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NONDEGENERACY OF GROUND STATES IN NONRELATIVISTIC QUANTUM FIELD THEORY

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ABSTRACT. We reconstruct and develop the abstract Perron–Frobenius theory studied by Faris. We apply the results to some models in nonrelativistic quantum field theory and show the nondegeneracy of the gound state if it exists. The Wigner–Weisskopf model, the spin-boson model, the Fröhlich polaron without ultraviolet cutoffs and the Fröhlich bipolaron without ultraviolet cutoffs are discussed.

KEYWORDS: *Ground states, non-relativistic quamtum field theory, self-dual cones, Perron–Frobenius theorem.*

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1. INTRODUCTION

In quantum field theory (QFT), a ground state is anticipated to be nondegenerate if it exists. The Perron–Frobenius theory has been first applied to show the uniqueness of the ground state in QFT by Glimm and Jaffe [16]. The Perron–Frobenius theory tells us that if the heat semi-group e^{-tK} generated by a Hamiltonian K in an L^2-space improves the positivity for all $t > 0$, then its lowest energy state or ground state is nondegenerate (if it exists). Here a sentence "A improves the positivity" is understood as follows: if $f \geqslant 0$ a.e. and $\|f\| \neq 0$, then $Af > 0$ a.e. In QFT, the Hamiltonian under consideration is living in the so-called Fock space which is identified with an L^2-space under the Schrödinger representation or Q-representation. Glimm and Jaffe considered their Hamiltonian H in the Schrödinger representation and proved that e^{-tH} improves the positivity for all $t > 0$. By the Perron–Frobenius theory, they obtained the nondegeneracy of the ground state as a direct consequence. This positivity techniques in the Schrödinger representation have been successfully used and developed by some authors [11], [37], [39], [40].

In nonrelativistic quantum field theory (NQFT), the Perron–Frobenius theory in the Schrödinger representation has also played important roles. This direction in NQFT has been established by Bach et al. in [5]. They investigated the Nelson model which describes a system consisting with quamtum mechanical particles coupled to a Bose field. Hence this system has components of not only the Bose field but also quantum particles governed by the Schrödinger operators. Accrodingly the Perron–Frobenius argument in [5] goes well by combining the Glimm and Jaffe's theory in QFT and the standard theory for the Schrödinger operators. There is another important model in NQFT, namely the Pauli–Fierz model which describes electrons moving in the quantized radiation fields. Roughly speaking the Hamiltonian of this model is expressed as a Pauli operator with the quantized vector potentials. Here a quantized vector potential is given by a field operator, while the standard vector potential is given by a real valued function. Hiroshima established a functional integral representation of the Pauli–Fierz model under the Schrödinger representation and proved that the heat semi-group generated by the (spinless) Hamiltonian improves the positivity which means the nondegeneracy of the ground state [21], [23]. The existence of the ground state for this model was established in [10].

On the other hand, the Perron–Frobenius theory is useful not only under the Schrödinger representation but also in more general situations. Gross has extended the theory to the second quantized fermion in [11]. In this case underlying Hilbert space is not standard L^2-space anymore. Faris constructed an abstract framework of the Perron–Frobenius theory which includes the theory in the Schrödinger representation mentioned in the above and Gross' fermion theory. In his abstract theory, no algebraic structure of the Hilbert space, but only a cone in the space is needed. The main purpose of this note is to develop further Faris' abstract theory and apply various obtained results to some typical models in NQFT, namely, to the Wigner–Weisskopf model, the spin-boson model, the Fröhlich polaron model without ultraviolet cutoffs and the Fröhlich bipolaron model without ultraviolet cutoffs.

In Section 2 we reconstruct and develop Faris' theory from a viewpoint of self-dual cones. Theorem 2.12 which was essentially proven by Faris [13] is our base. This theorem tells us the equivalence between the nondegeneracy of the ground state and the (generalized) positivity improving property of a Hamiltonian under consideration. It is worthy to remark that Theorem 2.16 is applicable to singular perturbative cases which will be discussed in later sections.

In Section 3, we discuss quantized operators in a Fock space in a light estabilshed in Section 2. We will see that the fundamental results in this section are crucial for applications to the concrete models in NQFT.

As a warmup, the Wigner–Weisskopf model is discussed in Section 4. This model describes one-mode fermion coupled to a Bose field. Since the Hamiltonian H strongly commutes with the total number operator, H has a direct sum decomposition $H = \bigoplus_{n=0}^{\infty} H_n$ associated with the total number of particles. (Here we say that two self-adjoint operators strongly commute with each other if their spectral projections commute.) We will show that, for all n, e^{-tH_n} improves the positivity in generalized sense. As a consequence, even if H has a degenerate ground states with α-fold degeneracy, only α Hamiltonians $H_{n_1}, \dots, H_{n_\alpha}$ have a nondegenerate ground state with energy which equals that of H. As to the existence of the ground states and physical arguments for this model, see [3], [19], [20] and references therein.

Section 5 is devoted to the spin-boson model. This model governs a two-level system coupled to a Bose field. Positivity improving property of the heat semi-group associated with the Hamiltonian is proven in the generalized sense. As an immediate consequence, overlap properties between the unique ground state and the vacuum which have been established by Hirokawa in [18] are rediscovered without any smalleness conditions of parameters. We can find the existence conditions of the ground state in [43]. (In [2], the existence of a ground state and its uniqueness for a generalized model have been also discussed by techniques unrelated to the methods here.) It should be noted that recently Hirokawa and Hiroshima have constructed a functional integral representation of the spin-boson model in [22] by using a Poission process. Applying this formula they proved the positivity improving property of the heat semi-group generated by the Hamiltonian. In this note we give a direct proof.

In Section 6, we treat the H. Fröhlich polaron without ultraviolet cutoffs. This model explains an electron in an ionic lattice. As for physical aspects of this model, Gerlach and Löwen's paper [9] is convenient for readers. (See also [42].) In his famous theses [14], [15], J. Fröhlich also studied similar model. He proved the (generalized) positivity improving property of the heat semi-group generated by the Hamiltonian of a fixed total momentum with finite cutoffs and it seems that the part of removal of cutoffs was explained in his unpublished note. In this section, we give a complete proof of this issue as an application of our abstract results. (In [14], [15] more singular models are investigated. Our methods cannot cover this strongly singular case.) We also remark that, in [29], [30], Møller has investigated a generalized polaron model with finite cutoffs. A complete proof of removal of cutoffs has been also done by Sloan [40]. More precisely, he established the theory of support maximizing operators [40] (cf. [12]) and applied it to the relativistic polaron without cutoffs. His method essentially works in the Schrödinger representation. By this restriction he only investigated the Hamiltonian with 0 total momentum. We argue the Hamiltonian of arbitrary total momentum in this note. An immediate consequence is a rotational symmetry of the unique ground state. (In [23], [25], similar kind of the rotational symmetry for the Pauli–Fierz Hamiltonian has been shown by a method different from ours.)

Finally we observe the Fröhlich bipolaron without ultraviolet cutoffs in Section 7. This model describes two electrons in an ionic crystal. It is the most complicated model in this note and diffcult to treat. On the one hand, the interaction between the electrons and the ionic crystal induces an attraction between the electrons, on the other hand a repulsion from the Coulomb force between the two is also effective. Therefore the existence of a ground state depends on the competition between the effective attraction and the Coulomb repulsion and this is a hard issue. Recently the author and Spohn proved that the bipolaron Hamiltonian actually has a groud state under some suitable conditions [28]. In this note, we show that the heat semi-group generated by the Hamiltonian of 0 total momentum improves the positivity in generalized sense. Removal of cutoffs is also studied. To extend the result to nonzero total momentum case, we apply analytic perturbation theory.

In Appendices A and B, we list preliminary results of removal of ultraviolet cutoffs needed in Section 6 and Section 7.

2. POSITIVITY PRESERVING AND IMPROVING OPERATORS ON A HILBERT SPACE

2.1. BASIC DEFINITIONS.

In general we denote the inner product and the norm of a Hilbert space $\mathfrak{h}$ by $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ and $\| \cdot \|_{\mathfrak{h}}$ respectively. If there is no danger of confusion, then we omit the subscript $\mathfrak{h}$ in $\langle \cdot, \cdot \rangle_{\mathfrak{h}}$ and $\| \cdot \|_{\mathfrak{h}}$. For a linear operator a on a Hilbert space, we denote its domain by $\mathrm{dom}(a)$. For a self-adjoint operator b on a Hilbert space, we denote its spectrum (respectively essential spectrum) by $\mathrm{spec}(b)$ (respectively $\mathrm{ess.\,spec}(b)$).

Let $\mathfrak{h}$ be a complex Hilbert space and $\mathfrak{p}$ be a convex cone in $\mathfrak{h}$. The *dual cone* $\mathfrak{p}^\dagger$ is defined by

$$\mathfrak{p}^\dagger = \{x \in \mathfrak{h} : \langle x, y \rangle \geqslant 0 \, \forall y \in \mathfrak{p}\}.$$

If $\mathfrak{p} = \mathfrak{p}^\dagger$, then $\mathfrak{p}$ is called *self-dual*. (The author learned the materials in this subsection from [8].)

PROPOSITION 2.1. *A self-dual cone $\mathfrak{p}$ has the following properties:*

(i) $\mathfrak{p} \cap (-\mathfrak{p}) = \{0\}$.

(ii) *There exists a unique involution J in $\mathfrak{h}$ such that $Jx = x$ for all $x \in \mathfrak{p}$.*

(iii) *Each element $x \in \mathfrak{h}$ with $Jx = x$ has a unique decomposition $x = x_+ - x_-$ where $x_+, x_- \in \mathfrak{p}$ and $\langle x_+, x_- \rangle = 0$.*

(iv) $\mathfrak{h}$ *is linearly spanned by* $\mathfrak{p}$.

For the proof see e.g. [8], [17].

Let $\mathfrak{h}^J = \{x \in \mathfrak{h} : Jx = x\}$. Then $\mathfrak{h}^J$ is a real closed subspace of $\mathfrak{h}$. By Proposition 2.1(iv), for each $x \in \mathfrak{h}$, there is a unique decomposition $x = \Re x + \mathrm{i}\Im x$ such that $\Re x, \Im x \in \mathfrak{h}^J$. Moreover $\Re x = (1/2)(\mathbb{1} + J)x, \Im x = (1/2\mathrm{i})(\mathbb{1} - J)x$ and $\|x\|^2 = \|\Re x\|^2 + \|\Im x\|^2$.

For each $x \in \mathfrak{h}^J$, the absolute value of x with respect to $\mathfrak{p}$ is defined as $|x|_{\mathfrak{p}} = x_+ + x_-$, where $x = x_+ - x_-$ is the decomposition of x in Proposition 2.1(iii). Clearly $\|x\| = \||x|_{\mathfrak{p}}\|$.

We will write $x \geqslant y$ (or $y \leqslant x$) with respect to $\mathfrak{p}$ if $x - y \in \mathfrak{p}$. The relation $\geqslant$ induces a structure of ordered Hilbert space on $\mathfrak{h}^J$. For $x, y \in \mathfrak{h}^J$, let us define $x \wedge y = y - (x-y)_-$ and $x \vee y = y + (x-y)_+$. We summarize some basic properties below:

LEMMA 2.2. *Let $x, y \in \mathfrak{h}^J$. The following properties hold:*

(i) $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$.

(ii) $-x_- = x \wedge 0$ *and* $x_+ = x \vee 0$.

(iii) $x \wedge y \leqslant x, y$ *with respect to* $\mathfrak{p}$ *and* $x \vee y \geqslant x, y$ *with respect to* $\mathfrak{p}$.

(iv) $x \vee y + x \wedge y = x + y$ *and* $x \vee y - x \wedge y = |x - y|_{\mathfrak{p}}$. *In particular* $x \wedge y \leqslant x \vee y$ *with respect to* $\mathfrak{p}$.

(v) $\|x \wedge y\|^2 + \|x \vee y\|^2 = \|x\|^2 + \|y\|^2$.

(iv) *Suppose that* $x, y \in \mathfrak{p}$. *Then* $\langle x, y\rangle = 0$ *if and only if* $x \wedge y = 0$.

Proof. (i) By the definition, $x \wedge y - y \wedge x = y - x - (x-y)_- + (y-x)_- = y - x - (y-x)_+ + (y-x)_- = 0$. Similary we can check that $x \vee y = y \vee x$.

(ii) is trivial.

(iii) From the definition it follows that $x \wedge y \leqslant y$ with respect to $\mathfrak{p}$. Moreover, by (i), we have $x \wedge y = y \wedge x = x - (y-x)_- \leqslant x$ with respect to $\mathfrak{p}$.

(iv) is easy to see and (v) is an immediate consequence of (iv).

(vi) By (iv) and (v), one has $\langle x, y\rangle = \langle x \wedge y, x \vee y\rangle$. Thus, if $\langle x, y\rangle = 0$, then $0 = \langle x \wedge y, x \vee y\rangle \geqslant \|x \wedge y\|^2$ which implies $x \wedge y = 0$. Conversely if $x \wedge y = 0$, then $\langle x, y\rangle = 0$ holds. ∎

Let A be a linear operator on $\mathfrak{h}$. We say that A is *J-real* if A satisfies $JA \subseteq AJ$. (For linear operators a and b, we write $a \subseteq b$ if $\mathrm{dom}(a) \subseteq \mathrm{dom}(b)$ and $ax = bx$ for all $x \in \mathrm{dom}(a)$.) For a self-adjoint operator A on $\mathfrak{h}$, one can check that A is J-real if and only if J commutes with the spectral measure for A: $E_A(S)J = JE_A(S)$ for all $S \in \mathbb{B}^1$, the Borel field of $\mathbb{R}$.

PROPOSITION 2.3. *Let A be a J-real operator on $\mathfrak{h}$. Assume that A is positive and self-adjoint. Set $A_J = A \upharpoonright \mathrm{dom}(A) \cap \mathfrak{h}^J$. Then the following properties hold:*

(i) A_J *is a positive self-adjoint operator on* $\mathfrak{h}^J$.

(ii) $E_{A_J}(S) = E_A(S) \upharpoonright \mathfrak{h}^J$ *for all* $S \in \mathbb{B}^1$. *Moreover, for any real Borel measurable function on* $\mathbb{R}$, $f(A_J) = f(A) \upharpoonright \mathfrak{h}^J$.

(iii) *If μ is an eigenvalue of A, then μ is also an eigenvalue of A_J. Moreover,* $\dim\ker(A - \mu) = \dim\ker(A_J - \mu)$.

(iv) *If A is bounded, so is A_J with* $\|A\| = \|A_J\|$.

Proof. (i) A is self-adjoint if and only if $\mathrm{ran}(A+1) = \mathfrak{h}$. From this it follows that $\mathrm{ran}(A_J + 1) = \mathfrak{h}^J$ which is equivalent to the self-adjointness of A_J.

(ii) For all $x, y \in \mathfrak{h}^J$, $\langle x, A_J y\rangle = \int \lambda \mathrm{d}\langle x, E_A(\lambda) y\rangle = \int \lambda \mathrm{d}\langle x, E_A(\lambda) \upharpoonright \mathfrak{h}^J y\rangle$. Thus $E_A \upharpoonright \mathfrak{h}^J$ is the spectral measure of A_J, i.e., $E_A \upharpoonright \mathfrak{h}^J = E_{A_J}$. The remaining assertion is a direct consequence of this fact.

(iii) Let x be a corresponding eigenvector: $Ax = \mu x$. Then we can write x as $x = \Re x + \mathrm{i}\Im x$. Since A is J-real, we can check that $A\Re x = \mu \Re x$ and $A\Im x = \mu\Im x$. Thus $\Re x$ and $\Im x$ are both eigenvectors of A_J, with the eigenvalue μ, which implies that $\dim\ker(A_J - \mu) \geqslant \dim\ker(A-\mu)$. The converse inequlity is trivial.

(iv) Notice that, since A is J-real, we have $\|Ax\|^2 = \|A\Re x\|^2 + \|A\Im x\|^2 \leqslant \|A_J\|^2\|x\|^2$. Hence $\|A\| \leqslant \|A_J\|$. The converse inequality is trivial. ∎

2.2. POSITIVITY PRESERVING OPERATORS. Let A and B be linear operators on $\mathfrak{h}$. If A and B satisfy $(A-B)[\mathfrak{p} \cap \mathrm{dom}(A) \cap \mathrm{dom}(B)] \subseteq \mathfrak{p}$, then we write $A \trianglerighteq B$ (or $B \trianglelefteq A$) with respect to $\mathfrak{p}$. If A satisfies $0 \trianglelefteq A$ with respect to $\mathfrak{p}$, then A is said to be *positivity preserving with respect to* $\mathfrak{p}$. Remark that a set of all positivity preserving operators $\mathfrak{B}(\mathfrak{h})^+_{\mathfrak{p}} = \{A \in \mathfrak{B}(\mathfrak{h}) : 0 \trianglelefteq A \text{ with respect to } \mathfrak{p}\}$ is a cone and closed under the weak operator topology, where $\mathfrak{B}(\mathfrak{h})$ is the set of all bounded operators on $\mathfrak{h}$.

This operator inequality was first introduced by Y. Miura [27]. Some interesting properties are investigated in [26], [27].

PROPOSITION 2.4. *Suppose that* $0 \trianglelefteq A_1 \trianglelefteq B_1$ *and* $0 \trianglelefteq A_2 \trianglelefteq B_2$ *with respect to* $\mathfrak{p}$. *The following are satisfied:*

(i) $0 \trianglelefteq A_1A_2$ *with respect to* $\mathfrak{p}$. *Moreover if* $A_1, B_1 \in \mathfrak{B}(\mathfrak{h})$, *then* $0 \trianglelefteq A_1A_2 \trianglelefteq B_1B_2$ *with respect to* $\mathfrak{p}$.

(ii) $0 \trianglelefteq aA_1 + bA_2 \trianglelefteq aB_1 + bB_2$ *with respect to* $\mathfrak{p}$, *for all* $a, b \in \mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$.

(iii) *Let* A *be positivity preserving:* $0 \trianglelefteq A$ *with respect to* $\mathfrak{p}$. *Suppose that* $\mathfrak{p} \cap \mathrm{dom}(A)$ *is dense in* $\mathfrak{p}$. *Then* $0 \trianglelefteq A^*$ *with respect to* $\mathfrak{p}$.

PROPOSITION 2.5. (i) *If* $A \in \mathfrak{B}(\mathfrak{h})^+_{\mathfrak{p}}$, *then* A *is* J*-real.*

(ii) *Let* A *be a positive self-adjoint operator. If* $0 \trianglelefteq \mathrm{e}^{-tA}$ *with respect to* $\mathfrak{p}$ *for all* $t \geqslant 0$, *then* A *is* J*-real.*

Proof. (i) Since $0 \trianglelefteq A$ with respect to $\mathfrak{p}$, we can show that $A\mathfrak{h}^J \subseteq \mathfrak{h}^J$. Thus for each $x \in \mathfrak{h}$, we have $AJx = A(\Re x - \mathrm{i}\Im x) = JAx$.

