J. OPERATOR THEORY
63:1(2010), 181–189

HIGHER-RANK NUMERICAL RANGES AND DILATIONS

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Communicated by Kenneth R. Davidson

ABSTRACT. For any n-by-n complex matrix A and any k, $1 \leqslant k \leqslant n$, let $\Lambda_k(A) = \{\lambda \in \mathbb{C} : X^*AX = \lambda I_k$ for some n-by-k X satisfying $X^*X = I_k\}$ be its rank-k numerical range. It is shown that if A is an n-by-n contraction, then

$$\Lambda_k(A) = \bigcap\{\Lambda_k(U) : U \text{ is an } (n+d_A)\text{-by-}(n+d_A) \text{ unitary dilation of } A\},$$

where $d_A = \operatorname{rank}(I_n - A^*A)$. This extends and refines previous results of Choi and Li on constrained unitary dilations, and a result of Mirman on S_n-matrices.

KEYWORDS: *Higher-rank numerical range, unitary dilation.*

MSC (2000): 15A21, 15A24, 15A60, 15A90, 81P68.

1. INTRODUCTION

We say that the operator A on space H *dilates* to B on K or B *compresses* to A if there is an isometry V from H to K such that $A = V^*BV$. It is easily seen that this is equivalent to B being unitarily similar to a 2-by-2 operator matrix of the form $\begin{bmatrix} A & * \\ * & * \end{bmatrix}$. The classical dilation result of Halmos asserts that *every contraction A, i.e., an A with $\|A\| \leqslant 1$, can be dilated to the unitary operator*

$$\begin{bmatrix} A & (I - AA^*)^{1/2} \\ (I - A^*A)^{1/2} & -A^* \end{bmatrix}$$

(cf. Problem 222(a) of [11]). With more care, the unitary dilation can be achieved in a most economical way: *if A is a contraction on H, then A can be dilated to a unitary operator U from $H \oplus K_1$ to $H \oplus K_2$ with K_1 and K_2 of dimensions $d_{A^*} \equiv \dim \overline{\operatorname{ran}(I - AA^*)^{1/2}}$ and $d_A \equiv \dim \overline{\operatorname{ran}(I - A^*A)^{1/2}}$, respectively, and, moreover, in this case d_{A^*} and d_A are the smallest dimensions of such spaces K_1 and K_2.* Here d_A and d_{A^*} are called the *defect indices* of the contraction A. They provide a measure on how far A deviates from the unitary operators and play a prominent role in the unitary dilation theory. Note that $d_{A^*} = d_A$ if H is finite-dimensional.

Let M_n be the algebra of n-by-n complex matrices. In [4], the authors introduced the notion of the *rank-k numerical range* of $A \in M_n$ in connection to the study of quantum error correction; see [5]. This can be defined equivalently as

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : X^*AX = \lambda I_k, \text{ for some } n\text{-by-}k \ X \text{ satisfying } X^*X = I_k\}.$$

Evidently, $\lambda \in \Lambda_k(A)$ if and only if λI_k dilates to A. When $k = 1$, this concept reduces to the classical numerical range. Many properties of the classical numerical range have been extended to the higher-rank numerical range; see [2], [3], [4], [5], [20]. In particular, it was shown in [13] that

$$(1.1) \quad \Lambda_k(A) = \{\mu \in \mathbb{C} : e^{it}\mu + e^{-it}\overline{\mu} \leqslant \lambda_k(e^{it}A + e^{-it}A^*) \text{ for all } t \in [0, 2\pi)\}.$$

Here $\lambda_1(X) \geqslant \cdots \geqslant \lambda_n(X)$ denote the eigenvalues of a Hermitian $X \in M_n$. In particular, $\Lambda_k(A)$ is the intersection of closed half planes in $\mathbb{C}$, and therefore is always convex. If $N \in M_n$ is normal with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$(1.2) \qquad \Lambda_k(N) = \bigcap_{1 \leqslant j_1 < \cdots < j_{n-k+1} \leqslant n} \operatorname{conv}\{\lambda_{j_1}, \ldots, \lambda_{j_{n-k+1}}\}$$

is a polygon (including interior). In [12], it was shown that for a given positive integer n, $\Lambda_k(A)$ is nonempty for every $A \in M_n$ if and only if $n \geqslant 3k - 2$.

In this paper, we refine and extend a result in [6] on constrained unitary dilation by proving the following.

