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GENERALIZED INVERSES AND POLAR DECOMPOSITION OF UNBOUNDED REGULAR OPERATORS ON HILBERT C^*-MODULES

MICHAEL FRANK and KAMRAN SHARIFI

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ABSTRACT. In this note we show that an unbounded regular operator t on Hilbert C^*-modules over an arbitrary C^*-algebra $\mathcal{A}$ has polar decomposition if and only if the closures of the ranges of t and $|t|$ are orthogonally complemented, if and only if the operators t and t^* have unbounded regular generalized inverses. For a given C^*-algebra $\mathcal{A}$ any densely defined $\mathcal{A}$-linear closed operator t between Hilbert C^*-modules has polar decomposition, if and only if any densely defined $\mathcal{A}$-linear closed operator t between Hilbert C^*-modules has generalized inverse, if and only if $\mathcal{A}$ is a C^*-algebra of compact operators.

KEYWORDS: *Hilbert C^*-module, unbounded operator, polar decomposition, generalized inverses, C^*-algebras of compact operator.*

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1. INTRODUCTION

In the theory of C^*-algebras, an important role is played by the spaces which are modules over a C^*-algebra and are equipped with a structure which is like an inner product but which, instead of being scalar-valued as in the case of Hilbert spaces, takes its values in the C^*-algebra. Such modules are called (pre-)Hilbert C^*-modules. Let us quickly recall the definition of a Hilbert C^*-module.

A (left) *pre-Hilbert C^*-module* over a (not necessarily unital) C^*-algebra $\mathcal{A}$ is a left $\mathcal{A}$-module E equipped with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{A}$, which is $\mathcal{A}$-linear in the first variable and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*; \quad \langle x, x \rangle \geqslant 0 \quad \text{with equality if and only if } x = 0.$$

We always suppose that the linear structures of $\mathcal{A}$ and E are compatible.

A pre-Hilbert $\mathcal{A}$-module E is called a *Hilbert $\mathcal{A}$-module* if E is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|_{\mathcal{A}}^{1/2}$. If E, F are two Hilbert $\mathcal{A}$-modules then the set of all ordered pairs of elements $E \oplus F$ from E and F is a Hilbert $\mathcal{A}$-module with respect to the $\mathcal{A}$-valued inner product $\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle_E + \langle y_1, y_2 \rangle_F$. It is called the *orthogonal sum of E and F*. A pre-Hilbert $\mathcal{A}$-module E of a pre-Hilbert $\mathcal{A}$-module F is an orthogonal summand if $E \oplus E^{\perp} = F$, where $E^{\perp}$ denotes the orthogonal complement of E in F. If F is a Hilbert $\mathcal{A}$-module and $F = E \oplus E^{\perp}$ then E and $E^{\perp}$ are necessarily Hilbert $\mathcal{A}$-submodules (cf. Lemma 15.3.4 of [20]). Some interesting results about orthogonally complemented submodules can be found in [5], [4], [14], [18]. For the basic theory of Hilbert C^*-modules we refer to the books [13], [15] and to some chapters of [20].

As a convention, throughout the present paper we assume $\mathcal{A}$ to be an arbitrary C^*-algebra (i.e. not necessarily unital). Since we deal with bounded and unbounded operators at the same time we simply denote bounded operators by capital letters and unbounded operators by lower case letters. We use the denotations $\mathrm{Dom}(\cdot)$, $\mathrm{Ker}(\cdot)$ and $\mathrm{Ran}(\cdot)$ for domain, kernel and range of operators, respectively.

Suppose $E,\ F$ are Hilbert $\mathcal{A}$-modules. We denote the set of all bounded $\mathcal{A}$-linear maps $T : E \to F$ for which there is a map $T^* : F \to E$ such that the equality $\langle Tx, y \rangle_F = \langle x, T^* y \rangle_E$ holds for any $x \in E,\ y \in F$ by $B(E, F)$. The operator T^* is called the *adjoint operator* of T.

The polar decomposition is a useful tool that represents an operator as a product of a partial isometry and a positive element. It is well known that every bounded operator on Hilbert spaces has polar decomposition. In general bounded adjointable operators on Hilbert C^*-modules do not have polar composition, but Wegge-Olsen has given a necessary and sufficient condition for bounded adjointable operators to admit polar decomposition. He has proved that a bounded adjointable operator T has polar decomposition if and only if $\overline{\mathrm{Ran}(T)}$ and $\overline{\mathrm{Ran}(|T|)}$ are orthogonal direct summands (cf. Theorem 15.3.7 of [20]).

