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PURELY INFINITE CORONA ALGEBRAS OF SIMPLE C^*-ALGEBRAS WITH REAL RANK ZERO

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ABSTRACT. In this paper we explore conditions on simple non-unital C^*-algebras with real rank zero and stable rank one under which their corona algebras are purely infinite and not necessarily simple. In particular, our results allow to characterize when the corona algebra of a simple AF-algebra is purely infinite in terms of continuity conditions on its scale.

KEYWORDS: *Corona algebras, purely infinite C^*-algebras, scales in C^*-algebras.*

MSC (2000): 46L35, 46L80, 46L05.

INTRODUCTION

The extent to which corona algebras are infinite has been of interest in different instances. It is of course well known that, in the case of the compact operators over a separable infinite dimensional Hilbert space, the corresponding corona algebra (i.e. the Calkin algebra) is in fact purely infinite simple.

Several analogies of this phenomenon have been studied, namely in the real rank zero situation. For example, it was shown by S. Zhang in [22] that if A is of real rank zero and $\mathcal{M}(A)/A$ is simple, then it is purely infinite simple. This has been further generalized by H. Lin in [15] (see also [14]) to encompass all simple and σ-unital C^*-algebras. More concretely, Lin proves that for a simple, σ-unital and non-unital C^*-algebra A with $A \not\cong \mathcal{K}$, simplicity of $\mathcal{M}(A)/A$ is actually equivalent to $\mathcal{M}(A)/A$ being purely infinite simple. In turn, this is equivalent to the base algebra A having a continous scale, a notion that is conveniently expressed in terms of Cuntz comparison of positive elements (see [14], [15] and the definitions below). The ideal structure of corona algebras of simple non-unital AF-algebras was first studied by Elliott in [4], followed by Lin in [13].

It is to be expected, however, that some form of infiniteness in the corona algebra prevades even when it is not simple. In this direction, it is natural to ask when such an algebra is going to be purely infinite, in the sense of Kirchberg and

Rørdam (see [10]). Recall that a C^*-algebra A is termed *purely infinite* if A has no characters and, whenever a, b are positive elements with a lying in the closed, two-sided ideal generated by b, one has $x_n b x_n^* \to a$ in norm, for a sequence (x_n) in A. In the particular case that A is simple, this notion agrees with Cuntz's definition of pure infiniteness, that required every non-zero hereditary algebra to contain a non-zero infinite projection (see [3]). The more general concept allows examples that lack (non-trivial) projections, as already proved in [10].

The above problem was already considered by the first author and P.W. Ng in Theorem 1.2 of [12], where they proved pure infiniteness of the corona algebra of a stable separable algebra A with real rank zero in the case that A has an AF-skeleton with finitely many extremal tracial rays (see also [12]).

In the present paper we will explore pure infiniteness in the corona algebra of a wide class of C^*-algebras. Our attention will be directed to those simple and separable algebras that have real rank zero, stable rank one and weakly unperforated K_0. One reason to do so is that the projection monoid of their multiplier algebras is completely described in terms of the projection monoid of the base algebra and a certain semigroup of lower semicontinuous affine functions defined on the state space (see [18] and Section 2). This semigroup contains (the class of) the unit of the multiplier algebra, and the fact that the base algebra has a continuous scale is translated, in this setting, to the fact that the function representing the unit is a continuous affine function. We are able then to weaken this concept to what we call *quasi-continuous scale* (see Definition 2.2), which covers a variety of situations.

The condition of quasi-continuous scale turns out to be what in fact characterizes pure infiniteness of the corona algebra. This is proved in Theorem 3.4, under the further assumption that the base algebra has finitely many *infinite* extremal quasi-traces (see below for the precise definitions), yet it can have infinitely many extremal quasi-traces. As a corollary we obtain that if A has quasi-continuous scale (hence $\mathcal{M}(A)/A$ is purely infinite), then the ideal lattice of the corona algebra is finite (Corollary 3.5). A further corollary shows that in the case $A\otimes\mathcal{K}$ where A is a unital algebra whose simplex of unital quasi-traces $\mathrm{QT}(A)$ is compact, pure infiniteness of $\mathcal{M}(A\otimes\mathcal{K})/A\otimes\mathcal{K}$ is equivalent to the fact that $\mathrm{QT}(A)$ is finite dimensional.

1. NOTATION AND PRELIMINARIES

If A is a C^*-algebra, we use $\mathrm{V}(A)$ to denote the abelian monoid of Murray–von Neumann equivalence classes of projections coming from matrices over A. We denote the class of a projection p by $[p]$ and addition in $\mathrm{V}(A)$ is defined as $[p]+[q] = \left[\left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right)\right]$. The importance of this monoid is that it captures structural aspects of the algebra that the ensuing Grothendieck group $\mathrm{K}_0(A)$ may fail to do if we do not have cancellation conditions.

A submonoid I of an abelian monoid M is said to be an *order-ideal* provided $x + y \in I$ exactly when x, $y \in I$. For example, if I is a closed two-sided ideal of a C^*-algebra A, then $\mathrm{V}(I)$ is an order-ideal of $\mathrm{V}(A)$. Given an order-ideal I of M, we may define a congruence $\sim$ on M by setting $x \sim y$ if there are elements z, w in I such that $x + z = y + w$. Denote by M/I the quotient of M modulo this congruence, which becomes an abelian monoid under the natural operations. In the case of interest, namely a C^*-algebra A and an ideal I, the quotient monoid $\mathrm{V}(A)/\mathrm{V}(I)$ does not always agree with $\mathrm{V}(A/I)$, but it does if A has real rank zero (see [1]), basically due to the fact that projections lift from quotients. Still, when they do not lift, the quotient monoid proves useful as we shall see below.

