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ENVELOPING ALGEBRAS OF PARTIAL ACTIONS AS GROUPOID C^*-ALGEBRAS

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ABSTRACT. We describe the enveloping C^*-algebra associated to a partial action of a countable discrete group on a locally compact space as a groupoid C^*-algebra (more precisely as a C^*-algebra from an equivalence relation) and we use our approach to show that, for a large class of partial actions of $\mathbb{Z}$ on the Cantor set, the enveloping C^*-algebra is an AF-algebra. We also completely characterize partial actions of a countable discrete group on a compact space such that the enveloping action acts in a Hausdorff space.

KEYWORDS: *Enveloping algebras, partial actions, groupoid C^*-algebras.*

MSC (2000): 46L05, 46L55.

INTRODUCTION

The concept of partial actions was introduced in [5] and [9] and it has been a very important tool in C^*-algebras and dynamical systems ever since. As the name suggests, partial actions generalize the notion of an action in a C^*-algebra or in a topological space. The problem of deciding whether or not a given partial action is the restriction of some global action (called enveloping action) was studied by F. Abadie in [1], where, among other things, he shows that the cross product of the enveloping C^*-algebra by the enveloping action is Morita–Rieffel equivalent (previously known as strongly Morita equivalent) to the partial cross product.

In this paper we are interested in partial actions of a countable discrete group G (in particular of $\mathbb{Z}$) on a locally compact, second countable space X (particularly on a Cantor set, that is, a compact, totally disconnected, with no isolated points, metric space). It is well known that there is a correspondence between partial actions on a Hausdorff locally compact space X and the partial actions on the C^*-algebra $C_0(X)$. In [1], it is shown that a partial action on a topological space always has an enveloping action, which may **not** act on a Hausdorff space (the odometer partial action for example). When the enveloping space is Hausdorff the notion of the enveloping action in the category of C^*-algebras is a rather natural one, but when the enveloping space is non Hausdorff the notion of the enveloping action has to be reformulated with the use of C^*-ternary rings and the introduction of the notion of strong Morita equivalence between partial actions. Although it seems that this approach can not be avoided in general, in the case of a partial action of a countable discrete group on a locally compact space we give a description of the Morita enveloping C^*-algebra as a C^*-algebra from an equivalence relation (viewed as a groupoid). Our approach has the advantage of working for either Hausdorff or non Hausdorff enveloping spaces. We also use our description of the enveloping algebra to show that it is an AF-algebra, provided we have a partial action of $\mathbb{Z}$ on the Cantor set with some mild assumptions, namely that it arises as a "restriction" of a global action (we should warn the reader that we use the word restriction here with a slight different meaning then what usually appears in the literature).

The paper is structured as follows. In Section 2 we make a quick review of the necessary notions on partial actions and enveloping actions. We completely characterize the partial actions of a countable discrete group on a second countable, compact space such that the enveloping space is Hausdorff in Section 3 and finally in Section 4 we describe the enveloping algebra as a groupoid C^*-algebra and show it is an AF-algebra under the assumptions mentioned above.

1. PARTIAL ACTIONS AND ENVELOPING ACTIONS

DEFINITION 1.1. A *partial action of a group* G on a set Ω is a pair $\theta = (\{\Delta_t\}_{t\in G}, \{h_t\}_{t\in G})$, where for each $t \in G$, Δ_t is a subset of Ω and $h_t : \Delta_{t^{-1}} \to \Delta_t$ is a bijection such that:

(i) $\Delta_e = \Omega$ and h_e is the identity in Ω;

(ii) $h_t(\Delta_{t^{-1}} \cap \Delta_s) = \Delta_t \cap \Delta_{ts}$;

(iii) $h_t(h_s(x)) = h_{ts}(x)$, $x \in \Delta_{s^{-1}} \cap \Delta_{s^{-1}t^{-1}}$.

If Ω is a topological space, we also require that each Δ_t is an open subset of Ω and that each h_t is a homeomorphism of $\Delta_{t^{-1}}$ onto Δ_t.

Analogously, a pair $\theta = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$ is a partial action of G on a C^*-algebra A if each D_t is a closed two sided ideal and each α_t is a $*$-isomorphism of $D_{t^{-1}}$ onto D_t.

Since we are very interested in partial actions of $\mathbb{Z}$, below we give the most important example of such partial actions.

EXAMPLE 1.2. Let X be a locally compact space, U and V open subsets of X and h a homeomorphism from U to V. Let $X_{-n} = \mathrm{dom}(h^n)$ and $h_n : X_{-n} \to X_n$ be defined by h^n, for $n \in \mathbb{Z}$. Then $\theta = (\{X_n\}_{n\in\mathbb{Z}}, \{h_n\}_{n\in\mathbb{Z}})$ is a partial action of $\mathbb{Z}$.

For the proof see [5].

