J. OPERATOR THEORY
64:2(2010), 401–416

SMALL PERTURBATIONS OF SELFADJOINT AND UNITARY OPERATORS IN KREIN SPACES

TOMAS YA. AZIZOV, PETER JONAS, and CARSTEN TRUNK

Communicated by Florian-Horia Vasilescu

ABSTRACT. We investigate the behaviour of the spectrum of selfadjoint operators in Krein spaces under perturbations with uniformly dissipative operators. Moreover we consider the closely related problem of the perturbation of unitary operators with uniformly bi-expansive. The obtained perturbation results give a new characterization of spectral points of positive type and of type π_+ of selfadjoint (respectively unitary) operators in Krein spaces.

KEYWORDS: *Selfadjoint operators, unitary operators, perturbation by uniformly dissipative operators, Krein spaces, perturbation by uniformly bi-expansive operators.*

MSC (2000): 47A55, 47B50, 46C20.

INTRODUCTION

A real point λ of the spectrum of a closed operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called a spectral point of positive (negative) type, if for every normed approximative eigensequence (x_n) corresponding to λ all accumulation points of the sequence $([x_n, x_n])$ are positive (respectively negative), see Definition 1.1 below. These spectral points were introduced by P. Lancaster, A. Markus and V. Matsaev in [24] for a bounded operator A which is selfadjoint in the Krein space $(\mathcal{H}, [\cdot,\cdot])$, i.e. the selfadjointness is understood with respect to $[\cdot,\cdot]$. In [26] the existence of a local spectral function was proved for intervals containing only spectral points of positive (negative) type or points of the resolvent set $\rho(A)$. Moreover it was shown that, if A is perturbed by a compact selfadjoint operator, a spectral point of positive type of A becomes either an inner point of the spectrum of the perturbed operator or it becomes an eigenvalue of type π_+. A point from the approximative point spectrum of A is of type π_+ if the abovementioned property of approximative eigensequences (x_n) holds only for sequences (x_n) belonging to some linear manifold of finite codimension (see Definition 1.2 below). Every spectral point of a selfadjoint operator in a Pontryagin space with finite rank of negativity is of type π_+. For a detailed study of the properties of the spectrum of type π_+ we refer to [3] and [7].

It is the main aim of this paper to consider perturbations of selfadjoint operators (unitary operators) in some Krein spaces with uniformly dissipative operators (respectively uniformly bi-expansive operators). Let A be a selfadjoint operator in the Krein space $\mathcal{H}$. Let λ_0 be no accumulation point of the non-real spectrum of A and let $(a,b) \setminus \{\lambda_0\}$ consists of spectral points of positive type or of points from the resolvent set of A only. In Section 2 below we show that λ_0 belongs to the spectrum of positive type of A if and only if there exists a fixed open neighbourhood $\mathcal{U}$ of λ_0 such that for all sufficiently small uniformly dissipative operators B the operator $A + B$ has no spectrum inside the intersection of $\mathcal{U}$ and the open lower half-plane. Moreover, the point λ_0 belongs to the spectrum of type π_+ if and only if for all sufficiently small uniformly dissipative operators B the operator $A + B$ has at most finitely many normal eigenvalues inside the intersection of $\mathcal{U}$ and the open lower half-plane. In particular, we are able to show that the sum of all spectral multiplicities within $\mathcal{U}$ intersected with the open lower half-plane equals the rank of negativity of $\kappa_-(E((a',b'))\mathcal{H})$, where $E(\cdot)$ denotes the local spectral function of A. On the other hand, if for every sufficiently small uniformly dissipative operator B the range of the Riesz–Dunford projector corresponding to $A + B$ and the intersection of $\mathcal{U}$ and the open lower half-plane is of infinite dimension, then λ_0 does not belong to $\sigma_{\pi_+}(A) \cup \rho(A)$.

In Section 3 we show that the above arguments hold true in a similar way for uniformly bi-expansive perturbations of unitary operators.

We view these perturbation results also as a new characterization of the spectral points of positive (respectively negative) type and of type π_+ (respectively π_-) of selfadjoint/unitary operators in Krein spaces. We mention that in the early work of L.S. Pontryagin such arguments were used in a similar manner, cf. [31].

Sign type spectrum is used in the theory of indefinite Sturm–Liouville operators, e.g. [6], [8], [12], [23]. Moreover, it is used in the theory of mathematical system theory, see e.g. [18], [19], [25] and in the study of $\mathcal{PT}$-symmetric problems [13], [14], [27].

We conclude this paper with an application of our results to a second order equation, cf. Section 4.

1. PRELIMINARIES

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space. Let A be a closed operator in $\mathcal{H}$. By $\mathcal{L}_\lambda(A)$ we denote the root subspace of A corresponding to λ, i.e. $\mathcal{L}_\lambda(A) = \bigcup_{n=1}^{\infty} \ker(A-\lambda)^n$. A point $\lambda_0 \in \mathbb{C}$ is said to belong to the *approximative point spectrum* $\sigma_{\rm ap}(A)$ of A if there exists a sequence $(x_n) \subset \mathcal{D}(A)$ with $\|x_n\| = 1$, $n = 1,2,\dots$, and $\|(A -$

$\lambda_0)x_n\| \to 0$ as $n \to \infty$. The boundary points of the spectrum of a closed operator belong to the approximative point spectrum. For a selfadjoint operator A in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ all real points of the spectrum of A belong to $\sigma_{\mathrm{ap}}(A)$ (see e.g. Corollary VI.6.2 of [10]). The operator A is called *Fredholm* if the dimension of the kernel of A and the codimension of the range of A are finite. The set

$$\sigma_{\mathrm{ess}}(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}$$

is called the *essential spectrum* of A.

The following definition is from [1]. In [24], [26] it was given for the case of a bounded selfadjoint operator.

DEFINITION 1.1. For a closed operator A in $\mathcal{H}$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of *positive* (*negative*) *type of A* if $\lambda_0 \in \sigma_{\mathrm{ap}}(A)$ and for every sequence $(x_n) \subset \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$, we have

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad (\text{respectively } \limsup_{n\to\infty} [x_n, x_n] < 0).$$

We denote the set of all points of positive (negative) type of A by $\sigma_{++}(A)$ (respectively $\sigma_{--}(A)$).

If the operator A is selfadjoint then the sets $\sigma_{++}(A)$ and $\sigma_{--}(A)$ are contained in $\mathbb{R}$ (cf. [26])

In a similar way as in Definition 1.1 we introduce now some subsets of $\sigma(A)$ containing $\sigma_{++}(A)$ and $\sigma_{--}(A)$, respectively, which will play an important role in the following (cf. [1] and for special case of a selfadjoint operator see [3]).