Recall that, given an abelian monoid M and an order-unit u in M, then a (normalized) *state* on M is a monoid morphism $s\colon M \to \mathbb{R}^+$ such that $s(u) = 1$. We shall denote the set of (normalized) states by $\mathrm{St}(M, u)$ or by S_u, which is a compact and convex set. For example, if A is a unital C^*-algebra, then $[1_A]$ is an order-unit for $\mathrm{V}(A)$ and $S_{[1_A]} = \mathrm{St}(K_0(A), [1_A])$. If, further, A has real rank zero, then in fact this state space is a Choquet simplex (see Theorem 1.2 in [7]).

For a compact convex set K, we shall use as customary $\mathrm{Aff}(K)$ to denote the group of all affine real-valued continuous functions defined on the space K. Its positive (respectively, strictly positive) elements, in the pointwise ordering, will be denoted by $\mathrm{Aff}(K)^+$ (respectively, $\mathrm{Aff}(K)^{++}$). We also denote $\mathrm{LAff}(K)^{++}$ the semigroup of lower semicontinuous functions with values on $\mathbb{R}^{++} \cup \{\infty\}$.

Given a C^*-algebra A and an order-unit u in $\mathrm{V}(A)$, we may represent $\mathrm{V}(A)$ as affine functions on its state space S_u via the map $\phi_u\colon \mathrm{V}(A) \to \mathrm{Aff}(S_u)^+$. This is just evaluation, namely $\phi_u(x)(s) = s(x)$ whenever $x \in \mathrm{V}(A)$ and $s \in S_u$.

2. QUASI-CONTINUOUS SCALES

Let A be a C^*-algebra. Recall that a *quasitrace* on A is a map $\tau\colon A_+ \to [0, \infty]$ which is linear on commutative subalgebras of A, satisfies the tracial property and extends to matrices over A. Let us denote by $\mathrm{LQT}(A)$ the set of all lower semicontinuous, densely defined quasi-traces whose restriction to the Pedersen ideal of A is finite. We say that a lower semicontinuous, densely defined quasitrace τ is *infinite* provided $\sup\limits_{\lambda} \tau(u_\lambda) = \infty$ for some approximate unit (u_λ).

For a C^*-algebra A, denote by $\mathrm{QT}(A)$ the convex set of normalized quasitraces, that is, those 2-quasitraces τ such that $\|\tau\| = 1$, and by $\mathrm{T}(A)$ the convex set of normalized traces. If A happens to be unital, then we have $1 = \tau(1) = \|\tau\|$ for any element τ in either $\mathrm{QT}(A)$ or $\mathrm{T}(A)$.

If p is a non-zero projection in A, set $\mathrm{Q}_p = \{\tau \in \mathrm{Q} : \tau(p) = 1\}$, which is a weakly compact convex set if A is simple. If A is exact, then since all quasitraces are traces, we shall write T_p instead of Q_p. Furthermore, if A is σ-unital and has real rank zero, a Blackadar–Handelman type theorem shows that Q_p is affinely homeomorphic to the space $S_u = \mathrm{St}(\mathrm{V}(A), u)$ where $u = [p]$ (see, e.g. Theorem 5.6 in [18]).

The following is possibly well known. We state it as a lemma for future reference.

LEMMA 2.1. *Let A be a simple, σ-unital C^*-algebra of real rank zero. For any non-zero projection p in A there is an affine homeomorphism $\mathrm{Q}_p \cong \mathrm{QT}(pAp)$.*

Proof. We have mentioned already that Q_p is affinely homeomorphic to $\mathrm{St}(V(A), [p])$. This, in turn, is affinely homeomorphic to $\mathrm{St}(V(pAp), [p])$ (in fact, there is an ordered group isomorphism from $\mathrm{K}_0(A)$ to $\mathrm{K}_0(pAp)$ — see, e.g. Lemma 14.4 in [8]). Now the Blackadar and Handelman Theorem (III.1.3 in [2]) gives us an affine homeomorphism between $\mathrm{QT}(pAp)$ and $\mathrm{St}(V(pAp), [p])$. ∎

Continue to assume that A is a simple, separable, C^*-algebra A with real rank zero and pick a non-zero projection p in A. Set $u = [p]$ in $\mathrm{V}(A)$ and define $d = \sup \phi_u([e_n])$ where (e_n) is an approximate unit for A consisting of projections and ϕ_u is the natural representation map of $\mathrm{V}(A)$ in the affine continuous functions on the state space S_u of $\mathrm{V}(A)$. Then $d \in \mathrm{LAff}(S_u)^{++}$, which in this paper will be referred to as the *scale* of A.

If we assume that A has moreover stable rank one and weak unperforation on $\mathrm{K}_0(A)$, then A has continuous scale in the sense defined by Lin in [14] and [15] if and only if the scale d as defined above is continuous (see Proposition 2.2 in [15]). One can also show that continuity or finiteness properties of the scale do not depend on the particular projection p or the approximate unit (e_n) chosen (see, e.g. [18]).