EXAMPLE 1.3 (The Odometer). Let $X = \{0,1\}^\infty = \prod_{\mathbb{N}}\{0,1\}$. Let $\max = 1^\infty$ (sequence of all 1's), $\min = 0^\infty$ (sequence of all 0's), $X_{-1} = X - \{\max\}$, $X_1 = X - \{\min\}$ and $h : X_{-1} \to X_1$ be addition of 1 with carryover to the right. Then $\theta = (\{X_n\}_{n\in\mathbb{Z}}, \{h_n\}_{n\in\mathbb{Z}})$, where $X_{-n} = \mathrm{dom}(h^n)$, is a topological partial action.

REMARK 1.4. With $D_t = \{f \in C_0(X) : f|_{X_t^c} = 0\}$, where X_t^c means the complement of X_t in X, and $\alpha_t : D_{t^{-1}} \to D_t$ defined by $\alpha_t(f) = f \circ h_t^{-1}$, we have a partial action on the C^*-algebra $C(X)$.

We recall the definition of the enveloping action in the topological sense.

DEFINITION 1.5. Let $\theta = (\{X_t\}_{t\in G}, \{h_t\}_{t\in G})$ be a partial action. The *enveloping space*, denoted by X^e, is the topological quotient space $\frac{(G\times X)}{\sim}$, where $\sim$ is the equivalence relation given by

$$(r,x) \sim (s,y) \Leftrightarrow x \in X_{r^{-1}s} \quad \text{and} \quad h_{s^{-1}r}(x) = y.$$

The *enveloping action*, denoted by h^e, is the action induced in X^e by the action $h^e_s(t,x) \mapsto (st,x)$.

Given a locally compact space X, the definition of the enveloping action and space in the C^*-algebraic sense is motivated by the 1-1 relation between partial actions on X and partial actions on $C_0(X)$. Basically, if a partial action $\theta = (\{X_t\}_{t\in G}, \{h_t\}_{t\in G})$ on X has a Hausdorff enveloping space X^e and enveloping action h^e, then $C_0(X^e)$ is the enveloping C^*-algebra and $\alpha^e(f) = f \circ (h^e)^{-1}$ is the induced global action associated to the partial action in $C_0(X)$. In [1], it is proved that when X^e is Hausdorff, the partial cross product of $C_0(X)$ by the partial action is Morita–Rieffel equivalent to the global cross product of the enveloping algebra $C_0(X^e)$ by the enveloping action. The problem arises when X^e is **not** Hausdorff (as for example in the Odometer partial action). The result mentioned above is not valid anymore, as we may have very few continuous functions in $C_0(X^e)$. Abadie, in [1], goes around this problem by making use of C^*-ternary rings and introducing the notion of Morita equivalence between partial actions. Below we completely characterize partial actions of a countable discrete group on a second countable, compact space for which the enveloping space is Hausdorff. Throughout the rest of the paper G will denote a countable discrete group and we will assume that the space X is always second countable.

2. PARTIAL ACTIONS OF G ON A COMPACT SPACE SUCH THAT THE ENVELOPING SPACE IS HAUSDORFF

In [1] it is shown that a partial action has a Hausdorff enveloping space if and only if the graph of the action is closed. Below we give a concrete characterization of partial actions of a countable discrete group on compact spaces for which the enveloping space is Hausdorff.

Let X be a second countable, compact space, $\{X_t, h_t\}$ a partial action of G on X and (X^e, h^e) the enveloping space and action respectively. Recall that X^e is the quotient of $G \times X$ by the equivalence relation $(r,x) \sim (s,y) \Leftrightarrow x \in X_{r^{-1}s}$ and $h_{s^{-1}r}(x) = y$, with the quotient topology. We denote the equivalence class of (n,x) in X_e by $[n,x]$.

PROPOSITION 2.1. *Let* G *be a countable discrete group with unit e, let* X *be a second countable, compact space, and let* $\{X_t, h_t\}$ *be a partial action of* G *on* X*. Then* X^e *is Hausdorff if and only if the partial action* $\{X_t, h_t\}$ *acts in clopen subsets of* X*, that is, if and only if* X_t *is clopen for each* $t \in G$*.*

Proof. First assume that X^e is Hausdorff. We will show that each X_t is closed (it is already open by the definition of a partial action).

Suppose there exists $t \in G$ such that X_t is not closed (we will show that this implies that X^e is non-Hausdorff).

Since X_t is not closed, there exists a sequence $(x_k)_{k\in\mathbb{N}}$ such that $x_k \in X_t$ for all k and such that $x_k \to x$, where $x \notin X_t$. By the compactness of X, $(h_{t^{-1}}(x_k))_{k\in\mathbb{N}}$ has a converging subsequence and we may pass to this subsequence. We may therefore assume that there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in X_t such that $x_k \to x$, where $x \notin X_t$ and such that $(h_{t^{-1}}(x_k))_{k\in\mathbb{N}}$ converges to a point $y \in X$.

We now have that the points $[t^{-1}, x]$ and $[e,y]$ can not be separated. Let's see why:

Suppose that U and V are open, $[t^{-1}, x] \in U$ and $[e,y] \in V$. Remember that U is open if and only if $q^{-1}(U)$ is open, where q is the quotient map.