Let us review the polar decomposition of densely defined closed operators on Hilbert spaces. Suppose H and H' are Hilbert spaces and $t : \mathrm{Dom}(t) \subseteq H \to H'$ is a densely defined closed operator, then there exists a partial isometry $\mathcal{V} \in B(H, H')$ such that

$$t = \mathcal{V}|t|, \quad \mathrm{Ker}(t) = \mathrm{Ker}(\mathcal{V}),$$

where $|t| := (t^* t)^{1/2}$ (cf. VI. Section 2.7 of [9]). Furthermore, every densely defined closed operator $t : \mathrm{Dom}(t) \subseteq H \to H'$ has a densely defined, closed generalized inverse, i.e. there exists a densely defined closed operator s such that $tst = t$, $sts = s$, $(ts)^* = \overline{ts}$ and $(st)^* = \overline{st}$ (cf. Lemma 12 of [17]).

In [8] Guljaš lifts the above facts to the densely defined closed operators on Hilbert C^*-modules over arbitrary C^*-algebras $K(H)$ of all compact operators on a Hilbert space H of arbitrary cardinality. In fact he has found a bijective operation-preserving map between the space of all densely defined closed operators on Hilbert $K(H)$-modules and the space of all densely defined closed operators on a suitable Hilbert space, so he may lift certain properties of operators from Hilbert spaces to Hilbert $K(H)$-modules (cf. Theorems 2.4, 3.1, 3.3 of [8]).

In the present note we give a necessary and sufficient condition for unbounded regular operators to admit polar decomposition. In fact we will prove that an unbounded regular operator t on Hilbert C^*-modules over an arbitrary C^*-algebra $\mathcal{A}$ has polar decomposition if and only if $\overline{\mathrm{Ran}(t)}$ and $\overline{\mathrm{Ran}(|t|)}$ are orthogonally complemented, if and only if the operators t and t^* have unbounded regular generalized inverses.

Some interesting characterizations of an arbitrary C^*-algebra of compact operators (i.e. of a C^*-algebra that admits a faithful $*$-representation in the set of all compact operators on a certain Hilbert space) have been given in [1], [6], [7], [14], [18]. Beside the work of these authors we give other descriptions of the C^*-algebra of compact operators via the above properties.

2. PRELIMINARIES

In this section we recall some definitions and basic facts about regular operators on Hilbert $\mathcal{A}$-modules. These operators were first introduced by Baaj and Julg in [2]. More details and properties can be found in Chapters 9 and 10 of [13], and in the papers [7], [10], [16], [11], [21].

Let E, F be Hilbert $\mathcal{A}$-modules; we will use the notation $t : \mathrm{Dom}(t) \subseteq E \to F$ to indicate that t is an $\mathcal{A}$-linear operator whose domain $\mathrm{Dom}(t)$ is a dense submodule of E (not necessarily identical with E) and whose range is in F. Given $t, s : \mathrm{Dom}(t), \mathrm{Dom}(s) \subseteq E \to F$, we write $s \subseteq t$ if $\mathrm{Dom}(s) \subseteq \mathrm{Dom}(t)$ and $s(x) = t(x)$ for all $x \in \mathrm{Dom}(s)$. A densely defined operator $t : \mathrm{Dom}(t) \subseteq E \to F$ is called *closed* if its graph $G(t) = \{(x, t(x)) : x \in \mathrm{Dom}(t)\}$ is a closed submodule of the Hilbert $\mathcal{A}$-module $E \oplus F$. If t is closable, the operator $s : \mathrm{Dom}(s) \subseteq E \to F$ with the property $G(s) = \overline{G(t)}$ is called the *closure* of t denoted by $s = \bar{t}$. The operator $\bar{t}$ is the smallest closed operator that contains t.

A densely defined operator $t : \mathrm{Dom}(t) \subseteq E \to F$ is called *adjointable* if it possesses a densely defined map $t^* : \mathrm{Dom}(t^*) \subseteq F \to E$ with the domain $\mathrm{Dom}(t^*) = \{y \in F : \text{there exists } z \in E \text{ such that } \langle t(x), y\rangle_F = \langle x, z\rangle_E \text{ for any } x \in \mathrm{Dom}(t)\}$ which satisfies the property $\langle t(x), y\rangle_F = \langle x, t^*(y)\rangle_E$, for any $x \in \mathrm{Dom}(t)$, $y \in \mathrm{Dom}(t^*)$. This property implies that t^* is a closed $\mathcal{A}$-linear map.