DEFINITION 1.2. For a closed operator A in $\mathcal{H}$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of *type* π_+ (*type* π_-) *of* A if $\lambda_0 \in \sigma_{\mathrm{ap}}(A)$ and if there exists a subspace $\mathcal{H}_0 \subset \mathcal{H}$ with $\operatorname{codim} \mathcal{H}_0 < \infty$ such that for every sequence $(x_n) \subset \mathcal{H}_0 \cap \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$, we have

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad (\text{respectively } \limsup_{n\to\infty} [x_n, x_n] < 0).$$

We denote the set of all points of type π_+ (type π_-) of A by $\sigma_{\pi_+}(A)$ (respectively $\sigma_{\pi_-}(A)$). We call $\mathcal{H}_0$ of *minimal codimension* if for each subspace $\mathcal{H}_1 \subset \mathcal{H}$ with $\operatorname{codim} \mathcal{H}_1 < \operatorname{codim} \mathcal{H}_0$ there exists a sequence $(x_n) \subset \mathcal{H}_1 \cap \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$, such that

$$\liminf_{n\to\infty} [x_n, x_n] \leqslant 0 \quad (\text{respectively } \limsup_{n\to\infty} [x_n, x_n] \geqslant 0).$$

Observe, that for a point $\lambda_0 \in \sigma_{\pi_+}(A)$ we have that $\lambda_0 \in \sigma_{++}(A)$ if and only if the subspace $\mathcal{H}_0$ from Definition 1.2 can be chosen as $\mathcal{H}_0 = \mathcal{H}$.

Recall that an operator C in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called *uniformly dissipative* if there exists some $\alpha > 0$ such that for $x \in \mathcal{D}(C)$ we have $\operatorname{Im}[Cx, x] \geqslant \alpha\|x\|^2$.

The second part of the following lemma is well-known, nevertheless we give a proof for the sake of completeness.

We set $\mathbb{C}^{\pm} := \{z \in \mathbb{C} : \pm\operatorname{Im} z > 0\}$.

LEMMA 1.3. *Let C be a closed uniformly dissipative operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$. Then*

$$\sigma_{\mathrm{ap}}(C) \cap \mathbb{C}^- \subset \sigma_{--}(C).$$

If $\lambda \in \sigma_p(C) \cap \mathbb{C}^-$ then for each $x \in \mathcal{L}_\lambda(C)$, $x \neq 0$, it follows

$$[x, x] < 0.$$

Proof. Let $\lambda_0 \in \sigma_{\mathrm{ap}}(C) \cap \mathbb{C}^-$. Then the first statement of Lemma 1.3 follows from the fact that for every sequence $(x_n) \subset \mathcal{D}(C)$ with $\|x_n\| = 1$ and $\|(C - \lambda_0)x_n\| \to 0$, $n \to \infty$, we have

$$|\mathrm{Im}\,[Cx_n, x_n] - \mathrm{Im}\,\lambda_0[x_n, x_n]| \leqslant \|(C - \lambda_0)x_n\| \to 0, \quad n \to \infty.$$

Let $\lambda \in \sigma_p(C) \cap \mathbb{C}^-$. It follows from Chapter 2, Corollary 2.17 of [2] that for each $y \in \mathcal{L}_\lambda(C)$ we have $[y, y] \leqslant 0$. Assume that there exists an $x \in \mathcal{L}_\lambda(C)$, $x \neq 0$, with $[x, x] = 0$. Then we have $[x, y] = 0$ for all $y \in \mathcal{L}_\lambda(C)$. Hence

$$0 = \mathrm{Im}\,[(C - \lambda)x, x],$$

which is a contradiction to the assumption that C is uniformly dissipative. ∎

2. UNIFORMLY DISSIPATIVE PERTURBATION OF SELFADJOINT OPERATORS IN KREIN SPACES

Let A be a selfadjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$, that is, $A = A^+$. Here we denote by A^+ the adjoint of a densely defined operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ with respect to $[\cdot,\cdot]$. Let B be a bounded uniformly dissipative operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$. Then the operator $A + B$, which is defined on $\mathcal{D}(A)$, is uniformly dissipative.

LEMMA 2.1. *Let A be a selfadjoint operator and let B be a bounded uniformly dissipative operator in $\mathcal{H}$. Then*

$$\mathbb{R} \subset \rho(A + B).$$

Proof. Set $C := A + B$. We choose $\alpha > 0$ such that $\mathrm{Im}\,[Bx, x] \geqslant \alpha\|x\|^2$, $x \in \mathcal{H}$. We have $\mathcal{D}(C) = \mathcal{D}(A) = \mathcal{D}(C^+)$ and, therefore, for $\lambda \in \mathbb{R}$ and $x \in \mathcal{D}(C)$, $x \neq 0$, it follows

$$\|x\|\|(C - \lambda)x\| \geqslant |[(C - \lambda)x, x]| \geqslant |\mathrm{Im}\,[Bx, x]| \geqslant \alpha\|x\|^2$$

and

$$\|(C^+ - \lambda)x\| \geqslant \alpha\|x\|.$$

As C is a closed operator the point λ belongs to $\rho(C)$. ∎

LEMMA 2.2. *Let $\mu \in \sigma_{++}(A)$. Then there exists a $\delta > 0$ and an $\varepsilon > 0$ such that for all bounded uniformly dissipative operators in $\mathcal{H}$ with $\|B\| \leqslant \varepsilon$ it follows that the intersection of $\mathbb{C}^-$ and the disc around μ with radius δ belongs to $\rho(A + B)$.*

Proof. Assume that the assertion of Lemma 2.2 is not true. Then there exist a sequence of bounded uniformly dissipative operators (B_n) in $\mathcal{H}$ with $\|B_n\| \to 0$, $n \to \infty$, and a sequence (λ_n) in $\sigma(A + B_n) \cap \mathbb{C}^-$ which converges to μ, $\mu \in \sigma_{++}(A)$. We assume $\lambda_n \in \sigma_{\mathrm{ap}}(A + B_n)$, $n \in \mathbb{N}$. In view of Lemma 2.1 this is no restriction. By Lemma 1.3 there exists a sequence (x_n), $x_n \in \mathcal{D}(A + B_n) = \mathcal{D}(A)$ with $\|x_n\| = 1$, $[x_n, x_n] < 0$ and $\|(A + B_n - \lambda_n)x_n\| \leqslant \frac{1}{n}$, $n \in \mathbb{N}$. Then $\liminf_{n\to\infty}[x_n, x_n] \leqslant 0$ and

$$(A - \mu)x_n = (A + B_n - \lambda_n)x_n + (\lambda_n - \mu)x_n - B_n x_n \to 0, \quad n \to \infty,$$

which contradicts $\mu \in \sigma_{++}(A)$. ∎

PROPOSITION 2.3. *Let A be a selfadjoint operator. Assume that λ_0, $\lambda_0 \in (a, b)$, is not an accumulation point of the non-real spectrum of A and that*

$$(a, b) \setminus \{\lambda_0\} \subset \sigma_{++}(A) \cup \rho(A) \tag{2.1}$$

holds. Let $a < a' < \lambda_0 < b' < b$. Then there exists a $\delta' > 0$ such that the strip