Well, since $x_k \to x$ and $h_{t^{-1}}(x_k) \to y$, there exists $N \in \mathbb{N}$ such that

$$(t^{-1}, x_k) \in q^{-1}(U) \quad \text{and} \quad (e, h_{t^{-1}}(x_k)) \in q^{-1}(V) \quad \text{for all } k > N.$$

Now notice that $(t^{-1}, x_k) \sim (e, h_{t^{-1}}(x_k))$ and hence $[t^{-1}, x_k] = [e, h_{t^{-1}}(x_k)]$ and $U \cap V \neq \emptyset$.

For the converse, we may now assume that each X_t is a clopen subset of X.

Let $[r,x] \neq [s,y]$ in X^e. So (r,x) is not equivalent to (s,y). We have two possibilities:

(i) If $x \notin X_{r^{-1}s}$ then there exists V_x such that $x \in V_x$ and $V_x \cap X_{r^{-1}s} = \emptyset$ (since $X_{r^{-1}s}$ is clopen).

We may assume $r \neq s$ (if $r = s$ then $x \neq y$ and we find the desired neighborhoods using the fact that X is Hausdorff).

Take $V = (r, V_x)$ and $U = (s, X)$. Then $i_r(V)$ and $i_s(U)$ are the desired open sets. (Where $i_r(x) = q(r,x)$, $i_s(x) = q(s,x)$ and q is the quotient map. Also notice that i_t is an open map and the proof is analogous to what is done in Theorem 2.1 of [1], for the map i.)

(ii) If $x \in X_{r^{-1}s}$ then $h_{s^{-1}r}(x) \neq y$.

Let $z = h_{s^{-1}r}(x)$. We have that $z \neq y$. Since X is Hausdorff, there exist open sets U_z and U_y such that $U_z \cap U_y = \emptyset$, $z \in U_z$ and $y \in U_y$. Take $V_x =$

$h^{-1}_{s^{-1}r}(U_z \cap X_{s^{-1}r})$, which is an open set. Then $i_r(V_x)$ and $i_s(U_y)$ have the desired properties. ∎

REMARK 2.2. Dokuchaev and Exel have a result in the more general context of partial actions on associative algebras, see Theorem 4.5 of [4], that implies the proposition above. Still we believe our proof above helps to give the reader a feeling for the space X^e.

Let X be a locally compact Hausdorff space with a countable basis of clopen sets. Following Danilenko [2], if X has no isolated points we call it a locally compact Cantor set. Recall [2] that any two non-compact locally compact Hausdorff Cantor set are homeomorphic.

PROPOSITION 2.3. *Let* X *be the Cantor set,* G *a countable discrete group and* $\{X_t, h_t\}$ *a partial action of* G *on* X *such that* X_t *is clopen for all* $t \in G$. *Then the enveloping space* X^e *is a locally compact Cantor set.*

Proof. By Proposition 2.1, X^e is Hausdorff. To prove that X^e is locally compact, let us first note that for each $t \in G$ the function $i_t(x) = q(t, x)$ is a continuous, open and closed map (we already know that i_t is continuous and open by an argument similar to what is done in Theorem 2.1 of [1]). To see that it is a closed map, let F be closed in X. Then F is compact and hence $i_t(F)$ is compact. Since X^e is Hausdorff we have that $i_t(F)$ is closed.

Let $[(r, x)]$ in X^e. Since X is compact, there exists a compact neighborhood, U_x, of x in X. But then $i_r(U_x)$ is a compact neighborhood of $[(r, x)]$ in X^e. Hence X^e is locally compact.

Now, if $\{U_n\}_{n\in G}$ be a countable basis of clopen sets of X then $\{i_t(U_n)\}_{n,t\in G}$ is a countable basis of clopen subsets of X^e.

Finally, we have that X^e has no isolated points, since if $[(t, x)] \in X^e$ and V is an open set that contains $[(t, x)]$ then $(t, x) \in q^{-1}(V)$ (we may assume that $q^{-1}(V)$ is of the form (t, U), where U is open in X). So there exists $(y, t) \neq (x, t)$ such that $(y, t) \in q^{-1}(V)$ and hence $[(y, t)] \in V$ and $[(y, t)] \neq [(x, t)]$. ∎

With the above propositions we completely characterized the enveloping actions of partial actions of G acting on clopen subsets of the Cantor set. The cross product of their enveloping C^*-algebra by the enveloping action is Morita–Rieffel equivalent to the partial cross product, see [1]. The problem is that most of the interesting examples, including the famous odometer (or adding machine), do not satisfy the conditions of the propositions above. Namely they fail to be partial actions on clopen sets. We note here that the majority of examples from partial actions arise as in Example 1.2. In the next section we show how to deal with these examples in a different (and we believe easier) way from what was done in [1]. As a consequence of our approach we show that the enveloping C^*-algebra associated to a partial action of $\mathbb{Z}$ on the Cantor set, as in Example 1.2, is an AF-algebra.