The observed affine homeomorphism between Q_p and S_u allows us to identify these spaces. Note that every infinite quasitrace in Q_p corresponds, in this setting, to a point of the space S_u at which the scale takes an infinite value. Denote $F_\infty = d^{-1}(\infty) \cap \partial_e S_u$, where $\partial_e S_u$ stands for the extreme boundary of S_u, identified with infinite extremal quasitraces in Q_p.

In the definition below, we shall need the notion of a *complementary face* on a Choquet simplex. Let us recall that, given a face F in a simplex K, the union F' of all the faces in K that are disjoint from F is the largest face of K that is disjoint from F (see, e.g. Proposition 10.12 in [6]) and is called the complementary face of F.

The following notion is key to this paper, hence it is conveniently highlighted:

DEFINITION 2.2. Let A be a simple C^*-algebra with real rank zero. Retaining the notations from the previous paragraphs, we say that A has *quasi-continuous scale* provided that

(i) F_∞ is finite,

(ii) the complementary face F'_∞ of the convex hull $\mathrm{conv}(F_\infty)$ is closed, and

(iii) $d_{|F'_\infty}$ is continuous.

LEMMA 2.3. *The notion of quasi-continuous scale does not depend on the particular projection chosen.*

Proof. Let p and q be projections in A and write $u = [p]$ and $v = [q]$ in $\mathrm{V}(A)$. There is then a homeomorphism $\alpha \colon S_u \to S_v$ given by $\alpha(s) = \frac{s}{s(v)}$, which is not affine but preserves extreme points and faces (see, e.g. Proposition 6.17 in [6]). It follows from this that if F is a face of S_u then $\alpha(F') = \alpha(F)'$.

Let (e_n) be an approximate unit consisting of projections, and write $d_u = \sup \phi_u(e_n)$, and $d_v = \sup \phi_v(e_n)$. Observe that, for any s in S_u, we have $d_u(s) = s(v)d_v(\alpha(s))$. If we denote by $F^u_\infty = \{s \in S_u : d_u(s) = \infty\}$ and F_∞ is defined accordingly, then it is clear that $\alpha(F^u_\infty) = F^v_\infty$ and $\alpha((F^u_\infty)') = (F^v_\infty)'$. Thus $(F^u_\infty)'$ is closed if and only if $(F^v_\infty)'$ is closed, and $d_{|(F^u_\infty)'}$ is continuous if and only if $d_{|(F^v_\infty)'}$ is continuous. ∎

The following lemma is possibly well-known. We include a proof for completeness.

LEMMA 2.4. *Let K be a Choquet simplex and let $s_1, \ldots, s_n$ be extreme points of K. Then the complementary face $\{s_1, \ldots, s_n\}'$ of* $\mathrm{conv}(s_1, \ldots, s_n)$ *is* $\{s_1\}' \cap \cdots \cap \{s_n\}'$.

Proof. By induction it is enough to prove the result for $n = 2$. It is clear that $\{s_1, s_2\}' \subseteq \{s_1\}' \cap \{s_2\}'$. Now let $x \in \{s_1\}' \cap \{s_2\}'$. There exist then faces F_i such that $s_i \notin F_i$ and $x \in F_1 \cap F_2$, which is also a face of K. If $F_1 \cap F_2 \cap \mathrm{conv}(s_1, s_2) \neq \emptyset$, take y in this intersection. Then $y = \alpha s_1 + (1-\alpha)s_2$ for α in $(0,1)$. But since $F_1 \cap F_2$ is a face of K we do get that, e.g. $s_1 \in F_1$, a contradiction. ∎

PROPOSITION 2.5. *Let A be a simple, separable, non-unital C^*-algebra with real rank zero, stable rank one and weakly unperforated* K_0. *If A has continuous scale, then A also has quasi-continuous scale and the converse does not hold in general. If $S_u = \mathrm{St}(\mathrm{V}(A), u)$ has compact extreme boundary (where $u = [p]$ and p is a non-zero projection), then A has quasi-continous scale if and only if the set F_∞ is finite and isolated, and restriction of the scale to its complement is continuous. Finally in the case where $A = B \otimes \mathcal{K}$ where B is unital and* $\mathrm{QT}(B)$ *has finitely many extremal points, then A has quasi-continuous scale.*

Proof. If A has continuous scale, then $F_\infty = \emptyset$. Examples of algebras having quasi-continuous but not continuous scale are easily constructed using the methods in [18].

Assume now $\partial_e S_u$ is compact. Then, by e.g. Corollary 11.20 in [6], there is an affine homeomorphism between S_u and $M_1^+(\partial_e S_u)$ that sends each extremal point t to ε_t (the point mass measure at t). Then the argument in Corollary 3.13 in [17] together with Lemma 2.4 show that, if F_∞ is finite, then the complementary face F'_∞ is closed if and only if $\partial_e S_u \setminus F_\infty$ is closed. And that amounts to saying that F_∞ is isolated.

Now suppose that $A = B \otimes \mathcal{K}$. Consider the element $p = 1 \otimes e_{11}$ in $B \otimes \mathcal{K}$ and set $u = [p]$ in $\mathrm{V}(B \otimes \mathcal{K})$. Observe that, via the natural isomorphism $\mathrm{V}(B \otimes \mathcal{K}) \cong \mathrm{V}(B)$, the element u corresponds to $[1]$. By III.1.3 in [2], the space of normalized states on $\mathrm{V}(B)$ is affinely homeomorphic to the simplex of unital (quasi)traces of B.

