \otimes \pi) = A(\lambda_p \otimes I_E)$ (this is also referred to as *Fell's absorption principle*). Furthermore, if $E \in SQ_p$ (or, in Herz's terminology, E is a *p-space*) then $A(\lambda_p \otimes I_E) = A(\lambda_p)$ ([18], Lemma 0).

For a commutative Banach algebra $\mathcal{A}$, we say that a linear map $T : \mathcal{A} \to \mathcal{A}$ is a *multiplier*, denoted by $T \in \mathcal{M}(\mathcal{A})$, if $T(ab) = aT(b)$ for $a, b \in \mathcal{A}$. Then $\mathcal{M}(\mathcal{A})$ becomes a Banach algebra with respect to the operator norm. For a locally compact group G, using the fact that $A_p(G)$ is a *regular tauberian algebra* (see [17], Section 3), we may use the Closed Graph Theorem to show that each multiplier on $A_p(G)$ is bounded, and furthermore, each multiplier is given by pointwise multiplication by some (necessarily continuous) function $u : G \to \mathbb{C}$. Henceforth we shall treat $\mathcal{M}(A_p(G))$ as a subspace of $C(G)$, with the norm

$$\|u\|_{\mathcal{M}} = \sup\{\|ua\|_{A_p} : a \in A_p(G), \|a\|_{A_p} \leqslant 1\} \quad (u \in \mathcal{M}(A_p(G))).$$

It is common in the literature to write $B_p(G)$ for $\mathcal{M}(A_p(G))$. This is confusing, as it is standard to denote by $B(G)$ the *Fourier–Stieltjes Algebra* of G. However, by results of Nebbia and Losert (see [24]) we have that $B_2(G) = B(G)$ if and only if G is amenable (see page 187 of [26] for an example where this confusion arises). To further confuse the issue, Herz himself defined a space $B_p(G)$ in [16], using a notion of Schur multipliers (which we shall study further below). Finally, Runde defined a generalisation of $B(G)$ in [33] which he, reasonably, denotes by $B_p(G)$. We shall stick to writing $\mathcal{M}(A_p(G))$.

In [6], De Cannière and Haagerup study *completely bounded multipliers* of $A_p(G)$, denoted by $\mathcal{M}_0(A_p(G))$. We have that $B_2(G) = \mathcal{M}_0(A_p(G))$ in Herz's notation (see [2] where unpublished results of J. Gilbert are used to show this). Similar ideas are explored [21]. We use [6] and [21] to motivate the following definitions.

DEFINITION 8.1. Let G be a locally compact group, let $1 < p < \infty$, and let $u \in \mathcal{M}(A_p(G))$. Then $u \in \mathcal{M}_{\mathrm{cb}}(A_p(G))$ if and only if u defines a member of $\mathcal{CB}_p(A_p(G))$ where $A_p(G)$ is given the dual p-operator space structure. We give $\mathcal{M}_{\mathrm{cb}}(A_p(G))$ the p-completely bounded norm.

We define $M_0(A_p(G))$ to be the space of those functions $u : G \to \mathbb{C}$ such that there exists $E \in SQ_p$ and bounded continuous maps $\alpha : G \to E$ and $\beta : G \to E'$ such that $u(ts^{-1}) = \langle \beta(t), \alpha(s)\rangle$ for $s, t \in G$. We give $\mathcal{M}_0(A_p(G))$ the obvious norm.

Then, for example, Jolissaint shows in [21] that $\mathcal{M}_0(A_2(G)) = \mathcal{M}_{\mathrm{cb}}(A_2(G))$.

LEMMA 8.2. *Let G be a locally compact group, and let $u : G \to \mathbb{C}$ be a function. Then the following are equivalent:*

(i) $u \in \mathcal{M}(A_p(G))$;

(ii) *There exists a bounded, weak*-continuous operator $M : PM_p(G) \to PM_p(G)$ such that $M(\lambda_p(s)) = u(s)\lambda_p(s)$ for $s \in G$.*

Proof. Suppose that (i) holds, let $m \in \mathcal{B}(A_p(G))$ be the operator defined by pointwise multiplication by u, and let $M = m' \in \mathcal{B}(PM_p(G))$. Then obviously $M(\lambda_p(s)) = u(s)\lambda_p(s)$ for $s \in G$.