3. THE ENVELOPING C^*-ALGEBRA AS A GROUPOID C^*-ALGEBRA

In this section, we start by showing that the enveloping C^*-algebra associated to a partial action of a countable discrete group G on a locally compact space can be seen as a C^*-algebra of an equivalence relation (seen as a groupoid in the usual way). Before we proceed we need to introduce the notion of core subalgebras, which will be used in our proof that the enveloping algebra can be realized as a groupoid C^*-algebra.

DEFINITION 3.1. Let A be a C^*-algebra and let $B \subseteq A$ be a (not necessarily closed) $*$-subalgebra. We shall say that B is a *core subalgebra* of A when every representation of B is continuous relative to the norm induced from A. (By a representation of a $*$-algebra B we mean a multiplicative, $*$-preserving, linear map $\pi : B \to \mathcal{B}(H)$, where H is a Hilbert space.)

Assuming that B is a core subalgebra of A, and given a representation π of B, we may therefore extend π to a representation $\overline{\pi}$ of $\overline{B}$ (the closure of B in A). Since $\overline{B}$ is a C^*-algebra we have by [3] that $\overline{\pi}$ is necessarily contractive. Therefore we have:

PROPOSITION 3.2. *B is a core subalgebra of A if and only if every representation of B is contractive.*

EXAMPLES 3.3. (i) Every closed $*$-subalgebra of a C^*-algebra is a core subalgebra by [3].

(ii) Let B be a $*$-subalgebra of a C^*-algebra A, such that $B = \bigcup_{i \in I} B_i$, where each B_i is a core subalgebra of A. Then B is a core subalgebra of A. This is because every representation of B must be contractive on each B_i.

(iii) If X is a locally compact space then $\mathrm{C_c(X)}$ is a core subalgebra of $\mathrm{C_0(X)}$. This follows from the fact that $\mathrm{C_c(X)}$ is the union of the closed $*$-subalgebras $\mathrm{C_0(U)}$, where U ranges in the collection of all relatively compact open subsets of X.

(iv) Let $\mathcal{G}$ be a groupoid satisfying the hypotheses of Corollary 1.22 of [11]. Then $\mathrm{C_c}(\mathcal{G})$ is a core subalgebra of $\mathrm{C}^*(\mathcal{G})$ by the same corollary of [11].

(v) Let B be a $*$-algebra such that

$$|||b||| := \sup \|\pi(b)\| < \infty, \quad \forall\, b \in B, \tag{3.1}$$

where the supremum is taken over the collection of all representations π of B. Then one may define the enveloping C^*-algebra, $\mathrm{C}^*(B)$, by moding out the elements b such that $|||b||| = 0$, and completing under $|||\cdot|||$. The image of B within $\mathrm{C}^*(B)$ is therefore a dense core subalgebra.

Let B be a core subalgebra of a C^*-algebra A. It is then evident that B satisfies (3.1) and moreover that $|||b||| = \|b\|$, where the right hand side refers to the norm of b computed as an element of A. Supposing that B is dense in A, it follows that A is isomorphic to the enveloping C^*-algebra $\mathrm{C}^*(B)$. From this one immediately has:

PROPOSITION 3.4. *Suppose that A_1 and A_2 are C^*-algebras, and that B_i is a dense core subalgebra of A_i, for $i = 1, 2$. If B_1 and B_2 are isomorphic as $*$-algebras, then A_1 and A_2 are isometrically $*$-isomorphic.*

THEOREM 3.5. *Let A be a C^*-algebra and let $\{p_i\}_i$ be a family of mutually orthogonal projections in the multiplier algebra of A, here denoted as $M(A)$. Also let B be a $*$-subalgebra of A such that B is contained in the algebraic direct sum*

$$\bigoplus_{i,j\in I} B \cap (p_i A p_j),$$

and such that $B\cap(p_i A p_i)$ is a core subalgebra for every $i \in I$. Then B is a core subalgebra.

Proof. Given $b \in B$, by hypothesis we have that

$$b = \sum_{k,l\in I} a_{kl},$$

where the nonzero summands are finite and each $a_{kl} \in B \cap (p_k A p_l)$. We therefore have for all $i, j \in I$, that

$$p_i b p_j = \sum_{k,l\in I} p_i a_{kl} p_j = a_{ij},$$

from where we see that $a_{ij} = p_i b p_j$. From now on we will adopt the notation

$$b_{ij} := p_i b p_j \quad \forall\, b \in A,\ \forall\, i, j \in I,$$

and hence we have for every $b \in B$ that $b_{ij} \in B \cap (p_i A p_j)$, while $b = \sum\limits_{i,j\in I} b_{ij}$, a sum with finitely many nonzero terms.