THEOREM 1.1. *Let $A \in M_n$ be a contraction, and $k \in \{1, \ldots, n\}$. Then A has a unitary dilation $U \in M_{n+d_A}$ such that $\lambda_k(A + A^*) = \lambda_k(U + U^*)$.*

When $k = 1$, our result improves Theorem 2.1 of [6] in the finite-dimensional case as Theorem 2.1 of [6] requires the use of unitary dilations of $A \in M_n$ of size $2n$. The authors of [6] gave examples to demonstrate that extending Theorem 2.1 of [6] in certain directions are impossible. Nevertheless, Theorem 1.1 shows that one can obtain useful generalizations of the result under a proper setting. In particular, Theorem 1.1 above can be used to deduce the following theorem, which extends a result on classical numerical range to the higher-rank numerical range.

THEOREM 1.2. *Let $A \in M_n$ be a contraction. Then, for each k, $1 \leqslant k \leqslant n$,*

$$\Lambda_k(A) = \bigcap \{\Lambda_k(U) : U \in M_{n+d_A} \text{ is a unitary dilation of } A\}.$$

When $k = 1$ and without the dimension assumption on the unitary U, Theorem 1.2 was conjectured by Halmos [10] and proved in [6]. Clearly, if $A \in M_n$ is nonzero then $A/\|A\|$ is a contraction. Thus, by Theorem 1.2, if $A \in M_n$ then $\Lambda_k(A)$ is the intersection of $\Lambda_k(\|A\|U)$, where $U \in M_{n+d_A}$ is a unitary dilation of $A/\|A\|$. Consequently, $\Lambda_k(A)$ is the intersection of polygons $\Lambda_k(N)$ of the form (1.2), where N is a (norm-preserving) normal dilation of A.

2. PROOFS

We begin with several lemmas. The first two are adaptations of Lemmas 3.2 and 3.3 in [6]. Part of the proofs are similar to those in [6]. We include the details for completeness.

LEMMA 2.1. *Let $H \in M_n$ be the leading principal submatrix of a Hermitian matrix $\widetilde{H} \in M_{n+1}$. Suppose there exists a unit vector $u \in \mathbb{C}^{n+1}$ with nonzero $(n+1)$st entry such that $\widetilde{H}u = \xi u$. For $1 \leqslant k \leqslant n$, if $\lambda_k(H) \leqslant \xi$, then $\lambda_k(\widetilde{H}) \leqslant \xi$.*

Proof. On the contrary, suppose that $\lambda_k(\widetilde{H}) > \xi$. Since ξ is an eigenvalue for $\widetilde{H}$, by the interlacing inequality ([1], Corollary III.1.5) we must have $\lambda_{k+1}(\widetilde{H}) = \xi = \lambda_k(H)$. Let $v_j \in \mathbb{C}^{n+1}$ be the unit eigenvector of $\widetilde{H}$ corresponding to the eigenvalue $\lambda_j(\widetilde{H})$ for $j = 1, 2, \ldots, k$, $M = \operatorname{span}\{u, v_1, \ldots, v_k\}$ and $N = M \cap (\mathbb{C}^n \oplus \{0\})$. Then $\dim N = k$, because $u \notin \mathbb{C}^n \oplus \{0\}$. Consider the compression A of $\widetilde{H}$ on N. Since $\Lambda_1(A) \subseteq \Lambda_1(\widetilde{H}|_M) = [\xi, \lambda_1(\widetilde{H})]$, it is clear that $\lambda_k(A) \geqslant \xi$. On the other hand, since $N \subseteq \mathbb{C}^n \oplus \{0\}$, we also have $\xi = \lambda_k(H) \geqslant \lambda_k(A)$. Thus $\lambda_k(A) = \xi$. Let $y \in N$ be a unit eigenvector of A corresponding to the eigenvalue ξ. Say, $y = c_0 u + c_1 v_1 + \cdots + c_k v_k$, where $\sum_{j=0}^{k} |c_j|^2 = 1$. Since $\xi = \langle Ay, y \rangle = \langle \widetilde{H}y, y \rangle = |c_0|^2 \xi + \sum_{j=1}^{k} |c_j|^2 \lambda_j(\widetilde{H})$ and $\lambda_1(\widetilde{H}) \geqslant \cdots \geqslant \lambda_k(\widetilde{H}) > \xi$, we infer that $|c_0| = 1$ and $c_1 = \cdots = c_k = 0$. This implies that $u \in N \subseteq \mathbb{C}^n \oplus \{0\}$, a contradiction. Hence $\lambda_k(\widetilde{H}) \leqslant \xi$ as asserted. ∎