Conversely, if (ii) holds, then as M is weak*-continuous, there exists $m \in \mathcal{B}(A_p(G))$ with $m' = M$. For $a \in A_p(G)$, we then have that

$$u(s)a(s) = \langle M(\lambda_p(s)), a\rangle = \langle \lambda_p(s), m(a)\rangle = m(a)(s) \quad (s \in G),$$

so that m is pointwise multiplication by u, and hence $u \in \mathcal{M}(A_p(G))$. ∎

When $p = 2$ the above can be significantly improved, essentially because $A(G)$ is a *closed* ideal in $B(G)$; see Proposition 1.2 of [6].

THEOREM 8.3. *Let G be a locally compact group, and let $1 < p < \infty$. Then $\mathcal{M}_0(A_p(G))$ and $\mathcal{M}_{\mathrm{cb}}(A_p(G))$ are commutative Banach algebras. Furthermore, $\mathcal{M}_{\mathrm{cb}}(A_p(G)) = \mathcal{M}_0(A_p(G))$ isometrically, and $\mathcal{M}_{\mathrm{cb}}(A_p(G)) \subseteq \mathcal{M}(A_p(G))$ contractively.*

Proof. For the proof, write $\mathcal{M}$ for $\mathcal{M}(A_p(G))$ and so forth. Obviously $\mathcal{M}_{\mathrm{cb}} \subseteq \mathcal{M}$ contractively, from which it follows easily that $\mathcal{M}_{\mathrm{cb}}$ is a commutative Banach algebra. For $E, F \in SQ_p$, by considering the space $E \oplus F$ with the ℓ_p norm $\|(x,y)\| = (\|x\|^p + \|y\|^p)^{1/p}$ for $x \in E, y \in F$, it follows that $\mathcal{M}_0$ is a vector space. Similarly, by considering the infinite ℓ_p sum of a countable family $(E_n)_{n=1}^\infty \subseteq SQ_p$, it follows that $\mathcal{M}_0$ is a Banach space. Finally, by using a suitable tensor product construction for SQ_p spaces (see Section 3 of [33]) it follows that $\mathcal{M}_0$ is a commutative Banach algebra.

Now let $u \in \mathcal{M}_0$ be defined by $u(ts^{-1}) = \langle \beta(t), \alpha(s)\rangle$, using some $E \in SQ_p$. Let $x \in L_p(G)$ and $\mu \in L_{p'}(G)$, and let $a = \Lambda_p(\mu \otimes x) \in A_p(G)$. Define $\widehat{x} \in L_p(G,E) = L_p(G) \otimes_p E$ and $\widehat{\mu} \in L_{p'}(G, E')$ by

$$\widehat{x}(s) = x(s)\alpha(s^{-1}), \quad \widehat{\mu}(s) = \mu(s)\beta(s^{-1}) \quad (s \in G).$$

Then $\|\widehat{x}\| \leqslant \|\alpha\|_\infty \|x\|$, $\|\widehat{\mu}\| \leqslant \|\beta\|_\infty \|\mu\|$, and for $s \in G$,

$$\begin{aligned}\langle \widehat{\mu}, (\lambda_p(s) \otimes I_E)(\widehat{x})\rangle &= \int_G \langle \widehat{\mu}(t), \widehat{x}(s^{-1}t)\rangle \,\mathrm{d}t = \int_G \langle \beta(t^{-1}), \alpha(t^{-1}s)\rangle \mu(t) x(s^{-1}t)\,\mathrm{d}t \\ &= \int_G u(tt^{-1}s)\mu(t)x(s^{-1}t)\,\mathrm{d}t = u(s)a(s).\end{aligned}$$

By Herz, we have that $A(\lambda_p \otimes I_E) = A_p(G)$, so that $ua \in A_p(G)$ and $\|ua\|_{A_p} \leqslant \|\alpha\|_\infty \|\beta\|_\infty \|x\| \|\mu\|$. As u was arbitrary, and by linearity and the definition of the norms on $A_p(G)$ and $\mathcal{M}_0$, we see that $\mathcal{M}_0 \subseteq \mathcal{M}$ is a norm-decreasing inclusion.