REMARK 2.1. Recall that the composition of two densely defined operators t, s is the unbounded operator ts with $\mathrm{Dom}(ts) = \{x \in \mathrm{Dom}(s) : s(x) \in \mathrm{Dom}(t)\}$ given by $(ts)(x) = t(s(x))$ for all $x \in \mathrm{Dom}(ts)$. The operator ts is not necessarily densely defined. Suppose two densely defined operators t, s are adjointable, then $s^*t^* \subseteq (ts)^*$. If T is a bounded adjointable operator, then $s^*T^* = (Ts)^*$.

A densely defined closed $\mathcal{A}$-linear map $t : \mathrm{Dom}(t) \subseteq E \to F$ is called *regular* if it is adjointable and the operator $1 + t^*t$ has a dense range. We denote the set of all regular operators from E to F by $R(E, F)$. A criterion of regularity via the graph of densely defined operators has been given in [7]. In fact a densely defined operator t is regular if and only if its graph is orthogonally complemented in $E \oplus F$ (cf. Corollary 2.2 of [7]). If t is regular then t^* is regular and $t = t^{**}$, moreover t^*t is regular and selfadjoint (cf. Corollaries 9.4, 9.6 and Proposition 9.9 of [13]). Define $Q_t = (1 + t^*t)^{-1/2}$ and $F_t = tQ_t$, then $\mathrm{Ran}(Q_t) = \mathrm{Dom}(t)$, $0 \leqslant Q_t \leqslant 1$ in $B(E, E)$ and $F_t \in B(E, F)$ (cf. Chapter 9 of [13]). The bounded operator F_t is called the bounded transform (or z-transform) of the regular operator t. The map $t \to F_t$ defines a bijection

$$R(E, F) \to \{T \in B(E, F) : \|T\| \leqslant 1 \text{ and } \mathrm{Ran}(1 - T^*T) \text{ is dense } in\ F\},$$

(cf. Theorem 10.4 of [13]). This map is adjoint-preserving, i.e. $F_t^* = F_{t^*}$, and for the bounded transform $F_t = tQ_t = t(1 + t^*t)^{-1/2}$ we have $\|F_t\| \leqslant 1$ and

$$t = F_t(1 - F_t^*F_t)^{-1/2} \quad \text{and} \quad Q_t = (1 - F_t^*F_t)^{1/2}.$$

For a regular operator $t \in R(E) := R(E, E)$ some usual properties may be defined. A regular operator t is called *normal* if and only if $\mathrm{Dom}(t) = \mathrm{Dom}(t^*)$ and $\langle t(x), t(x)\rangle = \langle t^*(x), t^*(x)\rangle$ for any $x \in \mathrm{Dom}(t)$. The operator t is called *selfadjoint* if and only if $t^* = t$, and t is called *positive* if and only if t is normal and $\langle t(x), x\rangle \geqslant 0$ for any $x \in \mathrm{Dom}(t)$. Remarkably, a regular operator t is selfadjoint (respectively, positive) if and only if its bounded transform F_t is selfadjoint (respectively, positive), cf. [11], [13]. Moreover, both t and F_t have the same range and the same kernel. A regular operator t has closed range if and only if its adjoint operator t^* has closed range, and then for $|t| := (t^*t)^{1/2}$ the orthogonal sum decompositions $E = \mathrm{Ker}(t) \oplus \mathrm{Ran}(t^*) = \mathrm{Ker}(|t|) \oplus \overline{\mathrm{Ran}(|t|)}$, $F = \mathrm{Ker}(t^*) \oplus \mathrm{Ran}(t) = \mathrm{Ker}(|t^*|) \oplus \overline{\mathrm{Ran}(|t^*|)}$ exist, cf. Proposition 1.2 of [7] and Result 7.19 of [11].

REMARK 2.2. Let t be a regular operator on an arbitrary Hilbert $\mathcal{A}$-module E, and F_t and Q_t be as above then one can see that $F_t \cdot p(F_t^*F_t) = p(F_tF_t^*) \cdot F_t$ for any polynomial p and, hence, by continuity for any p in $\mathbf{C}([0, 1])$. In particular, $F_t(1 - F_t^*F_t)^{1/2} = (1 - F_tF_t^*)^{1/2}F_t$ and so by the equalities $Q_t = (1 - F_t^*F_t)^{1/2}$ and $Q_{t^*} = (1 - F_tF_t^*)^{1/2}$ we have $tQ_t^2 = Q_{t^*}tQ_t$.

Before closing this section we would like to define the concept of generalized (or pseudo-) inverses of unbounded regular operators, which is motivated by the definitions of densely defined closed operators in [8] and [17].