LEMMA 2.2. *Let $A \in M_n$ be a contraction with $d_A \geqslant 1$ and denote $\lambda_k(A + A^*) = 2\cos\theta$ for some $\theta \in \mathbb{R}$. Suppose neither $\mathrm{e}^{\mathrm{i}\theta}$ nor $\mathrm{e}^{-\mathrm{i}\theta}$ is an eigenvalue for A. Then A has a contractive dilation $\widetilde{A} \in M_{n+1}$ such that $\lambda_k(\widetilde{A} + \widetilde{A}^*) = \lambda_k(A + A^*)$, $d_{\widetilde{A}} = d_A - 1$, and $\mathrm{e}^{\pm\mathrm{i}\theta}$ are two eigenvalues for $\widetilde{A}$.*

Proof. Let v be a unit vector such that $(A + A^*)v = (2\cos\theta)v$. By Lemma 3.1 of [6], we have $\|Av\| < 1$. Since

$$\begin{aligned} \|A^*v\|^2 - \|Av\|^2 &= v^*(AA^* - A^*A)v = v^*\{A(A + A^*) - (A + A^*)A\}v \\ &= v^*A(2\cos\theta)v - (2\cos\theta)v^*Av = 0, \end{aligned}$$

we have $\|A^*v\| = \|Av\|$. Let $\alpha = \sqrt{1 - \|Av\|^2} = \sqrt{1 - \|A^*v\|^2}$. Then $x = (I_n - A^*A)^{1/2}v/\alpha$ and $y = (I_n - AA^*)^{1/2}v/\alpha$ are unit vectors in $\mathbb{C}^n$. Write

$$X = \begin{bmatrix} I_n & \overrightarrow{0_n} \\ 0_n & x \end{bmatrix}, \quad Y = \begin{bmatrix} I_n & \overrightarrow{0_n} \\ 0_n & y \end{bmatrix}, \quad Z = \begin{bmatrix} A & -(I_n - AA^*)^{1/2} \\ (I_n - A^*A)^{1/2} & A^* \end{bmatrix},$$

and

$$\widetilde{A} = X^*ZY = \begin{bmatrix} A & -(I_n - AA^*)v/\alpha \\ v^*(I_n - A^*A)/\alpha & x^*A^*y \end{bmatrix} \in M_{n+1}.$$

Then X and Y are $2n$-by-$(n+1)$ matrices satisfying $X^*X = Y^*Y = I_{n+1}$, $Z^*Z = I_{2n}$ and $\widetilde{A}$ is a contractive dilation of A. Let $\widetilde{v} = \begin{bmatrix} v \\ 0 \end{bmatrix} \in \mathbb{C}^{n+1}$. Then

$$\widetilde{A}\widetilde{v} = \begin{bmatrix} Av \\ v^*(I_n - A^*A)v/\alpha \end{bmatrix} = \begin{bmatrix} Av \\ \alpha \end{bmatrix}$$

is a unit vector because $\alpha = \sqrt{1 - \|Av\|^2}$, and

$$(\widetilde{A} + \widetilde{A}^*)\widetilde{v} = \begin{bmatrix} (A+A^*)v \\ v^*(AA^* - A^*A)v/\alpha \end{bmatrix} = \begin{bmatrix} (2\cos\theta)v \\ 0 \end{bmatrix} = (2\cos\theta)\widetilde{v}$$

because $\|A^*v\| = \|Av\|$. It follows from Lemma 3.1 of [6] that $M = \operatorname{span}\{\widetilde{v}, \widetilde{A}\widetilde{v}\}$ is a reducing subspace of $\widetilde{A}$ and the restriction of $\widetilde{A}$ on M has $\mathrm{e}^{\pm \mathrm{i}\theta}$ as two of its eigenvalues. So, $\widetilde{A}\widetilde{v} = \begin{bmatrix} Av \\ \alpha \end{bmatrix}$ is also an eigenvector of $\widetilde{A} + \widetilde{A}^*$ corresponding to the eigenvalue $2\cos\theta$. Note that the last entry of $\widetilde{A}\widetilde{v}$ is $\alpha \neq 0$. Applying Lemma 2.1 with $H = A + A^*, \widetilde{H} = \widetilde{A} + \widetilde{A}^*$ and $\xi = 2\cos\theta$, we have $\lambda_k(\widetilde{A} + \widetilde{A}^*) \leqslant 2\cos\theta$. By the interlacing inequality ([1], Corollary III.1.5) we conclude that $\lambda_k(\widetilde{A} + \widetilde{A}^*) = 2\cos\theta$.