We can "amplify" this argument to show that $\mathcal{M}_0 \subseteq \mathcal{M}_{\mathrm{cb}}$ contractively. Let $u \in \mathcal{M}_0$ be as before, and let $M \in \mathcal{B}(PM_p(G))$ be induced by u, as given by the previous lemma. Given $x \in L_p(G)$ and $\mu \in L_{p'}(G)$, define $\widehat{x} \in L_p(G,E)$ and $\widehat{\mu} \in L_{p'}(G,E')$ as above. It is a simple calculation to show that

$$\langle \mu, M(T)(x) \rangle = \langle \widehat{\mu}, (T \otimes I_E)(\widehat{x}) \rangle \quad (T \in PM_p(G)).$$

Let $n \in \mathbb{N}$ and let $T = (T_{ij}) \in \mathbb{M}_n(PM_p(G))$. Let $\mu = (\mu_i)_{i=1}^n \in L_{p'}(G) \otimes_p \ell^n_{p'}$ and $x = (x_j)_{j=1}^n \in L_p(G) \otimes_p \ell^n_p$. Define $\widehat{x} \in L_p(G) \otimes_p E \otimes_p \ell^n_p$ by $\widehat{x} = \sum_{j=1}^n \widehat{x}_j \otimes \delta_j$, so that

$$\|\widehat{x}\| = \Big(\sum_{j=1}^n \|\widehat{x}_j\|^p \Big)^{1/p} \leqslant \|\alpha\|_\infty \Big(\sum_{j=1}^n \|x_j\|^p \Big)^{1/p} = \|\alpha\|_\infty \|x\|.$$

Similarly define $\widehat{\mu}$, so that $\|\widehat{\mu}\| \leqslant \|\beta\|_\infty \|\mu\|$. Finally, define $S \in \mathcal{B}(L_p(G) \otimes_p E \otimes_p \ell^n_p)$ by

$$S(x \otimes y \otimes \delta_j) = \sum_{i=1}^n T_{ij}(x) \otimes y \otimes \delta_i \quad (x \in L_p(G), y \in E, 1 \leqslant j \leqslant n).$$

If $\phi : L_p(G) \otimes_p E \otimes_p \ell^n_p \to L_p(G) \otimes_p \ell^n_p \otimes_p E$ is the canonical isometry, then $\phi S \phi^{-1} = T \otimes I_E$, so that $\|S\| = \|T\|$. Then

$$\begin{aligned} |\langle \mu, (M)_n(T)(x) \rangle| &= \Big| \sum_{i,j=1}^n \langle \mu_i, M(T_{ij})(x_j) \rangle \Big| = \Big| \sum_{i,j=1}^n \langle \widehat{\mu}_i, (T_{ij} \otimes I_E)(\widehat{x}_j) \rangle \Big| \\ &= |\langle \widehat{\mu}, S(\widehat{x}) \rangle| \leqslant \|\widehat{\mu}\| \|\widehat{x}\| \|S\| \leqslant \|\alpha\|_\infty \|\beta\|_\infty \|T\| \|\mu\| \|x\|, \end{aligned}$$

so that $M \in \mathcal{CB}_p(PM_p(G))$ with $\|M\|_{p\mathrm{cb}} \leqslant \|\alpha\|_\infty \|\beta\|_\infty$, as required.