(ii) By (i), $\mathrm{e}^{-tA}J = J\mathrm{e}^{-tA}$ for all $t \geqslant 0$. Thus $t^{-1}(\mathbb{1} - \mathrm{e}^{-tA})Jx = Jt^{-1}(\mathbb{1} - \mathrm{e}^{-tA})x$ for all $x \in \mathrm{dom}(A)$. Taking $t \downarrow 0$, we conclude that $JA \subseteq AJ$. ∎

PROPOSITION 2.6. *Let* A *be a positive self-adjoint operator. Then* $0 \trianglelefteq \mathrm{e}^{-tA}$ *for all* $t \geqslant 0$ *if and only if* $0 \trianglelefteq (A+s)^{-1}$ *for all* $s > 0$.

Proof. We just note the following two elementary facts: $(A+s)^{-1} = \int_0^\infty \mathrm{d}\lambda \cdot \mathrm{e}^{-\lambda(A+s)}$ and $\mathrm{e}^{-tA} = \text{s-}\lim_{n\to\infty} (\mathbb{1} + tA/n)^{-n}$. ∎

The following theorem is an abstract version of Beurling–Deny criterion [7].

THEOREM 2.7. *Let A be a positive self-adjoint operator on $\mathfrak{h}$. Assume A is J-real. Then the following are equivalent:*

(i) $0 \trianglelefteq \mathrm{e}^{-tA}$ *for all* $t \geqslant 0$.

(ii) *If* $x \in \mathrm{dom}(A) \cap \mathfrak{h}^J$, *then* $|x|_{\mathfrak{p}} \in \mathrm{dom}(A^{1/2}) \cap \mathfrak{h}^J$ *and* $\langle |x|_{\mathfrak{p}}, A|x|_{\mathfrak{p}} \rangle \leqslant \langle x, Ax \rangle$.

(iii) *If* $x \in \mathrm{dom}(A) \cap \mathfrak{h}^J$, *then* $x_+ \in \mathrm{dom}(A^{1/2}) \cap \mathfrak{h}^J$ *and* $\langle x_+, Ax_+ \rangle \leqslant \langle x, Ax \rangle$.

(iv) *If* $x \in \mathrm{dom}(A) \cap \mathfrak{h}^J$, *then* $x_\pm \in \mathrm{dom}(A^{1/2}) \cap \mathfrak{h}^J$ *and*

$$\langle x_+, Ax_+ \rangle + \langle x_-, Ax_- \rangle \leqslant \langle x, Ax \rangle.$$

Proof is a slight modification of Theorem XIII.50 in [35].

PROPOSITION 2.8. *Let E be an orthogonal projection on $\mathfrak{h}$. Assume that $0 \trianglelefteq E$ and $0 \trianglelefteq E^\perp$ with respect to $\mathfrak{p}$, where $E^\perp = \mathbb{1} - E$. Then $E\mathfrak{p}$ is a self-dual cone in $E\mathfrak{h}$. Moreover let A be a linear operator on $\mathfrak{h}$ which is reduced by $E\mathfrak{h}$, that is, $EA \subseteq AE$. If $0 \trianglelefteq A$ with respect to $\mathfrak{p}$, then $0 \trianglelefteq A_E$ with respect to $E\mathfrak{p}$, where $A_E = A \restriction E\mathfrak{h}$.*

Proof. For each $x \in \mathfrak{h}^J$, we will show that

$$(Ex)_+ = Ex_+, \quad (Ex)_- = Ex_-. \tag{2.1}$$

Since $0 = \langle x_+, x_- \rangle = \langle Ex_+, Ex_- \rangle + \langle E^\perp x_+, E^\perp x_- \rangle$ and $0 \trianglelefteq E, E^\perp$ with respect to $\mathfrak{p}$, we conclude that $\langle Ex_+, Ex_- \rangle = 0 = \langle E^\perp x_+, E^\perp x_- \rangle$. Thus $Ex = Ex_+ - Ex_-$ with $Ex_\pm \in \mathfrak{p}$ and $\langle Ex_+, Ex_- \rangle = 0$. By the uniqueness of the decomposition (Proposition 2.1 (iii)), We conclude (2.1).

To show that $E\mathfrak{p} \subseteq (E\mathfrak{p})^\dagger$ is easy. So we concentrate our attention to the converse inclusion. For each $x \in (E\mathfrak{p})^\dagger = \{x \in E\mathfrak{h} : \langle x, Ey \rangle \geqslant 0 \,\forall y \in \mathfrak{p}\}$, there exists $\varphi \in \mathfrak{h}$ such that $x = E\varphi$. Note that, since $0 \trianglelefteq E$, we see that $E\Im\varphi = 0$. Thus without loss of generality, we may assume that $\varphi \in \mathfrak{h}^J$ and we can write $\varphi = \varphi_+ - \varphi_-$ with $\varphi_\pm \in \mathfrak{p}$ and $\langle \varphi_+, \varphi_- \rangle = 0$. Since $\langle E\varphi, Ey \rangle \geqslant 0$ for all $y \in \mathfrak{p}$, we have $\langle (E\varphi)_+ - (E\varphi)_-, y \rangle \geqslant 0$. Thus $(E\varphi)_-$ must equal to 0. Hence, applying (2.1), we have $x = E\varphi = (E\varphi)_+ = E\varphi_+$ which means $(E\mathfrak{p})^\dagger \subseteq E\mathfrak{p}$.

Let A be a linear operator satisfying the assumptions in the above proposition. Then, for all $x \in \mathrm{dom}(A_E) \cap E\mathfrak{p} = \mathrm{dom}(A) \cap E\mathfrak{p}$ and $y = Ev \in E\mathfrak{p}$ ($v \in \mathfrak{p}$), $\langle A_E x, y \rangle = \langle E(Ax), v \rangle$. Since $0 \trianglelefteq A, E$ with respect to $\mathfrak{p}$, we get $0 \leqslant E(Ax)$ with respect to $\mathfrak{p}$. Hence $\langle A_E x, y \rangle \geqslant 0$ which means $0 \trianglelefteq A_E$ with respect to $E\mathfrak{p}$. ∎

THEOREM 2.9. *Let A and B be positive self-adjoint operators. We assume the following:*

(a) $\mathrm{dom}(A) = \mathrm{dom}(B)$.

(b) $(A+s)^{-1} \trianglerighteq 0$ *and* $(B+s)^{-1} \trianglerighteq 0$ *with respect to* $\mathfrak{p}$ *for all* $s > 0$.

Then the following are equivalent to each other:

(i) $B \trianglerighteq A$ *with respect to* $\mathfrak{p}$.

(ii) $(A+s)^{-1} \trianglerighteq (B+s)^{-1}$ *with respect to* $\mathfrak{p}$ *for all* $s > 0$.

(iii) $\mathrm{e}^{-tA} \trianglerighteq \mathrm{e}^{-tB}$ *with respect to* $\mathfrak{p}$ *for all* $t \geqslant 0$.

Proof. (i) ⇒ (ii) By the assumptions (a) and (b), we see that

$$(A+s)^{-1}-(B+s)^{-1}=(A+s)^{-1}(B-A)(B+s)^{-1}\trianglerighteq 0.$$

(ii) ⇒ (iii) One observes that $\mathrm{e}^{-tA} = \text{s-}\lim_{n\to\infty}(\mathbb{1}+tA/n)^{-n}\trianglerighteq \text{s-}\lim_{n\to\infty}(\mathbb{1}+tB/n)^{-n}=\mathrm{e}^{-tB}$.

(iii) ⇒ (ii) $(A+s)^{-1}=\int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{-\lambda(A+s)}\trianglerighteq\int_0^\infty \mathrm{d}\lambda\, \mathrm{e}^{-\lambda(B+s)}=(B+s)^{-1}$.

(iii) ⇒ (i) $A=\text{s-}\lim_{t\downarrow 0}(\mathbb{1}-\mathrm{e}^{-tA})/t\trianglelefteq\text{s-}\lim_{t\downarrow 0}(\mathbb{1}-\mathrm{e}^{-tB})/t=B$. ∎

THEOREM 2.10. *Let A be a positive self-adjoint operator and let B be a symmetric operator. Assume the following:*

(i) *B is A-bounded with relative bound $a<1$, i.e., $\mathrm{dom}(A)\subseteq\mathrm{dom}(B)$ and $\|Bx\|\leqslant a\|Ax\|+b\|x\|$ for all $x\in\mathrm{dom}(A)$.*

(ii) *$0\trianglelefteq\mathrm{e}^{-tA}$ with respect to $\mathfrak{p}$ for all $t\geqslant 0$.*

(iii) *$0\trianglelefteq -B$ with respect to $\mathfrak{p}$.*

Then $0\trianglelefteq\mathrm{e}^{-t(A+B)}$ with respect to $\mathfrak{p}$ for all $t\geqslant 0$.

Proof. Let $C=A+B$. Then by the assumptions we see that $\mathrm{dom}(A)=\mathrm{dom}(C)$ and $A-C=-B\trianglerighteq 0$. Thus applying Theorem 2.9, one obtains $\mathrm{e}^{-tC}\trianglerighteq\mathrm{e}^{-tA}\trianglerighteq 0$. ∎

Second proof. By (i) and the Kato–Rellich theorem [34], $A+B$ is self-adjoint and bounded from below. Applying the Duhamel formula, we have

$$\begin{aligned}(2.2)\quad &\mathrm{e}^{-t(A+B)}\\ &=\mathrm{e}^{-tA}+\sum_{n=1}^{\infty}\int_0^t\mathrm{d}s_1\int_0^{t-s_1}\mathrm{d}s_2\cdots\int_0^{t-\sum_{j=1}^{n-1}s_j}\mathrm{d}s_n\mathrm{e}^{-s_1A}(-B)\mathrm{e}^{-s_2A}(-B)\cdots\mathrm{e}^{-s_nA}(-B)\mathrm{e}^{-(t-\sum_{j=1}^{n}s_j)A}.\end{aligned}$$

Each term in the above expansion is positivity preserving with respect to $\mathfrak{p}$ by (ii) and (iii) which means $0\trianglelefteq\mathrm{e}^{-t(A+B)}$ with respect to $\mathfrak{p}$. ∎

2.3. POSITIVITY IMPROVING OPERATORS. $x\in\mathfrak{p}$ is *strictly positive* if $\langle y,x\rangle>0$ for all $y\in\mathfrak{p}\backslash\{0\}$ and we write this as $x>0$ (or $0<x$) with respect to $\mathfrak{p}$. Let $\mathfrak{p}_0=\{x\in\mathfrak{p}: x>0 \text{ with respect to } \mathfrak{p}\}$. Let A and B be bounded operators on $\mathfrak{h}$. If these operators satisfy $(A-B)\mathfrak{p}\backslash\{0\}\subseteq\mathfrak{p}_0$, then we will write $A\triangleright B$ (or $B\triangleleft A$) with respect to $\mathfrak{p}$. We say that A *improves the positivity* with respect to $\mathfrak{p}$ if $0\triangleleft A$ with respect to $\mathfrak{p}$.

PROPOSITION 2.11. *Let $A,B\in\mathfrak{B}(\mathfrak{h})$ with $0\triangleleft A$ and $0\trianglelefteq B$ with respect to $\mathfrak{p}$. Then we have the following properties:*

(i) *$0\triangleleft A^*$ with respect to $\mathfrak{p}$.*

(ii) *Suppose that $\ker B^{\#}=\{0\}$ with $a^{\#}=a$ or a^*. Then $0\triangleleft AB$ and $0\triangleleft BA$ with respect to $\mathfrak{p}$.*

(iii) *$0\triangleleft aA+bB$ with respect to $\mathfrak{p}$ for $a>0$ and $b\geqslant 0$.*

THEOREM 2.12 (Faris). *Let A be a positive self-adjoint operator on $\mathfrak{h}$. Suppose that $0 \trianglelefteq e^{-tA}$ with respect to $\mathfrak{p}$ for all $t \geqslant 0$ and* $\inf \mathrm{spec}(A)$ *is an eigenvalue. Then the following are equivalent:*

(i) $\inf \mathrm{spec}(A)$ *is a simple eigenvalue with a strictly positive eigenvector with respect to $\mathfrak{p}$.*

(ii) $0 \lhd (A+s)^{-1}$ *for some $s > 0$.*

(iii) *For all $x, y \in \mathfrak{p} \backslash \{0\}$, there exists a $t > 0$ such that $0 < \langle x, e^{-tA} y\rangle$.*

(iv) $0 \lhd (A+s)$ *for all $s > 0$.*

(v) $0 \lhd e^{-tA}$ *for all $t > 0$.*

Proof. Since A is J-real by Proposition 2.5(ii), we only consider the self-adjoint operator $A_J = A \restriction \mathfrak{h}^J$ by Proposition 2.3. For this A_J we can apply the Faris's results [13] and obtain the equivalence between (i), (ii) and (iii). The proof of (iv) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (iii) are trivial. To show (iii) $\Rightarrow$ (iv), we just note that $(A+s)^{-1} = \int_0^\infty e^{-\lambda(A+s)}\, d\lambda$.

(iii) $\Rightarrow$ (v) This part is a modification of [35]. For $x, y \in \mathfrak{p}\backslash\{0\}$, set $D_{x,y} = \{t > 0 : \langle x, e^{-tA} y\rangle > 0\}$. Then by the assumption $D_{x,y}$ is not empty. Choose $t \in D_{x,y}$ arbitrarily. Then by Lemma 2.2(vi), one has $x \wedge (e^{-tA} y) \neq 0$. Hence, for any $s > 0$, we have $\langle x, e^{-(s+t)A} y\rangle \geqslant \langle x \wedge (e^{-tA} y), e^{-sA} x \wedge (e^{-tA} y)\rangle = \|e^{-sA/2}\{x \wedge (e^{-tA} y)\}\|^2 > 0$. This means $s + t \in D_{x,y}$. Since s is arbitrary, we can conclude that $D_{x,y} = (a, \infty)$ with $a = \inf D_{x,y}$. Next we will show $a = 0$. Let $f(t) = \langle x, e^{-tA} y\rangle$. Then the function f is analytic in a neighborhood of the interval (a, ∞). Thus the point a must equal 0 otherwise f is zero on the connected set containing a which contradicts the obtained result $D_{x,y} = (a, \infty)$. ∎

PROPOSITION 2.13. *Let A be positive and self-adjoint. Assume that* (1) $0 \lhd e^{-tA}$ *for all $t > 0$,* (2) $Ax = \inf \mathrm{spec}(A) x$. *Let U be a positivity preserving, unitary operator commuting with A. Then $Ux = x$.*

Proof. We can assume that $x > 0$ with respect to $\mathfrak{p}$ by Proposition 2.12. Since U commutes with A, we have $AUx = \inf \mathrm{spec}(A) Ux$. By the uniqueness (Proposition 2.12(i)), $Ux = Cx$ with $C \in \mathbb{C}$ and $|C| = 1$. Since $0 \trianglelefteq U$, we can conclude $C = 1$. ∎

THEOREM 2.14. *Let H and H_0 be self-adjoint operators, bounded from below. Assume the following conditions:*

(i) *There exists a sequence of bounded operators V_n such that $H_0 + V_n$ converges to H in the strong resolvent sense and $H - V_n$ converges to H_0 in the strong resolvent sense.*

(ii) *For all $n \in \mathbb{N}$ and $t \geqslant 0$, $0 \trianglelefteq e^{-tV_n}$ with respect to $\mathfrak{p}$ holds.*

(iii) *For all $u, v \in \mathfrak{p}$ such that $\langle u, v\rangle = 0$, $\langle e^{-tV_n} u, v\rangle = 0$ holds for all $n \in \mathbb{N}$ and $t \geqslant 0$.*

(iv) $0 \lhd e^{-tH_0}$ *with respect to $\mathfrak{p}$ for all $t > 0$.*

Then we obtain $0 \lhd e^{-tH}$ with respect to $\mathfrak{p}$, for all $t > 0$.

The proof of this theorem is a slight modification of that of Theorem 3 in[13].

THEOREM 2.15. *Let A be a positive self-adjoint operator and let B be a symmetric operator. Set*

$$\mathcal{A}^{(n)}_{\varphi,\psi}(s_1,\dots,s_n;t) = \langle \varphi, \mathrm{e}^{-s_1 A}(-B)\mathrm{e}^{-s_2 A}(-B)\cdots \mathrm{e}^{-s_n A}(-B)\mathrm{e}^{-(t-\sum_{j=1}^n s_j)A}\psi\rangle$$

with $\mathcal{A}^{(0)}_{\varphi,\psi}(t) = \langle \varphi, \mathrm{e}^{-tA}\psi\rangle$. Assume the conditions (i)–(iii) *in Theorem* 2.10. *In addition we assume the following:*

(iv) *For each $x, y \in \mathfrak{p}\backslash\{0\}$ and $t > 0$, there exist an $n \in \mathbb{N}_0$ and $s_1,\dots,s_n \in \mathbb{R}_+$ with $0 \leqslant s_1 + \cdots + s_n \leqslant t$ such that $\mathcal{A}^{(n)}_{\varphi,\psi}(s_1,\dots,s_n;t) > 0$. (These n and $s_1,\dots,s_n$ could depend on φ and ψ.)*
Then $0 \lhd \mathrm{e}^{-t(A+B)}$ with respect to $\mathfrak{p}$ for all $t > 0$.

Proof. Note first that $\mathcal{A}^{(n)}_{\varphi,\psi}(s_1,\dots,s_n;t)$ is continuous in $s_1,\dots,s_n$. Thus, by the Duhamel formula (2.2), we have

$$\langle \varphi, \mathrm{e}^{-t(A+B)}\psi\rangle \geqslant \int_0^t \mathrm{d}s_1 \int_0^{t-s_1} \mathrm{d}s_2 \cdots \int_0^{t-\sum_{j=1}^{n-1} s_j} \mathrm{d}s_n \mathcal{A}^{(n)}_{\varphi,\psi}(s_1,\dots,s_n;t) > 0$$

or $\langle \varphi, \mathrm{e}^{-t(A+B)}\psi\rangle \geqslant \mathcal{A}^{(0)}_{\varphi,\psi}(t) > 0$, for all $t > 0$. ∎

THEOREM 2.16. *Let $\{A_n\}_{n\in\mathbb{N}}$ be a family of positive self-adjoint operators. Suppose that A_n converges to A in the strong resolvent sense as $n \to \infty$, where A is self-adjoint and positive. Moreover $\{A_n\}$ satisfies the following:*

(i) *For each $n \in \mathbb{N}$, $0 \lhd (A_n + \mathbb{1})^{-1}$ with respect to $\mathfrak{p}$.*
(ii) *For each $m, n \in \mathbb{N}$, $\mathrm{dom}(A_m) = \mathrm{dom}(A_n)$.*
(iii) *For each $m, n \in \mathbb{N}$ with $n \geqslant m$, $A_m - A_n \trianglerighteq 0$ with respect to $\mathfrak{p}$.*
Then we have $0 \lhd \mathrm{e}^{-tA}$ with respect to $\mathfrak{p}$ for all $t > 0$.