$$\{\lambda \in \mathbb{C}^- : a' \leqslant \operatorname{Re}\lambda \leqslant b', -\delta' \leqslant \operatorname{Im}\lambda < 0\}$$

belongs to the resolvent set of A. Moreover, if $\gamma_{\delta'}$ denotes the closed oriented curve in the complex plane which consists of the line segments connecting the points $b', b' - \mathrm{i}\delta', a' - \mathrm{i}\delta', a'$ and b' then there exists an $\varepsilon_0 > 0$ such that for all bounded uniformly dissipative operators B in $\mathcal{H}$ with $\|B\| \leqslant \varepsilon_0$ we have

$$\gamma_{\delta'} \subset \rho(A + B). \tag{2.2}$$

Proof. The first statement of Proposition 2.3 follows from [26] (or [3]). In order to show (2.2) we choose $\varepsilon_0 > 0$ so small that, cf. Lemma 2.2, for all bounded uniformly dissipative operators B in $\mathcal{H}$ with $\|B\| \leqslant \varepsilon_0$ the line segments connecting the points b' and $b' - \mathrm{i}\delta'$ and the points a' and $a' - \mathrm{i}\delta'$ belong to $\rho(A + B)$. Moreover, we choose ε_0 so small that

$$\varepsilon_0 < \frac{1}{\max_{\lambda \in \Gamma} \|(A - \lambda)^{-1}\|}$$

holds, where Γ is the line segment connecting the points $b' - \mathrm{i}\delta'$ and $a' - \mathrm{i}\delta'$. As $A + B - \lambda = (I + B(A - \lambda)^{-1})(A - \lambda)$, Γ is a subset of $\rho(A + B)$. Moreover, by Lemma 2.1, $\mathbb{R} \subset \rho(A + B)$, hence Proposition 2.3 is proved. ∎

The following theorem can be considered as the main result of this paper. Recall that for a selfadjoint operator satisfying (2.1) there exists a local spectral function E defined on subintervals of (a, b) with endpoints not equal to a, b or λ_0, cf. [3], [21]. In particular there exists the spectral projection $E((a', b'))$ corresponding to the interval (a', b') with $a < a' < \lambda_0 < b' < b$.

THEOREM 2.4. *Let A be a selfadjoint operator in the Krein space $\mathcal{H}$. Assume that λ_0, $\lambda_0 \in (a, b)$, is not an accumulation point of the non-real spectrum of A and that*

$$(a, b) \setminus \{\lambda_0\} \subset \sigma_{++}(A) \cup \rho(A). \tag{2.3}$$

Let a', b', δ', ε_0 and $\gamma_{\delta'}$ be as in Proposition 2.3. *Then the following assertions are valid.*

(i) *The point λ_0 belongs to $\sigma_{++}(A) \cup \rho(A)$ if and only if there exists an $\varepsilon_1 > 0$ such that for every uniformly dissipative operator B acting in $\mathcal{H}$ with $\|B\| < \varepsilon_1$ the operator $A + B$ has no spectrum inside the curve $\gamma_{\delta'}$.*

(ii) *The point λ_0 belongs to $\sigma_{\pi_+}(A)$ if and only if there exists an $\varepsilon_1 > 0$ such that for every uniformly dissipative operator B acting in $\mathcal{H}$ with $\|B\| < \varepsilon_1$ the spectrum of $A + B$ inside the curve $\gamma_{\delta'}$ consists of at most finitely many normal eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that*

$$\mathcal{M}_- := \operatorname{span}\{\mathcal{L}_{\lambda_j}(A+B) : 1 \leqslant j \leqslant k\}$$

is of finite dimension. Moreover, in this case, the dimension of $\mathcal{M}_-$ is equal to the rank of negativity $\kappa_-(E((a',b'))\mathcal{H})$ of the Pontryagin space $E((a',b'))\mathcal{H}$, that is

$$\dim \mathcal{M}_- = \kappa_-(E((a',b'))\mathcal{H}).$$

(iii) *The point λ_0 does not belong to $\sigma_{\pi_+}(A) \cup \rho(A)$ if and only if there exists an $\varepsilon_1 > 0$ such that for every uniformly dissipative operator B acting in $\mathcal{H}$ with $\|B\| < \varepsilon_1$ the range of the Riesz–Dunford projector corresponding to $A + B$ and $\gamma_{\delta'}$ is of infinite dimension.*

Proof. Let a', b', δ', ε_0 and $\gamma_{\delta'}$ be as in Proposition 2.3. Set $\mathcal{K} := (I - E((a',b')))\mathcal{H}$. Then the space $\mathcal{H}$ decomposes

$$\mathcal{H} = E((a',b'))\mathcal{H}\,[\dot{+}]\,\mathcal{K}, \tag{2.4}$$

where $[\dot{+}]$ denote the direct sum of spaces which are orthogonal with respect to $[\cdot,\cdot]$. Moreover,

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_0 & B_{01} \\ B_{10} & B_1 \end{bmatrix}$$

with respect to the decomposition (2.4). The operators B_0 and B_1 are uniformly dissipative operators in $E((a',b'))\mathcal{H}$ and $\mathcal{K}$, respectively. As E is the spectral function of A, we have

$$\sigma(A_0) \subset [a',b'] \quad \text{and} \quad \sigma(A_1) \subset \mathbb{R} \setminus (a',b').$$

By assumption, a' and b' belong to $\sigma_{++}(A) \cup \rho(A)$. Lemma 2.2 implies the existence of $\delta > 0$ and $\varepsilon > 0$ such that for all bounded uniformly dissipative operators in $\mathcal{H}$ with $\|B\| \leqslant \varepsilon$ it follows that the intersection of $\mathbb{C}^-$ and the discs around a' and b' with radius δ belong to the resolvent set of the operator

$$\begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix} + \begin{bmatrix} B_0 & 0 \\ 0 & B_1 \end{bmatrix}.$$

Denote by $\Gamma_{\delta'}$ the open set in $\mathbb{C}$ which has as its boundary the curve $\gamma_{\delta'}$, that is $\Gamma_{\delta'} = \{\lambda \in \mathbb{C} : a' < \operatorname{Re}\lambda < b', -\delta' < \operatorname{Im}\lambda < 0\}$. It follows from IV. Section 3.1 of [22] that there is an $\varepsilon_1 > 0$, $\varepsilon_1 < \min\{\varepsilon, \varepsilon_0\}$, such that for all uniformly dissipative operators B acting in $\mathcal{H}$ with $\|B\| < \varepsilon_1$ we have

$$\sigma(A_0 + B_0) \subset \{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, [a',b']) < \min\{\delta, \delta'\}\} \tag{2.5}$$

and

$$\overline{\Gamma_{\delta'}} \subset \rho(A_1 + B_1). \tag{2.6}$$

Then Lemma 2.1, Proposition 2.3 and (2.5) imply that for all uniformly dissipative operators B with $\|B\| < \varepsilon_1$