For each finite set of indices $F \subseteq I$, let

$$B_F = \bigoplus_{i,j\in F} B \cap (p_i A p_j).$$

It is easy to see that B_F is a $*$-subalgebra of A and we claim that it is a core subalgebra. In fact, given a representation π of B_F, we have for all $i, j \in F$, and all $b_{ij} \in B \cap (p_i A p_j)$ that

$$\|\pi(b_{ij})\|^2 = \|\pi(b_{ij} b_{ij}^*)\| \leqslant \|b_{ij} b_{ij}^*\| = \|b_{ij}\|^2,$$

where the crucial second step follows from the fact that $b_{ij} b_{ij}^* \in B \cap (p_i A p_i)$, the latter being a core subalgebra by hypothesis. Given any $b \in B_F$, we then have

$$\|\pi(b)\| \leqslant \sum_{i,j\in F} \|\pi(b_{ij})\| \leqslant \sum_{i,j\in F} \|b_{ij}\| \leqslant |F|^2 \|b\|.$$

This proves that π is bounded and hence that B_F is a core subalgebra, as claimed.

Now observe that $B = \bigcup\limits_{F} B_F$, where F ranges in the collection of all finite subsets of F, so the conclusion follows from Example 3.3(ii). ∎

We can now focus again on realizing the enveloping algebra as a groupoid C^*-algebra.

For this we fix, as before, a partial action of the discrete group G on a locally compact space X.

Recall that the enveloping space is the quotient of $\mathrm{G} \times \mathrm{X}$ by the equivalence relation $(r,x) \sim (s,y) \Leftrightarrow x \in \mathrm{X}_{r^{-1}s}$ and $h_{s^{-1}r}(x) = y$. Since this quotient may be non Hausdorff, instead of considering it, we will consider the equivalence relation $R \subseteq \mathrm{G} \times \mathrm{X} \times \mathrm{G} \times \mathrm{X}$ above, with the product topology.

Notice that a neighborhood base for $z = (t,x,s,y)$ is formed by neighborhoods of the following form, where $\mathrm{U}_x \subseteq X_{t^{-1}s}$ is open:

$$\mathrm{U}_{txs} = \{(t,x',s,h_{s^{-1}t}(x')) : x' \in \mathrm{U}_x\}.$$

Before we can consider the groupoid C^*-algebra of this equivalence relation we will show that R with this topology is étale, which, in the language of [11], means that R is an r-discrete groupoid with counting measure as a Haar system. In our context, this means that R can be equipped with two maps, called range and source, defined by $r(t,x,t',y) = (t,x)$ and $s(t,x,t',y) = (t',y)$ and such that R is σ-compact, $\Delta = \{(t,x,t,x) \in R : (t,x) \in \mathrm{G} \times \mathrm{X}\}$ is an open subset of R and for all $(t,x,t',y) \in R$, there exists a neighborhood U of (t,x,t',y) in R, such that r restricted to U and s restricted to U are homeomorphisms from U onto open subsets of $\mathrm{G} \times \mathrm{X}$, see also [10].

PROPOSITION 3.6. *R is étale.*

Proof. To see that R is sigma compact, we notice that for each fixed s and $t \in \mathrm{G}$, $\mathrm{X}_{t^{-1}s}$ is a countable union of compact sets and hence each U_{txs}, with $\mathrm{U}_x = \mathrm{X}_{t^{-1}s}$ is a countable union of compact sets.

For $(t,x,t,x) \in \Delta$, we take U_{txt} with $U_x = \mathrm{X}$ to see that Δ is open.

Finally, given $(t,x,s,y) \in R$, it is not hard to see that the range and source map are homeomorphisms, once restricted to U_{txs}, with $\mathrm{U}_x = \mathrm{X}_{t^{-1}s}$. ∎

We are now able to consider the full groupoid C^*-algebra of R, which we denote by $C^*(R)$ (see [8] or [10] for details on the groupoid C^*-algebra of étale equivalence relations). Next we show that $C^*(R)$ is isomorphic to the Morita enveloping algebra defined in [1]. In order to do so, we quickly remind the reader of the definitions in [1] (adapted to the case at hand).

Given a Fell bundle $B = (B_t)_{t \in \mathrm{G}}$ of a partial action $\{\mathrm{X}_t, h_t\}_{t \in \mathrm{G}}$, (in our case $B_t = \mathrm{C}_0(\mathrm{X}_t)\delta_t$), we consider the linear space, $k_c(B)$, of all functions $k : \mathrm{G} \times \mathrm{G} \to B$, with finite support and such that $k(r,s) \in B_{rs^{-1}}$. We now equip $k_c(B)$ with the involution $k^*(r,s) = k(s,r)^*, \forall k \in k_c(B)$, the multiplication $k_1 * k_2(r,s) = \sum_{t \in \mathrm{G}} k_1(r,t)k_2(t,s), \forall k_1, k_2 \in k_c(B)$ and the norm $\|k\| = \Big(\sum_{r,s \in \mathrm{G}} \|k(r,s)\|^2\Big)^{1/2}$. The universal C^*-algebra of the completion of $k_c(B)$ with respect to the norm above is the enveloping algebra, $k(B)$. Finally we notice that there exists a natural action of G on $k_c(B)$, which can be extended to $k(B)$, given by $\beta_t(k)(r,s) = k(rt, st)$. The pair $(k(B), \beta)$ is the enveloping action as in [1]. We can now prove our main result.