Proof. Choose $m, n \in \mathbb{N}$ as $n \geqslant m$ and write $B = A_m$ for simplicity. Applying Theorem 2.9 and assumptions, one has $(A_n + \mathbb{1})^{-1} \trianglerighteq (B + \mathbb{1})^{-1}$ for all $n \geqslant m$. Taking the limit $n \to \infty$, one sees

$$(A + \mathbb{1})^{-1} \trianglerighteq (B + \mathbb{1})^{-1}.$$

Since $0 \lhd (B + \mathbb{1})^{-1}$, we conclude that $0 \lhd (A + \mathbb{1})^{-1}$. ∎

2.4. DIRECT SUMS OF SELF-DUAL CONES. Let $\mathfrak{h} = \bigoplus_{n\in L} \mathfrak{h}_n$ and let $\mathfrak{p}_n$ $(n \in L)$ be a self-dual cone in $\mathfrak{h}_n$. Then, we can directly check that

$$\mathfrak{P} = \Big\{ x = \bigoplus_{n\in L} x_n \in \mathfrak{h} : x_n \in \mathfrak{p}_n \, \forall n \in L \Big\}$$

is also self-dual [8]. We write this dual cone as $\mathfrak{P} = \bigoplus_{n\in L} \mathfrak{p}_n$. We summarize the basic properties of $\bigoplus_{n\in L} \mathfrak{p}_n$ below.

PROPOSITION 2.17. *Let* $\mathfrak{P} = \bigoplus_{n\in L} \mathfrak{p}_n$. *Then we have the following properties:*

(i) *Let* J_n *be the involution with respect to* $\mathfrak{p}_n$. *Then* $J = \bigoplus_{n\in L} J_n$ *is the involution with respect to* $\mathfrak{P}$.

(ii) *Let* $x = \bigoplus_{n\in L} x_n \in \mathfrak{h}^J$ *and let* x_+ *and* x_- *be the positive and negative parts of* x *with respect to* $\mathfrak{P}$: $x = x_+ - x_-$ *with* $x_+, x_- \in \mathfrak{P}$ *and* $\langle x_+, x_-\rangle = 0$. *Then* $x_+ = \bigoplus_{n\in L} x_{n,+}$ *and* $x_- = \bigoplus_{n\in L} x_{n,-}$, *where* $x_{n,+}$ *and* $x_{n,-}$ *are positive and negative parts of* x_n *with respect to* $\mathfrak{p}_n$. *Moreover* $|x|_{\mathfrak{P}} = \bigoplus_{n\in L} |x_n|_{\mathfrak{p}_n}$.

(iii) *Let* $x = \bigoplus_{n\in L} x_n \in \mathfrak{h}$ *and let* $\Re x$ *and* $\Im x$ *be its real and imaginary parts with respect to* $\mathfrak{P}$ *respectively. Then* $\Re x = \bigoplus_{n\in L} \Re x_n$ *and* $\Im x = \bigoplus_{n\in L} \Im x_n$, *where* $\Re x_n$ *and* $\Im x_n$ *are real and imaginary parts of* x_n *with respect to* $\mathfrak{p}_n$. *Moreover,* $\|x\|^2 = \|\Re x\|^2 + \|\Im x\|^2 = \sum_{n\in L} (\|\Re x_n\|^2 + \|\Im x_n\|^2)$.

PROPOSITION 2.18. *Let* $\mathfrak{P} = \bigoplus_{n\in L} \mathfrak{p}_n$. *Let* A_n *be a linear operator on* $\mathfrak{h}_n$. *Then* $A = \bigoplus_{n\in L} A_n \trianglerighteq 0$ *with respect to* $\mathfrak{P}$ *if and only if* $A_n \trianglerighteq 0$ *with respect to* $\mathfrak{p}_n$ *for all* $n \in L$.

2.5. DIRECT INTEGRALS OF SELF-DUAL CONES. Let $(\mathcal{Z}, \mu, \mathcal{B})$ be a Borel space and let $\mathfrak{k}$ be a fixed Hilbert space. Let $\mathfrak{h} = L^2(\mathcal{Z}, \mathfrak{k}) = \int_{\mathcal{Z}}^{\oplus} \mathfrak{k}\, \mathrm{d}\mu$. For a self-dual cone $\mathfrak{p}$ in $\mathfrak{k}$, we set

$$\mathfrak{P} = \{x \in \mathfrak{h} : x(z) \in \mathfrak{p}\ \mu\text{-a.e.}\}.$$

Then $\mathfrak{P}$ is also self-dual and denoted by $\mathfrak{P} = \int_{\mathcal{Z}}^{\oplus} \mathfrak{p}\, \mathrm{d}\mu$. We restrict our attention to direct integrals of constant fields of a Hilbert space in this note, however we can also treat more general situation by using the terminologies developed in [8]. We summarize the fundamental properties of $\mathfrak{P} = \int_{\mathcal{Z}}^{\oplus} \mathfrak{p}\, \mathrm{d}\mu$.

PROPOSITION 2.19. (i) *Let* J *be the involution associated with* $\mathfrak{p}$. *Then* $J^{\oplus} = \int_{\mathcal{Z}}^{\oplus} J\, \mathrm{d}\mu$ *is the involution associated with* $\mathfrak{P} = \int_{\mathcal{Z}}^{\oplus} \mathfrak{p}\, \mathrm{d}\mu$.

(ii) *Let* $x = \int_{\mathcal{Z}}^{\oplus} x(z)\, \mathrm{d}\mu \in \mathfrak{h}^{J^{\oplus}}$ *and let* x_+ *and* x_- *be the positive and negative parts of* x *with respect to* $\mathfrak{P}$: $x = x_+ - x_-$. *Then* $x_\pm = \int_{\mathcal{Z}}^{\oplus} x(z)_\pm\, \mathrm{d}\mu$, *where* $x(z)_\pm$ *are positive and negative parts of* $x(z)$.

(iii) *Let* $x = \int_{\mathcal{Z}}^{\oplus} x(z)\,\mathrm{d}\mu \in \mathfrak{h}$ *and let* $\Re x$ *and* $\Im x$ *be its real and imaginary parts with respect to* $\mathfrak{P} = \int_{\mathcal{Z}}^{\oplus} \mathfrak{p}\,\mathrm{d}\mu$ *respectively. Then* $\Re x = \int_{\mathcal{Z}}^{\oplus} \Re x(z)$ *and* $\Im x = \int_{\mathcal{Z}}^{\oplus} \Im x(z)\,\mathrm{d}\mu$*. Moreover* $\|x\|^2 = \|\Re x\|^2 + \|\Im x\|^2 = \int_{\mathcal{Z}} (\|\Re x(z)\|^2 + \|\Im x(z)\|^2)\,\mathrm{d}\mu$.

A bounded operator A on $\mathfrak{h} = \int_{\mathcal{Z}}^{\oplus} \mathfrak{k}\,\mathrm{d}\mu$ is said to be *diagonalizable* if there exists a function $f \in L^\infty(\mathcal{Z})$ such that $(A\varphi)(z) = f(z)\varphi(z)$ μ-a.e., for each $\varphi \in \mathfrak{h}$.

Let $\mathfrak{A}$ be the abelian von Neumann algebra of diagonalizable operators. Let A be a closed operator on $\mathfrak{h} = \int_{\mathcal{Z}}^{\oplus} \mathfrak{k}\,\mathrm{d}\mu$. We say that A is *decomposable* if $BA \subseteq AB$ for all $B \in \mathfrak{A}$. If A is decomposable, then there exists a closed operator valued map $A(z)$ such that $(A\varphi)(z) = A(z)\varphi(z)$ μ-a.e., for all $\varphi \in \mathrm{dom}(A)$. We often write this as $A = \int_{\mathcal{Z}}^{\oplus} A(z)\,\mathrm{d}\mu$. Moreover A^* is also decomposable and $A^* = \int_{\mathcal{Z}}^{\oplus} A(z)^*\,\mathrm{d}\mu$. Readers can find more precise discussions of the decomposable operators in [36].

PROPOSITION 2.20. *Let* $A = \int_{\mathcal{Z}}^{\oplus} A(z)\,\mathrm{d}\mu$ *be a decomposable operator on* $\mathfrak{h} = \int_{\mathcal{Z}}^{\oplus} \mathfrak{k}\,\mathrm{d}\mu$*. If* $0 \trianglelefteq A(z)$ *with respect to* $\mathfrak{p}$ *for* μ*-a.e., then* $0 \trianglelefteq A$ *with respect to* $\mathfrak{P} = \int_{\mathcal{Z}}^{\oplus} \mathfrak{p}\,\mathrm{d}\mu$.

EXAMPLE 2.21. Let us consider a special case: $\mathfrak{h} = \int_{\mathbb{R}^d}^{\oplus} \mathfrak{k}\,\mathrm{d}x$. Let $x \to A(x)$ be a closed operator valued map with the following properties:

(i) There exists a dense subspace $\mathcal{D}$ of $\mathfrak{k}$ such that $\mathcal{D}$ is a common core of $A(x)^\#$ for all $x \in \mathbb{R}^d$.

(ii) For all $\varphi \in \mathcal{D}$, $A(x)^\#\varphi$ is strongly continuous in x.

Under these conditions, we define a linear operator A_0 by $(A_0\varphi)(x) = A(x)\varphi(x)$ for $\varphi \in \mathrm{dom}(A_0) = C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}$, where we use the identification $\mathfrak{h} = L^2(\mathbb{R}^d) \otimes \mathfrak{k}$. Clearly A_0 is closable. Now we define a closed operator A by $A = A_0^{**}$. Then A and A^* are both decomposable and $A^\# = \int_{\mathbb{R}^d}^{\oplus} A(x)^\#\,\mathrm{d}x$.

3. QUANTIZED OPERATORS

3.1. DEFINITIONS. Let $\mathfrak{h}$ be a complex Hilbert space. The Boson Fock space over $\mathfrak{h}$ is given by

$$\mathfrak{F}(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^n \mathfrak{h},$$

where $\otimes_s^n \mathfrak{h}$ denotes the n-fold symmetric tensor product of $\mathfrak{h}$ and $\otimes_s^0 \mathfrak{h} = \mathbb{C}$. The vector $\Omega = 1 \oplus 0 \oplus 0 \oplus \cdots \in \mathfrak{F}(\mathfrak{h})$ is called the Fock vacuum. We identify each vector $\varphi \in \otimes_s^n \mathfrak{h}$ with the corresponding vector $\bigoplus_{j=0}^{\infty} \delta_{j,n}\varphi$ in $\mathfrak{F}(\mathfrak{h})$. Under this identification, $\otimes_s^n \mathfrak{h}$ is a closed subspace of $\mathfrak{F}(\mathfrak{h})$. We denote by $a(f)$ $(f \in \mathfrak{h})$ the annihilation operator with index vector f on $\mathfrak{F}(\mathfrak{h})$. Its adjoint, called the creation operator, is given by

$$(a(f)^*\varphi)^{(n)} = \sqrt{n} S_n (f \otimes \varphi^{(n-1)}), \tag{3.1}$$

for $\varphi = \bigoplus_{n=0}^{\infty} \varphi^{(n)} \in \mathrm{dom}(a(f)^*)$, where S_n is the symmetrizer on $\otimes^n \mathfrak{h}$ and $\varphi^{(-1)} = 0$. The creation and annihilation operators satisfy the canonical commutation relations (CCRs):

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0 = [a(f)^*, a(g)^*],$$

on a suitable dense domain. In the case of $\mathfrak{h} = L^2(\mathbb{R}^d)$, we often use the symbolic notation for the annihilation and creation operators by the kernel:

$$a(f) = \int_{\mathbb{R}^d} \mathrm{d}k\, f(k)^* a(k), \quad a(f)^* = \int_{\mathbb{R}^d} \mathrm{d}k\, f(k) a(k)^*.$$

Let $\mathfrak{s}$ be a subspace of $\mathfrak{h}$. We define

$$\mathfrak{F}_{\mathrm{fin}}(\mathfrak{s}) = \mathrm{Lin}\{a(f_1)^* \cdots a(f_n)^*\Omega,\ \Omega : f_1, \dots, f_n \in \mathfrak{s},\ n \in \mathbb{N}\},$$

where $\mathrm{Lin}\{\cdots\}$ means the linear span of the set $\{\cdots\}$. If $\mathfrak{s}$ is dense in $\mathfrak{h}$, $\mathfrak{F}_{\mathrm{fin}}(\mathfrak{s})$ is also dense in $\mathfrak{F}(\mathfrak{h})$.

Let C be a contraction operator from $\mathfrak{h}_1$ to $\mathfrak{h}_2$, i.e., $\|C\| \leqslant 1$. The linear operator $\Gamma(C) : \mathfrak{F}(\mathfrak{h}_1) \to \mathfrak{F}(\mathfrak{h}_2)$ is defined by the following, with the convention $\otimes^0 C = \mathbb{1}$:

$$\Gamma(C) \restriction \otimes_s^n \mathfrak{h}_1 = \otimes^n C.$$

For a densely defined closable operator A on $\mathfrak{h}$, $\mathrm{d}\Gamma(A) : \mathfrak{F}(\mathfrak{h}) \to \mathfrak{F}(\mathfrak{h})$ is defined by

$$\mathrm{d}\Gamma(A) \restriction \otimes_s^n \mathrm{dom}(A) = \sum_{j=1}^{n} \mathbb{1} \otimes \cdots \otimes \underset{j\,\mathrm{th}}{A} \otimes \cdots \otimes \mathbb{1}$$

and $\mathrm{d}\Gamma(A)\Omega = 0$. Clearly $\mathrm{d}\Gamma(A)$ is closable and we denote its closure by the same symbol. Also remark that if A is self-adjoint, then $\mathrm{d}\Gamma(A)$ is essentially self-adjoint. As a typical example, the number operator N_{f} is given by $N_{\mathrm{f}} = \mathrm{d}\Gamma(\mathbb{1})$. Also we note the following relation, for A which is positive and self-adjoint:

$$\Gamma(\mathrm{e}^{-tA}) = \mathrm{e}^{-t\mathrm{d}\Gamma(A)}, \quad t \geqslant 0. \tag{3.2}$$

3.2. SELF-DUAL CONES IN A FOCK SPACE. Let $\mathfrak{p}$ be a self-dual cone in $\mathfrak{h}$. Let

$$\mathfrak{P}_n = \{\varphi \in \otimes_{\mathrm{s}}^n \mathfrak{h} : \langle \varphi, x_1 \otimes \cdots \otimes x_n \rangle \geqslant 0 \, \forall x_1, \ldots, x_n \in \mathfrak{p}\}.$$

It is not hard to see that $\mathfrak{P}_n^\dagger \subseteq \mathfrak{P}_n$. Throughout remainder of this section, we **assume** that $\mathfrak{P}_n$ is a self-dual cone and denote it by $\otimes_{\mathrm{s}}^n \mathfrak{p}$.

EXAMPLE 3.1. Let $\mathfrak{h} = L^2(\mathbb{R}^d)$ and let $\mathfrak{p} = L^2(\mathbb{R}^d)_+ := \{f \in L^2(\mathbb{R}^d) : f(x) \geqslant 0 \text{ a.e.}\}$. Then $\otimes_{\mathrm{s}}^n \mathfrak{p}$ is self-dual and under the natural identification $\otimes_{\mathrm{s}}^n L^2(\mathbb{R}^d) = L^2_{\mathrm{sym}}(\mathbb{R}^{nd})$, the symmetric L^2-space, we have $\otimes_{\mathrm{s}}^n \mathfrak{p} = L^2_{\mathrm{sym}}(\mathbb{R}^{nd})_+ := \{f \in L^2_{\mathrm{sym}}(\mathbb{R}^{nd}) : f(X) \geqslant 0 \, \text{a.e.}\}$.

EXAMPLE 3.2. Let $\mathfrak{h} = L^2(\mathbb{R}^d)$. Let M_1 and M_2 be subsets of $\mathbb{R}^d$ such that $M_1 \cup M_2 = \mathbb{R}^d$ and $M_1 \cap M_2 = \emptyset$. Set $\mathfrak{p} = \{f \in L^2(\mathbb{R}^d) : f(x) \geqslant 0 \text{ on } M_1 \text{ and } f(x) \leqslant 0 \text{ on } M_2\}$. Then $\mathfrak{p}$ is a self-dual cone. Moreover $\otimes_{\mathrm{s}}^n \mathfrak{p}$ is also self-dual.

Let

$$\mathfrak{F}(\mathfrak{p}) = \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^n \mathfrak{p}$$

with $\otimes_{\mathrm{s}}^0 \mathfrak{p} = \mathbb{R}_+$. Then $\mathfrak{F}(\mathfrak{p})$ is self-dual by Proposition 2.17. Let J_n be the involution associated with $\otimes_{\mathrm{s}} \mathfrak{p}$. Then the involution associated with $\mathfrak{F}(\mathfrak{p})$ is given by $\Gamma(J) := j \oplus \Big[\bigoplus_{n=1}^{\infty} J_n \Big]$ where j is the natural involution on $\mathbb{C}$: $jz = z^*$ for $z \in \mathbb{C}$.

THEOREM 3.3. (i) *Let A be a contraction on $\mathfrak{h}$. If $0 \trianglelefteq A$ with respect to $\mathfrak{p}$, then $0 \trianglelefteq \Gamma(A)$ with respect to $\mathfrak{F}(\mathfrak{p})$.*

(ii) *Let A be a positive self-adjoint operator on $\mathfrak{h}$. If $0 \trianglelefteq \mathrm{e}^{-tA}$ with respect to $\mathfrak{p}$ for all $t \geqslant 0$, then $0 \trianglelefteq \mathrm{e}^{-t\mathrm{d}\Gamma(A)}$ with respect to $\mathfrak{F}(\mathfrak{p})$ for all $t \geqslant 0$.*

(iii) *If $0 \leqslant f$ with respect to $\mathfrak{p}$, then $0 \trianglelefteq a(f)$ and $0 \trianglelefteq a(f)^*$ with respect to $\mathfrak{F}(\mathfrak{p})$.*

(iv) *Let $A \in \mathfrak{B}(\mathfrak{h})$. If $0 \trianglelefteq A$ with respect to $\mathfrak{p}$, then $\mathrm{d}\Gamma(A)^n \restriction \otimes_{\mathrm{s}}^n \mathfrak{h} \trianglerighteq \otimes^n A \trianglerighteq 0$ with respect to $\otimes_{\mathrm{s}}^n \mathfrak{p}$.*

(v) *For $f \in \mathfrak{p}$, $(a(f)a(f)^*)^n \restriction \otimes_{\mathrm{s}}^n \mathfrak{h} \trianglerighteq (a(f)^* a(f))^n \restriction \otimes_{\mathrm{s}}^n \mathfrak{h} \trianglerighteq \otimes^n |f\rangle\langle f| \trianglerighteq 0$ with respect to $\otimes_{\mathrm{s}}^n \mathfrak{p}$, where $|f\rangle\langle f| x := \langle f, x\rangle f$ for $x \in \mathfrak{h}$.*

Proof. (i) For all $\varphi \in \otimes_{\mathrm{s}}^n \mathfrak{p}$ and $x_1, \ldots, x_n \in \mathfrak{p}$, we have $\langle \Gamma(A)\varphi, x_1 \otimes \cdots \otimes x_n \rangle = \langle \varphi, (Ax_1) \otimes \cdots \otimes (Ax_n) \rangle \geqslant 0$, since $0 \trianglelefteq A$ with respect to $\mathfrak{p}$. Thus $0 \trianglelefteq \Gamma(A) \restriction \otimes_{\mathrm{s}}^n \mathfrak{h}$ with respect to $\otimes_{\mathrm{s}}^n \mathfrak{p}$ for all $n \in \mathbb{N}_0$. Now applying Proposition 2.18, we have the desired result. The assertion (ii) is a direct consequence of (i) by (3.2).