We now check that $d_{\widetilde{A}} = d_A - 1$. Note that the leading n-by-n principal submatrix of $\widetilde{A}^*\widetilde{A}$ equals $A^*A + ww^*$ with $w = (I_n - A^*A)v/\alpha$. Thus,

$$d_{\widetilde{A}} = \operatorname{rank}(I_{n+1} - \widetilde{A}^*\widetilde{A}) \geqslant \operatorname{rank}(I_n - A^*A - ww^*) \geqslant \operatorname{rank}(I_n - A^*A) - 1 = d_A - 1.$$

It remains to show that $d_{\widetilde{A}} \leqslant d_A - 1$. Let K be the eigenspace of A^*A corresponding to the eigenvalue 1. Then K has dimension $m = n - d_A$, and there is an orthonormal basis $\{u_1, \dots, u_m\}$ for K such that $\|Au_j\| = 1$ for all $j = 1, \dots, m$. Now, consider the vectors of the form $\widetilde{u}_j = \begin{bmatrix} u_j \\ 0 \end{bmatrix} \in \mathbb{C}^{n+1}$ for $j = 1, \dots, m$, and let $\widetilde{K}$ be the space spanned by them. Clearly, $\widetilde{v} \notin \widetilde{K}$ and $\widetilde{A}\widetilde{v}$ does not lie in the span of $\widetilde{K} \cup \{\widetilde{v}\}$. Now, $\|\widetilde{A}w\| = 1$ for all $w \in \{\widetilde{u}_1, \dots, \widetilde{u}_m, \widetilde{v}, \widetilde{A}\widetilde{v}\}$, which spans an $(m+2)$-dimensional subspace. Thus $\widetilde{A}^*\widetilde{A}$ has at least $m+2$ linearly independent eigenvectors for 1. So, $d_{\widetilde{A}} \leqslant n + 1 - (m+2) = d_A - 1$. ∎

LEMMA 2.3. *Let $A \in M_n$ be a contraction with $d_A \geqslant 1$ such that $\lambda_n(A + A^*) \geqslant \gamma$ for some $\gamma > -2$. Then A has a contractive dilation $\widetilde{A} \in M_{n+1}$ such that $d_{\widetilde{A}} = d_A - 1$, $\lambda_n(\widetilde{A} + \widetilde{A}^*) \geqslant \gamma$ and $-1, \mathrm{e}^{\mathrm{i}\theta}$ are two eigenvalues for $\widetilde{A}$, where $2\cos\theta \geqslant \gamma$.*