THEOREM 3.7. *Given a partial action h of a countable discrete group* G *on a locally compact space* X, *the groupoid C^*-algebra $C^*(R)$, as defined above, is isomorphic to the enveloping C^*-algebra $k(B)$.*

Proof. Initially let us observe that, given any element $(r, x, s, y) \in R$, one has that $y = h_{s^{-1}r}(x)$. Therefore the fourth variable "y" is a function of the first three, and hence we may discard it. In more precise terms we have that

$$(r, x, s, y) \mapsto (x, r, s)$$

establishes a one-to-one correspondence from R to the set

$$R' = \{(x, r, s) \in X \times G \times G : x \in X_{r^{-1}s}\}.$$

Moreover this correspondence is seen to be a homeomorphism if R' is viewed as a subspace of the topological product space $X \times G \times G$.

Borrowing the groupoid structure from R we have that R' itself becomes an étale groupoid under the multiplication operation

$$(x, r, s) \cdot (y, t, u) = (x, r, u),$$

defined if and only if $y = h_{s^{-1}r}(x)$, and $s = t$, while the inversion operation is given by

$$(x, r, s)^{-1} = (h_{s^{-1}r}(x), s, r).$$

Since R and R' are isomorphic topological groupoids, it is enough to show that $C^*(R')$ and $k(B)$ are isomorphic C^*-algebras. We will derive this result from Proposition 3.4, by showing the existence of two isomorphic dense core subalgebras of $C^*(R')$ and $k(B)$, respectively.

On the one hand recall that $C_c(R')$ is a dense core subalgebra of $C^*(R')$, as observed in Example 3.3(iv). To define the relevant dense core subalgebra of $k(B)$, recall that B is the Fell bundle with fibers $B_t = C_0(X_t)\delta_t$.

Denoting by $C_c(X_t)$ the set of all continuous compactly supported functions on X whose support is contained in X_t, put $D_t = C_c(X_t)\delta_t$, so that each D_t is a dense linear subspace of B_t. Moreover it is easy to see that, for all $r, s \in G$,

$$D_r D_s \subseteq D_{rs} \quad \text{and} \quad D_r^* = D_{r^{-1}}. \tag{3.2}$$

Denote by $k_c(D)$ the subset of $k_c(B)$ formed by all $k \in k_c(B)$ such that $k(r,s) \in D_{rs^{-1}}$, for all r and s. As a consequence of (3.2) one easily proves that $k_c(D)$ is a $*$-subalgebra of $k(B)$, which is also easily seen to be dense.

We will next show that $k_c(D)$ is a core subalgebra of $k(B)$ by using Theorem 3.5. With this in mind we must first define a family of projections $\{p_t\}_{t\in G}$ in the multiplier algebra $M(k(B))$. Given $t \in G$, consider the maps

$$L_t, R_t : k_c(B) \to k_c(B),$$

given, for every $k \in k_c(B)$, by

$$L_t(k)(r,s) = \begin{cases} k(r,s) & \text{if } r = t, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall r,s \in G;$$

$$R_t(k)(r,s) = \begin{cases} k(r,s) & \text{if } s = t, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall r,s \in G.$$

One may then prove that both L_t and R_t extend continuously to $k(B)$, giving a multiplier

$$p_t = (L_t, R_t) \in M(k(B)),$$

which is also self-adjoint and idempotent, thus producing a family $\{p_t\}_{t\in G}$ of mutually orthogonal projections.

For every $r,s \in G$ one has that $p_r k_c(D) p_s$ consists of all the $k \in k_c(D)$ which are supported on the singleton $\{(r,s)\}$. In particular $p_t k_c(D) p_t \simeq C_c(\mathrm{X})$.

On the other hand notice that, by similar reasons, $p_t k_c(B) p_t \simeq C_0(X)$. As a C^*-algebra this is complete and hence it coincides with its closure which is clearly $p_t k(B) p_t$.

Therefore the inclusion of $p_t k_c(D) p_t$ within $p_t k(B) p_t$ is, modulo a canonical isomorphism, the same as the inclusion of $C_c(X)$ within $C_0(X)$, and hence $p_t k_c(D) p_t$ is a core subalgebra of $p_t k(B) p_t$, as desired. Theorem 3.5 therefore applies and hence we deduce that $k_c(D)$ is a core subalgebra of $k(B)$.