DEFINITION 2.3. Let $t \in R(E, F)$ be a regular operator between two Hilbert $\mathcal{A}$-modules E, F over some fixed C^*-algebra $\mathcal{A}$. A regular operator $s \in R(F, E)$ is called the generalized inverse of t if $tst = t$, $sts = s$, $(ts)^* = \overline{ts}$ and $(st)^* = \overline{st}$.

If a regular operator t has a generalized inverse s, then the above definition implies that $\mathrm{Ran}(t) \subseteq \mathrm{Dom}(s)$ and $\mathrm{Ran}(s) \subseteq \mathrm{Dom}(t)$. Note that bounded $\mathcal{A}$-linear operators may admit generalized inverses in the set of regular operators even if they do not admit any bounded generalized inverse operator. For examples, consider contractive operators on Hilbert spaces with dense, but non-closed range. Moreover, for bounded linear operators on Hilbert spaces, for example, the property to admit polar decomposition does not imply the property to admit a bounded generalized inverse. More surprising are the results for unbounded operators described in the next section.

3. THE POLAR DECOMPOSITION AND GENERALIZED INVERSES

THEOREM 3.1. *If E, F are arbitrary Hilbert $\mathcal{A}$-modules over a C^*-algebra of coefficients $\mathcal{A}$ and $t \in R(E,F)$ denotes a regular operator then the following conditions are equivalent:*

(i) *t has a unique polar decomposition $t = \mathcal{V}|t|$, where $\mathcal{V} \in B(E,F)$ is a partial isometry for which $\mathrm{Ker}(\mathcal{V}) = \mathrm{Ker}(t)$, $\mathrm{Ker}(\mathcal{V}^*) = \mathrm{Ker}(t^*)$, $\mathrm{Ran}(\mathcal{V}) = \overline{\mathrm{Ran}(t)}$, $\mathrm{Ran}(\mathcal{V}^*) = \overline{\mathrm{Ran}(|t|)}$. That is $\overline{\mathrm{Ran}(t)}$ and $\overline{\mathrm{Ran}(|t|)} = \overline{\mathrm{Ran}(t^*)}$ are final and initial submodules of the partial isometry $\mathcal{V}$, respectively.*

(ii) *$E = \mathrm{Ker}(|t|) \oplus \overline{\mathrm{Ran}(|t|)}$ and $F = \mathrm{Ker}(t^*) \oplus \overline{\mathrm{Ran}(t)}$.*

(iii) *t and t^* have unique generalized inverses which are adjoint to each other, s and s^*. In this situation, $\mathcal{V}^*\mathcal{V} = \overline{t^*s^*}$ is the projection onto $\overline{\mathrm{Ran}(|t|)} = \overline{\mathrm{Ran}(t^*)}$, $\mathcal{V}\mathcal{V}^* = \overline{ts}$ is the projection onto $\overline{\mathrm{Ran}(t)}$, and $\mathcal{V}^*\mathcal{V}|t| = |t|$, $\mathcal{V}^*t = |t|$ and $\mathcal{V}\mathcal{V}^*t = t$.*

Proof. (i) $\Rightarrow$ (ii) Let $\mathcal{V} \in B(E,F)$ be a partial isometry satisfying condition (i). Then the identity map of E can be written as the sum of two orthogonal projections $I - \mathcal{V}^*\mathcal{V}$ and $\mathcal{V}^*\mathcal{V}$. By Result 7.19 of [11] we have

$$\mathrm{Ran}(I-\mathcal{V}^*\mathcal{V}) = \mathrm{Ker}(\mathcal{V}) = \mathrm{Ker}(t) = \mathrm{Ker}(|t|), \quad \mathrm{Ran}(\mathcal{V}^*\mathcal{V}) = \mathrm{Ran}(\mathcal{V}^*) = \overline{\mathrm{Ran}(|t|)}.$$

So we get $E = \mathrm{Ran}(I - \mathcal{V}^*\mathcal{V}) \oplus \mathrm{Ran}(\mathcal{V}^*\mathcal{V}) = \mathrm{Ker}(|t|) \oplus \overline{\mathrm{Ran}(|t|)}$. Similarly, F can be decomposed as $F = \mathrm{Ker}(t^*) \oplus \overline{\mathrm{Ran}(t)}$.