Since $B \otimes \mathcal{K}$ is stable, its scale d defined on S_u is identically infinite, and we have that $F_\infty = \partial_e S_u$, a finite set under our assumption. Thus $F'_\infty = \emptyset$ and clearly A has quasi-continuous scale. ∎

3. PURELY INFINITE CORONA ALGEBRAS

In this section we shall use the notion of quasi-continuous scale to analyze pure infiniteness in corona algebras. As mentioned in the introduction we consider simple, separable algebras with real rank zero, stable rank one and weakly unperforated K_0.

If such an algebra is not unital and not of type one, it was shown by the second author in Theorem 3.9 in [18] that, for any non-zero projection p in A, there is an isomorphism of monoids

$$\varphi\colon \mathrm{V}(\mathcal{M}(A)) \cong \mathrm{V}(A) \sqcup W_\sigma^d(S_u)\,, \tag{†}$$

where $u = [p] \in \mathrm{V}(A)$ and, by definition,

$$W_\sigma^d(S_u) = \{f \in \mathrm{LAff}(S_u)^{++} : f + g = nd\,,\ \text{for some } n \text{ in } \mathbb{N}, g \text{ in } \mathrm{LAff}(S_u)^{++}\}\,.$$

In short, $\varphi([p]) = [p]$ if $p \in A$ and $\varphi([p]) = \sup\{\phi_u([q]) : [q] \in \mathrm{V}(A) \text{ and } q \lesssim p\}$, if $p \notin A$. Note that addition on the right hand side is defined as $x + f = \phi_u(x) + f$ if $x \in \mathrm{V}(A)$ and $f \in \mathrm{LAff}(S_u)^{++}$. We will identify $\mathrm{V}(\mathcal{M}(A))$ with its image under φ.

Recall that, if A is a C^*-algebra and p is a non-zero projection in A, we say that p is *properly infinite* if $p \oplus p \lesssim p$.

LEMMA 3.1. *Let A be a non-unital, simple, separable C^*-algebra with real rank zero, stable rank one and with $K_0(A)$ weakly unperforated. Let p be a non-zero projection. Assume that A has finitely many infinite extremal quasi-traces in Q_p and that A has quasi-continuous scale. Then every non-zero projection in $\mathcal{M}(A) \setminus A$ has an image that is properly infinite in the corona algebra.*

Proof. We may of course assume that A is not of type I.

Adopt the notation from the preceding paragraphs. Let n be the number of extreme points of S_u where d is infinite, and label them as $F_\infty = \{s_1, \dots, s_n\}$ (the possibility $n = 0$ is not excluded). Via the identifications just made before, each such point corresponds to a infinite extremal quasi-trace.

Next, if p is a projection in $\mathcal{M}(A) \setminus A$, then its equivalence class is a function f in $W_\sigma^d(S_u)$. We may of course assume that f is not identically infinite (even if d is). By assumption F'_∞ is a closed subset of S_u, hence compact. Since there is g in $W_\sigma^d(S_u)$ and m in $\mathbb{N}$ such that $f + g = md$ and $d_{|F_\infty}$ is continuous, we find that $f_{|F'_\infty}$ is continuous and bounded, say $f_{|F'_\infty} \ll k$ for some k.

Without loss of generality we have that $f(s_i) = \infty$ for $i = 1, \dots, l$ for some $l \leqslant n$ (again, the possibility $l = 0$ is not excluded). Take x in $\mathrm{V}(A)$ such that $a_s = \phi_u(x)(s) - f(s) > 0$ for s in $F_\infty \setminus \{s_1, \dots, s_l\}$ and $\phi_u(x) \gg k$. Let $h \in \mathrm{LAff}(S_u)^{++}$ be defined by $h(s) = a_s$ if $s \in F_\infty \setminus \{s_1, \dots, s_l\}$, $h(s_i) = 1$ for $i = 1, \dots, l$ and $h_{|F'_\infty} = (\phi_u(x) - f)_{|F'_\infty}$. Note that this can be done because S_u is the convex direct sum of the convex hull of $\{s_1, \dots, s_n\}$ and F'_∞ (see, e.g. Theorem 11.28 of [6] and also Corollary 11.27 of [6]).

Restricting to the extreme boundary $\partial_e S_u$ of S_u, we find that $(f + \phi_u(x))_{|\partial_e S_u} = (2f + h)_{|\partial_e S_u}$. Therefore $f + x = 2f + h$. This implies that $h \in W_\sigma^d(S_u)$ and also that $[f] = 2[f] + [h]$ in $\mathrm{V}(\mathcal{M}(A))/\mathrm{V}(A)$. Note that h does not represent a projection in A, but in $\mathcal{M}(A)$. Consider the natural monoid morphism $\pi_* \colon \mathrm{V}(\mathcal{M}(A))/\mathrm{V}(A) \to \mathrm{V}(\mathcal{M}(A)/A)$ defined using the quotient map $\pi \colon \mathcal{M}(A) \to \mathcal{M}(A)/A$. It then follows that $\pi_*([f])$ is properly infinite. ∎

We now analyse the purely infinite property of multiplier algebras of simple C^*-algebras with real rank zero. Recall that, if A is a σ-unital, simple, non-elementary C^*-algebra, then $\mathcal{M}(A)$ contains a closed ideal $L(A)$ that properly contains A and is minimal with this property (see Remark 2.9 in [14]). Roughly speaking, it consists of the elements that have continuous scale.