$$\sigma(A_0 + B_0) \cap \mathbb{C}^- \subset \Gamma_{\delta'}. \tag{2.7}$$

Now we assume that λ_0 belongs to $\sigma_{\pi_+}(A)$. Then $(E((a', b')), [\cdot, \cdot])$ is a Pontryagin space with a finite rank of negativity and, if $\lambda_0 \in \sigma_{++}(A)$, it is even a Hilbert space (cf. Theorems 23 and 24 of [3]). An application of Theorem 11.6 in [16] implies that $\sigma(A_0 + B_0) \cap \mathbb{C}^-$ consists of at most finitely many eigenvalues and that

$$\mathcal{M}_- := \operatorname{span}\{\mathcal{L}_\lambda(A_0 + B_0) : \lambda \in \sigma(A_0 + B_0) \cap \mathbb{C}^-\}$$

is a maximal uniformly negative subspace of $E((a', b'))\mathcal{H}$ invariant under $A_0 + B_0$. Therefore

$$\dim \mathcal{M}_- = \kappa_-(E((a', b'))\mathcal{H})$$

and relations (2.7) and (2.6) imply that the operator $A + B$ has the properties stated in assertions (i) and (ii) if $B_{01} = B_{10} = 0$. If $B_{01} \neq 0$ or $B_{10} \neq 0$ we consider the operators

$$\begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix} + \begin{bmatrix} B_0 & tB_{01} \\ tB_{10} & B_1 \end{bmatrix},$$

where t runs through $[0, 1]$. Then by IV. Section 3.4 of [22], Lemma 1.3 and Proposition 2.3 the operator $A + B$ has the properties stated in assertions (i) and (ii).

It remains to consider the case $\lambda_0 \notin \sigma_{\pi_+}(A)$. Assume that the range of the Riesz–Dunford projector P_- corresponding to $A_0 + B_0$ and $\gamma_{\delta'}$ is of finite dimension. Then, by Lemma 1.3, it is a uniformly negative subspace of $E((a', b'))\mathcal{H}$. Moreover the range of the Riesz–Dunford projector P_+ corresponding to $A_0 + B_0$ and $\sigma(A_0 + B_0) \cap \mathbb{C}^+$ is a nonnegative subspace (cf. [2]) and we have

$$E((a', b'))\mathcal{H} = P_+ E((a', b'))\mathcal{H} [\dot{+}] P_- E((a', b'))\mathcal{H}.$$

We claim that $P_- E((a', b'))\mathcal{H}$ is a maximal uniformly negative subspace of the Krein space $E((a', b'))\mathcal{H}$. Indeed, assume that there exists a maximal uniformly negative subspace $\widetilde{\mathcal{M}}_-$ with $P_- E((a', b'))\mathcal{H} \subset \widetilde{\mathcal{M}}_-$ and there exists some x, $x \in \widetilde{\mathcal{M}}_- \setminus P_- E((a', b'))\mathcal{H}$. Then $[x - P_- x, x - P_- x] < 0$ holds. But this is a contradiction to $x - P_- x = P_+ x \in P_+ E((a', b'))\mathcal{H}$.

Therefore the Krein space $E((a', b'))\mathcal{H}$ has a finite dimensional maximal uniformly negative subspace, hence $E((a', b'))\mathcal{H}$ is a Pontryagin space. But this is impossible as $\lambda_0 \notin \sigma_{\pi_+}(A)$ (cf. Theorem 24 of [3]) and the operator $A + B$ has the properties stated in assertions (iii) if $B_{01} = B_{10} = 0$. If $B_{01} \neq 0$ or $B_{10} \neq 0$ then a similar reasoning as above shows that assertion (iii) holds and Theorem 2.4 is proved. ∎

COROLLARY 2.5. *Let λ_0, $\lambda_0 \in (a,b)$, belongs to $\sigma_{\pi_+}(A) \setminus \sigma_{++}(A)$ and choose $\mathcal{H}_0$ as in Definition* 1.2 *such that $\mathcal{H}_0$ is of minimal codimension. Assume that λ_0 is not an accumulation point of the non-real spectrum of A and that* (2.1) *holds. Let a', b', ε_1 and $\mathcal{M}_-$ be as in Theorem* 2.4. *Then we have for every uniformly dissipative operator B acting in $\mathcal{H}$ with $\|B\| < \varepsilon_1$*

$$\operatorname{codim} \mathcal{H}_0 \leqslant \dim \mathcal{M}_- = \kappa_-(E((a',b'))\mathcal{H}). \tag{2.8}$$

Moreover, let

$$\ker (A - \lambda_0) = \mathcal{N}_0[\dot{+}]\mathcal{N}_+[\dot{+}]\mathcal{N}_- \quad \text{and} \quad \mathcal{L}_{\lambda_0}(A) = \mathcal{L}_0[\dot{+}]\mathcal{L}_+[\dot{+}]\mathcal{L}_-$$

be fundamental decompositions of $\ker (A - \lambda_0)$ *and $\mathcal{L}_{\lambda_0}(A)$, respectively, that is, $\mathcal{N}_0 =$* $\ker (A - \lambda_0) \cap (\ker (A - \lambda_0))^{[\perp]}$, $\mathcal{L}_0 = \mathcal{L}_{\lambda_0}(A) \cap (\mathcal{L}_{\lambda_0}(A))^{[\perp]}$, *$\mathcal{N}_+$, $\mathcal{L}_+$ are positive subspaces of $E((a',b'))\mathcal{H}$ and $\mathcal{N}_-$, $\mathcal{L}_-$ are negative subspace of $E((a',b'))\mathcal{H}$. We have equality in* (2.8), *that is,*

$$\operatorname{codim} \mathcal{H}_0 = \dim \mathcal{M}_- = \kappa_-(E((a',b'))\mathcal{H})$$

if and only if

$$\dim \mathcal{N}_0 + \dim \mathcal{N}_- = \dim \mathcal{L}_0 + \dim \mathcal{L}_-.$$

In this case we have

$$\dim \mathcal{N}_0 + \dim \mathcal{N}_- = \dim \mathcal{L}_0 + \dim \mathcal{L}_- = \operatorname{codim} \mathcal{H}_0 = \dim \mathcal{M}_- = \kappa_-(E((a',b'))\mathcal{H}).$$

Proof. We choose a fundamental decomposition for the Pontryagin space $E((a',b'))\mathcal{H}$, $E((a',b'))\mathcal{H} = \Pi_+ [\dot{+}] \Pi_-$. Then $\Pi_+ [\dot{+}] (I - E((a',b')))\mathcal{H}$ is of finite codimension in $\mathcal{H}$ and an easy calculation shows that (2.8) holds. The remaining statements of Corollary 2.5 follows from Theorem 3.6 of [7]. ∎

We refer to [7] for an example such that the inequality in (2.8) is strict.

3. UNIFORMLY BI-EXPANSIVE PERTURBATION OF UNITARY OPERATORS IN KREIN SPACES

A bounded operator U in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called unitary if U is surjective and $[Ux, Ux] = [x, x]$ for all $x \in \mathcal{H}$.