(ii) $\Rightarrow$ (i) Let $t \in R(E,F)$ be a regular operator and F_t be its bounded transform. Then Proposition 1.2 of [7], Result 7.19 of [11] and Proposition 3.7 of [13] imply that

$$\mathrm{Ker}(t) = \mathrm{Ker}(F_t) = \mathrm{Ker}(|F_t|) = \mathrm{Ker}(|t|), \quad \mathrm{Ker}(t^*) = \mathrm{Ker}(F_{t^*}),$$
$$\overline{\mathrm{Ran}(|F_t|)} = \overline{\mathrm{Ran}(F_t^*)} = \overline{\mathrm{Ran}(t^*)} = \overline{\mathrm{Ran}(|t|)}, \quad \overline{\mathrm{Ran}(t)} = \overline{\mathrm{Ran}(F_t)}.$$

From the above equalities and (ii) we have $E = \mathrm{Ker}(|F_t|) \oplus \overline{\mathrm{Ran}(|F_t|)}$, $F = \mathrm{Ker}(F_t^*) \oplus \overline{\mathrm{Ran}(F_t)}$. Now Theorem 15.3.7 of [20] implies that there exists a unique partial isometry $\mathcal{V} \in B(E,F)$ such that $F_t = \mathcal{V}|F_t|$ and

$$\mathrm{Ker}(\mathcal{V}) = \mathrm{Ker}(F_t), \quad \mathrm{Ker}(\mathcal{V}^*) = \mathrm{Ker}(F_t^*), \quad \mathrm{Ran}(\mathcal{V}) = \overline{\mathrm{Ran}(F_t)}, \quad \mathrm{Ran}(\mathcal{V}^*) = \overline{\mathrm{Ran}(|F_t|)}.$$

Therefore

$$\mathrm{Ker}(\mathcal{V})=\mathrm{Ker}(t), \quad \mathrm{Ker}(\mathcal{V}^*)=\mathrm{Ker}(t^*), \quad \mathrm{Ran}(\mathcal{V})=\overline{\mathrm{Ran}(t)}, \quad \mathrm{Ran}(\mathcal{V}^*)=\overline{\mathrm{Ran}(|t|)}.$$

Furthermore, by Remark 2.2 we have $F_t = \mathcal{V}|F_t| = \mathcal{V}(t^*Q_{t^*}tQ_t)^{1/2} = \mathcal{V}(t^*tQ_t^2)^{1/2}$ that is $tQ_t = \mathcal{V}(t^*t)^{1/2}Q_t$. But $Q_t : E \longrightarrow \mathrm{Ran}(Q_t) = \mathrm{Dom}(t)$ is invertible, so $t = \mathcal{V}(t^*t)^{1/2} = \mathcal{V}|t|$.

(ii) $\Rightarrow$ (iii) Recall, that $\mathrm{Ker}(|t|) = \mathrm{Ker}(t)$ and $\mathrm{Ran}(|t|) = \mathrm{Ran}(t^*)$. We set $\mathrm{Dom}(s) := \mathrm{Ran}(t) \oplus \mathrm{Ker}(t^*)$ and define $s : \mathrm{Dom}(s) \subseteq F \longrightarrow E$ by $s(t(x_1 + x_2) + x_3) = x_1$, for all $x_1 \in \mathrm{Dom}(t) \cap \overline{\mathrm{Ran}(t^*)}$, $x_2 \in \mathrm{Dom}(t) \cap \mathrm{Ker}(t)$ and $x_3 \in \mathrm{Ker}(t^*)$. This definition is correct since $E = \mathrm{Ker}(t) \oplus \overline{\mathrm{Ran}(t^*)}$ by supposition. Then s is an $\mathcal{A}$-linear module map the domain of which is a dense $\mathcal{A}$-submodule of F, since $F = \overline{\mathrm{Ran}(t)} \oplus \mathrm{Ker}(t^*)$.

For each $x \in \mathrm{Dom}(t)$ with $x = x_1 + x_2$, $x_1 \in \mathrm{Dom}(t) \cap \overline{\mathrm{Ran}(t^*)}$, $x_2 \in \mathrm{Dom}(t) \cap \mathrm{Ker}(t)$ we have $tst(x) = ts(t(x_1 + x_2) + 0) = t(x_1) = t(x_1 + x_2)$, i.e. $tst = t$. Similarly, for each $x = t(x_1 + x_2) + x_3 \in \mathrm{Dom}(s)$ such that $x_1 \in \mathrm{Dom}(t) \cap \overline{\mathrm{Ran}(t^*)}$, $x_2 \in \mathrm{Dom}(t) \cap \mathrm{Ker}(t)$ and $x_3 \in \mathrm{Ker}(t^*)$, we have $sts(x) = st(x_1 + x_2) = x_1 = s(x)$, and so $sts = s$. Now we are going to derive the properties of Definition 2.3 to demonstrate that s is a regular operator and the generalized inverse of the operator t.