We will next prove that $C_c(R')$ and $k_c(D)$ are isomorphic as $*$-algebras and hence the result will follow from Proposition 3.4. Given $r,s \in G$, let

$$R'_{r,s} = R' \cap (X \times \{r\} \times \{s\})$$

or, equivalently,

$$R'_{r,s} = \{(x,r,s) : x \in \mathrm{X}_{r^{-1}s}\},$$

so $R'_{r,s}$ naturally identifies with $\mathrm{X}_{r^{-1}s}$. Given $f \in C_c(R')$, denote by $f_{r,s}$ the restriction of f to $R'_{r,s}$, seen as an element of $C_c(\mathrm{X}_{r^{-1}s})$. Alternatively (and more precisely) one may define $f_{r,s}$ as follows:

$$f_{r,s}(x) = \begin{cases} f(x,r,s) & \text{if } x \in X_{r^{-1}s}, \\ 0 & \text{otherwise.} \end{cases}$$

Since f is compactly supported, only finitely many $f_{r,s}$ will be nonzero. Define $\psi : C_c(R') \to k_c(D)$ by

$$\psi(f)(r,s) = f_{r^{-1},s^{-1}} \delta_{rs^{-1}}, \quad \forall f \in C_c(R'), \ \forall r,s \in G.$$

Observing that R' is the disjoint union of the $R'_{r,s}$, it should be obvious that ψ is a well defined vector space isomorphism. The proof will then be concluded once we show that ψ is a $*$-homomorphism.

In order to prove that $\psi(f * g) = \psi(f)\psi(g)$, we may suppose without loss of generality that f is supported in $R'_{r,s}$ and that g is supported in $R'_{t,u}$. When $s \neq t$, the product in R' of (x,r,s) and (y,t,u) is never defined, so $f * g = 0$.

Otherwise, if $s = t$, we have that $f * g$ is supported in $R'_{r,u}$. Moreover, given $(x, r, u) \in R'_{r,u}$, the only way of writting (x, r, u) as a product of an element of $R'_{r,s}$ and an element of $R'_{s,u}$ is

$$(x, r, u) = (x, r, s)(h_{s^{-1}r}(x), s, u),$$

as long as $x \in X_{r^{-1}s}$. Thus

$$(f * g)(x, r, u) = f(x, r, s)g(h_{s^{-1}r}(x), s, u) = f_{r,s}(x)g_{s,u}(h_{s^{-1}r}(x)).$$

On the other hand, since $\psi(f)$ is supported on the singleton $\{(r^{-1}, s^{-1})\}$, and $\psi(g)$ is supported on $\{(s^{-1}, u^{-1})\}$, we have that $\psi(f)\psi(g)$ is supported on $\{(r^{-1}, u^{-1})\}$, and

$$\begin{aligned}(\psi(f)\psi(g))(r^{-1}, u^{-1}) &= \psi(f)(r^{-1}, s^{-1})\ \psi(g)(s^{-1}, u^{-1}) \\ &= (f_{r,s}\delta_{r^{-1}s})\ (g_{s,u}\delta_{s^{-1}u}) = f_{r,s}(g_{s,u} \circ h_{s^{-1}r})\delta_{r^{-1}u},\end{aligned}$$

from where it is easily seen that $\psi(f * g) = \psi(f)\psi(g)$. We leave it for the reader to prove that ψ preserves the adjoint operation. ∎

COROLARY 3.8. *Let α be the action on $C^*(R)$ given by $\alpha_t(f)(r, x, s, y) = f(rt, x, st, y)$. Then $C^*(R) \rtimes_\alpha G$ is isomorphic to $k(B) \rtimes_\beta G$, which is strong Morita equivalent to the partial cross product $C(X) \rtimes G$.*

Proof. It is clear that the actions α and β are intertwined by the isomorphism $C^*(R) \cong k(b)$ of Theorem 3.7 and hence the isomorphism follows. The second part is done in [1]. ∎

We finish the paper showing that for the partial actions of $\mathbb{Z}$ on the Cantor set, X, as in Example 1.2, with $X_{-1} \neq X$, R is an approximately proper equivalence relation and $C^*_r(R)$ (and hence the enveloping C^*-algebra) is an AF-algebra.

Recall that an equivalence relation is said to be proper when the quotient space is Hausdorff. In [12], Renault defines approximately proper and approximately finite equivalence relations as below.

DEFINITION 3.9. An equivalence relation R, on a locally compact, second countable, Hausdorff space X, is said to be *approximately proper* if there exists an increasing sequence of proper equivalence relations $\{R_n\}_{n\in\mathbb{N}}$ such that $R = \bigcup_{n\in\mathbb{N}} R_n$. An approximately proper equivalence relation on a totally disconected space is called an AF equivalence relation.

REMARK 3.10. In [7], Giordano, Putnam and Skau define an AF equivalence relation as an equivalence relation that can be written as an increasing union of compact open étale sub-equivalence relations. They also mention that their definition is equivalent to the definition above.

To prove that R is approximately proper we will come up with a sequence of partial actions by clopen sets (so that their enveloping space is Hausdorff) such that the union of the induced equivalence relations is R.

Recall that R is associated with a partial action $\theta = \{X_{-n}, h^n\}$ on X, as in Example 1.2. That is, h is a homeomorphism from U to V (where U is a proper open subset of X), $X_{-n} = \mathrm{dom}(h^n)$ and $h_n = h^n$.