A bounded operator V is said to be bi-expansive if both V and V^+ are non-contractive with respect to $[\cdot,\cdot]$, that is,

$$[Vx, Vx] \geqslant [x, x] \quad \text{and} \quad [V^+x, V^+x] \geqslant [x, x] \quad \text{for all } x \in \mathcal{H}.$$

The operator V is called uniformly bi-expansive if the operator V is bi-expansive and there is an $\alpha_V > 0$ such that $[Vx, Vx] \geqslant [x, x] + \alpha_V \|x\|^2$. If V is uniformly bi-expansive then also V^+ is uniformly bi-expansive and $\alpha_{V^+} = \alpha_V$.

For every uniformly bi-expansive operator V we have

$$\mathbb{T} \subset \rho(V), \tag{3.1}$$

where $\mathbb{T}$ denote the unit circle $\mathbb{T} = \{\lambda : |\lambda| = 1\}$ (see, e.g., Theorem 2.4.31 of [2]). The operator

$$A := \mathrm{i}(V+1)(V-1)^{-1} \tag{3.2}$$

is called the Cayley–Neumann transformation of V. If V is a uniformly bi-expansive operator then we have for $x \in \mathcal{H}$ with $y := (V-1)x$,

$$\begin{aligned} \operatorname{Im}[Ay,y] &= \operatorname{Re}([(V+I)x,(V-I)x]) = \operatorname{Re}([Vx,Vx] + [x,Vx] - [Vx,x] - [x,x]) \\ &= [Vx,Vx] - [x,x] \end{aligned}$$

and A is uniformly dissipative.

It is well-known that the classes of selfadjoint and unitary operators (as well as the classes of bounded uniformly dissipative operators and uniformly bi-expansive operators) are closely connected via Cayley–Neumann transformation. It is a natural idea to prove similar results as in the previous sections using Cayley–Neumann transformation for bi-expansive perturbations of unitary operators. But this does not work in general since the image of an unbounded uniformly dissipative operator $A + B$ need not to be a uniformly bi-expansive operator. Because of this in this section we follow the same ideas as in the previous sections replacing dissipative perturbations of selfadjoint operators by bi-expansive perturbations of unitary operators.

The following lemma is an analog of Lemma 1.3. Let $\mathbb{D}$ denotes the open unit disc,

$$\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}.$$

LEMMA 3.1. *Let V be a uniformly bi-expansive operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$. Then*

$$\sigma_{\mathrm{ap}}(V) \cap \mathbb{D} \subset \sigma_{--}(V);$$

If $\lambda \in \sigma_p(V) \cap \mathbb{D}$ then for each $x \in \mathcal{L}_\lambda(V)$, $x \neq 0$, it follows $[x,x] < 0$.

Proof. Let $\lambda_0 \in \sigma_{\mathrm{ap}}(V) \cap \mathbb{D}$. Let (x_n) be a sequence with $\|x_n\| = 1$, $n \in \mathbb{N}$, and $(V - \lambda_0)x_n \to 0$ as $n \to \infty$. Since

$$[Vx_n, Vx_n] - |\lambda_0|^2[x_n,x_n] = [(V-\lambda_0)x_n, Vx_n] + \lambda_0[x_n,(V-\lambda_0)x_n],$$

we have

$$0 = \liminf_{n\to\infty}[Vx_n,Vx_n] - |\lambda_0|^2 \liminf_{n\to\infty}[x_n,x_n] \geqslant (1-|\lambda_0|^2)\liminf_{n\to\infty}[x_n,x_n] + \alpha_V.$$

Hence

$$\liminf_{n\to\infty}[x_n,x_n] \leqslant -\frac{\alpha_v}{1-|\lambda_0|^2} < 0.$$

Now we show that $\mathcal{L}_\lambda(V)$ is a negative subspace, i.e. $[x,x] < 0$ for all non-zero $x \in \mathcal{L}_\lambda(V)$. By (3.1), $1 \in \rho(V)$ and we consider the Cayley–Neumann transformation A of V, cf. (3.2). The operator A is uniformly dissipative. Since $\mathcal{L}_\lambda(V) = \mathcal{L}_\mu(A)$ for $\mu = \mathrm{i}\frac{\lambda+1}{\lambda-1}$, the statement follows from Lemma 1.3. ∎

LEMMA 3.2. *Let U be a unitary operator and let $\mu \in \sigma_{++}(U)$. Then $|\mu| = 1$ and there exist a $\delta > 0$ and an $\varepsilon > 0$ such that for all uniformly bi-expansive operators V with $\|I - V\| \leqslant \varepsilon$ it follows that the intersection of $\mathbb{D}$ and the disc around μ with radius δ belongs to $\rho(UV)$.*

Proof. First we show that $|\mu| = 1$. Assume the contrary: $|\mu| \neq 1$. Let $\|z_n\| = 1$, $n \in \mathbb{N}$, and $(U - \mu)z_n \to 0$ as $n \to \infty$. Since

$$(1 - |\mu|^2)[z_n, z_n] = [Uz_n, Uz_n] - |\mu|^2[z_n, z_n] = [(U - \mu)z_n, Uz_n] + \mu[z_n, (U - \mu)z_n]$$

we have $\lim_{n\to\infty}[z_n, z_n] = 0$ which contradicts to $\mu \in \sigma_{++}(U)$.

Assume now that the second assertion of the lemma is not true. Then there exists a sequence of uniformly bi-expansive operators V_n in $\mathcal{H}$ with $V_n \to I$ as $n \to \infty$ and a sequence $(\lambda_n) \subset \sigma(UV_n) \cap \mathbb{D}$ which converges to $\mu \in \sigma_{++}(U)$. In view of (3.1) it is no restriction if we assume that $\lambda_n \in \sigma_{\mathrm{ap}}(UV_n)$, $n \in \mathbb{N}$. By Lemma 3.1 there exists a sequence (x_n) with $\|x_n\| = 1$, $[x_n, x_n] < 0$ and $\|(UV_n - \lambda_n)x_n\| \leqslant \frac{1}{n}$. Then $\liminf_{n\to\infty}[x_n, x_n] \leqslant 0$ and as $n \to \infty$ we have

$$(U - \mu)x_n = (UV_n - \lambda_n)x_n + (\lambda_n - \mu)x_n - U(V_n - I)x_n \to 0$$

which contradicts $\mu \in \sigma_{++}(U)$. ∎

Assume

$$\varphi, \psi \in [0, 2\pi), \quad \varphi < \psi \quad \text{and} \quad \delta \in (0, 1).$$

Denote by $\omega_{\varphi,\psi}$ the open arc of the unit circle given by

$$\omega_{\varphi,\psi} := \{\lambda = \mathrm{e}^{\mathrm{i}\eta} : \varphi < \eta < \psi\},$$

by $\Omega_{\varphi,\psi,\delta}$ the part of the sector generated by $\omega_{\varphi,\psi}$,

$$\Omega_{\varphi,\psi,\delta} := \{\lambda = r\mathrm{e}^{\mathrm{i}\eta} : \varphi \leqslant \eta \leqslant \psi,\ 1 - \delta \leqslant r < 1\},$$

and by $\gamma_{\varphi,\psi,\delta}$ the boundary of $\Omega_{\varphi,\psi,\delta}$.

