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SOME REMARKS ON HAAGERUP'S APPROXIMATION PROPERTY

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ABSTRACT. A finite von Neumann algebra $\mathcal{M}$ with a faithful normal trace τ has Haagerup's approximation property if there exists a pointwise deformation of the identity in 2-norm by subtracial compact completely positive maps. In this paper we prove that the subtraciality condition can be removed. This enables us to provide a description of Haagerup's approximation property in terms of correspondences. We also show that if $\mathcal{N} \subset \mathcal{M}$ is an amenable inclusion of finite von Neumann algebras and $\mathcal{N}$ has Haagerup's approximation property, then $\mathcal{M}$ also has Haagerup's approximation property.

KEYWORDS: *Von Neumann algebras, Haagerup's approximation property, relative Haagerup's approximation property, relative amenability.*

MSC (2000): 46L35; 46L10.

INTRODUCTION

The objective of this paper is to solve two problems regarding Haagerup's approximation property posed by Sorin Popa in [13]. Specifically, Theorem 3.2 answers a problem posed in Remark 2.6 of [13], while Theorem 5.1 answers a problem stated at the end of Remark 3.5.2° in [13].

A finite von Neumann algebra $\mathcal{M}$ with a faithful normal trace τ has Haagerup's approximation property if there exists a net $(\varphi_\alpha)_{\alpha\in\Lambda}$ of normal completely positive maps from $\mathcal{M}$ to $\mathcal{M}$ that satisfy the subtracial condition $\tau \circ \varphi_\alpha \leqslant \tau$, the extension operators T_{φ_α} are bounded compact operators on $L^2(\mathcal{M},\tau)$, and pointwise approximate the identity map in the trace-norm, i.e., $\lim\limits_{\alpha} \|\varphi_\alpha(x) - x\|_2 = 0$ for all $x \in \mathcal{M}$. In this paper we will prove that the subtracial condition can in fact be removed from the above definition. Precisely, we will prove the following result (Theorem 2.2): Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ and $\mathcal{N}$ a von Neumann subalgebra. Denote by $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ the associated basic construction semifinite von Neumann algebra and $\mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$

its compact ideal space. Suppose $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ is a net of normal completely positive $\mathcal{N}$-bimodular maps from $\mathcal{M}$ to $\mathcal{M}$ such that $\lim_\alpha \|\varphi_\alpha(x) - x\|_2 = 0$ for all $x \in \mathcal{M}$, and the extension operators T_{φ_α} are bounded operators in $\mathcal{J}(\langle\mathcal{M}, e_\mathcal{N}\rangle)$ for every $\alpha \in \Lambda$. Then there exists a net $\{\psi_\beta\}_{\beta\in\Gamma}$ of normal completely positive $\mathcal{N}$-bimodular maps from $\mathcal{M}$ to $\mathcal{M}$ satisfying:

(i) $\psi_\beta(1) = 1$ and $\tau \circ \psi_\beta = \tau$, $\forall\beta \in \Gamma$;
(ii) $T_{\psi_\beta} \in \mathcal{J}(\langle\mathcal{M}, e_\mathcal{N}\rangle)$, $\forall\beta \in \Gamma$;
(iii) $\lim_\beta \|\psi_\beta(x) - x\|_2 = 0$, $\forall x \in \mathcal{M}$.

As the first application of Theorem 2.2, we provide a description of Haagerup's approximation property in the language of correspondences [6], [12]. We then prove that a finite von Neumann algebra $\mathcal{M}$ has Haagerup's approximation property if and only if the identity correspondence of $\mathcal{M}$ is weakly contained in some C_0-correspondence of $\mathcal{M}$ (see Section 4). We also show that if $\mathcal{M}$ is a finite von Neumann algebra with Haagerup's approximation property, then the set of classes of C_0-correspondences of $\mathcal{M}$ is dense in $\mathrm{Corr}(\mathcal{M})$, the space of all classes of correspondences of $\mathcal{M}$.

In [11], Jolissaint proved that if the basic construction $\langle\mathcal{M}, e_\mathcal{N}\rangle$ is a finite von Neumann algebra and $\mathcal{N}$ has Haagerup's approximation property, then $\mathcal{M}$ also has Haagerup's approximation property. In [1], Anantharaman-Delaroche proved that if $\mathcal{L}_H \subset \mathcal{L}_G$ is an inclusion of group von Neumann algebras which is amenable in the sense of Popa [13] and $\mathcal{L}_H$ has Haagerup's approximation property, then $\mathcal{L}_G$ also has Haagerup's approximation property. As the second application of Theorem 2.2, we prove the following general result: If $\mathcal{N} \subset \mathcal{M}$ is an amenable inclusion of finite von Neumann algebras in the sense of Popa [12] and $\mathcal{N}$ has Haagerup's approximation property, then $\mathcal{M}$ also has Haagerup's approximation property.

1. PRELIMINARIES

1.1. THE HAAGERUP PROPERTY FOR GROUPS. Recall that a locally compact group G has the *Haagerup property* if there is a sequence of continuous normalized positive definite functions vanishing at infinity on G that converges to 1 uniformly on compact subsets of G. In [9], Haagerup established the seminal result that free groups have the Haagerup property. Now we know that the class of groups having the Haagerup property is quite large (see [4]. Choda proved in [5] that a discrete group has the Haagerup property if and only if its associated group von Neumann algebra has Haagerup's approximation property.