With the isomorphism provided in (†) at hand (and for the class to which it applies), we have that $L(A)$ is the unique closed ideal such that $\mathrm{V}(L(A)) \cong \mathrm{V}(A) \sqcup \mathrm{Aff}(S_u)^{++}$. Then we can define $I_{\mathrm{fin}}(A)$ as the unique closed ideal of $\mathcal{M}(A)$ such that $\varphi(\mathrm{V}(I_{\mathrm{fin}}(A))) = \mathrm{V}(A) \sqcup \{f \in W_\sigma^d(S_u) : f_{|\partial_e S_u} < \infty\}$. As it turns out, $I_{\mathrm{fin}}(A)/L(A)$ is the maximal ideal with the condition that $\mathrm{V}(I_{\mathrm{fin}}(A))/\mathrm{V}(L(A))$ is a cancellative monoid; and if furthermore $\mathcal{M}(A)$ has real rank zero, then $I_{\mathrm{fin}}(A)/L(A)$ is the maximal ideal of stable rank one in $\mathcal{M}(A)/L(A)$ (see Proposition 6.1 in [18]).

PROPOSITION 3.2. *Let A be a separable and simple C^*-algebra with real rank zero. Then $\mathcal{M}(A)/A$ is purely infinite if and only if $\mathcal{M}(A)/L(A)$ is.*

Proof. We may of course assume at the ouset that A is not unital. We know from Theorem 4.19 in [10] that pure infinitess has a 2-of-3 property. Hence, considering

$$0 \to L(A)/A \to \mathcal{M}(A)/A \to \mathcal{M}(A)/L(A) \to 0\,,$$

we see that we need only prove that $L(A)/A$ is purely infinite. Since A has real rank zero, we know that every hereditary algebra in $\mathcal{M}(A)/A$ is the closed linear span of projections coming from $\mathcal{M}(A)$ (by Theorem 1.1 in [22]), and all (nonzero) projections in $\mathcal{M}(A)/A$ are infinite, by Theorem 1.3 in [22]. This implies that $L(A)/A$ is purely infinite simple. ∎

For the proof of our main result below, we need to use Cuntz comparison of positive elements. We recall the definition for the convenience of the reader.

Given positive elements a and b in a C^*-algebra A, we say that a is *Cuntz subequivalent* to b, in symbols $a \lesssim b$, in case there is a sequence (x_n) in A such that $x_n b x_n^* \to a$ in norm. This can be extended to the (local) algebra $M_\infty(A)$ in the obvious way. Given a positive a in A and $\varepsilon > 0$, write $(a-\varepsilon)_+$ as the positive part of $a - \varepsilon \cdot 1$. In other words, $(a-\varepsilon)_+ = f(a)$, where $f\colon \mathbb{R} \to \mathbb{R}$ is given by $f(t) = \max(t-\varepsilon, 0)$. We say that a is *Cuntz equivalent* to b in case $a \lesssim b$ and $b \lesssim a$. It is well known that the relation $\lesssim$ gives, when restricted to projections, the usual Murray von Neumann subequivalence, although the relation $\sim$ does not restrict to equivalence of projections unless the algebra is stably finite.

One of the useful characterizations of the relation $\lesssim$ is proved in Proposition 2.6 in [10] (see also Proposition 2.4 in [21]): $a \lesssim b$ if and only if for any $\varepsilon > 0$, there is $\delta > 0$ and x in A such that $(a-\varepsilon)_+ = x(b-\delta)_+x^*$. This is also equivalent to saying that for any $\varepsilon > 0$, there is $\delta > 0$ such that $(a-\varepsilon)_+ \lesssim (b-\delta)_+$. Note that, in case we have a projection p, then $(p-\varepsilon)_+ \sim p$ for $\varepsilon < 1$, so that in this case $p \lesssim b$ if and only if there is $\delta > 0$ for which $p \lesssim (b-\delta)_+$.

LEMMA 3.3. *Let A be a non-unital, separable C^*-algebra with real rank zero. Let p, q, r be projections in $\mathcal{M}(A)$ and let $a \in \mathcal{M}(A)_+$. If $p \oplus q \lesssim r \oplus a$, then there are projections $q_1, \dots, q_m$ in $\overline{a\mathcal{M}(A)a}$ such that $p \oplus q \lesssim r \oplus \bigoplus_{i=1}^{m} q_i$.*