By the definition of s, the equality $ts(t(x_1 + x_2) + x_3) = t(x_1) = t(x_1 + x_2)$ holds. Consequently, the operator ts acts on $\mathrm{Ran}(t)$ as the identity operator, and on the orthogonal complement $\mathrm{Ran}(t)^\perp$ as the zero operator. By continuity, the closure $\overline{ts}$ of ts is the projection onto the orthogonal summand $\overline{\mathrm{Ran}(t)}$ of F. So, $(ts)^* = \overline{(ts)^*} = (\overline{ts})^* = \overline{ts}$. Analogously, the operator equality $(st)^* = \overline{st}$ can be derived, and $(st)^*$ can be shown to be the projection onto the orthogonal summand $\overline{\mathrm{Ran}(s)}$ of E.

We set $\mathrm{Dom}(\widetilde{s}) := \mathrm{Ran}(t^*) \oplus \mathrm{Ker}(t)$ and define the module map $\widetilde{s} : \mathrm{Dom}(\widetilde{s}) \subseteq E \longrightarrow F$ by $\widetilde{s}(t^*(y_1 + y_2) + y_3) = y_1$, for any $y_1 \in \mathrm{Dom}(t^*) \cap \overline{\mathrm{Ran}(t)}$, $y_2 \in \mathrm{Dom}(t^*) \cap \mathrm{Ker}(t^*)$ and $y_3 \in \mathrm{Ker}(t)$. Then $\widetilde{s}$ is an $\mathcal{A}$-linear module map which domain $\mathrm{Dom}(\widetilde{s})$ is a dense $\mathcal{A}$-submodule of E since $E = \overline{\mathrm{Ran}(|t|)} \oplus \mathrm{Ker}(|t|) = \overline{\mathrm{Ran}(t^*)} \oplus \mathrm{Ker}(t)$. We also have $t^*\widetilde{s}t^* = t^*$ and $\widetilde{s}t^*\widetilde{s} = \widetilde{s}$. Similarly, $\overline{t^*\widetilde{s}} = (t^*\widetilde{s})^*$ and $\overline{\widetilde{s}t^*} = (\widetilde{s}t^*)^*$ are orthogonal projections onto $\overline{\mathrm{Ran}(t^*)}$ and $\overline{\mathrm{Ran}(\widetilde{s})}$, respectively.

We prove that s is a regular operator and $s^* = \widetilde{s}$. Consider the isometry $U \in B(E \oplus F, F \oplus E)$ by $U(x, y) = (y, x)$, then by Proposition 9.3 of [13] we have $F \oplus E = UG(t) \oplus G(-t^*)$ and so

$$\begin{aligned} F \oplus E &= \{(t(x_1), x_1) : x_1 \in \mathrm{Dom}(t) \cap \overline{\mathrm{Ran}(t^*)}\} \oplus \{(0, y_3) : y_3 \in \mathrm{Ker}(t)\} \\ &\quad \oplus \{(y_1, -t^*(y_1)) : y_1 \in \mathrm{Dom}(t^*) \cap \overline{\mathrm{Ran}(t)}\} \oplus \{(x_3, 0) : x_3 \in \mathrm{Ker}(t^*)\} \\ &= \{(t(x_1) + x_3, x_1) : x_1 \in \mathrm{Dom}(t) \cap \overline{\mathrm{Ran}(t^*)},\ x_3 \in \mathrm{Ker}(t^*)\} \\ &\quad \oplus \{(y_1, -t^*(y_1) - y_3) : y_1 \in \mathrm{Dom}(t^*) \cap \overline{\mathrm{Ran}(t)},\ y_3 \in \mathrm{Ker}(t)\} = G(s) \oplus VG(\widetilde{s}), \end{aligned}$$

where $V \in B(E \oplus F, F \oplus E)$ is an isometry defined by $V(x,y) = (y,-x)$. The equality $F \oplus E = G(s) \oplus VG(\widetilde{s})$ and Corollary 2.2 of [7] imply that the operator s is adjointable, closed and the range of $1 + s^*s$ is dense in F. In particular $s^* = \widetilde{s}$. Clearly, the regular operators s and s^* with the properties of generalized inverses of the operators t and t^*, respectively, are unique.

(iii) $\Rightarrow$ (ii) Let s be a generalized inverse of t, so that $tst = t$, $sts = s$, $(ts)^* = \overline{ts}$ and $(st)^* = \overline{st}$. Therefore $(ts)^2 = \underline{(ts)}$ and $\mathrm{Ran}(ts) = \mathrm{Ran}(t)$, what implies that $\overline{ts}$ is an orthogonal projection on $\overline{\mathrm{Ran}(t)}$, i.e. $\overline{\mathrm{Ran}(t)}$ is orthogonally complemented. By the hypothesis s^* is the generalized inverse of t^*, therefore $\overline{t^*s^*}$ is an orthogonal projection onto $\overline{\mathrm{Ran}(t^*)} = \overline{\mathrm{Ran}(|t|)}$, i.e. $\overline{\mathrm{Ran}(|t|)}$ is orthogonally complemented.