(iii) Let P_n ($n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$) be the orthogonal projection onto the subspace $\bigoplus_{j=0}^{n} \otimes_{\mathrm{s}}^j \mathfrak{h}$. For each $\psi \in \mathfrak{F}(\mathfrak{p})$ and $\varphi \in \mathrm{dom}(a(f)) \cap \mathfrak{F}(\mathfrak{p})$, we have $\langle a(f)\varphi, P_n \psi \rangle = \langle \varphi, a(f)^* P_n \psi \rangle = \sum_{j=1}^{n} \sqrt{j} \langle \varphi^{(j)}, f \otimes \psi^{(j-1)} \rangle \geqslant 0$ for all $n \in \mathbb{N}_0$, by using (3.1).

Thus, noting $\text{s-}\lim_{n\to\infty} P_n = \mathbb{1}$, we have $\langle a(f)\varphi, \psi\rangle \geqslant 0$. Similarly we can see that $0 \trianglelefteq a(f)^*$ with respect to $\mathfrak{F}(\mathfrak{p})$.

Proof of (iv) is easy. For the proof of (v), we note that

$$\mathrm{d}\Gamma(|f\rangle\langle f|) = a(f)^* a(f)$$

for $f \in \mathfrak{p}$. Thus, applying (iv), we have $(a(f)^*a(f))^n \restriction \otimes_{\mathrm{s}}^n \mathfrak{h} \trianglerighteq \otimes^n |f\rangle\langle f| \trianglerighteq 0$. Moreover, by the CCRs, we have $a(f)a(f)^* = a(f)^*a(f) + \|f\|^2 \trianglerighteq a(f)^*a(f)$. ∎

4. WIGNER–WEISSKOPF MODEL

4.1. MAIN RESULTS IN SECTION 4. Let σ_+, σ_- and σ_3 be 2×2 matrices on $\mathbb{C}^2$ given by

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The Hamiltonian H of the Wigner–Weisskopf model is defined by

$$H = \frac{\mu}{2}(\mathbb{1} + \sigma_3)\otimes \mathbb{1} + \mathbb{1}\otimes \mathrm{d}\Gamma(\omega) - g\{\sigma_+ \otimes a(\varrho) + \sigma_- \otimes a(\varrho)^*\}$$

acting in $\mathbb{C}^2 \otimes \mathfrak{F}(L^2(\mathbb{R}^3_k))$, with $\mu > 0$, $g \in \mathbb{R}\backslash\{0\}$, $\omega(k) = |k|$ and $\varrho, \varrho/\omega^{1/2} \in L^2(\mathbb{R}^3_k)$. By the well-known bound $\|a(f)^\#(\mathrm{d}\Gamma(\omega) + \mathbb{1})^{-1/2}\| \leqslant \|\omega^{-1/2} f\|$ and the Kato–Rellich theorem, H is self-adjoint on $\mathrm{dom}(\mathbb{1}\otimes \mathrm{d}\Gamma(\omega))$, bounded from below for all μ and g. Without loss of generality, we can assume that $g > 0$ (because $\mathbb{1}\otimes \Gamma(\mathrm{e}^{\mathrm{i}\pi}) H_g \mathbb{1}\otimes \Gamma(\mathrm{e}^{-\mathrm{i}\pi}) = H_{-g}$).

Let N_{tot} be the total number operator defined by

$$N_{\mathrm{tot}} = \sigma_+\sigma_- \otimes \mathbb{1} + \mathbb{1}\otimes N_{\mathrm{f}}.$$

For each $n \in \mathbb{N}_0 := \{0\}\cup \mathbb{N}$, let $\mathcal{H}_n = \ker(N_{\mathrm{tot}} - n)$. Then we have the decomposition

$$\mathbb{C}^2 \otimes \mathfrak{F}(L^2(\mathbb{R}^3_k)) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n. \tag{4.1}$$

We can directly check that H srongly commutes with N_{tot}, that is, $\exp(\mathrm{i}sN_{\mathrm{tot}}) \exp(\mathrm{i}tH) = \exp(\mathrm{i}tH)\exp(\mathrm{i}sN_{\mathrm{tot}})$ for all $s, t \in \mathbb{R}$. Thus H is represented as a direct sum associated with (4.1):

$$H = \bigoplus_{n=0}^{\infty} H_n \quad \text{with } H_n = H \restriction \mathcal{H}_n.$$

Let $\mathfrak{p} = L^2(\mathbb{R}^3_k)_+$ and let $\eta_\uparrow = \binom{1}{0}$ and $\eta_\downarrow = \binom{0}{1}$. We introduce a subset $\mathfrak{P}_n$ of $\mathcal{H}_n$ by

$$\mathfrak{P}_n = \{\varphi \in \mathcal{H}_n : \varphi = \eta_\uparrow \otimes \varphi_{n-1} + \eta_\downarrow \otimes \varphi_n \text{ with } \varphi_{n-1} \in \otimes_{\mathrm{s}}^{n-1}\mathfrak{p} \text{ and } \varphi_n \in \otimes_{\mathrm{s}}^n \mathfrak{p}\}$$

with $\mathfrak{P}_0 = \{a\eta_\downarrow \otimes \Omega : a \in \mathbb{R}^+\}$. Then $\mathfrak{P}_n$ is a self-dual cone in $\mathcal{H}_n$ for all $n \in \mathbb{N}_0$. (For the proof note that, any element $F \in \mathcal{H}_n$ has a unique expression $F = \eta_\uparrow \otimes$

$F_\uparrow + \eta_\downarrow \otimes F_\downarrow$ with $F_\uparrow \in \otimes_{\mathrm{s}}^{n-1} L^2(\mathbb{R}_k^3)$ and $F_\downarrow \in \otimes_{\mathrm{s}}^{n} L^2(\mathbb{R}_k^3)$. Using this representation and self-duality of $\otimes_{\mathrm{s}}^n \mathfrak{p}$, F is in $\mathfrak{P}_n^\dagger$ if and only if $F_\uparrow \in \otimes_{\mathrm{s}}^{n-1}\mathfrak{p}$ and $F_\downarrow \in \otimes_{\mathrm{s}}^{n}\mathfrak{p}$.)

THEOREM 4.1. *Suppose that $\varrho, \varrho/\omega^{1/2} \in L^2(\mathbb{R}_k^3)$ and $\mu, g > 0$. Assume that $\varrho(k) > 0$ a.e. k. Then, for all $n \in \mathbb{N}_0$, $0 \lhd \mathrm{e}^{-tH_n}$ with respect to $\mathfrak{P}_\mathfrak{n}$ for all $t > 0$.*

COROLLARY 4.2. *Under the conditions in Theorem* 4.1, *assume that H has degenerate ground states with α-fold degeneracy. Then there exist $n_1, \dots, n_\alpha \in \mathbb{N}_0$ with $n_1 < n_2 < \cdots < n_\alpha$ such that each H_{n_j} $(j = 1, \dots, \alpha)$ has a unique ground state which is strictly positive with respect to $\mathfrak{P}_{n_j}$, and* $\inf\mathrm{spec}(H) = \inf\mathrm{spec}(H_{n_1}) = \cdots = \inf\mathrm{spec}(H_{n_\alpha})$.

COROLLARY 4.3. *Under the conditions in Theorem* 4.1, *assume that H has a ground state φ. Moreover assume that $\varphi \in \mathcal{H}_n$ for some $n \in \mathbb{N}_0$. Then it is a unique ground state for H_n and can be chosen to be strictly positive with respect to $\mathfrak{P}_n$.*

Combining this result with [19], we obtain the following.

COROLLARY 4.4. *Under the conditions in Theorem* 4.1, *assume that*

$$g^2 \int_{\mathbb{R}^3} \mathrm{d}k\, \frac{\varrho(k)^2}{|k|} \gg 1.$$

Then there exists an $n \geqslant 2$ such that H_n has a unique ground state $\varphi \in \mathcal{H}_n$. Moreover we can choose φ to be strictly positive with respect to $\mathfrak{P}_n$.

REMARK 4.5. The assumption $\varrho(k) > 0$ a.e. is just for the simplicity of our proof. We can treat more general functions. Namely let ϱ be a *real* valued function with $\varrho, \varrho/\omega^{1/2} \in L^2(\mathbb{R}_k^3)$. Set $S = \mathrm{supp}\varrho$ and write $L^2(\mathbb{R}_k^3) = L^2(S) \oplus L^2(S^{\mathrm{c}})$, where S^{c} is the complement of S. Then we have the natural identification $\mathfrak{F}(L^2(\mathbb{R}_k^3)) = \mathfrak{F}(L^2(S)) \otimes \mathfrak{F}(L^2(S^{\mathrm{c}}))$, thus $\mathbb{C}^2 \otimes \mathfrak{F}(L^2(\mathbb{R}_k^3)) = \mathbb{C}^2 \otimes \mathfrak{F}(L^2(S)) \otimes \mathfrak{F}(L^2(S^{\mathrm{c}}))$. Under this identification, we can represent H as

$$H = H_S \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega \restriction S^{\mathrm{c}})$$

with

$$H_S = \frac{\mu}{2}(\mathbb{1} + \sigma_3) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega \restriction S) - g\{\sigma_+ \otimes a_S(\varrho) + \sigma_- \otimes a_S(\varrho)^*\},$$

where $a_S(\cdot)$ and $a_S^*(\cdot)$ are the annihilation and creation operators on $\mathfrak{F}(L^2(S))$ respectively. Note that, to show the uniqueness of a ground state of H_n, it suffices to show that of $H_S \restriction \mathcal{H}_n(S)$ with $\mathcal{H}_n(S) = \ker(\sigma_+\sigma_- \otimes \mathbb{1} + \mathbb{1} \otimes N_S - n)$ where N_S is the number operator on $\mathfrak{F}(L^2(S))$. To this end, let $S_- = \{k \in S : \varrho(k) < 0\}$ and let χ_{S_-} be the characteristic function of the set S_-. Observe that

$$\begin{aligned}\mathbb{1} \otimes & \mathrm{e}^{\mathrm{i}\pi \mathrm{d}\Gamma(\chi_{S_-})} H_S \mathbb{1} \otimes \mathrm{e}^{-\mathrm{i}\pi \mathrm{d}\Gamma(\chi_{S_-})} \\ &= \frac{\mu}{2}(\mathbb{1} + \sigma_3) \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega \restriction S) - g\{\sigma_+ \otimes a_S(|\varrho|) + \sigma_- \otimes a_S(|\varrho|)^*\}.\end{aligned}$$

One can apply all arguments in this section to $\mathbb{1} \otimes e^{i\pi d\Gamma(\chi_{S_-})} H_S \mathbb{1} \otimes e^{-i\pi d\Gamma(\chi_{S_-})}$ because $|\varrho(k)| > 0$ a.e. k on S and obtain the corresponding uniqueness theorems.

4.2. PROOF OF THEOREM 4.1. Let $\mathfrak{p}_{\mathbb{C}^2} = \mathbb{R}^2_+$ be a natural self-dual cone in $\mathbb{C}^2$. Take a self-dual cone $\mathfrak{F}(\mathfrak{p})$ in $\mathfrak{F}(L^2(\mathbb{R}^3_k))$ with $\mathfrak{p} = L^2(\mathbb{R}^3_k)_+$. Now we choose a self-dual cone in $\mathbb{C}^2 \otimes \mathfrak{F}(L^2(\mathbb{R}^3_k))$ as

$$\mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p}) := \{\varphi \in \mathbb{C}^2 \otimes \mathfrak{F}(L^2(\mathbb{R}^3_k)) : \varphi = \eta_\uparrow \otimes \varphi_\uparrow + \eta_\downarrow \otimes \varphi_\downarrow \text{ with } \varphi_\uparrow, \varphi_\downarrow \in \mathfrak{F}(\mathfrak{p})\}.$$

(Indeed the reader can directly check the self-duality of $\mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})$ using the fact that, for each $\varphi \in \mathbb{C}^2 \otimes \mathfrak{F}(L^2(\mathbb{R}^3_k))$, there exist $\varphi_\uparrow, \varphi_\downarrow \in \mathfrak{F}(L^2(\mathbb{R}^3))$ such that $\varphi = \eta_\uparrow \otimes \varphi_\uparrow + \eta_\downarrow \otimes \varphi_\downarrow$.) Moreover we have the decomposition

$$\mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p}) = \bigoplus_{n=0}^{\infty} \mathfrak{P}_n.$$

LEMMA 4.6. *Let E_n $(n \in \mathbb{N}_0)$ be the orthogonal projection onto $\mathcal{H}_n$. Then we obtain the following:*

(i) $0 \trianglelefteq E_n, E_n^\perp$ *with respect to* $\mathfrak{F}(\mathfrak{p})$ *for all* $n \in \mathbb{N}_0$.

(ii) $\mathfrak{P}_n = E_n \mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})$ *for all* $n \in \mathbb{N}_0$.

Proof. (i) For each $\varphi \in \mathbb{C}^2 \otimes \mathfrak{F}(L^2(\mathbb{R}^3_k))$, we have the representation $\varphi = \eta_\uparrow \otimes \varphi_\uparrow + \eta_\downarrow \otimes \varphi_\downarrow$ with $\varphi_\uparrow, \varphi_\downarrow \in \mathfrak{F}(L^2(\mathbb{R}^3_k))$. Then we have

$$E_n \varphi = \eta_\uparrow \otimes \Big(\bigoplus_{j=0}^{\infty} \delta_{j,n-1} \varphi_\uparrow^{(j)}\Big) + \eta_\downarrow \otimes \Big(\bigoplus_{j=0}^{\infty} \delta_{j,n} \varphi_\downarrow^{(j)}\Big) \tag{4.2}$$

with $E_0 \varphi = \eta_\downarrow \otimes \Big(\bigoplus_{j=0}^{\infty} \delta_{j,0} \varphi^{(j)}\Big)$. On the other hand, $\varphi \in \mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})$ if and only if $\varphi_\uparrow, \varphi_\downarrow \in \mathfrak{F}(\mathfrak{p})$. Thus, by the formula (4.2), if $\varphi \in \mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})$, then $E_n \varphi \in \mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})$ which means $0 \trianglelefteq E_n$ with respect to $\mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})$. Similarly we can prove that $0 \trianglelefteq E_n^\perp$ with respect to $\mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})$. (ii) also follows from (4.2). ∎

LEMMA 4.7. *Let $K = \mu(\mathbb{1} + \sigma_3)/2 \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(\omega)$ and let $L = \sigma_+ \otimes a(\varrho) + \sigma_- \otimes a(\varrho)^*$. For each $n \in \mathbb{N}_0$, we have the following:*

(i) $0 \trianglelefteq e^{-tK} \restriction \mathcal{H}_n$ *with respect to* $\mathfrak{P}_n$ *for all* $t \geqslant 0$.

(ii) $0 \trianglelefteq L \restriction \mathcal{H}_n$ *with respect to* $\mathfrak{P}_n$.

(iii) $0 \trianglelefteq e^{-tH_n}$ *with respect to* $\mathfrak{P}_n$ *for all* $t \geqslant 0$.

Proof. Since K, L and H are reduced by $\mathrm{ran}(E_n)$, it suffices to show the corresponding properties with respect to $\mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})$ by Proposition 2.8 and Lemma 4.6.

(i) It is not hard to show that $\langle |\varphi|_{\mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})}, K |\varphi|_{\mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})}\rangle = \langle \varphi, K\varphi \rangle$ for all $\varphi \in \mathrm{dom}(K) \cap (\mathbb{C}^2 \otimes \mathfrak{F}(L^2(\mathbb{R}^3_k)))^J$ where J is the involution associated with $\mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})$. Thus applying Theorem 2.7, $0 \trianglelefteq e^{-tK}$ with respect to $\mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})$ for all $t \geqslant 0$.

(ii) Since $0 \trianglelefteq a(\varrho)^\#$ with respect to $\mathfrak{F}(\mathfrak{p})$ and $0 \trianglelefteq \sigma_\pm$ with respect to $\mathfrak{p}_{\mathbb{C}^2}$, we can check that $0 \trianglelefteq L$ with respect to $\mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})$.

(iii) By Theorem 2.10 with $A = K$ and $B = -gL$, we have $0 \trianglelefteq \mathrm{e}^{-tH}$ with respect to $\mathfrak{p}_{\mathbb{C}^2} \otimes \mathfrak{F}(\mathfrak{p})$. ∎

LEMMA 4.8. *For each $\varphi, \psi \in \mathfrak{P}_n \backslash \{0\}$, there exists an $N \in \mathbb{N}_0$ such that $\langle \varphi, L^N \psi \rangle > 0$.*

Proof. It suffices to show the following:

(i) There exists $N_1 \in \mathbb{N}_0$ such that $\langle \eta_\uparrow \otimes F_{n-1}, L^{N_1} \eta_\uparrow \otimes G_{n-1} \rangle > 0$ for all $F_{n-1}, G_{n-1} \in (\otimes_{\mathrm{s}}^{n-1} \mathfrak{p}) \backslash \{0\}$.

(ii) There exists $N_2 \in \mathbb{N}_0$ such that $\langle \eta_\uparrow \otimes F_{n-1}, L^{N_2} \eta_\downarrow \otimes G_n \rangle > 0$ for all $F_{n-1} \in (\otimes_{\mathrm{s}}^{n-1} \mathfrak{p}) \backslash \{0\}$ and $G_n \in (\otimes_{\mathrm{s}}^{n} \mathfrak{p}) \backslash \{0\}$.

(iii) There exists $N_3 \in \mathbb{N}_0$ such that $\langle \eta_\downarrow \otimes F_n, L^{N_1} \eta_\downarrow \otimes G_n \rangle > 0$ for all $F_n, G_n \in (\otimes_{\mathrm{s}}^{n} \mathfrak{p}) \backslash \{0\}$.

Proof of (i) *and* (iii). Note that

$$L^{2m} = (\sigma_+ \sigma_-)^m \otimes (a(\varrho) a(\varrho)^*)^m + (\sigma_- \sigma_+)^m \otimes (a(\varrho)^* a(\varrho))^m.$$

Thus, applying Proposition 3.3(v),

$$\begin{aligned} L^{2n-2} \eta_\uparrow \otimes F_{n-1} &\geqslant (\sigma_+ \sigma_-)^{n-1} \eta_\uparrow \otimes (a(\varrho) a(\varrho)^*)^{n-1} F_{n-1} \\ &\geqslant \langle F_{n-1}, \otimes_{\mathrm{s}}^{n-1} \varrho \rangle \eta_\uparrow \otimes (\otimes_{\mathrm{s}}^{n-1} \varrho) \quad \text{with respect to } \mathfrak{P}_n. \end{aligned}$$

By the assumption $\varrho(k) > 0$ a.e. k, we have $\langle F_{n-1}, \otimes_{\mathrm{s}}^{n-1} \varrho \rangle > 0$. Thus

$$\langle \eta_\uparrow \otimes F_{n-1}, L^{4n-4} \eta_\uparrow \otimes G_{n-1} \rangle \geqslant \langle F_{n-1}, \otimes_{\mathrm{s}}^{n-1} \varrho \rangle \langle G_{n-1}, \otimes_{\mathrm{s}}^{n-1} \varrho \rangle \|\varrho\|^{2n-2} > 0$$

which completes the proof of (i). Similarly we can prove (iii).