Proof. Let $0 < \varepsilon < 1$. Then $p \oplus q \sim ((p \oplus q) - \varepsilon)_+ \lesssim ((r \oplus a) - \delta)_+ \lesssim r \oplus (a-\delta)_+$ for some $\delta > 0$. Since $\overline{a\mathcal{M}(A)a}$ is the closed linear span of its projections we have that, given $\delta > 0$, there are projections $p_1, \dots, p_n$ in $\overline{a\mathcal{M}(A)a}$ and complex scalars $\lambda_1, \dots, \lambda_n$ such that with $b_n = \sum_{i=1}^{n} \lambda_i p_i$,

$$\|a^{1/2} - b_n\| < \sqrt{\delta}\,.$$

Thus $\|(a^{1/2} - b_n)(a^{1/2} - b_n^*)\| < \delta$, which yields

$$\delta_0 := \|a - (a^{1/2}b_n^* + b_n a^{1/2} - b_n b_n^*)\| < \delta\,,$$

so if we put $c_n = a^{1/2}b_n^* + b_n a^{1/2} - b_n b_n^*$, the same argument as in Proposition 2.2 in [21] shows that

$$\begin{aligned}(\delta - \delta_0)(a-\delta)_+ &\leqslant (a-\delta)_+^{1/2}(a-\delta_0)(a-\delta)_+^{1/2} \leqslant (a-\delta)_+^{1/2} c_n (a-\delta)_+^{1/2} \\ &\leqslant (a-\delta)_+^{1/2}(a^{1/2}b_n^* + b_n a^{1/2})(a-\delta)_+^{1/2}\,.\end{aligned}$$

From this it follows easily that $(a-\delta)_+ \precsim 2 \cdot \bigoplus_{i=1}^{n} p_i$.

Hence,

$$p \oplus q \precsim r \oplus (a-\delta)_+ \precsim r \oplus \Big(2 \cdot \bigoplus_{i=1}^{n} p_i\Big)\,,$$

as was to be shown. ∎

THEOREM 3.4. *Let A be a non-unital, simple, separable C^*-algebra with real rank zero, stable rank one and weakly unperforated K_0. Let p be a non-zero projection and suppose that A has finitely many infinite extremal quasi-traces in Q_p. Then the following conditions are equivalent:*

(i) *A has quasi-continuous scale;*

(ii) *$\mathcal{M}(A)/A$ is purely infinite;*

(iii) *$\mathcal{M}(A)/L(A)$ is purely infinite;*

(iv) *$I_{\mathrm{fin}}(A) = L(A)$.*

Proof. (i) ⇒ (ii) This follows using the argument in Theorem 2.5 in [12]. We provide some details for the convenience of the reader. Using the work of Zhang (Theorem 1.1 in [22]) we know that every hereditary algebra of the corona is the closed linear span of some projections coming from the multipliers, and so the same will be true for a hereditary algebra in any quotient of the corona. Since by Lemma 3.1, all projections in $\mathcal{M}(A) \setminus A$ have a properly infinite image in the corona, we conclude using Proposition 4.7 in [10] that $\mathcal{M}(A)/A$ is purely infinite.

(ii) ⇔ (iii) This is Proposition 3.2.

(iii) ⇒ (iv) Suppose that $I_{\mathrm{fin}}(A)/L(A)$ is not zero. Then as A has real rank zero we have a non-zero projection p in $I_{\mathrm{fin}}(A) \setminus L(A)$. If we denote by $\pi_L\colon \mathcal{M}(A) \to \mathcal{M}(A)/L(A)$ the natural quotient map, we see that (iii) together with Theorem 4.16 in [10] imply that $\pi_L(p) \oplus \pi_L(p) \precsim \pi_L(p)$. Hence, there is an element a in $L(A)_+$ such that

$$p \oplus p \precsim p \oplus a\,,$$

(see, e.g. Lemma 4.12 in [10]).

Using Lemma 3.3 we find projections $p_1, \dots, p_n$ in the hereditary subalgebra of $\mathcal{M}(A)$ generated by a, hence also in $L(A)$, such that $p \oplus p \precsim p \oplus \bigoplus_{i=1}^{n} p_i$.

Denote also by $\pi_L\colon \mathrm{V}(\mathcal{M}(A)) \to \mathrm{V}(\mathcal{M}(A))/\mathrm{V}(L(A))$ the natural monoid map. We then get that $\pi_L([p]) + \pi_L([p]) \leqslant \pi_L([p])$. We have observed in the comments preceding this theorem that this is a cancellative monoid, hence $\pi_L([p]) = 0$. This entails that $p \in L(A)$, a contradiction.

(iv) $\Rightarrow$ (i) Use the notation in Lemma 3.1. We may of course assume that A does not have continuous scale. Then, because of our assumption there is an extremal state s in S_u such that $d(s) = \infty$. Consider the complementary face $\{s\}'$ of $\{s\}$. We claim that $\{s\}'$ is closed.

Suppose, by way of contradiction, that this is not the case. By using Proposition 4.10 in [18] we can construct a function g in $\mathrm{LAff}(S_u)^{++}$ such that $g(s) = 1$ and $g_{|\{s\}'} = 2$. Moreover $d + g = 2 + d$, and so $g \in W_\sigma^d(S_u)$ and its restriction to $\partial_e S_u$ is finite. Then g is continuous by (iv).

Take x in $\overline{\{s\}'} \setminus \{s\}'$. Then x can be approximated by a net (x_λ) in $\{s\}'$. Since g is continuous we have that $g(x) = \lim g(x_\lambda) = 2$. On the other hand, since S_u is the convex direct sum of $\{s\}$ and $\{s\}'$ there exist α in $(0, 1]$ and t in $\{s\}'$ satisfying $x = \alpha s + (1 - \alpha)t$. Hence $g(x) = 2 - \alpha \neq 2$, a contradiction. This establishes the claim.