PROPOSITION 3.3. *Let U be a unitary operator. Assume $\lambda_0 = \mathrm{e}^{\mathrm{i}\eta_0}$, $\lambda_0 \in \omega_{\varphi,\psi}$, is not an accumulation point of $\sigma(U) \setminus \mathbb{T}$ and that*

$$\omega_{\varphi,\psi} \setminus \{\lambda_0\} \subset \sigma_{++}(U) \cup \rho(U). \tag{3.3}$$

Let

$$\varphi < \varphi' < \eta_0 < \psi' < \psi.$$

Then there exists a $\delta' > 0$ such that $\Omega_{\varphi',\psi',\delta'} \subset \rho(U)$.

Moreover, there exists an $\varepsilon_0 > 0$ such that for all uniformly bi-expansive operators V with $\|I - V\| < \varepsilon_0$ we have $\gamma_{\varphi',\psi',\delta'} \subset \rho(UV)$.

Proof. We omit the proof since it repeats similar arguments as we used in the proof of Proposition 2.3 . ∎

A unitary operator in a Krein space satisfying (3.3) has a local spectral function E defined on subarcs of $\omega_{\varphi,\psi}$ with endpoints not equal to $e^{i\varphi}, e^{i\psi}$ or λ_0, cf. [21]. In particular there exists the spectral projection $E(\omega_{\varphi',\psi'})$ corresponding to the subarc $\omega_{\varphi',\psi'}$ with $\varphi < \varphi' < \eta_0 < \psi' < \psi$.

THEOREM 3.4. *Let U be a unitary operator in the Krein space $\mathcal{H}$. Assume that $\lambda_0 = e^{i\eta_0}$, $\lambda_0 \in \omega_{\varphi,\psi}$, is not an accumulation point of $\sigma(U) \setminus \mathbb{T}$ and that*

$$\omega_{\varphi,\psi} \setminus \{\lambda_0\} \subset \sigma_{++}(U) \cup \rho(U).$$

Let a', b', δ', ε_0 and $\gamma_{\varphi',\psi',\delta'}$ be as in Proposition 3.3. *Then the following assertions are valid.*

(i) *The point λ_0 belongs to $\sigma_{++}(U) \cup \rho(U)$ if and only if there exists an $\varepsilon_1 > 0$ such that for every uniformly bi-expansive operator V acting in $\mathcal{H}$ with $\|I - V\| < \varepsilon_1$ the operator UV has no spectrum inside the curve $\gamma_{\varphi',\psi',\delta'}$.*

(ii) *The point λ_0 belongs to $\sigma_{\pi_+}(U)$ if and only if there exists an $\varepsilon_1 > 0$ such that for every uniformly bi-expansive operator V acting in $\mathcal{H}$ with $\|I - V\| < \varepsilon_1$ the spectrum of UV inside the curve $\gamma_{\varphi',\psi',\delta'}$ consists of at most finitely many normal eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then*

$$\mathcal{M}_- := \operatorname{span}\{\mathcal{L}_{\lambda_j}(UV) : 1 \leqslant j \leqslant k\}$$

is of finite dimension and moreover, in this case, the dimension of $\mathcal{M}_-$ is equal to the rank of negativity $\kappa_-(E(\omega_{\varphi',\psi'})\mathcal{H})$ of the Pontryagin space $E(\omega_{\varphi',\psi'})\mathcal{H}$, that is

$$\dim \mathcal{M}_- = \kappa_-(E(\omega_{\varphi',\psi'})\mathcal{H}).$$

(iii) *The point λ_0 does not belong to $\sigma_{\pi_+}(U) \cup \rho(U)$ if and only if there exists an $\varepsilon_1 >$ 0 such that for every uniformly bi-expansive operator V acting in $\mathcal{H}$ with $\|I - V\| < \varepsilon_1$ the range of the Riesz–Dunford projector corresponding to UV and $\gamma_{\varphi',\psi',\delta'}$ is of infinite dimension.*

Proof. Similar to the proof of Theorem 2.4. ∎

We left it to the reader to formulate and to prove statements like Corollary 2.5 for operators UV, where U is a unitary and V is a bi-expansive operator.

4. AN APPLICATION: SECOND ORDER SYSTEMS

A linear equation describing transverse motions of a thin beam can be written in the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial r^2}\Big[\frac{\partial^2 u}{\partial r^2} + \alpha \frac{\partial^3 u}{\partial r^2 \partial t}\Big] = 0, \quad r \in (0,1), t > 0,$$

where $u(r,t)$ is the transverse displacement of the beam at time t and position r and α is a constant. The existence of solutions depends also on boundary and initial conditions. Identifying the function $u(\cdot,t)$ with an element $z(t) \in L^2(0,1)$ by

$z(t)(r) = u(r,t)$ we obtain from the partial differential equation above a second order equation in $L^2(0,1)$ of the form

$$\ddot{z}(t) + A_0 z(t) + D\dot{z}(t) = 0, \tag{4.1}$$

where $A_0 = \frac{\partial^4}{\partial r^4}$, $D = \alpha A_0$ acting in $L^2(0,1)$ with appropriate domains encoding the boundary conditions under consideration.

In the following we study second order equations of type (4.1) in an abstract Hilbert space H where the operator A_0 is an unbounded, uniformly positive operator on H and the operator $D : H_{1/2} \to H_{-1/2}$ is a bounded operator such that $A_0^{-1/2} D A_0^{-1/2}$ is a bounded non-negative operator in H. Here $H_{1/2}$ is the domain of the positive square root of A_0 equipped with the norm $\| \cdot \|_{H_{1/2}} := \|A_0^{1/2} \cdot \|_H$ and $H_{-1/2}$ is the completion of H with respect to the norm $\|z\|_{H_{-1/2}} = \|A_0^{-1/2} z\|_H$. Thus A_0 restricts to a bounded operator $A_0 : H_{1/2} \to H_{-1/2}$.

The second order equation (4.1) is equivalent to the standard first order equation $\dot{x}(t) = \mathcal{A}x(t)$ where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H_{1/2} \times H \to H_{1/2} \times H$ is given by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix}, \quad \mathcal{D}(\mathcal{A}) = \{ \left(\begin{smallmatrix} z \\ w \end{smallmatrix} \right) \in H_{1/2} \times H_{1/2} : A_0 z + Dw \in H \}.$$

This operator matrix has been studied in the literature for more than 20 years. Interest in this particular model is motivated by various problems such as stabilization, see for example [9], [29], [30], [32], solvability of Riccati equations [15] and minimum-phase property [17]. It is well-known that $\mathcal{A}$ generates a C_0-semigroup of contractions in $H_{1/2} \times H$ and thus the spectrum of $\mathcal{A}$ is located in the closed left half plane. This goes back to [4], [28], see also [5], [11]. Moreover, $0 \in \rho(\mathcal{A})$, see [33].