Proof. Since A is a contraction, it is unitarily similar to $U_0 \oplus A_0$, where $U_0 \in M_{n-m}$ $(1 \leqslant m \leqslant n)$ is unitary and $A_0 \in M_m$ is a contraction with no eigenvalue on the unit circle. Clearly, $d_{A_0} = d_A$. Note that $\Lambda_1(A_0)$ is a compact convex set contained in the open unit disc, and $-1 \notin \Lambda_1(A_0)$. Hence there are two chords $[-1, \mathrm{e}^{\mathrm{i}\theta}]$ and $[-1, \mathrm{e}^{\mathrm{i}\phi}]$ which are tangent to $\partial\Lambda_1(A_0)$, where $-\pi < \phi \leqslant \theta < \pi$. It is clear that $2\cos\theta \geqslant \gamma$, because $\Lambda_1(A_0)$ is contained in the closed half plane $\{z \in \mathbb{C} : z + \overline{z} \geqslant \gamma\}$. Let $A_0' = \mathrm{e}^{-\mathrm{i}(\theta+\pi)/2}A_0$. Then the line segment $[\mathrm{e}^{\mathrm{i}(\pi-\theta)/2}, \mathrm{e}^{\mathrm{i}(\theta-\pi)/2}]$ is tangent to $\partial\Lambda_1(A_0')$, and $\Lambda_1(A_0')$ is contained in the closed half plane $\{z \in \mathbb{C} : z + \bar{z} \leqslant 2\cos((\pi-\theta)/2)\}$. That is, $\lambda_1(A_0' + {A_0'}^*) = 2\cos((\pi - \theta)/2)$. By Lemma 2.2 for $k = 1$, A_0' has a contractive dilation $\widetilde{A_0'} \in M_{m+1}$ such that $d_{\widetilde{A_0'}} = d_{A_0'} - 1 = d_A - 1$, $\lambda_1(\widetilde{A_0'} + \widetilde{A_0'}^*) = 2\cos((\pi-\theta)/2)$ and $\mathrm{e}^{\pm \mathrm{i}(\pi-\theta)/2}$ are two eigenvalues for $\widetilde{A_0'}$. Let $\widetilde{A_0} = \mathrm{e}^{\mathrm{i}(\theta+\pi)/2}\widetilde{A_0'}$ and $\widetilde{A} = U_0 \oplus \widetilde{A_0}$. We deduce that $\widetilde{A}$ is a contractive dilation of A, $d_{\widetilde{A}} = d_{\widetilde{A_0'}} = d_A - 1$ and $-1, \mathrm{e}^{\mathrm{i}\theta}$ are two eigenvalues for $\widetilde{A}$. By the interlacing inequality, it is clear that $\lambda_n(\widetilde{A} + \widetilde{A}^*) \geqslant \lambda_n(A + A^*) \geqslant \gamma$ as desired. ∎

We are now ready for the

Proof of Theorem 1.1. We prove the result by induction on d_A. If $d_A = 0$, then $U = A$ as asserted. Assume $d_A \geqslant 1$ and the result holds if d_A is smaller. For convenience, say, $\lambda_k(A + A^*) = 2\cos\theta$, where $\theta \in \mathbb{R}$. It suffices to show that A has a contractive dilation $A_1 \in M_{n+1}$ such that $\lambda_k(A_1 + A_1^*) = \lambda_k(A + A^*)$ and $d_{A_1} = d_A - 1$. The result will then follow from the induction hypothesis.

Since A is a contraction, it is unitarily similar to $U_0 \oplus A_0$, where $U_0 \in M_{n-m}$ $(1 \leqslant m \leqslant n)$ is unitary and $A_0 \in M_m$ is a contraction with no eigenvalue on the unit circle. Clearly, $d_{A_0} = d_A \geqslant 1$. Let

$$j_0 = \max\{j : \lambda_j(A_0 + A_0^*) > 2\cos\theta\} \quad \text{and} \quad j_1 = \max\{j : \lambda_j(U_0 + U_0^*) > 2\cos\theta\}$$

with the convention that $j_0 = 0$ and $j_1 = 0$ when the corresponding set of indices is empty. Then

$$j_0 \leqslant m, \quad j_0 + j_1 < k \quad \text{and} \quad \lambda_{j_0+j_1+1}(A + A^*) = 2\cos\theta.$$

We consider two cases.

Case 1. Suppose $j_0 < m$. Then $2\cos\theta \geqslant \lambda_{j_0+1}(A_0 + A_0^*) = 2\cos\theta_0$. Note that neither $\mathrm{e}^{\mathrm{i}\theta_0}$ nor $\mathrm{e}^{-\mathrm{i}\theta_0}$ is an eigenvalue for A_0. By Lemma 2.2, A_0 has a contractive dilation $\widetilde{A}_0 \in M_{m+1}$ such that $\lambda_{j_0+1}(\widetilde{A}_0 + \widetilde{A}_0^*) = \lambda_{j_0+1}(A_0 + A_0^*) = 2\cos\theta_0 \leqslant 2\cos\theta$, $d_{\widetilde{A}_0} = d_{A_0} - 1$ and $\mathrm{e}^{\pm\mathrm{i}\theta_0}$ are two eigenvalues for $\widetilde{A}_0$. Moreover, by the interlacing inequality, $\lambda_j(\widetilde{A}_0 + \widetilde{A}_0^*) \geqslant \lambda_j(A_0 + A_0^*) > 2\cos\theta$ for $j \leqslant j_0$. Consequently, $\max\{j : \lambda_j(\widetilde{A}_0 + \widetilde{A}_0^*) > 2\cos\theta\} = j_0$. Thus, $A_1 = U_0 \oplus \widetilde{A}_0 \in M_{n+1}$ is a contractive dilation of A satisfying $d_{A_1} = d_{\widetilde{A}_0} = d_{A_0} - 1 = d_A - 1$ and $\max\{j : \lambda_j(A_1 + A_1^*) > 2\cos\theta\}$ equal to

$$\max\{j : \lambda_j(U_0 + U_0^*) > 2\cos\theta\} + \max\{j : \lambda_j(\widetilde{A}_0 + \widetilde{A}_0^*) > 2\cos\theta\} = j_1 + j_0.$$