Proof of (ii). By a similar argument to that above, we have

$$L^{2n-2} \eta_\uparrow \otimes F_{n-1} \geqslant \langle F_{n-1}, \otimes_{\mathrm{s}}^{n-1} \varrho \rangle \eta_\uparrow \otimes (\otimes_{\mathrm{s}}^{n-1} \varrho) \quad \text{with respect to } \mathfrak{P}_n.$$

Since $L \restriction \mathcal{H}_n \trianglerighteq \sigma_- \otimes a(\varrho)^* \restriction \mathcal{H}_n \trianglerighteq 0$ with respect to $\mathfrak{P}_n$, we have

$$\begin{aligned} L^{2n-1} \eta_\uparrow \otimes F_{n-1} &\geqslant \langle F_{n-1}, \otimes_{\mathrm{s}}^{n-1} \varrho \rangle (\sigma_- \otimes a(\varrho)^*) \eta_\uparrow \otimes (\otimes_{\mathrm{s}}^{n-1} \varrho) \\ &= \sqrt{n} \langle F_{n-1}, \otimes_{\mathrm{s}}^{n-1} \varrho \rangle \eta_\downarrow \otimes (\otimes_{\mathrm{s}}^{n} \varrho) \quad \text{with respect to } \mathfrak{P}_n. \end{aligned}$$

On the other hand, by the similar way to the proof of (i), we obtain

$$L^{2n} \eta_\downarrow \otimes G_n \geqslant \langle G_n, \otimes_{\mathrm{s}}^{n} \varrho \rangle \eta_\downarrow \otimes (\otimes_{\mathrm{s}}^{n} \varrho) \quad \text{with respect to } \mathfrak{P}_n.$$

Combining these estimates, we have

$$\langle \eta_\uparrow \otimes F_{n-1}, L^{4n-1} \eta_\downarrow \otimes G_n \rangle \geqslant \sqrt{n} \langle F_{n-1}, \otimes_{\mathrm{s}}^{n-1} \varrho \rangle \langle G_n, \otimes_{\mathrm{s}}^{n} \varrho \rangle \|\varrho\|^{2n} > 0.$$

This completes the proof. ∎

For the proof of Theorem 4.1 note Lemmas 4.7 and 4.8, and we can apply Theorem 2.15.

5. SPIN-BOSON MODEL

5.1. MAIN RESULTS IN SECTION 5. The spin-boson Hamiltonian is given by

$$H_{\mathrm{SB}} = \frac{\mu}{2}\sigma_3 \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \alpha\sigma_1 \otimes (a(\varrho) + a(\varrho)^*)$$

acting in $\mathbb{C}^2 \otimes \mathfrak{F}(L^2(\mathbb{R}^3_k))$, with $\mu > 0$, $\alpha \in \mathbb{R}\backslash\{0\}$, $\omega(k) = |k|$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We assume that $\varrho, \varrho/\omega^{1/2} \in L^2(\mathbb{R}^3_k)$ and that $\varrho(-k)^* = \varrho(k)$. Then H_{SB} is self-adjoint on $\mathrm{dom}(\mathbb{1} \otimes \mathrm{d}\Gamma(\omega))$ and bounded from below.

Let us consider the Schrödinger representation of the Fock space $\mathfrak{F}(L^2(\mathbb{R}^3_k)) = L^2(\mathcal{Q}, \mathrm{d}\mu)$, where μ is a Gaussian probability measure. The points of this representation are the following facts [38]:

(a) $\phi(\varrho) = 2^{-1/2}(a(\varrho) + a(\varrho)^*)^{**}$ is a real multiplication operator,

(b) $0 \trianglelefteq \Gamma(\mathrm{e}^{-t\omega})$ with respect to $L^2(\mathcal{Q}, \mathrm{d}\mu)_+ = \{F \in L^2(\mathcal{Q}, \mathrm{d}\mu) : F \geqslant 0\ \mu\text{-a.e.}\}$ for all $t \geqslant 0$,

(c) the Fock vacuum Ω is the constant function identically one.

Let $x_1 = (1/\sqrt{2})\binom{1}{1}$ and $x_2 = (1/\sqrt{2})\binom{-1}{1}$. We choose the following self-dual cone in $\mathbb{C}^2 \otimes \mathfrak{F}(L^2(\mathbb{R}^3_k)) = \mathbb{C}^2 \otimes L^2(\mathcal{Q}, \mathrm{d}\mu)$:

$$\mathfrak{P}_{\mathrm{SB}} = \{\varphi \in \mathbb{C}^2 \otimes L^2(\mathcal{Q}, \mathrm{d}\mu) : \varphi = x_1 \otimes \varphi_1 + x_2 \otimes \varphi_2 \text{ with } \varphi_1, \varphi_2 \in L^2(\mathcal{Q}, \mathrm{d}\mu)_+\}.$$

THEOREM 5.1. *Assume that* $\varrho, \varrho/\omega^{1/2} \in L^2(\mathbb{R}^3_k)$ *and* $\varrho(-k)^* = \varrho(k)$. *Then, under the Schrödinger representation, we have* $0 \triangleleft \mathrm{e}^{-tH_{\mathrm{SB}}}$ *with respect to* $\mathfrak{P}_{\mathrm{SB}}$ *for all* $t > 0$.

COROLLARY 5.2. *Under the conditions in Theorem* 5.1, *assume that* H_{SB} *has a ground state* φ_{GS}. *Then it is nondegenerate and strictly positive with respect to* $\mathfrak{P}_{\mathrm{SB}}$. *Thus, for any* $\Psi \in \mathfrak{P}_{\mathrm{SB}}\backslash\{0\}$, *we have* $\langle \varphi_{\mathrm{GS}}, \Psi\rangle > 0$. *In particular* $\langle \varphi_{\mathrm{GS}}, x_1 \otimes \Omega\rangle > 0$ *and* $\langle \varphi_{\mathrm{GS}}, x_2 \otimes \Omega\rangle > 0$.

5.2. PROOF OF THEOREM 5.1. Let U be a unitary operator on $\mathbb{C}^2$ given by $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. We introduce a new Hamiltonian by

$$\widehat{H}_{\mathrm{SB}} = U \otimes \mathbb{1} H_{\mathrm{SB}} U \otimes \mathbb{1}.$$

Using the formulas $U\sigma_3 U = \sigma_1$ and $U\sigma_1 U = \sigma_3$, we have

$$\widehat{H}_{\mathrm{SB}} = \frac{\mu}{2}\sigma_1 \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{d}\Gamma(\omega) + \sqrt{2}\alpha\sigma_3 \otimes \phi(\varrho).$$

We also remark that

$$\widehat{\mathfrak{P}}_{\mathrm{SB}} = U \otimes \mathbb{1}\mathfrak{P}_{\mathrm{SB}} = \{\varphi \in \mathbb{C}^2 \otimes L^2(\mathcal{Q}, \mathrm{d}\mu) : \varphi = \eta_\uparrow \otimes \varphi_\uparrow - \eta_\downarrow \otimes \varphi_\downarrow \text{ with } \varphi_\uparrow, \varphi_\downarrow \in L^2(\mathcal{Q}, \mathrm{d}\mu)_+\},$$

where $\eta_\uparrow = \binom{1}{0}$ and $\eta_\downarrow = \binom{0}{1}$.

LEMMA 5.3. *Let* $\widehat{H}_0 = (\mu/2)\sigma_1 \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma(\omega)$. *Then* $0 \lhd e^{-t\widehat{H}_0}$ *with respect to* $\widehat{\mathfrak{P}}_{SB}$ *for all* $t > 0$.

Proof. Since $e^{-t\mu\sigma_1/2} = \cosh(\mu t/2) - \sinh(\mu t/2)\sigma_1$, we have, for $\varphi = \eta_\uparrow \otimes \varphi_\uparrow - \eta_\downarrow \otimes \varphi_\downarrow \in \widehat{\mathfrak{P}}_{SB}$,

$$\begin{aligned} e^{-t\widehat{H}_0}\varphi &= \cosh(\mu t/2)\eta_\uparrow \otimes \Gamma(e^{-t\omega})\varphi_\uparrow + \sinh(\mu t/2)\eta_\uparrow \otimes \Gamma(e^{-t\omega})\varphi_\downarrow \\ &\quad - (\sinh(\mu t/2)\eta_\downarrow \otimes \Gamma(e^{-t\omega})\varphi_\uparrow + \cosh(\mu t/2)\eta_\downarrow \otimes \Gamma(e^{-t\omega})\varphi_\downarrow). \end{aligned}$$

Note that $0 \lhd \Gamma(e^{-t\omega})$ with respect to $L^2(\mathcal{Q}, d\mu)_+$. (For the proof note that $d\Gamma(\omega)$ has a unique ground state Ω and it is identically one in the Schrödinger representation, hence we can apply Theorem 2.12 to conclude that $0 \lhd \Gamma(e^{-t\omega})$ with respect to $L^2(\mathcal{Q}, d\mu)_+$.) Thus if $\varphi \neq 0$, then $0 < e^{-t\widehat{H}_0}\varphi$ with respect to $\widehat{\mathfrak{P}}_{SB}$. ∎

LEMMA 5.4. *Let* $\widehat{V}_n = \sqrt{2}\alpha\sigma_3 \otimes \phi(\varrho)\chi_{\{|\phi(\varrho)| \leqslant n\}}$, *where* $\chi_{\{f<a\}}$ *is the characteristic function of the set* $\{f < a\}$. *Then we have the following properties:*

(i) *For all* $n \in \mathbb{N}$ *and* $t \geqslant 0$, $0 \trianglelefteq e^{-t\widehat{V}_n}$ *with respect to* $\widehat{\mathfrak{P}}_{SB}$.

(ii) *For* $u, v \in \widehat{\mathfrak{P}}_{SB}$ *with* $\langle u, v\rangle = 0$, *we have* $\langle e^{-t\widehat{V}_n}u, v\rangle = 0$ *for all* $n \in \mathbb{N}$.

(iii) $\widehat{H}_0 + \widehat{V}_n$ *converges to* $\widehat{H}_{SB}$ *in the strong resolvent sense,* $\widehat{H}_{SB} - \widehat{V}_n$ *converges to* $\widehat{H}_0$ *in the strong resolvent sense as* $n \to \infty$.

Proof. (i) For $\varphi \in (\mathbb{C}^2 \otimes L^2(\mathcal{Q}, d\mu))^J$, we have the representation $\varphi = \eta_\uparrow \otimes \varphi_\uparrow + \eta_\downarrow \otimes \varphi_\downarrow$ with $\varphi_\uparrow, \varphi_\downarrow \in L^2_{\rm real}(\mathcal{Q}, d\mu)$, the space of real valued L^2-functions. Under this representation, we have $|\varphi|_{\widehat{\mathfrak{P}}_{SB}} = \eta_\uparrow \otimes |\varphi_\uparrow| - \eta_\downarrow \otimes |\varphi_\downarrow|$, where $|\cdot|$ means $|\cdot|_{L^2(\mathcal{Q},d\mu)_+}$, that is, the standard absolute value. Let $\phi_n = \phi(\varrho)\chi_{\{|\phi(\varrho)| \leqslant n\}}$. Since ϕ_n is a real multiplication operator, we obtain

$$\begin{aligned} \langle |\varphi|_{\widehat{\mathfrak{P}}_{SB}}, \sigma_3 \otimes \phi_n |\varphi|_{\widehat{\mathfrak{P}}_{SB}}\rangle &= \langle |\varphi_\uparrow|, \phi_n|\varphi_\uparrow|\rangle - \langle |\varphi_\downarrow|, \phi_n|\varphi_\downarrow|\rangle \\ &= \langle \varphi_\uparrow, \phi_n\varphi_\uparrow\rangle - \langle \varphi_\downarrow, \phi_n\varphi_\downarrow\rangle = \langle \varphi, \sigma_3 \otimes \phi_n\varphi\rangle. \end{aligned}$$

Thus applying Theorem 2.7, we obtain the desired assertion.

(ii) Note first that $u, v \in \widehat{\mathfrak{P}}_{SB}$ have representations $u = \eta_\uparrow \otimes u_\uparrow - \eta_\downarrow \otimes u_\downarrow$ and $v = \eta_\uparrow \otimes v_\uparrow - \eta_\downarrow \otimes v_\downarrow$ with $u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow \in L^2(\mathcal{Q}, d\mu)_+$. Thus $\langle u, v\rangle = 0$ if and only if $\langle u_\uparrow, v_\uparrow\rangle = 0 = \langle u_\downarrow, v_\downarrow\rangle$ which means $u_\uparrow v_\uparrow = 0 = u_\downarrow v_\downarrow$ μ-a.e. Hence, for $N = 2m$, we have

$$\langle u, (\sigma_3 \otimes \phi_n)^N v\rangle = \langle u_\uparrow, \phi_n^{2m} v_\uparrow\rangle + \langle u_\downarrow, \phi_n^{2m} v_\downarrow\rangle = 0.$$

Similarly, for $N = 2m + 1$, we have

$$\langle u, (\sigma_3 \otimes \phi_n)^N v\rangle = \langle u_\uparrow, \phi_n^{2m+1} v_\uparrow\rangle - \langle u_\downarrow, \phi_n^{2m+1} v_\downarrow\rangle = 0.$$

Thus we conclude the desired result.

(iii) For all $n \in \mathbb{N}$, $\widehat{H}_{SB}$ and $\widehat{H}_0 + \widehat{V}_n$ are self-adjoint on a common domain $\mathrm{dom}(\mathbb{1} \otimes d\Gamma(\omega))$, and for all $\varphi \in \mathrm{dom}(\mathbb{1} \otimes d\Gamma(\omega))$, we see that $(\widehat{H}_0 + \widehat{V}_n)\varphi \to \widehat{H}_{SB}\varphi$ strongly as $n \to \infty$. Applying Theorem VIII. 25(a) of [33], we obtain that

$\widehat{H}_0 + \widehat{V}_n$ converges to $\widehat{H}_{\rm SB}$ in the strong resolvent sense. Similarly we can show the remainder assertion. ∎

For the proof of Theorem 5.1 note Lemmas 5.3 and 5.4. We can apply Theorem 2.14 and conclude that $0 \lhd e^{-t\widehat{H}_{\rm SB}}$ with respect to $\widehat{\mathfrak{P}}_{\rm SB}$.

6. FRÖHLICH POLARON WITHOUT ULTRAVIOLET CUTOFFS

6.1. MAIN RESULTS IN SECTION 6. For each $P \in \mathbb{R}^3$, the Fröhlich polaron Hamiltonian of a fixed total momentum P with an ultraviolet cutoff κ is defined by

$$H_\kappa(P) = \frac{1}{2}(P - P_{\rm f})^2 + \sqrt{\alpha}\lambda_0 \int\limits_{|k|\leqslant\kappa} \frac{{\rm d}k}{(2\pi)^{3/2}|k|}\,[a(k) + a(k)^*] + N_{\rm f}$$

which is acting in $\mathfrak{F}(L^2(\mathbb{R}^3_k))$, where $\lambda_0 = (2\sqrt{2}\pi)^{1/2}$ and $P_{\rm f}$ is the field momentum operator defined by $P_{\rm f} = (P_{{\rm f},1}, P_{{\rm f},2}, P_{{\rm f},3}) = ({\rm d}\Gamma(k_1), {\rm d}\Gamma(k_2), {\rm d}\Gamma(k_3))$. Applying the bound $\|a(f)^\#(N_{\rm f} + \mathbb{1})^{-1/2}\| \leqslant \|f\|$ and the Kato–Rellich theorem, $H_\kappa(P)$ is self-adjoint on ${\rm dom}(P_{\rm f}^2) \cap {\rm dom}(N_{\rm f})$ and bounded from below for all $\kappa < \infty$, $P \in \mathbb{R}^3$ and $\alpha < \infty$.

PROPOSITION 6.1. *For all $P \in \mathbb{R}^3$, there exists a self-adjoint operator $H(P)$ such that $H_\kappa(P)$ converges to $H(P)$ in the strong resolvent sense as $\kappa \to \infty$.*

REMARK 6.2. Applying the arguments in [1], we can show the norm resolvent convergence. In this note, the strong convergence is enough for our purpose. This remark also goes to Propositions 6.5, 7.1 and 7.4.

For the proof see Appendix A.

Let $\mathfrak{p} = L^2(\mathbb{R}^3_k)_+$. In this case we can define a self-dual cone $\mathfrak{F}(\mathfrak{p})$ in $\mathfrak{F}(L^2(\mathbb{R}^3_k))$.

THEOREM 6.3. *For all $P \in \mathbb{R}^3$ and $t \geqslant 0$,we have $0 \lhd e^{i\pi N_{\rm f}} e^{-tH(P)} e^{-i\pi N_{\rm f}}$ with respect to $\mathfrak{F}(\mathfrak{p})$.*

Let l_k be the angular momentum operators in $L^2(\mathbb{R}^3_k)$ given by $l_k = k \times (-i\nabla_k)$ and let $L_{\rm f}$ be its second quantization: $L_{\rm f} = {\rm d}\Gamma(l_k)$.

THEOREM 6.4. *For $|P| < \sqrt{2}$, $H(P)$ has a unique ground state φ_P such that $e^{i\pi N_{\rm f}}\varphi_P$ is strictly positive with respect to $\mathfrak{F}(\mathfrak{p})$. Moreover φ_P has the following properties:*

(i) *For all $\theta \in \mathbb{R}$ and $\omega \in \mathbb{S}^2 = \{\omega \in \mathbb{R}^3 : |\omega| = 1\}$, we have $e^{i\theta\omega\cdot L_{\rm f}}\varphi_0 = \varphi_0$.*

(ii) *Let $P \neq 0$ with $|P| < \sqrt{2}$. Then, for all $\theta \in \mathbb{R}$, we have $e^{i\theta\omega_P\cdot L_{\rm f}}\varphi_P = \varphi_P$ with $\omega_P = P/|P|$.*

6.2. PROOF OF THEOREM 6.3. Let $\varrho_m(k) = e^{-|k|/m}$ for $m > 0$ and define a new Hamiltonian

$$H_{\varrho_m}(P) = \frac{1}{2}(P - P_{\rm f})^2 + \sqrt{\alpha}\lambda_0 \int\limits_{\mathbb{R}^3} {\rm d}k\, \frac{\varrho_m(k)}{(2\pi)^{3/2}|k|}\,[a(k) + a(k)^*] + N_{\rm f}.$$

$H_{\varrho_m}(P)$ is self-adjoint on $\mathrm{dom}(P_{\mathrm{f}}^2) \cap \mathrm{dom}(N_{\mathrm{f}})$, bounded from below. In Appendix A, we show the following.