To show that $\mathcal{M}_{\mathrm{cb}} \subseteq \mathcal{M}_0$, one can easily adapt Jolissaint's proof in [21] by combining it with Pisier's representation theorem for p-completely bounded maps (Theorem 4.1), a task we now sketch. Let $u \in \mathcal{M}_{\mathrm{cb}} \subseteq \mathcal{M}$, and let $M \in \mathcal{B}(PM_p(G))$ be given as in the lemma above. By definition, $M \in \mathcal{CB}_p(PM_p(G))$, so as $PM_p(G)$ is a unital algebra, by the comment after Theorem 4.1, there exists $E \in SQ_p$, a p-representation $\widehat{\pi} : PM_p(G) \to \mathcal{B}(E)$ and $U : L_p(G) \to E$ and $V : E \to L_{p'}(G)$ with $\|U\| \|V\| \leqslant \|M\|_{p\mathrm{cb}}$, such that $M(T) = V \widehat{\pi}(T) U$ for $T \in PM_p(G)$. It is clear from the definitions that $\widehat{\pi}$ is a norm-decreasing algebra homomorphism, and so $\widehat{\pi} \circ \lambda_p : L_1(G) \to \mathcal{B}(E)$ is a norm-decreasing algebra homomorphism. By the discussion at the beginning of this section, there hence exists a one-complemented subspace F of E and a group representation $\sigma : G \to \mathcal{B}(F)$. As the action of $\widehat{\pi} \circ \lambda_p$ is only non-trivial of F, and F is one-complemented, we loose nothing by assuming that actually $E = F$. We then notice that

$$V \sigma(s) U = u(s) \lambda_p(s) \quad (s \in G).$$

Choose $\mu_0 \in L_{p'}(G)$ and $x_0 \in L_p(G)$ with $\|x_0\| = \|\mu_0\| = \langle \mu_0, x_0 \rangle = 1$, and define $\alpha : G \to E$ and $\beta : G \to E'$ by

$$\alpha(s) = \sigma(s^{-1})U\lambda_p(s)(x_0), \quad \beta(s) = \sigma(s)'V'\lambda_p(s^{-1})'(\mu_0) \quad (s \in G),$$

so that $\|\alpha\|_\infty \leqslant \|U\|$ and $\|\beta\|_\infty \leqslant \|V\|$. Hence, for $s, t \in G$, we have that

$$\begin{aligned}\langle \beta(t), \alpha(s) \rangle &= \langle \sigma(t)'V'\lambda_p(t^{-1})'(\mu_0), \sigma(s^{-1})U\lambda_p(s)(x_0) \rangle \\ &= \langle \mu_0, \lambda_p(t^{-1})V\sigma(ts^{-1})U\lambda_p(s)(x_0) \rangle \\ &= u(ts^{-1})\langle \mu_0, \lambda_p(t^{-1})\lambda_p(ts^{-1})\lambda_p(s)(x_0) \rangle = u(ts^{-1}).\end{aligned}$$

It remains to show that α and β are continuous. However, this follows immediately, as a weakly-continuous group representation is strongly continuous. Thus $\mathcal{M}_{\mathrm{cb}} \subseteq \mathcal{M}_0$ contractively, completing the proof. ∎

8.1. HERZ'S MULTIPLIER ALGEBRAS. We shall now show how these ideas relate to Herz's algebras $B_p(G)$. To avoid confusion, we shall write instead $HS_p(G)$, for *Herz–Schur* multiplier. Let I be an index set, and let $\psi : I \times I \to \mathbb{C}$ be a function. We say that $\psi \in V_p(I)$ if and only if, for each $T \in \mathcal{B}(\ell_p(I))$, we have that $T\psi \in \mathcal{B}(\ell_p(I))$, where $T\psi$ is defined by

$$\langle \delta_i^*, (T\psi)(\delta_j) \rangle = \psi(i,j)\langle \delta_i^*, T(\delta_j) \rangle \quad (i, j \in I).$$

By the closed-graph theorem, $V_p(I) \subseteq \mathcal{B}(\mathcal{B}(\ell_p(I)))$, which gives the obvious norm on $V_p(I)$.

Let X be a separable locally compact space, and let X_{d} be the space X equipped with the discrete topology. Then we set $V_p(X)$ to be $C(X \times X) \cap V_p(X_{\mathrm{d}})$. Finally, suppose that G is a separable locally compact group, and let $u \in HS_p(G)$ if and only if $\psi \in V_p(G)$ where ψ is defined by $\psi(s,t) = u(st^{-1})$ for $s, t \in G$. For an arbitrary G, recall that there is an open and closed separable subgroup H such that G is the union of left cosets of H. As such, we can reduce topological questions about G to questions about H, as G/H has the discrete topology. To avoid tedious calculations, we shall not mention such topological issues further.