Note that $\mathcal{V}^*\mathcal{V}$ is the orthogonal projection onto $\overline{\mathrm{Ran}(|t|)}$ so $|t| = \mathcal{V}^*\mathcal{V}|t|$. This together with the polar decomposition of t, gives $\mathcal{V}^*t = |t|$ and $t = \mathcal{V}\mathcal{V}^*t$. ∎

The previous theorem and its proof imply some interesting results as follows:

COROLLARY 3.2. *If $t \in R(E,F)$ and F_t is its bounded transform, then t has polar decomposition $t = \mathcal{V}|t|$ if and only if F_t has polar decomposition $F_t = \mathcal{V}|F_t|$, if and only if F_t has polar decomposition $F_t = \mathcal{V}F_{|t|}$, for the partial isometry $\mathcal{V}$ which was introduced in Theorem* 3.1.

For the proof, just recall that t and F_t have the same kernel and the same range and that $F_{t^*} = F_t^*$. Note that $Q_{|t|} = Q_t$ and so $F_{|t|} = |F_t|$.

COROLLARY 3.3. *An operator $t \in R(E,F)$ has polar decomposition $t = \mathcal{V}|t|$ if and only if its adjoint t^* has polar decomposition $t^* = \mathcal{V}^*|t^*|$, for the partial isometry $\mathcal{V}$ which was introduced in Theorem* 3.1.

Proof. If $t = \mathcal{V}|t|$ then $F_t = \mathcal{V}|F_t|$, so $F_t^* = |F_t|\mathcal{V}^*$. Define $G := \mathcal{V}|F_t|\mathcal{V}^*$, then G is selfadjoint and $G^2 = G \cdot G = \mathcal{V}|F_t|\mathcal{V}^*\mathcal{V}|F_t|\mathcal{V}^* = \mathcal{V}|F_t||F_t|\mathcal{V}^* = F_tF_t^* = |F_t^*|^2$, i.e. $G = |F_t^*| = \mathcal{V}|F_t|\mathcal{V}^*$. Thus $F_t^* = |F_t|\mathcal{V}^* = \mathcal{V}^*\mathcal{V}|F_t|\mathcal{V}^* = \mathcal{V}^*|F_t^*|$. Following the proof of Theorem 3.1, (ii) $\Rightarrow$ (i), we get $t^* = \mathcal{V}^*|t^*|$. The converse direction can be shown taking into account that $\mathcal{V}^{**} = \mathcal{V}$ and interchanging the roles of t and t^* in the first part of the proof. ∎

COROLLARY 3.4. *If $t \in R(E,F)$ has closed range, then t has polar decomposition. In this case the generalized inverse of t is a bounded adjointable operator.*

Proof. If t has closed range then Proposition 1.2 of [7] implies that $E = \mathrm{Ran}(t^*) \oplus \mathrm{Ker}(t)$ and $F = \mathrm{Ran}(t) \oplus \mathrm{Ker}(t^*)$, so t has polar decomposition by Theorem 3.1. The operators s and $\widetilde{s}$ were defined in part "(ii) $\Rightarrow$ (iii)". They are bounded because $\mathrm{Dom}(s) = \mathrm{Ran}(t) \oplus \mathrm{Ker}(t^*) = F$ and $\mathrm{Dom}(\widetilde{s}) = \mathrm{Ran}(t^*) \oplus \mathrm{Ker}(t) = E$, i.e. the generalized inverse of t is a bounded adjointable operator. ∎

Theorem 3.1 and Corollary 3.2 together with a recent result by Lun Chuan Zhang [23] and by Qingxiang Xu and Lijuan Sheng [22] give us the opportunity to derive a criterion for bounded C^*-linear operators between Hilbert C^*-modules to admit a generalized inverse in the sense of Banach algebra theory. These authors proved independently that a bounded adjointable C^*-linear operator between two Hilbert C^*-modules admits a bounded generalized inverse if and only if the operator has closed range.