PROPOSITION 6.5. *For all $P \in \mathbb{R}^3$ and $\alpha < \infty$, $H_{\varrho_m}(P)$ converges to $H(P)$ in the strong resolvent sense as $m \to \infty$.*

Let us define $\widehat{H}_{\varrho_m}(P) = \mathrm{e}^{\mathrm{i}\pi N_{\mathrm{f}}} H_{\varrho_m}(P)\, \mathrm{e}^{-\mathrm{i}\pi N_{\mathrm{f}}}$ and $\widehat{H}(P) = \mathrm{e}^{\mathrm{i}\pi N_{\mathrm{f}}} H(P)\, \mathrm{e}^{-\mathrm{i}\pi N_{\mathrm{f}}}$. We can easily check that, for $m > 0$,

$$\widehat{H}_{\varrho_m}(P) = \frac{1}{2}(P - P_{\mathrm{f}})^2 - \sqrt{\alpha}\lambda_0 \int_{\mathbb{R}^3} \mathrm{d}k\, \frac{\varrho_m(k)}{(2\pi)^{3/2}|k|}\, [a(k) + a(k)^*] + N_{\mathrm{f}}. \tag{6.1}$$

LEMMA 6.6. *Let*

$$L(P) = \frac{1}{2}(P - P_{\mathrm{f}})^2 + N_{\mathrm{f}} \quad \textit{and} \quad B_m = \sqrt{\alpha}\lambda_0 \int_{\mathbb{R}^3} \mathrm{d}k\, \frac{\varrho_m(k)}{(2\pi)^{3/2}|k|} a(k).$$

(i) *For all $P \in \mathbb{R}^3$ and $t \geqslant 0$, we have $0 \trianglelefteq \mathrm{e}^{-tL(P)}$ with respect to $\mathfrak{F}(\mathfrak{p})$.*
(ii) *For all $m > 0$, we have $0 \trianglelefteq B_m, B_m^*$ with respect to $\mathfrak{F}(\mathfrak{p})$.*
(iii) *For each $m > 0, P \in \mathbb{R}^3$ and $t \geqslant 0$, we have $0 \trianglelefteq \mathrm{e}^{-t\widehat{H}_{\varrho_m}(P)}$ with respect to $\mathfrak{F}(\mathfrak{p})$.*

Proof. (i) Note that, for all $\varphi \in \otimes_{\mathrm{s}}^n L^2(\mathbb{R}_k^3) \cap \mathrm{dom}(P_{\mathrm{f}}^2)$, $\langle |\varphi|_{\otimes_{\mathrm{s}}^n \mathfrak{p}}, L(P)|\varphi|_{\otimes_{\mathrm{s}}^n \mathfrak{p}}\rangle = \langle \varphi, L(P)\varphi\rangle$. Thus, by Theorem 2.7, we have $0 \trianglelefteq \mathrm{e}^{-tL(P)} \restriction \otimes_{\mathrm{s}}^n L^2(\mathbb{R}_k^3)$ with respect to $\otimes_{\mathrm{s}}^n \mathfrak{p}$ for all $n \in \mathbb{N}_0$. By Proposition 2.18, we conclude that $0 \trianglelefteq \mathrm{e}^{-tL(P)}$ with respect to $\mathfrak{F}(\mathfrak{p})$.

(ii) This is a direct consequence of Theorem 3.3(iii).

(iii) Note that, $B_m^{\#}$ is infinitesimally small with respect to $L(P)$. Therefore, noting (i) and (ii), we can apply Theorem 2.10 with $A = L(P)$ and $B = -B_m - B_m^*$. ∎

LEMMA 6.7. *For all $\varphi \in (\otimes_{\mathrm{s}}^p \mathfrak{p})\backslash\{0\}$ and $\psi \in (\otimes_{\mathrm{s}}^q \mathfrak{p})\backslash\{0\}$, there exists an $N \in \mathbb{N}_0$ such that $\langle \varphi, (B_m + B_m^*)^N \psi\rangle > 0$.*

Proof. By Theorem 3.3(v), for each $n \in \mathbb{N}$, we have

$$(B_m + B_m^*)^{2n} \restriction \otimes_{\mathrm{s}}^n L^2(\mathbb{R}_k^3) \trianglerighteq (B_m^* B_m)^n \restriction \otimes_{\mathrm{s}}^n L^2(\mathbb{R}_k^3) \trianglerighteq \otimes^n |\xi_m\rangle\langle \xi_m| \quad \text{with respect to } \otimes_{\mathrm{s}}^n \mathfrak{p}$$

where $\xi_m(k) = \sqrt{\alpha}\lambda_0 \varrho_m(k)/(2\pi)^{3/2}|k|$. Also note that $B_m^n \otimes_{\mathrm{s}}^n \xi_m = \sqrt{(n+1)!}\, \|\xi_m\|^{2n}\Omega$. Thus, for $\Psi \in \otimes_{\mathrm{s}}^n \mathfrak{p}$, we have

$$\begin{aligned}(B_m + B_m^*)^{3n}\Psi &\geqslant B_m^n (B_m^* B_m)^n \Psi \geqslant \langle \otimes_{\mathrm{s}}^n \xi_m, \Psi\rangle B_m^n \otimes_{\mathrm{s}}^n \xi_m \\ &= \sqrt{(n+1)!}\langle \otimes_{\mathrm{s}}^n \xi_m, \Psi\rangle \|\xi_m\|^{2n}\Omega \quad \text{with respect to } \mathfrak{F}(\mathfrak{p}).\end{aligned}$$

Since $0 < \xi_m$ with respect to $\mathfrak{p}$, we have $\langle \otimes_{\mathrm{s}}^n \xi_m, \Psi\rangle > 0$ if $\Psi \neq 0$. Thus $\langle \varphi, (B_m + B_m^*)^{3p+3q}\psi\rangle \geqslant \sqrt{(p+1)!}\sqrt{(q+1)!}\langle \varphi, \otimes_{\mathrm{s}}^p \xi_m\rangle \langle \otimes_{\mathrm{s}}^q \xi_m, \psi\rangle \|\xi_m\|^{2p+2q} > 0$. ∎

PROPOSITION 6.8. *For any $m > 0, P \in \mathbb{R}^3$ and $t > 0$, we have $0 \triangleleft \mathrm{e}^{-t\widehat{H}_{\varrho_m}(P)}$ with respect to $\mathfrak{F}(\mathfrak{p})$.*

The proof follows by Lemmas 6.6 and 6.7, and we can apply Theorem 2.15.

LEMMA 6.9. *One has the following:*

(i) *For each m, n, one has*

$$\mathrm{dom}(\widehat{H}_{\varrho_m}(P)) = \mathrm{dom}(\widehat{H}_{\varrho_n}(P)) = \mathrm{dom}(P_{\mathrm{f}}^2) \cap \mathrm{dom}(N_{\mathrm{f}}).$$

(ii) *For m, n with $n > m$, one has $\widehat{H}_{\varrho_m}(P) - \widehat{H}_{\varrho_n}(P) \trianglerighteq 0$ with respect to $\mathfrak{F}(\mathfrak{p})$.*

Proof. (i) is trivial. As to (ii), we remark that

$$\widehat{H}_{\varrho_m}(P) - \widehat{H}_{\varrho_n}(P) = (B_n - B_m) + (B_n^* - B_m^*).$$

Since $\varrho_m \leqslant \varrho_n$ with respect to $\mathfrak{p}$, one sees that $B_n - B_m \trianglerighteq 0$ and $B_n^* - B_m^* \trianglerighteq 0$ with respect to $\mathfrak{F}(\mathfrak{p})$. ∎

Proof of Theorem 6.3 follows by Proposition 6.8 and Lemma 6.9, and we can apply Theorem 2.16.

6.3. PROOF OF THEOREM 6.4. First we note H. Spohn's result [43] (see also [29]):

$$\inf \mathrm{ess.spec}(H_\kappa(P)) - \inf \mathrm{spec}(H_\kappa(P)) = \inf \mathrm{spec}(H_\kappa(0)) - \inf \mathrm{spec}(H_\kappa(P)) + 1$$

for all $\kappa < \infty$ and $P \in \mathbb{R}^3$. Also note the following inequality

$$\inf \mathrm{spec}(H_\kappa(P)) \leqslant \inf \mathrm{spec}(H_\kappa(0)) - \frac{P^2}{2}$$

for all $\kappa < \infty$ and $P \in \mathbb{R}^3$, see e.g., [14]. Applying Propositions A.1 and A.4 we arrive at

$$\inf \mathrm{ess.spec}(H(P)) - \inf \mathrm{spec}(H(P)) = \inf \mathrm{spec}(H(0)) - \inf \mathrm{spec}(H(P)) + 1$$

and

$$\inf \mathrm{spec}(H(P)) \leqslant \inf \mathrm{spec}(H(0)) - \frac{P^2}{2}.$$

Now we have a unique ground state φ_P with $|P| < \sqrt{2}$ by Theorem 6.3. On the other hand, $0 \trianglelefteq \mathrm{e}^{\mathrm{i}\theta\omega\cdot L_{\mathrm{f}}}$ with respect to $\mathfrak{F}(\mathfrak{p})$ because $0 \trianglelefteq \mathrm{e}^{\mathrm{i}\theta\omega\cdot l_k}$ with respect to $\mathfrak{p}$. For $\theta \in \mathbb{R}$ and $\omega \in \mathbb{S}^2$, let $g(\theta, \omega) \in SO(3)$ be the rotation around ω with angle θ. We can confirm that $\mathrm{e}^{\mathrm{i}\theta\omega\cdot L_{\mathrm{f}}} H_\kappa(P) \mathrm{e}^{-\mathrm{i}\theta\omega\cdot L_{\mathrm{f}}} = H_\kappa(g(\theta,\omega)^{-1}P)$ which is equivalent to $\mathrm{e}^{\mathrm{i}\theta\omega\cdot L_{\mathrm{f}}} \mathrm{e}^{\mathrm{i}sH_\kappa(P)} \mathrm{e}^{-\mathrm{i}\theta\omega\cdot L_{\mathrm{f}}} = \mathrm{e}^{\mathrm{i}sH_\kappa(g(\theta,\omega)^{-1}P)}$ for all $s \in \mathbb{R}$. Taking $\kappa \to \infty$ and applying Propositions A.1 and A.4, we have $\mathrm{e}^{\mathrm{i}\theta\omega\cdot L_{\mathrm{f}}} \mathrm{e}^{\mathrm{i}sH(P)} \mathrm{e}^{-\mathrm{i}\theta\omega\cdot L_{\mathrm{f}}} = \mathrm{e}^{\mathrm{i}sH(g(\theta,\omega)^{-1}P)}$. Now we obtain the following: (a) $\mathrm{e}^{\mathrm{i}\theta\omega\cdot L_{\mathrm{f}}} \mathrm{e}^{\mathrm{i}sH(0)} \mathrm{e}^{-\mathrm{i}\theta\omega\cdot L_{\mathrm{f}}} = \mathrm{e}^{\mathrm{i}sH(0)}$, (b) for $P \neq 0$, $\mathrm{e}^{\mathrm{i}\theta\omega_P\cdot L_{\mathrm{f}}} \mathrm{e}^{\mathrm{i}sH(P)} \mathrm{e}^{-\mathrm{i}\theta\omega_P\cdot L_{\mathrm{f}}} = \mathrm{e}^{\mathrm{i}sH(P)}$ where $\omega_P = P/|P|$. Accordingly we can apply Propoistion 2.13 to conclude the result.

7. FRÖHLICH BIPOLARON WITHOUT ULTRAVIOLET CUTOFFS

7.1. MAIN RESULTS IN SECTION 7. The Hamiltonian of the Fröhlich bipolaron of a fixed total momentum P with an ultraviolet cutoff κ is defined by

$$H_{\mathrm{bp},\kappa}(P)=\frac{1}{4}(P-\mathbb{1}\otimes P_{\mathrm{f}})^2+\Big(-\Delta_x+\frac{U\alpha}{|x|}\Big)\otimes\mathbb{1}+\mathbb{1}\otimes N_{\mathrm{f}}$$
$$+2\sqrt{\alpha}\lambda_0\int\limits_{|k|\leqslant\kappa}\frac{\mathrm{d}k}{(2\pi)^{3/2}|k|}\cos(k\cdot x/2)\otimes[a(k)+a(k)^*]$$

which is acting in $L^2(\mathbb{R}^3_x)\otimes\mathfrak{F}(L^2(\mathbb{R}^3_k))$, with $P\in\mathbb{R}^3, 0\leqslant\alpha<\infty, 0\leqslant U$ and $\lambda_0=(2\sqrt{2}\pi)^{1/2}$. The field-particle interaction term is understood as follows: For each $x\in\mathbb{R}^3$, let us introduce $A(x)=\int\limits_{|k|\leqslant\kappa}\frac{\mathrm{d}k}{(2\pi)^{3/2}|k|}\cos(k\cdot x/2)a(k)$. Then under the identification $L^2(\mathbb{R}^3_x)\otimes\mathfrak{F}(L^2(\mathbb{R}^3_k))=\int^{\oplus}_{\mathbb{R}^3_x}\mathfrak{F}(L^2(\mathbb{R}^3_k))\,\mathrm{d}k$, we can define a closed operator $A=\int^{\oplus}_{\mathbb{R}^3_x}A(x)\,\mathrm{d}x$ via similar arguments in Example 2.21. From this view point, the interaction term is given by $2\sqrt{\alpha}\lambda_0(A+A^*)$.

By the bounds $\|a(f)^{\#}(N_{\mathrm{f}}+\mathbb{1})^{-1/2}\|\leqslant\|f\|$ and $\||x|^{-1}\varphi\|\leqslant\varepsilon\|\Delta_x\varphi\|+b_\varepsilon\|\varphi\|$, $\varphi\in\mathrm{dom}(\Delta_x)$ for any $\varepsilon>0$, we can apply the Kato–Rellich theorem, and conclude that $H_{\mathrm{bp},\kappa}(P)$ is self-adjoint on $\mathrm{dom}(\Delta_x\otimes\mathbb{1})\cap\mathrm{dom}(\mathbb{1}\otimes N_{\mathrm{f}})\cap\mathrm{dom}(\mathbb{1}\otimes P_{\mathrm{f}}^2)$, bounded from below for all $P\in\mathbb{R}^3, 0\leqslant U<\infty, 0\leqslant\alpha<\infty$ and $\kappa<\infty$. In our previous work [28], we have shown the following.

PROPOSITION 7.1. *For all $P\in\mathbb{R}^3, 0\leqslant U<\infty$ and $0\leqslant\alpha<\infty$, there exists a self-adjoint operator $H_{\mathrm{bp}}(P)$, bounded from below, such that $H_{\mathrm{bp},\kappa}(P)$ converges to $H_{\mathrm{bp}}(P)$ in the strong resolvent sense as $\kappa\to\infty$.*

Let $\mathcal{F}:L^2(\mathbb{R}^3_k)\to L^2(\mathbb{R}^3_y)$ be the Fourier transformation on $L^2(\mathbb{R}^3_k)$, where $L^2(\mathbb{R}^3_y)$ is the configuration L^2-space. Then $\Gamma(\mathcal{F})$ is a unitary operator from $\mathfrak{F}(L^2(\mathbb{R}^3_k))$ onto $\mathfrak{F}(L^2(\mathbb{R}^3_y))$. Let $\mathfrak{p}_x:=L^2(\mathbb{R}^3_x)_+$ be a self-dual cone in $L^2(\mathbb{R}^3_x)$ and let $\mathfrak{p}_y:=L^2(\mathbb{R}^3_y)_+$ be a self-dual cone in $L^2(\mathbb{R}^3_y)$. Now we choose the following self-dual cone in $L^2(\mathbb{R}^3_x)\otimes\mathfrak{F}(L^2(\mathbb{R}^3_y))$:

$$\mathfrak{p}_x\otimes\mathfrak{F}(\mathfrak{p}_y):=\{\varphi\in L^2(\mathbb{R}^3_x)\otimes\mathfrak{F}(L^2(\mathbb{R}^3_y)):\langle\varphi,u\otimes v\rangle\geqslant 0\ \forall u\in\mathfrak{p}_x\forall v\in\mathfrak{F}(\mathfrak{p}_y)\}.$$

Note that, under the identification $L^2(\mathbb{R}^3_x)\otimes\mathfrak{F}(L^2(\mathbb{R}^3_y))=\int^{\oplus}_{\mathbb{R}^3_x}\mathfrak{F}(L^2(\mathbb{R}^3_y))\,\mathrm{d}x$, we obtain $\mathfrak{p}_x\otimes\mathfrak{F}(\mathfrak{p}_y)=\int^{\oplus}_{\mathbb{R}^3_x}\mathfrak{F}(\mathfrak{p}_y)\,\mathrm{d}x$.

THEOREM 7.2. *Let us define $\vartheta = \mathbb{1} \otimes e^{i\pi N_f}\Gamma(\mathcal{F})$. For all $0 \leqslant U < \infty, 0 < t$ and $\alpha < \infty$, we have $0 \lhd \vartheta e^{-tH_{bp}(0)}\vartheta^*$ with respect to $\mathfrak{p}_x \otimes \mathfrak{F}(\mathfrak{p}_y)$.*

Let us define the *binding energy* by

$$E_{bin}(\alpha, U) = 2 \inf \mathrm{spec}(H(0)) - \inf \mathrm{spec}(H_{bp}(0)),$$

where $H(0)$ is the Fröhlich polaron Hamiltonian of 0 total momentum without ultraviolet cutoffs discussed in Section 6. If $E_{bin}(\alpha, U) > 0$ holds, then we say that the *binding condition* is satisfied. A set of (α, U) satisfying the binding condition is denoted by Λ_{bin}. Namely

$$\Lambda_{bin} = \{(\alpha, U) \in \mathbb{R}_+ \times \mathbb{R}_+ : E_{bin}(\alpha, U) > 0\}.$$

THEOREM 7.3. *Assume that $(\alpha, U) \in \Lambda_{bin}$. Then there exists a $P_c > 0$ such that $H_{bp}(P)$ has a nondegenerate ground state φ_P for $|P| < P_c$. Moreover we can choose φ_0 such that $\vartheta\varphi_0$ is strictly positive with respect to $\mathfrak{p}_x \otimes \mathfrak{F}(\mathfrak{p}_y)$. Let $L_{tot} = l_x \otimes \mathbb{1} + \mathbb{1} \otimes L_f$ be the total angular momentum operator. Then we obtain the following:*

(i) $e^{i\phi\omega\cdot L_{tot}}\varphi_0 = \varphi_0$ *for all $\omega \in \mathbb{S}^2$ and $\phi \in \mathbb{R}$.*

(ii) *For $P \neq 0$ with $|P| < P_c$, set $\omega_P = P/|P|$. Then $e^{i\phi\omega_P\cdot L_{tot}}\varphi_P = \varphi_P$ for all $\phi \in \mathbb{R}$.*

7.2. PROOF OF THEOREM 7.2. Let us consider a new Hamiltonian

$$H_{bp,\varrho_m}(P) = \frac{1}{4}(P - \mathbb{1} \otimes P_f)^2 + \Big(-\Delta_x + \frac{\alpha U}{|x|}\Big) \otimes \mathbb{1} + \mathbb{1} \otimes N_f + 2\sqrt{\alpha}\lambda_0 \int_{\mathbb{R}^3_k} dk\, \frac{\varrho_m(k)}{(2\pi)^{3/2}|k|} \cos(k \cdot x/2) \otimes [a(k) + a(k)^*]$$

with $\varrho_m(k) = e^{-|k|/m}, m > 0$. Let $b(y)$ and $b(y)^*$ be the annihilation and creation operators in the configuration Fock space $\mathfrak{F}(L^2(\mathbb{R}^3_y))$. Then, the transformed Hamiltonian $\widehat{H}_{bp,\varrho_m}(P) = \vartheta H_{bp,\varrho_m}(P)\vartheta^*$ is given by

$$\widehat{H}_{bp,\varrho_m}(P) = \frac{1}{4}(P - \mathbb{1} \otimes \widehat{P}_f)^2 + \Big(-\Delta_x + \frac{\alpha U}{|x|}\Big) \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{N}_f - \int_{\mathbb{R}^3_y} dy\, G_m(x,y) \otimes [b(y) + b(y)^*],$$

where $\widehat{P}_f = d\Gamma(-i\nabla_y)$, $\widehat{N}_f$ is the number operator on $\mathfrak{F}(L^2(\mathbb{R}^3_y))$ and

$$G_m(x,y) = \frac{\sqrt{\alpha}\lambda_0}{8\pi^2}\Big\{\frac{1}{(y + x/2)^2 + 1/m^2} + \frac{1}{(y - x/2)^2 + 1/m^2}\Big\}.$$

PROPOSITION 7.4. *Let $\widehat{H}_{bp}(P) = \vartheta H_{bp}(P)\vartheta^*$. Then $\widehat{H}_{bp,\varrho_m}(P)$ converges to $\widehat{H}_{bp}(P)$ in the strong resolvent sense as $m \to \infty$.*

This can be proven by modifying arguments in Appendix A of [28].