PROPOSITION 8.4. *Let I be an index set, let $\psi : I \times I \to \mathbb{C}$ be a function, and let $C > 0$. Then the following are equivalent:*

(i) *$\psi \in V_p(I)$ and $\|\psi\|_{V_p} \leqslant C$.*

(ii) *There is a measure space (Ω, ν) and elements $(x_j)_{j \in I} \subseteq L_p(\nu)$ and $(\mu_i)_{i \in I} \subseteq L_{p'}(\nu)$ such that $\psi(i,j) = \langle \mu_i, x_j \rangle$ for each $i, j \in I$, and $\sup_i \|\mu_i\| \sup_j \|x_j\| \leqslant C$;*

(iii) *ψ is a p-completely bounded multiplier on $\mathcal{B}(\ell_p(I))$, with $\|\psi\|_{p\mathrm{cb}} \leqslant C$.*

Proof. These follow from Theorems 5.11 and 8.2 in [28]. ∎

Notice that if G is a discrete group, then using conditions (ii) and (iii) above, it is easy to show that $HS_p(G) = \mathcal{M}_0(A_p(G))$ with equal norms. However, for general G, we have the problem that the above proposition works with G_{d}, hence losing continuity conditions.

Herz shows in Lemme 1 and Lemme 2 of [16] that we have the following alternative definition of $V_p(X)$.

PROPOSITION 8.5. *Let X be a separable locally compact space, and let μ be a Radon measure on X such that each non-empty open subset of X has non-zero μ-measure. Then $\psi \in V_p(X)$ if and only if ψ is continuous and there exists $C > 0$ such that for $\lambda \in L_{p'}(X,\mu)$ and $x \in L_p(X,\mu)$, there exists $(\mu_n)_{n=1}^\infty \subseteq L_{p'}(X,\mu)$ and $(x_n)_{n=1}^\infty \subseteq L_p(X,\mu)$ with the following almost everywhere in μ, and $\sum_{n=1}^\infty \|\mu_n\|_{p'}\|x_n\|_p \leqslant C\|\lambda\|_{p'}\|x\|_p$:*

$$\lambda(s)x(t)\psi(s,t) = \sum_{n=1}^\infty \mu_n(s)x_n(t) \quad (s,t \in X).$$

That is, $V_p(X)$ coincides with the space of continuous multipliers of $L_{p'}(X,\mu)\widehat{\otimes}L_p(X,\mu)$, once we have made sense of what this means. Let G be a locally compact group with the Haar measure. Then the above applies to $V_p(G)$, and hence also to $HS_p(G)$.

Let G be a locally compact group, let $\psi \in V_p(G)$, and let $n \in \mathbb{N}$. Let $G_n = G \times \{1,2,\dots,n\}$ where $\{1,2,\dots,n\}$ is given the counting measure, so that $L_p(G \times \{1,2,\dots,n\}) = L_p(G) \otimes_p \ell_p^n$. Define $\psi_n : G_n \times G_n \to \mathbb{C}$ by

$$\psi_n((s,i),(t,j)) = \psi(s,t) \quad (s,t \in G, 1 \leqslant i,j \leqslant n),$$

so that ψ_n is continuous. We shall now show that $\psi_n \in V_p(G_n)$, using the original definition of V_p. Let $T \in \mathcal{B}(\ell_p(G_n))$, so we may also view T as a member of $\mathbb{M}_n(\mathcal{B}(\ell_p(G)))$, say $T = (T_{ij})$, where

$$\langle \delta_s^*, T_{ij}(\delta_t)\rangle = \langle \delta_s^* \otimes \delta_i^*, T(\delta_t \otimes \delta_j)\rangle \quad (s,t \in G, 1 \leqslant i,j \leqslant n).$$

Let $S = \psi_n \cdot T$, so viewing $S \in \mathbb{M}_n(\mathcal{B}(\ell_p(G)))$,

$$\begin{aligned}\langle \delta_s^*, S_{ij}(\delta_t)\rangle &= \langle \delta_s^* \otimes \delta_i^*, (\psi_n \cdot T)(\delta_t \otimes \delta_j)\rangle \\ &= \psi_n((s,i),(t,j))\langle \delta_s^* \otimes \delta_i^*, T(\delta_t \otimes \delta_j)\rangle = \psi(s,t)\langle \delta_s^*, T_{ij}(\delta_t)\rangle,\end{aligned}$$

for $s,t \in G$ and $1 \leqslant i,j \leqslant n$. By Proposition 8.4, as ψ is automatically p-completely bounded, we see that $\psi_n \in V_p(G_n)$ with $\|\psi_n\|_{V_p} \leqslant \|\psi\|_{V_p}$.