We have then proved that for any s in F_∞, the complementary face $\{s\}'$ is closed. Now F_∞ is a finite intersection of closed faces, hence will be closed by Lemma 2.4.

Finally, since F_∞ is finite by hypothesis, there exists by Corollary 4.12 in [18] a lower semicontinous and affine function d' such that $d'_{|F_\infty} = 1$ and $d'_{|F'_\infty} = d_{|F'_\infty}$. In particular $d + d' = 2d$ so $d' \in W_\sigma^d(S_u)$. By construction $d'_{|\partial_e S_u}$ is finite, hence it is continuous by (iv). Therefore A has quasi-continuous scale. ∎

The fact that continuous scale characterizes simplicity of the corona (by the results of Lin [15]), suggests the possibility that finiteness of the ideal lattice in the corona may be closely tied up with the property of being purely infinite and having quasi-continuous scale. This possibility is explored in the next results.

COROLLARY 3.5. *Let A be a non-unital, simple, separable C^*-algebra with real rank zero, stable rank one, and weakly unperforated K_0. If A has quasi-continuous scale then the ideal lattice of $\mathcal{M}(A)$ is finite.*

Proof. By Theorem 3.4 we know that $I_{\text{fin}}(A) = L(A)$. Since by Theorem 6.3 in [18] the ideal lattice of $\mathcal{M}(A)/I_{\text{fin}}(A)$ is finite, the result follows. ∎

THEOREM 3.6. *Let A be a simple, unital C^*-algebra with real rank zero, stable rank one and with weakly unperforated K_0. Assume that the simplex of unital quasitraces $\mathrm{QT}(A)$ is a Bauer simplex. Then the following conditions are equivalent:*

(i) $\mathrm{QT}(A)$ *has finitely many extreme points;*

(ii) $\mathcal{M}(A \otimes \mathcal{K})/A \otimes \mathcal{K}$ *is purely infinite;*

(iii) $\mathcal{M}(A \otimes \mathcal{K})/L(A \otimes \mathcal{K})$ *is purely infinite;*

(iv) $I_{\text{fin}}(A \otimes \mathcal{K}) = L(A \otimes \mathcal{K})$.

Any of these conditions imply

(v) $\mathcal{M}(A \otimes \mathcal{K})/A \otimes \mathcal{K}$ *has finitely many ideals,*

and if A is furthermore exact, then all five conditions are equivalent.

Proof. As in the proof of Proposition 2.5, we have that if $p = 1 \otimes e_{11} \in A \otimes \mathcal{K}$ then we have $\mathrm{V}(A \otimes \mathcal{K}) \cong \mathrm{V}(A)$ with $u = [p]$ corresponding to $[1]$. Also, the state space of $\mathrm{V}(A)$ (normalized at [1]) is homeomorphic to $\mathrm{QT}(A)$.

The scale d defined on S_u is identically infinite by stability, hence $F_\infty = \partial_{\mathrm{e}} S_u$. Thus the monoid isomorphism in (†) says in this case

$$\varphi\colon \mathrm{V}(\mathcal{M}(A \otimes \mathcal{K})) \cong \mathrm{V}(A \otimes \mathcal{K}) \sqcup \mathrm{LAff}(S_u)^{++}\,.$$

Assume that (i) holds. Then the argument in Proposition 2.5 shows that $A \otimes \mathcal{K}$ has quasi-continuous scale and by Theorem 3.4 we obtain (ii).

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) follow exactly as in Theorem 3.4.

Assume now (iv) and also that S_u has infinitely many extremal points. Since $\partial_{\mathrm{e}} S_u$ is compact, this implies that there is an accumulation point s (in $\partial_{\mathrm{e}} S_u$). Denote by $\{s\}'$ the complementary face of $\{s\}$, and define an affine function f on S_u by setting $f_{|\{s\}'} = 2$ and $f(s) = 1$. Then $f \in \mathrm{LAff}(S_u)^{++}$, represents a projection in $I_{\mathrm{fin}}(A \otimes \mathcal{K})$, and is not continuous. To see the latter, note that since we can find (s_n) in $\partial_{\mathrm{e}} S_u$ such that $s_n \to s$, then lower semicontinuity entails

$$1 = f(s) \leqslant \liminf f(s_n) = 2\,.$$

Hence we have that the inclusion $\mathrm{V}(L(A \otimes \mathcal{K})) \subseteq \mathrm{V}(I_{\mathrm{fin}}(A \otimes \mathcal{K}))$ is strict, and so $L(A \otimes \mathcal{K})$ and $I_{\mathrm{fin}}(A \otimes \mathcal{K})$ are distinct.

That condition (v) is implied by any of the previous conditions follows, for example, by assuming (i) and using Corollary 3.5. For the converse direction, assume further that A is exact and let us show (v) $\Rightarrow$ (ii).

Let τ be an extremal element in $\mathrm{T}(A)$, and put $I_\tau = \{x \in \mathcal{M}(A \otimes \mathcal{K}) : \tau(x^*x) < \infty\}^-$ as in [20], which is a closed, two-sided ideal. If, for τ_1 and τ_2 we have $I_{\tau_1} = I_{\tau_2}$, then Proposition 4.1 in [20] implies that there are positive numbers c and d such that $c\tau_1 \leqslant \tau_2$ and $d\tau_2 \leqslant \tau_1$. We claim that $c = 1$, whence $\tau_1 = \tau_2$. Indeed, if $c < 1$, then we have that $\sigma := \frac{1}{1-c}(\tau_2 - c\tau_1) \in \mathrm{T}(A)$. Since $(1-c)\sigma + c\tau_1 = \tau_2$ and τ_2 is extremal, this implies $c = 1$, impossible.