It follows that

$$2\cos\theta \geqslant \lambda_{j_0+j_1+1}(A_1 + A_1^*) \geqslant \lambda_k(A_1 + A_1^*) \geqslant \lambda_k(A + A^*) = 2\cos\theta,$$

because $j_0 + j_1 < k$. Hence $\lambda_k(A_1 + A_1^*) = \lambda_k(A + A^*)$ and A_1 is a desired dilation.

Case 2. Suppose $j_0 = m$. Then $\lambda_m(A_0 + A_0^*) > 2\cos\theta$. By Lemma 2.3, A_0 has a contractive dilation $\widetilde{A}_0 \in M_{m+1}$ such that

$$\lambda_m(\widetilde{A}_0 + \widetilde{A}_0^*) > 2\cos\theta, \quad d_{\widetilde{A}_0} = d_{A_0} - 1 = d_A - 1 \quad \text{and} \quad \lambda_{m+1}(\widetilde{A}_0 + \widetilde{A}_0^*) = -2.$$

Then $A_1 = U_0 \oplus \widetilde{A}_0 \in M_{n+1}$ is a contractive dilation of A satisfying $d_{A_1} = d_{\widetilde{A}_0} = d_A - 1$ and

$$\begin{aligned}&\max\{j : \lambda_j(A_1 + A_1^*) > 2\cos\theta\}\\ &= \max\{j : \lambda_j(U_0 + U_0^*) > 2\cos\theta\} + \max\{j : \lambda_j(\widetilde{A}_0 + \widetilde{A}_0^*) > 2\cos\theta\} = j_1 + m = j_1 + j_0.\end{aligned}$$

It follows that

$$2\cos\theta \geqslant \lambda_{j_0+j_1+1}(A_1 + A_1^*) \geqslant \lambda_k(A_1 + A_1^*) \geqslant \lambda_k(A + A^*) = 2\cos\theta,$$

because $j_0 + j_1 < k$. Hence $\lambda_k(A_1 + A_1^*) = \lambda_k(A + A^*)$ and A_1 is a desired dilation. ∎

We can now use Theorem 1.1 to prove Theorem 1.2. The proof depends heavily on (1.1) and is similar to the proof of Theorem 2.4 in [6].

Proof of Theorem 1.2. Let $A \in M_n$ be a contraction. It is obvious that $\Lambda_k(A) \subseteq \Lambda_k(B)$ if B is a dilation of A. Thus, we have

$$\Lambda_k(A) \subseteq \bigcap\{\Lambda_k(U) : U \in M_{n+d_A} \text{ is a unitary dilation of } A\}.$$

To prove the reverse inclusion, we consider any particular $\zeta \notin \Lambda_k(A)$. Since $\Lambda_k(A)$ is a compact convex set, there exists $\theta \in [0, 2\pi)$ and $\mu \in \mathbb{R}$ such that $e^{i\theta}\zeta + e^{-i\theta}\overline{\zeta} > \mu$, while $e^{i\theta}\Lambda_k(A) = \Lambda_k(e^{i\theta}A)$ is included in the closed half plane $\{z \in \mathbb{C} : z + \overline{z} \leqslant \mu\}$. From (1.1), we see that $\lambda_k(e^{i\theta}A + e^{-i\theta}A^*) \leqslant \mu$. By Theorem 1.1, there is a unitary dilation $U \in M_{n+d_A}$ of A such that $\lambda_k(e^{i\theta}U + e^{-i\theta}U^*) \leqslant \mu$. By (1.1) again, $\Lambda_k(e^{i\theta}U) \subseteq \{z \in \mathbb{C} : z + \overline{z} \leqslant \mu\}$. Hence $e^{i\theta}\zeta \notin \Lambda_k(e^{i\theta}U)$ and $\zeta \notin \Lambda_k(U)$. This completes the proof. ∎