We will apply our results from Section 2 to the operator matrix $\mathcal{A}$. For this we introduce an inner product on $H_{1/2} \times H$ via

$$\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right] = (x_1, x_2)_{H_1/2} - (y_1, y_2) \quad \text{for } \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in H_{1/2} \times H.$$

It is well-known (e.g. [18], [34]) that $\mathcal{A}$ is selfadjoint operator in the Krein space $(H_{1/2} \times H, [\cdot,\cdot])$, and, hence, $\mathbb{R} \subset \sigma_{\text{ap}}(\mathcal{A}) \cup \rho(\mathcal{A})$.

We denote by E_{A_0} the spectral function of the selfadjoint operator A_0 in the Hilbert space H.

PROPOSITION 4.1. *Assume, in addition, that the operator A_0^{-1} is compact in H. Then*

$$\sigma_{\text{ess}}(\mathcal{A}) = \left\{ \lambda \in \mathbb{C} \backslash \{0\} : \frac{1}{\lambda} \in \sigma_{\text{ess}}(-A_0^{-1} D) \right\}, \tag{4.2}$$

where $A_0^{-1} D$ is considered as an operator acting in $H_{1/2}$. Let $\lambda_0 \in \sigma(\mathcal{A}) \cap \mathbb{R}$ and denote by κ the number of eigenvalues of A_0 below or equal to λ_0^2 (counted according to their *multiplicities), that is*

$$\kappa = \dim E_{A_0}((0, \lambda_0^2])H.$$

Moreover, we assume

$$\|A_0^{1/2}x\| \neq |\lambda_0|\|x\| \quad \textit{for all } \begin{pmatrix} x \\ \lambda_0 x \end{pmatrix} \in \ker(\mathcal{A} - \lambda_0) \textit{ with } x \neq 0. \tag{4.3}$$

Then there exists an interval (a, b), $\lambda_0 \in (a, b)$, such that $\mathcal{A}$ satisfies (2.3) *and, with δ' and $\gamma_{\delta'}$ as in Proposition* 2.3*, there exists an $\varepsilon_1 > 0$ such that for every uniformly dissipative operator $\mathcal{B}$ acting in $H_{1/2} \times H$ with $\|\mathcal{B}\| < \varepsilon_1$ the spectrum of $\mathcal{A} + \mathcal{B}$ inside $\gamma_{\delta'}$ consists of at most finitely many normal eigenvalues with*

$$2\kappa \geqslant \dim \operatorname{span} \{\mathcal{L}_\lambda(\mathcal{A} + \mathcal{B}) : \lambda \text{ inside } \gamma_{\delta'}\}.$$

Proof. Relation (4.2) is proved in Theorem 4.1 of [18]. The space $E_{A_0}((0, \lambda_0^2])H$ is finite dimensional and a subset of $\mathcal{D}(A_0)$. We set

$$\mathcal{H}_0 := (E_{A_0}((0, \lambda_0^2])H \times E_{A_0}((0, \lambda_0^2])H)^\perp,$$

where $\perp$ denotes the orthogonal complement in $H_{1/2} \times H$ with respect to the usual Hilbert scalar product in $H_{1/2} \times H$. Then

$$\operatorname{codim} \mathcal{H}_0 = 2\kappa. \tag{4.4}$$

For every sequence $\left(\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right)$ in $\mathcal{D}(\mathcal{A}) \cap \mathcal{H}_0$ with

$$\left\|\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\|^2_{H_{1/2} \times H} = 1 \quad \text{and} \quad \lim_{n \to \infty} \left\|(\mathcal{A} - \lambda_0)\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\|_{H_{1/2} \times H} = 0$$

we have

$$\|y_n - \lambda_0 x_n\|_{H_{1/2}} \to 0 \quad \text{as } n \to \infty.$$

This gives

$$\begin{aligned} \liminf_{n \to \infty} \left[\begin{pmatrix} x_n \\ y_n \end{pmatrix}, \begin{pmatrix} x_n \\ y_n \end{pmatrix}\right] &= \liminf_{n \to \infty} ((x_n, x_n)_{H_{1/2}} - (y_n, y_n)) \\ &= \liminf_{n \to \infty} ((A_0 x_n, x_n) - \lambda_0^2 (x_n, x_n)) > 0, \end{aligned}$$

hence $\lambda_0 \in \sigma_{\pi_+}(\mathcal{A})$.

By Corollary 5.2 of [18] the non-real spectrum of $\mathcal{A}$ does not accumulate to a real point. Then, together with Theorem 18 of [3] we find real numbers such that $\mathcal{A}$ satisfies (2.3). Moreover, for every vector $\begin{pmatrix} x \\ \lambda_0 x \end{pmatrix} \in \ker(\mathcal{A} - \lambda_0)$ with $x \neq 0$ we deduce from (4.3)

$$\left[\begin{pmatrix} x_n \\ \lambda_0 x_n \end{pmatrix}, \begin{pmatrix} x_n \\ \lambda_0 x_n \end{pmatrix}\right] = \|x_n\|^2_{H_{1/2}} - |\lambda_0|^2 \|x_n\|^2 \neq 0,$$

that is, there exists no Jordan chain of $\mathcal{A}$ corresponding to λ_0 of length greater than one. Therefore $\ker(\mathcal{A} - \lambda_0) = \mathcal{L}_{\lambda_0}(A)$ and, by Theorem 2.4, Corollary 2.5 and (4.3), we obtain

$$\dim \operatorname{span} \{\mathcal{L}_\lambda(\mathcal{A} + \mathcal{B}) : \lambda \text{ inside } \gamma_{\delta'}\} \leqslant 2\kappa. \quad \blacksquare$$

REMARK 4.2. Choose λ_0, ε_1 as in Proposition 4.1 and let us assume that the uniformly dissipative operator $\mathcal{B}$ in Proposition 4.1 is of the following form with respect to $H_{1/2} \times H$

$$\mathcal{B} = \begin{bmatrix} \beta_0 I & 0 \\ 0 & B_1 \end{bmatrix},$$

where β_0, $|\beta_0| \leqslant \varepsilon_1$, is a complex number with positive imaginary part and $-B_1$ is a bounded, uniformly dissipative operator in the Hilbert H with norm less or equal to ε_1. Then the first order equation $\dot{x}(t) = (\mathcal{A} + \mathcal{B})x(t)$ is equivalent to the second order equation

$$\ddot{z}(t) + (D - \beta_0 I - B_1)\dot{z}(t) + (A_0 - \beta_0 D + \beta_0 B_1)z(t) = 0.$$

Acknowledgements. Sadly, our colleague and friend Peter Jonas passed away on July, 18th 2007.

The research of T.Ya. Azizov is partially supported by RFBR grant 08-01-00566-a.

REFERENCES

[1] T.YA. AZIZOV, J. BEHRNDT, P. JONAS, C. TRUNK, Spectral points of type π_+ and type π_- for closed linear relations in Krein spaces, *J. London Math. Soc.*, to appear.