Thus $\tau_1 \neq \tau_2$ implies $I_{\tau_1} \neq I_{\tau_2}$. As by assumption there are finitely many ideals in the corona algebra, we conclude that there must be finitely many extremal points in $\mathrm{T}(A)$. ∎

COROLLARY 3.7. *Let A be a unital* AF*-algebra whose simplex of unital traces has compact extreme boundary. Then $\mathcal{M}(A \otimes \mathcal{K})/A \otimes \mathcal{K}$ is purely infinite if and only if $\mathcal{M}(A \otimes \mathcal{K})/A \otimes \mathcal{K}$ has finitely many ideals.* ∎

COROLLARY 3.8. *Let A be a simple, separable and non-unital C^*-algebra with real rank zero, stable rank one and with weakly unperforated K_0. Assume that for some (hence any) non-zero projection p in A the simplex Q_p is a Bauer simplex. Then the following conditions are equivalent:*

(i) *Q_p has finitely many extreme points;*

(ii) *$\mathcal{M}(A \otimes \mathcal{K})/A \otimes \mathcal{K}$ is purely infinite;*

(iii) *$\mathcal{M}(A \otimes \mathcal{K})/L(A \otimes \mathcal{K})$ is purely infinite;*

(iv) $I_{\rm fin}(A\otimes\mathcal{K})=L(A\otimes\mathcal{K})$.
Any of these conditions imply
(v) $\mathcal{M}(A\otimes\mathcal{K})/A\otimes\mathcal{K}$ *has finitely many ideals,*
and if A *is furthermore exact, then all five conditions are equivalent.*

Proof. Notice that A will be Morita equivalent to pAp, hence $A\otimes\mathcal{K}\cong pAp\otimes\mathcal{K}$.

By Lemma 2.1, we have $\mathrm{Q}_p\cong\mathrm{QT}(pAp)$, and since there is an ordered group isomorphism from $\mathrm{K}_0(A)$ to $\mathrm{K}_0(pAp)$, the latter will also be weakly unperforated. We may then apply Theorem 3.6 to pAp and $\mathrm{QT}(pAp)$ to obtain the result. ∎

Recall that a given trace τ on a C^*-algebra A can be extended to a trace τ' on the multiplier algebra $\mathcal{M}(A)$, by

$$\tau'(a)=\sup\{\tau(b):b\in A_+\text{ and }b\leqslant a\}\,,$$

whenever $a\in\mathcal{M}(A)_+$. If τ is normalized, then also τ' is normalized (see Section 13 in [8] and also [16]). This defines an embedding of $\mathrm{T}(A)$ into $\mathrm{T}(\mathcal{M}(A))$.

In the theorem below, recall that F_∞ is (identified with) the set of infinite extremal (quasi)traces.

THEOREM 3.9. *Let* A *be a simple, exact, non-unital, separable,* C^**-algebra with real rank zero, stable rank one and weakly unperforated* K_0. *Assume* A *is not of type* I. *Let* p *be a non-zero projection of* A *and assume that* T_p *has compact extreme boundary. Then the following conditions are equivalent:*
(i) $\mathcal{M}(A)/A$ *has finitely many ideals and* $\mathrm{T}(A)$ *is compact;*
(ii) $\mathcal{M}(A)/A$ *is purely infinite and* F_∞ *is a finite set.*

Proof. By setting $u=[p]$ in $\mathrm{V}(A)$ for a non-zero projection p and $S_u=\mathrm{St}(\mathrm{V}(A),u)$, the hypotheses ensure that $\partial_{\rm e}S_u$ is a compact space.

(i) $\Rightarrow$ (ii) If $\mathcal{M}(A)/A$ has finitely many ideals, then F_∞ is a finite set of $\partial_{\rm e}S_u$ which is also isolated, by Theorem 6.6 and Theorem 6.8 in [18]. The fact that $\mathrm{T}(A)$ is compact means exactly that the restriction of d to the set where it is finite is actually continuous (by Theorem 14.6 in [8]), hence $d_{|\partial_{\rm e}S_u\setminus F_\infty}$ is continuous. But this means exactly that A has quasi-continuous scale and so $\mathcal{M}(A)/A$ is purely infinite, by Theorem 3.4.

(ii) $\Rightarrow$ (i) Clearly, if $\mathcal{M}(A)/A$ is purely infinite, then $\mathrm{T}(\mathcal{M}(A)/A)=\emptyset$. It then follows from standard arguments that $\mathrm{T}(A)$ is compact. Indeed, in this case the embedding of $\mathrm{T}(A)$ into $\mathrm{T}(\mathcal{M}(A))$ by extension is onto, and restriction of traces in $\mathcal{M}(A)$ to A, which is w*-continuous, maps $\mathrm{T}(A)$ to $\mathrm{T}(A)$ (see p. 108 in [5]).

The fact that the ideal lattice of $\mathcal{M}(A)/A$ is finite follows from Corollary 3.5 once we notice that our hypotheses imply that A has quasi-continuous scale. ∎

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