Note first that, under the natural identification: $L^2(\mathbb{R}^3_x) \otimes \mathfrak{F}(L^2(\mathbb{R}^3_y)) = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^3_x) \otimes L^2_{sym}(\mathbb{R}^{3n}_y)$, we have that $\mathfrak{p}_x \otimes \mathfrak{F}(\mathfrak{p}_y) = \bigoplus_{n=0}^{\infty} \mathfrak{P}_n$ with $\mathfrak{P}_n = \mathfrak{p}_x \otimes$

$(\otimes_{\rm s}^n \mathfrak{p}_y)$. Let E_n be the orthogonal projection onto $L^2(\mathbb{R}_x^3)\otimes L^2_{\rm sym}(\mathbb{R}_y^{3n})$, then E_n and $E_n^\perp$ are both positivity preserving with respect to $\mathfrak{p}_x\otimes\mathfrak{F}(\mathfrak{p}_y)$.

LEMMA 7.5. *Let* $\widehat{L}_{\rm bp}(P)=\frac{1}{4}(P-\mathbb{1}\otimes\widehat{P}_{\rm f})^2+\Big(-\Delta_x+\frac{\alpha U}{|x|}\Big)\otimes\mathbb{1}+\mathbb{1}\otimes\widehat{N}_{\rm f}$ *and let* $C_m=\int_{\mathbb{R}_x^3}^{\oplus} C_m(x)\,\mathrm{d}x$ *with* $C_m(x)=\int_{\mathbb{R}_y^3}\mathrm{d}y\,G_m(x,y)b(y)$. *Then we have the following properties:*

(i) $0\trianglelefteq \mathrm{e}^{-t\widehat{L}_{\rm bp}(0)}$ *with respect to* $\mathfrak{p}_x\otimes\mathfrak{F}(\mathfrak{p}_y)$ *for all* $t\geqslant 0$.
(ii) $0\trianglelefteq C_m$ *and* $0\trianglelefteq C_m^*$ *with respect to* $\mathfrak{p}_x\otimes\mathfrak{F}(\mathfrak{p}_y)$.
(iii) $0\trianglelefteq \mathrm{e}^{-t\widehat{H}_{{\rm bp},\varrho_m}(0)}$ *with respect to* $\mathfrak{p}_x\otimes\mathfrak{F}(\mathfrak{p}_y)$ *for all* $t\geqslant 0$ *and* $m>0$.

Proof. (i) Since $\widehat{L}_{\rm bp}(0)$ is reduced by $L^2(\mathbb{R}_x^3)\otimes L^2_{\rm sym}(\mathbb{R}_y^{3n})$ for all $n\in\mathbb{N}_0$, it sufficies to show that $0\trianglelefteq \mathrm{e}^{-t\widehat{L}_{\rm bp}(0)}\restriction L^2(\mathbb{R}_x^3)\otimes L^2_{\rm sym}(\mathbb{R}_y^{3n})$ with respect to $\mathfrak{P}_n$ for all $n\in\mathbb{N}_0$ by Proposition 2.8. Note first that $\mathrm{e}^{-t\widehat{P}_{\rm f}^2}\restriction L^2_{\rm sym}(\mathbb{R}_y^{3n})=\mathrm{e}^{-t(\sum_{j=1}^n(-\mathrm{i}\nabla_{y_j}))^2}$ and clearly the right hand side is positivity preserving with respect to $\otimes_{\rm s}^n\mathfrak{p}_y$. On the other hand, with the notation $h=-\Delta_x+U\alpha/|x|$, we have $\mathrm{e}^{-th}\trianglerighteq 0$ for all $t\geqslant 0$. Thus $\mathrm{e}^{-t\widehat{L}_{\rm bp}(0)}\restriction L^2(\mathbb{R}_x^3)\otimes L^2_{\rm sym}(\mathbb{R}^{3n})=\mathrm{e}^{-th}\otimes\mathrm{e}^{-t\{(\sum_{j=1}^n(-\mathrm{i}\nabla_{y_j}))^2+n\}}\trianglerighteq 0\,\forall t\geqslant 0$.

(ii) Note that $G_m(x,y)>0\,\forall x,y$. Hence, for all $x\in\mathbb{R}_x^3$, we have $0\trianglelefteq C_m(x)$ and $0\trianglelefteq C_m(x)^*$ with respect to $\mathfrak{F}(\mathfrak{p}_y)$ by Theorem 3.3(iii). Thus we have the desired result by Proposition 2.20.

(iii) This is a direct consequence of Theorem 2.10. ∎

PROPOSITION 7.6. *For all* $m>0$ *and* $t>0$, *we have* $0\triangleleft \mathrm{e}^{-t\widehat{H}_{{\rm bp},\varrho_m}(0)}$ *with respect to* $\mathfrak{p}_x\otimes\mathfrak{F}(\mathfrak{p}_y)$.

Proof. Choose $\varphi,\psi\in(\mathfrak{p}_x\otimes\mathfrak{F}(\mathfrak{p}_y))\backslash\{0\}$. Then there exist $p,q\in\mathbb{N}_0$ such that $\varphi^{(p)}\in\mathfrak{P}_p\backslash\{0\}$ and $\psi^{(q)}\in\mathfrak{P}_q\backslash\{0\}$. Let

$$\mathcal{A}^{(n)}_{\psi^{(p)},\varphi^{(q)}}(s_1,\dots,s_n;t)=\langle\psi^{(p)},\mathrm{e}^{-s_1\widehat{L}_{\rm bp}(0)}(C_m+C_m^*)\cdots(C_m+C_m^*)\mathrm{e}^{-(t-\sum_{j=1}^n s_j)\widehat{L}_{\rm bp}(0)}\varphi^{(q)}\rangle.$$

Taking Theorem 2.15 into consideration, it suffices to show

$$\mathcal{A}^{(p+q)}_{\psi^{(p)},\varphi^{(q)}}(0,\dots,0,s_p,0,\dots,0;t)>0$$

for any $0<s_p\leqslant t$. To this end, observe that

$$\begin{aligned}&\mathcal{A}^{(p+q)}_{\psi^{(p)},\varphi^{(q)}}(0,\dots,0,s_p,0,\dots,0;t)\\ &\geqslant\langle\mathrm{e}^{-s_p\widehat{L}_{\rm bp}(0)/2}C_m^p\psi^{(p)},\mathrm{e}^{-s_p\widehat{L}_{\rm bp}(0)/2}C_m^q\mathrm{e}^{-(t-s_p)\widehat{L}_{\rm bp}(0)}\varphi^{(q)}\rangle.\end{aligned}\tag{7.1}$$

For any $x\in\mathbb{R}_x^3$ we have

$$(C_m^p\psi^{(p)})(x)=\sqrt{(p+1)!}\langle\psi^{(p)}(x,\cdot),\otimes_{\rm s}^p G_m(x,\cdot)\rangle_{\mathfrak{F}(L^2(\mathbb{R}_y^3))}$$

$$= \sqrt{(p+1)!}\int dy_1\cdots dy_p\,\psi^{(p)}(x,y_1,\ldots,y_n)G_m(x,y_1)\cdots G_m(x,y_p).$$

Since $G_m(x,y) > 0\,\forall x,y \in \mathbb{R}^3$, we conclude that $C_m^p\psi^{(p)} \in \mathfrak{p}_x\backslash\{0\}$ as a function on $\mathbb{R}^3_x$. Let $h = -\Delta_x + U\alpha/|x|$. Since $0 \lhd e^{-th}$ with respect to $\mathfrak{p}_x = \mathfrak{P}_0$ for all $t > 0$, we obtain $e^{-s_p\widehat{L}_{\rm bp}(0)/2}C_m^p\psi^{(p)} = e^{-s_ph/2}C_m^p\psi^{(p)} > 0$ with respect to $\mathfrak{p}_x$. Similarly $e^{-s_p\widehat{L}_{\rm bp}(0)/2}C_m^q e^{-(t-s_p)\widehat{L}_{\rm bp}(0)}\varphi^{(q)} > 0$ with respect to $\mathfrak{p}_x$. Thus the right hand side of (7.1) is strictly positive. ∎

For the proof of Theorem 7.2 choose m,n as $n > m$. Then one can check the all conditions in Theorem 2.16. (Remark that $\widehat{H}_{{\rm bp},\varrho_m}(0) - \widehat{H}_{{\rm bp},\varrho_n}(0) = (C_n - C_m) + (C_n^* - C_m^*) \trianglerighteq 0$ because $G_n(x,y) \geqslant G_m(x,y)$.)

7.3. PROOF OF THEOREM 7.3.

PROPOSITION 7.7. *There exists a $P_{\rm c} > 0$ such that, for $|P| < P_{\rm c}$, $H_{\rm bp}(P)$ has a unique ground state φ_P. Moreover $\vartheta\varphi_0$ is strictly positive with respect to $\mathfrak{p}_x \otimes \mathfrak{F}(\mathfrak{p}_y)$.*

Proof. In [28], it has been established that

$$\inf {\rm ess.spec}(H_{\rm bp}(P)) - \inf {\rm spec}(H_{\rm bp}(P)) \geqslant \min\{1, E_{\rm bin}(\alpha,U)\} - \frac{P^2}{4}.$$

Thus $H_{\rm bp}(P)$ has a ground state φ_P for $|P| < 2\min\{\sqrt{E_{\rm bin}(\alpha,U)}, 1\}$ under the binding condition $(\alpha,U) \in \Lambda_{\rm bin}$. By Theorem 7.2, φ_0 is nondegenerate and $\vartheta\varphi$ is strictly positive with respect to $\mathfrak{p}_x \otimes \mathfrak{F}(\mathfrak{p}_y)$.

Fix $\omega \in \mathbb{S}^2$ arbitrarily. Then $H_{\rm bp}(\beta\omega)$ is an analytic family of type (B) with respect to $\beta \in \mathbb{C}$ by Lemma B.1. In particular $H_{\rm bp}(\beta\omega)$ is an analytic family in the sense of Kato [35]. Set $E(P) = \inf {\rm spec}(H_{\rm bp}(P))$ and take $\varepsilon > 0$ as $\varepsilon < {\rm dist}\{E(0), {\rm spec}(H_{\rm bp}(0))\backslash\{E(0)\}\}$. Then we can choose $P_{\rm c} > 0$ so that $E \notin {\rm spec}(H_{\rm bp}(\beta\omega))$ if $|E - E(0)| = \varepsilon$ and $|\beta| < P_{\rm c}$. Then

$$P(\beta) = -(2\pi {\rm i})^{-1} \oint_{|E-E(0))|=\varepsilon} dE\,(H_{\rm bp}(\beta\omega) - E)^{-1}$$

exists and is analytic for β with $|\beta| \leqslant P_{\rm c}$. Hence $\dim {\rm ran}(P(|P|)) = 1$ for $|P| < P_{\rm c}$ because $\dim {\rm ran}(P(0)) = 1$. Since ω is arbitrary, we have the desired result. ∎

PROPOSITION 7.8. *One has the following properties:*

(i) $e^{i\phi\omega\cdot L_{\rm tot}}\varphi_0 = \varphi_0$ *for all $\omega \in \mathbb{S}^2$ and $\phi \in \mathbb{R}$.*

(ii) *For $P \neq 0$ with $|P| < P_{\rm c}$, set $\omega_P = P/|P|$. Then $e^{i\phi\omega_P\cdot L_{\rm tot}}\varphi_P = \varphi_P$ for all $\phi \in \mathbb{R}$.*

Proof. (i) Observe that $0 \unlhd e^{i\phi\omega\cdot L_{\rm tot}}$ with respect to $\mathfrak{p}_x \otimes \mathfrak{F}(\mathfrak{p}_y)$ for all $\phi \in \mathbb{R}$ and $\omega \in \mathbb{S}^2$. In addition, we see that $e^{i\phi\omega\cdot L_{\rm tot}}e^{is\widehat{H}_{{\rm bp},\rho_m}(0)}e^{-i\phi\omega\cdot L_{\rm tot}} = e^{is\widehat{H}_{{\rm bp},\rho_m}(0)}$ for all ϕ, s and ω. Thus one concludes that $\widehat{H}_{\rm bp}(0)$ commutes with $e^{i\phi\omega\cdot L_{\rm tot}}$ by taking $m \to \infty$. Now we can apply Proposition 2.13 and conclude the rotational symmetry of the ground state φ_0.

(ii) The basic idea of our proof is essentially comming from [23]. First note that $\mathrm{spec}(\omega_P \cdot L_{\mathrm{tot}}) = \mathbb{Z}$. Thus we have the decomposition:

$$L^2(\mathbb{R}^3_x) \otimes \mathfrak{F}(L^2(\mathbb{R}^3_k)) = \bigoplus_{n=-\infty}^{\infty} \mathcal{H}_n(P) \quad \text{with } \mathcal{H}_n(P) = \ker(\omega_P \cdot L_{\mathrm{tot}} - n).$$

We also remark that $\mathrm{e}^{\mathrm{i}\phi\omega_P \cdot L_{\mathrm{tot}}} H_{\mathrm{bp}}(P) \mathrm{e}^{-\mathrm{i}\phi\omega_P \cdot L_{\mathrm{tot}}} = H_{\mathrm{bp}}(P)$. (Indeed we can check the similar relation for any finite ultraviolet cutoff. Hence we can extend the result to the above one by Proposition 7.1.) Accordingly the Hamiltonian $H_{\mathrm{bp}}(P)$ has a corresponding decomposition:

$$H_{\mathrm{bp}}(P) = \bigoplus_{n=-\infty}^{\infty} H^{(n)}_{\mathrm{bp}}(P) \quad \text{with } H^{(n)}_{\mathrm{bp}}(P) = H_{\mathrm{bp}}(P) \restriction \mathcal{H}_n(P).$$

Step 1. We will show that, for any $n \in \mathbb{Z}\setminus\{0\}$, there exists a unitary operator $U_n(P)$ from $\mathcal{H}_n(P)$ to $\mathcal{H}_{-n}(P)$ such that

$$U_n(P) H^{(n)}_{\mathrm{bp}}(P) U_n(P)^* = H^{(-n)}_{\mathrm{bp}}(P).$$

It suffices to consider the case $P_0 = (0, 0, |P|)$ because we have the following relation:

$$\mathrm{e}^{\mathrm{i}\phi\omega \cdot L_{\mathrm{tot}}} H_{\mathrm{bp}}(P) \mathrm{e}^{-\mathrm{i}\phi\omega_P \cdot L_{\mathrm{tot}}} = H_{\mathrm{bp}}(g^{-1}(\phi, \omega) P)$$

for all $\phi \in \mathbb{R}$ and $\omega \in \mathbb{S}^2$, where $g(\phi, \omega) \in SO(3)$ is the rotation around ω with angle ϕ. Let u_q be a unitary operator on $L^2(\mathbb{R}^3_q)$ given by $(u_q f)(q_1, q_2, q_3) = f(-q_1, q_2, q_3)$ for $f \in L^2(\mathbb{R}^3_q)$. Then, with $\omega_0 := \omega_{P_0} = (0, 0, 1)$, we see that $u_x \otimes \Gamma(u_k)(\omega_0 \cdot L_{\mathrm{tot}}) u_x^* \otimes \Gamma(u_k)^* = -\omega_0 \cdot L_{\mathrm{tot}}$ which means $u_x \otimes \Gamma(u_k)\mathcal{H}_n(P_0) = \mathcal{H}_{-n}(P_0)$ for all n. Moreover it is verified that $u_x \otimes \Gamma(u_k) H_{\mathrm{bp}}(P_0) u_x^* \otimes \Gamma(u_k)^* = H_{\mathrm{bp}}(P_0)$. (For the proof check first the above relation with finite cutoffs, then extend the result to the case without cutoffs by Proposition 7.1.) Thus by setting $U_n(P_0) = u_x \otimes \Gamma(u_k)$, we have the desired result.