PROPOSITION 3.5. *Let $T \in R(E,F)$ be a bounded $\mathcal{A}$-linear operator between two Hilbert $\mathcal{A}$-modules E, F over some fixed C^*-algebra $\mathcal{A}$. Suppose, T has polar decomposition. Then T admits a regular operator s as its generalized inverse, and vice versa. Moreover, s is bounded if and only if the range of T is closed.*

COROLLARY 3.6. *For $t \in R(E,F)$ the bounded transform F_t has a bounded generalized inverse if and only if F_t has closed range, if and only if t has closed range.*

Magajna and Schweizer have shown, respectively, that C^*-algebras of compact operators can be characterized by the property that every norm closed (coinciding with its biorthogonal complement, respectively) submodule of every Hilbert C^*-module over them is automatically an orthogonal summand, cf. [14], [18]. Recently further generic properties of the category of Hilbert C^*-modules over C^*-algebras which characterize precisely the C^*-algebras of compact operators have been found by the authors in [6] and [7]. All in all, C^*-algebras of compact operators turn out to be of unique interest in Hilbert C^*-module theory.

THEOREM 3.7. *Let $\mathcal{A}$ be a C^*-algebra. The following conditions are equivalent, among others:*

(i) *$\mathcal{A}$ is an arbitrary C^*-algebra of compact operators.*

(ii) *For every (maximal) norm closed left ideal I of $\mathcal{A}$ the corresponding open projection $p \in \mathcal{A}^{**}$ is an element of the multiplier C^*-algebra $M(\mathcal{A})$ of $\mathcal{A}$.*

(iii) *For every Hilbert $\mathcal{A}$-module E every Hilbert $\mathcal{A}$-submodule $F \subseteq E$ is automatically orthogonally complemented, i.e. F is an orthogonal summand.*

(iv) *For every Hilbert $\mathcal{A}$-module E every Hilbert $\mathcal{A}$-submodule $F \subseteq E$ that coincides with its biorthogonal complement $F^{\perp\perp} \subseteq E$ is automatically orthogonally complemented in E.*

(v) *For every Hilbert $\mathcal{A}$-module E every Hilbert $\mathcal{A}$-submodule is automatically topologically complemented there, i.e. it is a topological direct summand.*

(vi) *For every pair of Hilbert $\mathcal{A}$-modules E, F, every densely defined closed operator $t : \mathrm{Dom}(t) \subseteq E \to F$ possesses a densely defined adjoint operator $t^* : \mathrm{Dom}(t^*) \subseteq F \to E$.*

(vii) *For every pair of Hilbert $\mathcal{A}$-modules E, F, every densely defined closed operator $t : \mathrm{Dom}(t) \subseteq E \to F$ is regular.*

(viii) *The kernels of all densely defined closed operators between arbitrary Hilbert $\mathcal{A}$-modules are orthogonal summands.*

(ix) *The images of all densely defined closed operators with norm closed range between arbitrary Hilbert $\mathcal{A}$-modules are orthogonal summands.*

COROLLARY 3.8. *Let $\mathcal{A}$ be a C^*-algebra. The following conditions are equivalent:*

(i) *$\mathcal{A}$ is an arbitrary C^*-algebra of compact operators.*

(x) *For every pair of Hilbert $\mathcal{A}$-modules E, F, every densely defined closed operator $t : \mathrm{Dom}(t) \subseteq E \to F$ has polar decomposition, i.e. there exists a unique partial isometry $\mathcal{V}$ with initial set $\overline{\mathrm{Ran}(|t|)}$ and the final set $\overline{\mathrm{Ran}(t)}$ such that $t = \mathcal{V}|t|$.*

(xi) *For every pair of Hilbert $\mathcal{A}$-modules E, F, every densely defined closed operator $t : \mathrm{Dom}(t) \subseteq E \to F$ and its adjoint have generalized inverses.*

Proof. The statements are deduced from Theorem 3.1 and from the conditions (vii), (viii) of Theorem 3.7. ∎

After searching in A.M.S.' MathSciNet data base we believe that part of the results of Theorem 3.1, Proposition 3.5, and Corollaries 3.4, 3.6 and 3.8 are essentially new even in the case of Hilbert spaces.

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MICHAEL FRANK, HOCHSCHULE FÜR TECHNIK, WIRTSCHAFT UND KULTUR (HTWK) LEIPZIG, FACHBEREICH IMN, GUSTAV-FREYTAG-STRASSE 42A, D-04277 LEIPZIG, GERMANY

E-mail address: mfrank@imn.htwk-leipzig.de

KAMRAN SHARIFI, DEPARTMENT OF MATHEMATICS, SHAHROOD UNIVERSITY OF TECHNOLOGY, P. O. BOX 3619995161-316, SHAHROOD, IRAN

E-mail address: sharifi.kamran@gmail.com and sharifi@shahroodut.ac.ir

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