Now let $u \in HS_p(G)$, so that when $\psi(s,t) = u(st^{-1})$ for $s,t \in G$, we have that $\psi \in V_p(G)$. Let $M_u \in \mathcal{B}(L_{p'}(G)\widehat{\otimes}L_p(G))$ be the multiplier defined by ψ, using Herz's alternative definition of $V_p(G)$ as shown in Proposition 8.5. Let $x \in L_p(G)$ and $\mu \in L_{p'}(G)$, so that $a = \Lambda_p(\mu \otimes x) \in A_p(G)$. Then

$$\Lambda_p(M_u(\mu \otimes x))(s) = \int_G u(tt^{-1}s)\mu(t)x(s^{-1}t)\,\mathrm{d}t = u(s)a(s) \quad (s \in G),$$

so that M_u drops under Λ_p to pointwise multiplication of $A_p(G)$ by u. We hence immediately see that $HS_p(G) \subseteq \mathcal{M}(A_p(G))$ contractively. Combining this observation with the previous paragraph, we immediately have the following.

THEOREM 8.6. *Let G be a locally compact group. Then $HS_p(G) = \mathcal{M}_{\mathrm{cb}}(A_p(G))$ isometrically.*

8.2. ALGEBRAIC DEFINITIONS. In [6], a more group-theoretic characterisation of $\mathcal{M}_{\mathrm{cb}}(A_p(G))$ is shown, and this is used in [2] to show that $HS_2(G) = \mathcal{M}_{\mathrm{cb}}(A(G))$ (which we generalised above, using another method).

Given sets I and J and functions $u : I \to \mathbb{C}, v : J \to \mathbb{C}$, let $u \times v : I \times J \to \mathbb{C}$ be defined by $(u \times v)(i, j) = u(i)v(j)$ for $i \in I$ and $j \in J$.

PROPOSITION 8.7. *Let G be a locally compact group, let $1 < p < \infty$, and let $u \in \mathcal{M}_{\mathrm{cb}}(A_p(G))$. Then, for every locally compact group H, $u \times 1_H \in \mathcal{M}(A_p(G \times H))$ and $\|u \times 1_H\|_{\mathcal{M}} \leqslant \|u\|_{p\mathrm{cb}}$.*

Proof. By Proposition 7.2, we know that $PM_p(G)\overline{\otimes}PM_p(H) = PM_p(G \times H)$. By the above lemma, there exists a weak*-continuous map $M \in \mathcal{B}(PM_p(G))$ such that $M(\lambda_p(s)) = u(s)\lambda_p(s)$ for $s \in G$. Again, by the lemma, we wish to show that there exists a weak*-continuous map $\widehat{M} \in PM_p(G \times H)$, such that

$$\widehat{M}(T \otimes S) = M(T) \otimes S \quad (T \in PM_p(G), S \in PM_p(H)).$$

However, this follows immediately from Theorem 6.4, which also shows that $\|u \times 1_H\|_{\mathcal{M}} = \|\widehat{M}\| \leqslant \|M\|_{p\mathrm{cb}} = \|u\|_{p\mathrm{cb}}$. ∎

In [6], the converse to the above is shown in the case $p = 2$. Furthermore, to check that u is completely-bounded, it suffices to check that $u \times 1_K \in \mathcal{M}(A_p(G \times K))$ in the special case that $K = SU(2)$. However, we do not have a simple description of what $PM_p(SU(2))$ is, unless $p = 2$.