To create the partial actions, let $\{U_k\}_{k=0,1,\dots}$ be an increasing sequence of clopen sets such that their union is $X_{-1} = U \neq X$. For each U_k, denote the partial action by clopen sets obtained by restricting h to U_k and proceeding as in Example 1.2 by $\theta_k = \{X^k_{-n}, h_n\}_{n\in\mathbb{Z}}$, where $X^k_{-1} = U_k$ and h_1 is h restricted to U_k.

Now, we consider the sub equivalence relation $R_k \subseteq \mathbb{Z}\times X\times\mathbb{Z}\times X$ given by $(r,x) \sim_k (s,y) \Leftrightarrow x \in X^k_{r^{-1}s}$ and $h_{s^{-1}r}(x) = y$. Since each R_k is associated to a partial action on clopen sets, we have by Proposition 2.1 that the quotient $\frac{\mathbb{Z}\times X}{\sim_k}$ is Hausdorff for every k. With this set up we can prove that R is approximately proper.

PROPOSITION 3.11. *R is approximately proper.*

Proof. It remains to show that $R = \bigcup_{k\in\mathbb{N}} R_k$ (it is clear that $R_k \subseteq R_{k+1}$ for $k = 0,1,\dots$). It follows promptly that $R_k \subseteq R$ for all k. Next we show that $R \subseteq \bigcup_{k\in\mathbb{N}} R_k$.

Let $(r,x,s,y) \in R$ (which happens if and only if $x \in X_{r^{-1}s}$ and $h_{s^{-1}r}(x) = y$). All we need to do is find a K such that $x \in X^K_{r^{-1}s}$, since this would imply that $(r,x,s,y) \in R_K$. Now recall that $X_{-n} = \mathrm{dom}(h^n)$ and assuming that $r^{-1}s \geqslant 0$ (the case $r^{-1}s \leqslant 0$ is analogous) we have that

$$X_{r^{-1}s} = \mathrm{dom}(h^{r^{-1}s}) = U \cap h^{-1}(U) \cap \cdots \cap h^{s^{-1}r+1}(U).$$

So $x, h(x), h^2(x), \dots, h^{r^{-1}s+1}$ belong to U and hence we can find a K such that $x, h(x), h^2(x), \dots, h^{r^{-1}s+1}$ all belong to the same U_K, since $U = \bigcup_{k\in\mathbb{N}} U_k$ with $U_k \subseteq U_{k+1}$. We conclude that $x \in U_K \cap h^{-1}(U_K) \cap \cdots \cap h^{s^{-1}r+1}(U_K) = X^K_{r^{-1}s}$ as desired. ∎

Since we have shown that R is approximately proper, it is natural to consider $R = \bigcup R_k$ with the inductive limit topology. This approach will allow us to write $C^*_r(R)$ as an inductive limit C^*-algebra. But first we need to show that the inductive limit and product topology agree on R.

PROPOSITION 3.12. *Let $R = \bigcup_{k\in\mathbb{Z}} R_k$ above. Then the inductive limit topology and the product topology on R are the same.*

Proof. Suppose $U \neq \emptyset$ is open in the inductive limit topology. Then $U \cap R_k$ is open for all k. Let $(t,x,s,y) \in U$ and find K such that $(t,x,s,y) \in R_K$. Then $U \cap R_K$ contains an open neighborhood of (t,x,s,y) of the form

$$\{(t,z,s,h_{s^{-1}t}(z)) : z \in U^K \subseteq X^K_{t^{-1}s} \subseteq X_{t^{-1}s}\},$$

where U^K is open in $X^K_{t^{-1}s}$ and hence open in $X_{t^{-1}s}$. So U is open in the product topology.

Now, notice that

$$U_{txs} \cap R_k = \{(t, x, s, h_{s^{-1}t}(x)) : x \in U_x \subseteq X_{t^{-1}s}, \text{where } U_x \text{ is open}\} \cap R_k$$

is open in R_k for all k and hence U_{txs} is open in the inductive limit topology. ∎

COROLARY 3.13. $C^*_r(R) = \varinjlim C^*_r(R_k)$.

The proof is analogous to what is done in [8] for C^*-algebras from substitution tilings.

PROPOSITION 3.14. *$C^*_r(R)$ is an* AF-*algebra.*

Proof. We already know, by proposition 3.11, that R is approximately proper. Also, R is an equivalence relation in $\mathbb{Z} \times X$ and since X is the Cantor set it is clear that R is an AF equivalence relation. Then by Theorem 3.9 of [7] we have that R is isomorphic to tail equivalence in some Bratteli diagram and by [6] we have that the associated C^*-algebra is an AF-algebra.

Another way to prove this proposition would be to show that each R_k is an AF equivalence relation, (as defined in [7]), so that $C^*_r(R_k)$ is an AF C^*-algebra. Since inductive limits of AF C^*-algebras are again AF (via the local characterization of AF-algebras) this will yield that $C^*_r(R)$ is also AF. To see that each R_k is AF, notice that $\bigcup\limits_n R^n_k = R_k$, where $R^n_k = \{(r, x, s, y) \in \mathbb{Z} \times X \times \mathbb{Z} \times X : |r|, |s| \leqslant n\} \cap R_k$. ∎

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