Step 2. We will prove the rotation symmetry in (ii). Since φ_P is a unique ground state, it must belong to $\mathcal{H}_n(P)$ for some $n \in \mathbb{Z}$. If $n \neq 0$, $U_n(P)\varphi_P$ is ground state for $H_{\mathrm{bp}}(P)$ too and in $\mathcal{H}_{-n}(P)$ by Step 1. This means $H_{\mathrm{bp}}(P)$ has at least two ground states φ_P and $U_n(P)\varphi_P$ which contradicts the uniqueness. Thus n must be 0: $\omega_P \cdot L_{\mathrm{tot}} \varphi_P = 0$. This completes the proof. ∎

Appendix A. REMOVAL OF ULTRAVIOLET CUTOFFS. I

The basic idea in this appendix is essentially due to Nelson [31]. For $\rho \in \mathcal{E} := \{\rho \in L^\infty(\mathbb{R}^3_k) : 0 \leqslant \rho(k) \leqslant 1 \text{ a.e. } k \text{ and } \rho(k)/|k| \in L^2(\mathbb{R}^3_k)\}$, we introduce

$$H_\rho(P) = \frac{1}{2}(P - P_{\mathrm{f}})^2 + \sqrt{\alpha}\lambda_0 \int_{\mathbb{R}^3_k} \mathrm{d}k \, \frac{\rho(k)}{(2\pi)^{3/2}|k|} [a(k) + a(k)^*] + N_{\mathrm{f}}.$$

Then, by the Kato–Rellich theorem, $H_\rho(P)$ is self-adjoint on $\mathrm{dom}(N_\mathrm{f}) \cap \mathrm{dom}(P_\mathrm{f}^2)$ and bounded from below, for all $0 \leqslant \alpha < \infty$, $P \in \mathbb{R}^3$ and $\rho \in \mathcal{E}$. Let

$$T_{K,\rho} = \int_{\mathbb{R}^3_k} \mathrm{d}k\, \beta_{K,\rho}(k)[a(k) - a(k)^*], \quad \beta_{K,\rho}(k) = -\frac{\sqrt{\alpha}\lambda_0}{(2\pi)^{3/2}} \frac{\rho(k)}{|k|(1+k^2/2)}(1-\chi_K(k)),$$

where $\chi_K(k) = 1$ if $|k| < K$, $\chi_K(k) = 0$ otherwise. Then $T_{K,\rho}$ is essentially skew-adjoint: $T^*_{K,\rho} = -T^{**}_{K,\rho}$. Henceforth we denote the closure of $T_{K,\rho}$ by the same symbol (hence $T_{K,\rho}$ is skew-adjoint). Using the formula

$$\mathrm{e}^{T_{K,\rho}} a(k) \mathrm{e}^{-T_{K,\rho}} = a(k) + \beta_{K,\rho}(k),$$

we obtain

$$\langle \varphi, \widetilde{H}_{K,\rho}(P)\psi\rangle = \langle \varphi, \mathrm{e}^{T_{K,\rho}} H_\rho(P) \mathrm{e}^{-T_{K,\rho}}\psi\rangle = \langle \varphi, H_0(P)\psi\rangle + B_{K,\rho}(\varphi,\psi)$$

for each $\varphi, \psi \in \mathrm{dom}(H_0(P)^{1/2}) \times \mathrm{dom}(H_0(P)^{1/2})$, where $H_0(P) = (1/2)(P-P_\mathrm{f})^2 + N_\mathrm{f}$ and

$$\begin{aligned} &B_{K,\rho}(\varphi,\psi) \\ &= -\langle (P-P_\mathrm{f})\varphi, A_{K,\rho}\psi\rangle - \langle A_{K,\rho}\varphi, (P-P_\mathrm{f})\psi\rangle \\ &\quad + \frac{1}{2}\langle A_{K,\rho}\varphi, A^*_{K,\rho}\psi\rangle + \frac{1}{2}\langle A^*_{K,\rho}\varphi, A_{K,\rho}\psi\rangle + \langle A_{K,\rho}\varphi, A_{K,\rho}\psi\rangle + \langle \varphi, H_{IK}\psi\rangle + E_{K,\rho}\langle\varphi,\psi\rangle \end{aligned}$$

with

$$A_{K,\rho} = \int_{\mathbb{R}^3_k} \mathrm{d}k\, k\beta_{K,\rho}(k) a(k), \quad H_{IK} = \sqrt{\alpha}\lambda_0 \int_{|k|\leqslant K} \frac{\mathrm{d}k}{(2\pi)^{3/2}|k|}[a(k) + a(k)^*],$$

$$E_{K,\rho} = -\alpha\lambda_0^2 \int_{|k|\leqslant K} \mathrm{d}k \frac{\rho(k)^2}{(2\pi)^3|k|^2(1+k^2/2)}.$$

PROPOSITION A.1. *Assume that $\rho \in \mathcal{E}$. Then, the following operator identity holds for all $P \in \mathbb{R}^3$ and $0 \leqslant K < \infty$:*

$$\begin{aligned} \mathrm{e}^{T_{K,\rho}} H_\rho(P) \mathrm{e}^{-T_{K,\rho}} &= H_0(P) - (P-P_\mathrm{f})\cdot A_{K,\rho} - A^*_{K,\rho}\cdot(P-P_\mathrm{f}) \\ &\quad + \frac{1}{2}A^*_{K,\rho}\cdot A^*_{K,\rho} + \frac{1}{2}A_{K,\rho}\cdot A_{K,\rho} + A^*_{K,\rho}\cdot A_{K,\rho} + H_{IK} + E_{K,\rho}. \end{aligned} \tag{A.1}$$

Proof. Let us denote the right hand side of (A.1) by $\mathcal{H}_{K,\rho}(P)$. Then, by the ineqaulities $\|a(f)^\#(N_\mathrm{f}+\mathbb{1})^{-1/2}\| \leqslant \|f\|$ and $\|a(f)^{\#_1}a(g)^{\#_2}(N_\mathrm{f}+\mathbb{1})^{-1}\| \leqslant C\|f\|\|g\|$, we have $\|\mathcal{H}_{K,\rho}(P)\varphi\| \leqslant \mathrm{const.}(\|H_0(P)\varphi\| + \|\varphi\|)$ $\forall \varphi \in \mathrm{dom}(H_0(P))$ under the assumption $|k|\rho(k)/(1+k^2/2) \in L^2(\mathbb{R}^3_k)$, because, in order to estimate the term $P_\mathrm{f}\cdot A_{K,\rho}\varphi$, we used the commutation relation between P_f and $A_{K,\rho}$ which induces an extra term $\int \mathrm{d}k\, k^2\beta_{K,\rho}(k)a(k)\varphi$. Thus we have

$$\|H_\rho(P)\mathrm{e}^{T_{K,\rho}}\varphi\| = \|\mathrm{e}^{T_{K,\rho}} H_\rho(P) \mathrm{e}^{-T_{K,\rho}}\varphi\| \leqslant \mathrm{const.}(\|H_0(P)\varphi\| + \|\varphi\|) \tag{A.2}$$

for all $\varphi \in \mathfrak{F}_{\rm fin}(C_0^\infty(\mathbb{R}_k^3))$. Since $\mathrm{dom}(H_\rho(P)) = \mathrm{dom}(H_0(P))$, we have

$$\|H_0(P)\mathrm{e}^{-T_{K,\rho}}\varphi\| \leqslant \mathrm{const.}(\|H_0(P)\varphi\| + \|\varphi\|)$$

for all $\varphi \in \mathfrak{F}_{\rm fin}(C_0(\mathbb{R}_k^3))$ by the closed graph theorem and (A.2). Hence we conclude that $\mathrm{e}^{-T_{K,\rho}}\mathrm{dom}(H_0(P)) \subseteq \mathrm{dom}(H_0(P))$. Similarly we have $\mathrm{e}^{T_{K,\rho}}\mathrm{dom}(H_0(P)) \subseteq \mathrm{dom}(H_0(P))$. Summarizing the results, we have $\mathrm{dom}(\mathrm{e}^{T_{K,\rho}}H_\rho(P)\mathrm{e}^{-T_{K,\rho}}) = \mathrm{dom}(\mathrm{e}^{T_{K,\rho}}H_0(P)\mathrm{e}^{-T_{K,\rho}}) = \mathrm{dom}(H_0(P)) = \mathrm{dom}(H_\rho(P))$. Since $\mathrm{e}^{T_{K,\rho}}H_\rho(P)\mathrm{e}^{-T_{K,\rho}} = \mathcal{H}_{K,\rho}(P)$ on $\mathfrak{F}_{\rm fin}(C_0^\infty(\mathbb{R}_k^3))$ which is a core for $\mathrm{e}^{T_{K,\rho}}H_\rho(P)\mathrm{e}^{-T_{K,\rho}}$, we have the operator equality (A.1). ∎

Let $\mathcal{E}_0 := \{\rho \in L^\infty(\mathbb{R}_k^3) : 0 \leqslant \rho(k) \leqslant 1 \text{ a.e. } k\}$. Even if $\rho \notin \mathcal{E}$ but $\rho \in \mathcal{E}_0$, the linear operators $A_{K,\rho}$ and $T_{K,\rho}$ are well-defined because $\beta_{K,\rho}$ and $|k|\beta_{K,\rho}$ are in $L^2(\mathbb{R}_k^3)$. Hence the form $B_{K,\rho}(\cdot,\cdot)$ is also well-defined on $\mathrm{dom}(H_0(P)^{1/2}) \times \mathrm{dom}(H_0(P)^{1/2})$ in this case: $\rho \in \mathcal{E}_0 \backslash \mathcal{E}$.

LEMMA A.2. *Let $\rho \in \mathcal{E}_0$. There exists $0 < C_{\varepsilon,K} < \infty$, for any $\varepsilon > 0$ and $\varphi \in \mathrm{dom}(H_0(P)^{1/2}) = \mathrm{dom}(H_0(0)^{1/2})$, such that*

$$|B_{k,\rho}(\varphi,\varphi)| \leqslant (4C(K)^2 + 2C(K) + \varepsilon)\|(H_0(P) + 1\!\!1)^{1/2}\varphi\|^2 + C_{\varepsilon,K}\|\varphi\|^2$$

with

$$C(K)^2 = \int_{|k|>K} \mathrm{d}k\, \frac{\alpha\lambda_0^2}{(2\pi)^3(1+k^2/2)^2}.$$

Proof. For each $\varphi \in \mathrm{dom}(H_0(P)^{1/2})$, we have

$$\|(P - P_{\rm f})\varphi\| \leqslant \|(H_0(P) + 1\!\!1)^{1/2}\varphi\|, \quad \|A^{\#}_{K,\rho}\varphi\| \leqslant C(K)\|(H_0(P) + 1\!\!1)^{1/2}\varphi\|.$$

Using these formulas, we obtain

$$|\langle (P - P_{\rm f})\varphi, A_{K,\rho}\varphi\rangle| \leqslant 2C(K)\|(H_0(P) + 1\!\!1)^{1/2}\varphi\|^2,$$
$$|\langle A^{\#_1}_{K,\rho}\varphi, A^{\#_2}_{K,\rho}\varphi\rangle| \leqslant C(K)^2\|(H_0(P) + 1\!\!1)^{1/2}\varphi\|^2,$$
$$|\langle \varphi, H_{IK}\varphi\rangle| \leqslant \varepsilon\|(H_0(P) + 1\!\!1)^{1/2}\varphi\|^2 + \frac{4}{\varepsilon}C_2(K)\|\varphi\|^2.$$

By these estimates, we obtain the result. ∎

Choose K sufficiently large as $4C(K)^2 + 2C(K) < 1$. Then, by the above lemma and the KLMN theorem [34], there exists a unique self-adjoint operator $\widetilde{H}_{K,\rho}(P)$ such that, for every $\rho \in \mathcal{E}_0$,

$$\langle \varphi, \widetilde{H}_{K,\rho}(P)\varphi\rangle = \langle \varphi, H_0(P)\varphi\rangle + B_{K,\rho}(\varphi,\varphi).$$

LEMMA A.3. *For $\rho_1, \rho_2 \in \mathcal{E}_0$, we obtain*

$$\begin{aligned}&|B_{K,\rho_1}(\varphi,\varphi) - B_{K,\rho_2}(\varphi,\varphi)|\\ &\leqslant \{4C(\rho_1-\rho_2;K)^2 + 4C(K)C(\rho_1-\rho_2;K) + |E_{K,\rho_1} - E_{K,\rho_2}|\} \times \|(H_0(P)+1\!\!1)^{1/2}\varphi\|^2,\end{aligned}$$

where

$$C(\rho_1-\rho_2;K)^2=\alpha\lambda_0^2\int\limits_{|k|>K}\mathrm{d}k\,\frac{(\rho_1(k)-\rho_2(k))^2}{(2\pi)^3(1+k^2/2)^2}.$$

By the similar arguments in the proof of Lemma A.2, we see the result in the lemma.

As a corollary of the above lemma, we obtain the following.

PROPOSITION A.4. *Let $\rho_n\in\mathcal{E}_0$ be a sequence such that $\rho_n(k)\to 1$ a.e. k as $n\to\infty$. For K with $4C(K)^2+2C(K)<1$, $\widetilde{H}_{K,\rho_n}(P)$ converges to $\widetilde{H}_{K,1}(P)$ as $n\to\infty$ in the norm resolvent sense. Moreover*

$$\lim_{n\to\infty}\inf\mathrm{spec}(\widetilde{H}_{K,\rho_n}(P))=\inf\mathrm{spec}(\widetilde{H}_{K,1}(P)),$$
$$\lim_{n\to\infty}\inf\mathrm{ess.spec}(\widetilde{H}_{K,\rho_n}(P))=\inf\mathrm{ess.spec}(\widetilde{H}_{K,1}(P)).$$

Proof. By Lemma A.3, we can show the following:

$$\lim_{n\to\infty}B_{K,\rho_n}(\varphi,\varphi)=B_{K,1}(\varphi,\varphi)$$

uniformly on any set of φ in $\mathrm{dom}(H_0(P)^{1/2})$ for which $\|(H_0(P)+\mathbb{1})^{1/2}\varphi\|$ is bounded. Thus applying Theorem VIII.25 of [34], we see that $\widetilde{H}_{K,\rho_n}(P)$ converges to $\widetilde{H}_{K,1}(P)$ as $n\to\infty$ in the norm resolvent sense.

By Lemma A.3, we have $\widetilde{H}_{K,\rho_n}(P)\leqslant\widetilde{H}_{K,1}(P)+D_n(H_0(P)+\mathbb{1})$ with $\lim_{n\to\infty}D_n=0$. On the other hand, by Lemma A.2, one has $H_0(P)+\mathbb{1}\leqslant C(\widetilde{H}_{K,\rho_n}(P)+\mathbb{1})$ with C independent of n. Summarizing these inequalities, one arrives at $\widetilde{H}_{K,\rho_n}(P)\leqslant\widetilde{H}_{K,1}(P)+CD_n(\widetilde{H}_{K,\rho_n}(P)+\mathbb{1})$. By exchanging the roles of $\widetilde{H}_{K,\rho_n}(P)$ and $\widetilde{H}_{K,1}(P)$, one also obtains $\widetilde{H}_{K,1}(P)\leqslant\widetilde{H}_{K,\rho_n}(P)+CD_n(\widetilde{H}_{K,1}(P)+\mathbb{1})$. Combining these estimates and the min-max principle Theorem XIII.2 of [35], we can conclude the remainder assertions. ∎

Proof of Propositions 6.1 *and* 6.5. Choose ρ as $\rho=\chi_\kappa$. Set K as $4C(K)^2+2C(K)<1$. Then, by Proposition A.1, we have

$$\mathrm{e}^{T_{K,\chi_\kappa}}H_\kappa(P)\mathrm{e}^{-T_{K,\chi_\kappa}}=\widetilde{H}_{K,\chi_\kappa}(P).\tag{A.3}$$

Let $H(P)=\mathrm{e}^{-T_{K,1}}\widetilde{H}_{K,1}\mathrm{e}^{T_{K,1}}$. Since $\mathrm{e}^{\pm T_{K,\chi_\kappa}}$ strongly converges to $\mathrm{e}^{\pm T_{K,1}}$, we can show Proposition 6.1 by (A.3) and Proposition A.4. Similarly we can prove Proposition 6.5. ∎

Appendix B. REMOVAL OF ULTRAVIOLET CUTOFFS. II

In the case of the bipolaron, ultraviolet cutoffs can be removed as we did in Appendix A. In this appendix, we will explain how to carry out this briefly. For more details we refer to [28].

For $\rho \in \mathcal{E}$, we introduce

$$H_{\mathrm{bp},\rho}(P) = \frac{1}{4}(P - \mathbb{1}\otimes P_{\mathrm{f}})^2 + \Big(- \Delta_x + \frac{U\alpha}{|x|}\Big) \otimes \mathbb{1} + \mathbb{1}\otimes N_{\mathrm{f}} + 2\sqrt{\alpha}\lambda_0 \int_{\mathbb{R}^3_k} \frac{\mathrm{d}k}{(2\pi)^{3/2}|k|}\rho(k)\cos(k\cdot x/2)\otimes [a(k)+a(k)^*].$$

Let

$$W_{\rho,K} = \exp\Big\{ \sum_{j=1,2} \int \mathrm{d}k\, \beta_{K,\rho}(k)[\mathrm{e}^{\mathrm{i}k\cdot(-1)^{j-1}x/2}\otimes a(k) - \mathrm{e}^{-\mathrm{i}k\cdot(-1)^{j-1}x/2}\otimes a(k^*)]\Big\}.$$

A direct calculation yields $W_{K,\rho}H_{\mathrm{bp},\rho}(P)W_{K,\rho}^* = \widetilde{H}^{\mathrm{bp}}_{K,\rho}(P)$ with

$$\begin{aligned}\widetilde{H}^{\mathrm{bp}}_{K,\rho}(P) =& \frac{1}{4}(P - \mathbb{1}\otimes P_{\mathrm{f}})^2 - \Delta_x\otimes\mathbb{1} + \frac{\alpha U}{|x|}\otimes\mathbb{1} + \mathbb{1}\otimes N_{\mathrm{f}} \\ &+ \sum_{j=1,2}\Big\{ - \Big[(-1)^{j-1}(-\mathrm{i}\nabla_x)\otimes\mathbb{1} + \frac{1}{2}(P-\mathbb{1}\otimes P_{\mathrm{f}})\Big]\cdot A_{K,\rho}\Big((-1)^{j-1}\frac{x}{2}\Big) \\ &- A_{K,\rho}\Big((-1)^{j-1}\frac{x}{2}\Big)^*\cdot\Big[(-1)^{j-1}(-\mathrm{i}\nabla_x)\otimes\mathbb{1} + \frac{1}{2}(P-\mathbb{1}\otimes P_{\mathrm{f}})\Big] \\ &+ \frac{1}{2}A_{K,\rho}\Big((-1)^{j-1}\frac{x}{2}\Big)^2 + \frac{1}{2}A_{K,\rho}\Big((-1)^{j-1}\frac{x}{2}\Big)^{*2} \\ &+ A_{K,\rho}\Big((-1)^{j-1}\frac{x}{2}\Big)^*\cdot A_{K,\rho}\Big((-1)^{j-1}\frac{x}{2}\Big)\Big\} \\ &+ 2\sqrt{\alpha}\lambda_0\int_{|k|\leqslant K}\frac{\mathrm{d}k}{(2\pi)^{3/2}|k|}\rho(k)\cos(k\cdot x/2)\otimes[a(k)+a(k)^*] \\ &+ V_{K,\rho}(x)\otimes\mathbb{1} + E_{K,\rho}, \end{aligned} \tag{B.1}$$

where

$$A_{K,\rho}(x) = \int_{\mathbb{R}^3_k} \mathrm{d}k\, k\beta_{K,\rho}(k)\mathrm{e}^{\mathrm{i}k\cdot x}\otimes a(k),$$

$$V_{K,\rho}(x_1 - x_2) = \sum_{i\neq j}\int_{\mathbb{R}^3_k}\mathrm{d}k\Big\{\beta_{K,\rho}(k)^2 + \frac{2\sqrt{\alpha}\lambda_0}{(2\pi)^{3/2}|k|}\beta_{K,\rho}(k)\Big\}\mathrm{e}^{-\mathrm{i}k\cdot(x_i-x_j)},$$

$$E_{K,\rho} = -2\alpha\lambda_0^2\int_{K\leqslant|k|}\mathrm{d}k\,\frac{\rho(k)^2}{(2\pi)^3(1+k^2/2)|k|^2}.$$

Even for $\rho \in \mathcal{E}_0$, the right hand side of (B.1) can be defined as the self-adjoint operator associated with the form for sufficiently large K. Moreover by noting the fact $1 \in \mathcal{E}_0$, the Hamiltonian without ultraviolet cutoff $H_{\mathrm{bp}}(P)$ is concretely given by

$$H_{\mathrm{bp}}(P) = W_{K,1}\widetilde{H}^{\mathrm{bp}}_{K,1}(P)W_{K,1}^*$$

with K sufficiently large. The arguments about removal of cutoffs are in parallel to Appendix A.

LEMMA B.1. *Fix $\omega \in \mathbb{S}^2$ arbitrarily. We have the following:*

(i) *For $\beta \in \mathbb{C}$, the form domain of $\widetilde{H}^{\mathrm{bp}}_{K,1}(\beta\omega)$ is given by* $\mathrm{dom}(H_{\mathrm{bp},0}(0)^{1/2})$ *with* $H_{\mathrm{bp},0}(0) = (1/4)\mathbb{1} \otimes P_{\mathrm{f}}^2 + (-\Delta_x) \otimes \mathbb{1} + \mathbb{1} \otimes N_{\mathrm{f}}$.

(ii) $\langle \varphi, \widetilde{H}^{\mathrm{bp}}_{K,1}(\beta\omega)\varphi\rangle$ *is an analytic function for each* $\varphi \in \mathrm{dom}(H_{\mathrm{bp},0}(0)^{1/2})$.

Proof. The proof of (i) is almost same as that of Lemma A.2. (ii) follows from (B.1) directly. ∎

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