8.3. MULTIPLIERS AND AMENABILITY. In [33], Runde suggests a definition of a p-generalisation of the Fourier–Stieltjes algebra, which he denotes by $B_p(G)$. In what follows, we shall follow the conventions of Herz, which means that we sometimes swap p with p' as compared to Runde. We define $B_p(G) \subseteq C(G)$ to be functions of the form

$$a(s) = \langle \mu, \pi(s)(x) \rangle \quad (s \in G),$$

where $\pi : G \to \mathcal{B}(E)$ is a representation on some $E \in SQ_p$, and $x \in E$, $\mu \in E'$. We set $\|a\|_{B_p} = \inf\{\|\mu\|\|x\|\}$ where the infimum runs over all representations. Runde shows that $B_p(G)$ is a commutative Banach algebra. It is immediate that $B_p(G) \subseteq \mathcal{M}_0(A_p(G))$ contractively.

It is shown in Corollary 5.3 of [33] that when G is an amenable locally compact group, we have that $\mathcal{M}(A_p(G)) = B_p(G)$ isometrically, where $B_p(G)$ is Runde's generalisation of the Fourier–Stieltjes algebra. We thus immediately have the following.

PROPOSITION 8.8. *Let G be an amenable locally compact group, and let $1 < p < \infty$. Then $B_p(G) = \mathcal{M}_{\mathrm{cb}}(A_p(G)) = \mathcal{M}(A_p(G))$ isometrically.*

As stated above, Nebbia and Losert (see [24]) show that $\mathcal{M}(A(G)) = B(G)$ if and only if G is amenable. In [1], Bożejko showed that for a discrete group G, $\mathcal{M}_{\mathrm{cb}}(A(G)) = B(G)$ if and only if G is amenable. A key point in the proof is that, as a Banach space, $B(G)$ has cotype 2. We conjecture that Runde's algebra $B_p(G)$ has cotype $\max(p, p')$, but we seem to be rather far from having the tools available to prove this.

In unpublished lecture notes, [23], Losert shows in full generality that $\mathcal{M}_{\mathrm{cb}}(A(G)) = B(G)$ only when G is amenable. The arguments are very close to those used in [24], but it appears that it is not possible to simply take the result of [24] and directly deduce the corresponding result for $\mathcal{M}_{\mathrm{cb}}(A(G))$. Furthermore, Losert's arguments in [24] seem to depend upon the Hilbert space basis of $A(G)$ much more than Nebbia's and Bożejko's arguments. We hence seem to be rather far from being able to show that $\mathcal{M}_{\mathrm{cb}}(A_p(G)) = B_p(G)$ only when G is amenable, when $p \neq 2$.

9. CONCLUSIONS

Compared to the operator space structure on $A_p(G)$ considered in [22], we get a contractive quantised Banach algebra, and not just a bounded algebra product. It could also be argued that our approach is more natural, as $A_p(G)$ is an L_p-space generalisation of $A(G)$, so arguably L_p spaces should be used to define a quantised structure on $A_p(G)$. However, our approach seems to require amenability to be introduced to get the theory to work perfectly. We are not aware of anyone considering multipliers in the framework of [22]. It would be interesting to see if Herz's ideas appear naturally in that setting, as they do in our setting.

It would be interesting to investigate if Theorem 7.3 holds for any non-amenable groups, when $p \neq 2$. Furthermore, it would be interesting to try to extend the tentative results in Section 8.2. Surely a first step in this programme would be to study the algebras $PM_p(G)$ for, say, $G = SU(2)$. Finally, surely the ideas in Section 7.1 have scope for further study.

We have hinted that perhaps the definition of a p-operator space is not correct. To be precise, for operators spaces, we consider not just a space E, but also the spaces $\ell_2^n \otimes E$. This is reasonable, as ℓ_2^n is (up to isometric isomorphism) the only n-dimensional Hilbert space. For p-operator spaces, we replace ℓ_2^n with ℓ_p^n, but we have less justification for this, as there are many n-dimensional SQ_p spaces. Of course, Pisier's and Le Merdy's results suggest that maybe this is enough, as we do get an intrinsic characterisation of SQ_p spaces, for example. A more technical problem here is seemingly we do not have a well-defined way to define a tensor product of two SQ_p spaces. In Section 3 of [33], Runde shows that given $E, F \in SQ_p$, we may define a completion of $E \otimes F$ in such a way as to get another SQ_p space, and with a suitable mapping property holding. However, it seems that Runde's construction depends upon the chosen representation of E and F as subspaces of quotients of L_p spaces.

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