We end this paper by relating the rank-k numerical ranges of S_n-matrices to the Poncelet property. An n-by-n complex matrix A is said to be of *class* S_n if (i) A is a contraction, (ii) the eigenvalues of A are all in the open unit disc $\mathbb{D}$, and (iii) $d_A = 1$. In recent years, properties of the classical numerical ranges of S_n-matrices have been intensely studied (cf. [7], [8], [9], [15], [16], [17], [18], [19], [21]). Among other things, it was obtained that the boundary of the classical numerical range $\Lambda_1(A)$ of an S_n-matrix A has the $(n+1)$-Poncelet property. This means that there are infinitely many $(n+1)$-gons interscribing between the unit circle $\partial\mathbb{D}$ and the boundary $\partial\Lambda_1(A)$ or, put more precisely, for any point a on $\partial\mathbb{D}$ there is a (unique) $(n+1)$-gon with a as one of its vertices such that all its $n+1$ vertices are in $\partial\mathbb{D}$ and all its $n+1$ edges are tangent to $\partial\Lambda_1(A)$ (cf. Theorem 2.1 of [7] or Theorem 1 of [15]).

If A is in S_n, so is $\mathrm{e}^{-\mathrm{i}t}A$ for any real t. Hence the eigenvalues of $(\mathrm{e}^{-\mathrm{i}t}A + \mathrm{e}^{\mathrm{i}t}A^*)/2$ are all distinct by Corollary 2.7 of [7]. The curve Γ_j, $j = 1, \ldots, n$, is the envelope of chords

$$x \cos t + y \sin t = \lambda_j(t),$$

where $\lambda_j(t) = \lambda_j((\mathrm{e}^{-\mathrm{i}t}A + \mathrm{e}^{\mathrm{i}t}A^*)/2)$. Equations for the curves Γ_j are described by $\alpha_j(t) = (x_j(t), y_j(t))$ with

$$x_j(t) = \lambda_j(t) \cos t - \lambda_j'(t) \sin t, \quad y_j(t) = \lambda_j(t) \sin t + \lambda_j'(t) \cos t.$$

These curves Γ_j are expected to have a Poncelet-type property just as $\Gamma_1 = \partial\Lambda_1(A)$ does. This is indeed the case and is proved in Theorem 8 of [15]. Note that, in this case, Γ_j and Γ_{n-j+1} coincide for any j, and if $U = \mathrm{diag}\,(b_1, \ldots, b_{n+1})$ is a unitary dilation of A, where the b_j's are arranged counterclockwise around $\partial\mathbb{D}$, then, for each j, the not-necessarily-convex $(n+1)$-gon $b_1\ b_{j+1}\ b_{2j+1}\ \cdots\ b_{nj+1}$ ($b_p = b_q$ if $p \equiv q \pmod{n+1}$) has all its sides $[b_{kj+1}, b_{(k+1)j+1}]$ tangent to Γ_j. A detailed analysis of such curves, called *a package of Poncelet curves*, has been carried out by Mirman [15], [16], [18]. Note that the curve Γ_1 is convex and $\Lambda_1(A)$ is equal to the convex hull of Γ_1. Other curves Γ_j's ($2 \leqslant j \leqslant n-1$) are not necessarily convex (cf. Example 7 of [15]), and hence $\Lambda_j(A)$ does not necessarily coincide with the convex hull of Γ_j. However, by Theorem 1.2 and Theorem 8 of [15], the former is always contained in the latter and when Γ_j ($1 \leqslant j \leqslant n/2$) is convex, they are equal to each other.

Acknowledgements. The research was started while Li was visiting National Central University in April of 2007 supported by Center for Theoretical Sciences, Mathematical Division, NCU. Li would like to thank the warm hospitality of the colleagues at the mathematics department of National Central University. We would like to thank the authors of [2] and [20] for sending us their preprints.

Research of Gau was partially supported by the National Science Council of the Republic of China and Center for Theoretical Sciences, Mathematical Division, NCU, Taiwan. Research of Li was partially supported by a USA NSF grant and a HK RGC grant. He is an honorary professor of the University of Hong Kong. Research of Wu was partially supported by the National Science Council of the Republic of China, and by the MOE-ATU project.

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Received May 10, 2007.