[2] T.YA. AZIZOV, I.S. IOKHVIDOV, *Linear Operators in Spaces with an Indefinite Metric*, John Wiley & Sons, Chichester 1989.

[3] T.YA. AZIZOV, P. JONAS, C. TRUNK, Spectral points of type π_+ and π_- of selfadjoint operators in Krein spaces, *J. Funct. Anal.* **226**(2005), 114–137.

[4] H.T. BANKS, K. ITO, A unified framework for approximation in inverse problems for distributed parameter systems, *Control Theory Adv. Tech.* **4**(1988), 73–90.

[5] H.T. BANKS, K. ITO, Y. WANG, Well posedness for damped second order systems with unbounded input operators, *Differential Integral Equations* **8**(1995), 587–606.

[6] J. BEHRNDT, On the spectral theory of singular indefinite Sturm–Liouville operators, *J. Math. Anal. Appl.* **334**(2007), 1439–1449.

[7] J. BEHRNDT, F. PHILIPP, C. TRUNK, Properties of the spectrum of type π_+ and type π_- of self-adjoint operators in Krein spaces, *Methods Funct. Anal. Topology* **12**(2006), 326–340.

[8] J. BEHRNDT, C. TRUNK, On the negative squares of indefinite Sturm–Liouville operators, *J. Differential Equations* **238**(2007), 491–519.

[9] C.D. BENCHIMOL, A note on weak stabilizability of contraction semigroups, *SIAM J. Control Optim.* **16**(1978), 373–379.

[10] J. BOGNAR, *Indefinite Inner Product Spaces*, Springer-Verlag, Berlin-Heidelberg-New York 1974.

[11] S. CHEN, K. LIU, Z. LIU, Spectrum and stability for elastic systems with global or local Kelvin–Voigt damping, *SIAM J. Appl. Math.* **59**(1999), 651–668.

[12] B. ĆURGUS, H. LANGER, A Krein space approach to symmetric ordinary differential operators with an indefinite weight function, *J. Differential Equations* **79**(1989), 31–61.

[13] U. GÜNTHER, O.N. KIRILLOV, A Krein space related perturbation theory for MHD α^2-dynamos and resonant unfolding of diabolical points, *J. Phys. Math. Gen.* **39**(2006), 10057–10076.

[14] U. GÜNTHER, F. STEFANI, M. ZNOJIL, MHD α^2-dynamo, squire equation and $\mathcal{PT}$-symmetric interpolation between square well and harmonic oscillator, *J. Math. Phys.* **46**(2005), 063504, 22p.

[15] E. HENDRICKSON, I. LASIECKA, Numerical approximations and regularizations of Riccati equations arising in hyperbolic dynamics with unbounded control operators, *Comput. Optim. Appl.* **2**(1993), 343–390.

[16] I.S. IOKHVIDOV, M.G. KREIN, H. LANGER, *Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric*, Akademie-Verlag, Berlin 1982.

[17] B. JACOB, K. MORRIS, C. TRUNK, Minimum-phase infinite-dimensional second-order systems, *IEEE Trans. Automat. Control* **52**(2007), 1654–1665.

[18] B. JACOB, C. TRUNK, Location of the spectrum of operator matrices which are associated to second order equations, *Operators and Matrices* **1**(2007), 45–60.

[19] B. JACOB, C. TRUNK, M. WINKLMEIER, Analyticity and Riesz basis property of semigroups associated to damped vibrations, *J. Evolution Equations* **8**(2008), 263–281.

[20] P. JONAS, On a class of selfadjoint operators in Krein space and their compact perturbations, *Integral Equations Operator Theory* **11**(1988), 351–384

[21] P. JONAS, On locally definite operators in Krein spaces, in *Spectral Theory and Applications*, Theta Ser. Adv. Math., vol. 2, Theta, Bucharest 2003, pp. 95–127.

[22] T. KATO, *Perturbation Theory for Linear Operators*, Second Edition, Springer-Verlag, Berlin-Heidelberg-New York 1976.

[23] I.M. KARABASH, C. TRUNK, Spectral properties of singular Sturm–Liouville operators with indefinite weight sign x, *Proc. Roy. Soc. Edinburgh Sect. A* **139**(2009), 483–503.

[24] P. LANCASTER, A.S. MARKUS, V.I. MATSAEV, Definitizable operators and quasihyperbolic operator polynomials, *J. Funct. Anal.* **131**(1995), 1–28.

[25] P. LANCASTER, A. SHKALIKOV, Damped vibrations of beams and related spectral problems, *Canad. Appl. Math. Quart.* **2**(1994), 45–90.

[26] H. LANGER, A.S. MARKUS, V.I. MATSAEV, Locally definite operators in indefinite inner product spaces, *Math. Ann.* **308**(1997), 405–424.

[27] H. LANGER, C. TRETTER, A Krein space approach to *PT* symmetry, *Czech. J. Phys.* **54**(2004), 1113–1120.

[28] I. LASIECKA, Stabilization of wave and plate-like equations with nonlinear dissipation on the boundary, *J. Differential Equations* **79**(1989), 340–381.

[29] I. LASIECKA, R. TRIGGIANI, Uniform exponential energy decay of wave equations in a bounded region with $L_2(0,\infty;L_2(\Gamma))$-feedback control in the Dirichlet boundary conditions, *J. Differential Equations* **66**(1987), 340–390.

[30] N. LEVAN, The stabilizability problem: a Hilbert space operator decomposition approach, *IEEE Trans. Circuits Systems* **25**(1978), 721–727.

[31] L.S. PONTRYAGIN, Hermitian operators in spaces with indefinite metric [Russian], *Izv. Akad. Nauk USSR Ser. Mat.* **8**(1944), 243–280.

[32] M. SLEMROD, Stabilization of boundary control systems, *J. Differential Equations* **22**(1976), 402–415.

[33] M. TUCSNAK, G. WEISS, How to get a conservative well-posed system out of thin air. I, *ESAIM Control Optim. Calc. Var.* **9**(2003), 247–274.

[34] M. TUCSNAK, G. WEISS, How to get a conservative well-posed system out of thin air. II, *SIAM J. Control Optim.* **42**(2003), 907–935.

TOMAS YA. AZIZOV, DEPARTMENT OF MATHEMATICS, VORONEZH STATE UNIVERSITY, UNIVERSITETSKAYA PL. 1, 394006 VORONEZH, RUSSIA
E-mail address: azizov@math.vsu.ru

PETER JONAS, TECHNISCHE UNIVERSITÄT BERLIN, INSTITUT FÜR MATHEMATIK, MA 6–4, STR. DES 17. JUNI 136, D–10623 BERLIN, GERMANY

CARSTEN TRUNK, TECHNISCHE UNIVERSITÄT ILMENAU, INSTITUT FÜR MATHEMATIK, POSTFACH 10 05 65, D–98684 ILMENAU, GERMANY
E-mail address: carsten.trunk@tu-ilmenau.de

Received January 18, 2008; revised May 8, 2008.