1.2. EXTENSION OF COMPLETELY POSITIVE MAPS TO HILBERT SPACE OPERATORS. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ, and $\Omega_\mathcal{M}$ be the standard trace vector in $L^2(\mathcal{M}, \tau)$ corresponding to $1 \in \mathcal{M}$. For $x, y \in \mathcal{M}$,

$\langle x\Omega_{\mathcal{M}}, y\Omega_{\mathcal{M}}\rangle_\tau$ is defined to be $\tau(y^*x)$ and $\|x\|_{2,\tau} = \tau(x^*x)^{1/2}$. When no confusion arises, we simply write Ω instead of $\Omega_{\mathcal{M}}$, and $\|x\|_2$ instead of $\|x\|_{2,\tau}$.

Suppose φ is a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$. Recall that if there is a $c > 0$ such that $\|\varphi(x)\|_2 \leqslant c\|x\|_2$ for all $x \in \mathcal{M}$, then there is a (unique) bounded operator T_φ on $L^2(\mathcal{M}, \tau)$ such that

$$T_\varphi(x\Omega) = \varphi(x)\Omega \quad \forall x \in \mathcal{M},$$

where T_φ is called *the extension operator* of φ. If $\tau \circ \varphi \leqslant c_0\tau$ for some $c_0 > 0$, then $\|\varphi(x)\|_2 \leqslant c_0\|\varphi(1)\|^{1/2}\ \|x\|_2$ (see Lemma 1.2.1 of [13]) and so there is a bounded operator T_φ on $L^2(\mathcal{M}, \tau)$ such that $T_\varphi(x\Omega) = \varphi(x)\Omega$ for all $x \in \mathcal{M}$.

1.3. THE BASIC CONSTRUCTION AND ITS COMPACT IDEAL SPACE. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ, and $\mathcal{N}$ a von Neumann subalgebra of $\mathcal{M}$. The basic construction $\langle \mathcal{M}, e_{\mathcal{N}}\rangle$ is the von Neumann algebra on $L^2(\mathcal{M}, \tau)$ generated by $\mathcal{M}$ and the orthogonal projection $e_{\mathcal{N}}$ from $L^2(\mathcal{M}, \tau)$ onto $L^2(\mathcal{N}, \tau)$. The basic construction $\langle \mathcal{M}, e_{\mathcal{N}}\rangle$ is a semi-finite von Neumann algebra with a faithful normal semi-finite tracial weight Tr such that

$$\mathrm{Tr}(xe_{\mathcal{N}}y) = \tau(xy), \quad \forall x, y \in \mathcal{M}.$$

Recall that $\langle \mathcal{M}, e_{\mathcal{N}}\rangle = J\mathcal{N}'J$, where J is the conjugate linear isometry defined by $J(x\Omega) = x^*\Omega$, $\forall x \in \mathcal{M}$. The compact ideal space of $\langle \mathcal{M}, e_{\mathcal{N}}\rangle$, denoted by $\mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}}\rangle)$, is the norm-closed two-sided ideal generated by finite projections of $\langle \mathcal{M}, e_{\mathcal{N}}\rangle$. Since $e_{\mathcal{N}}$ is a finite projection in $\langle \mathcal{M}, e_{\mathcal{N}}\rangle$, it follows that $e_{\mathcal{N}} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}}\rangle)$. We refer the reader to [10], [13] for more details on the basic construction and its compact ideal space.

1.4. CORRESPONDENCES. Let $\mathcal{N}$ and $\mathcal{M}$ be von Neumann algebras. A correspondence between $\mathcal{N}$ and $\mathcal{M}$ is a Hilbert space $\mathcal{H}$ with a pair of commuting normal representations $\pi_{\mathcal{N}}$ and $\pi_{\mathcal{M}^\circ}$ of $\mathcal{N}$ and $\mathcal{M}^\circ$ (the opposite algebra of $\mathcal{M}$) on $\mathcal{H}$, respectively. Usually, the triple $(\mathcal{H}, \pi_{\mathcal{N}}, \pi_{\mathcal{M}^\circ})$ will be denoted by $\mathcal{H}$. For $x \in \mathcal{N}$, $y \in \mathcal{M}$ and $\xi \in \mathcal{H}$, we shall write $x\xi y$ instead of $\pi_{\mathcal{N}}(x)\pi_{\mathcal{M}^\circ}(y)\xi$. For two vectors $\xi, \eta \in \mathcal{H}$, we denote by $\langle \xi, \eta\rangle_{\mathcal{H}}$ the inner product of vectors ξ and η. If $\mathcal{N} = \mathcal{M}$, then we simply say $\mathcal{H}$ is a correspondence of $\mathcal{M}$.

Two correspondences $\mathcal{H}, \mathcal{K}$ between $\mathcal{N}$ and $\mathcal{M}$ are *equivalent*, denoted by $\mathcal{H} \cong \mathcal{K}$, if they are unitarily equivalent as $\mathcal{N} - \mathcal{M}$ bimodules (see [12]).

1.5. CORRESPONDENCES ASSOCIATED TO COMPLETELY POSITIVE MAPS. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ, and φ be a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$. Define on the linear space $\mathcal{H}_0 = \mathcal{M} \otimes \mathcal{M}$ the sesquilinear form

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_\varphi = \tau(\varphi(x_2^*x_1)y_1y_2^*), \quad \forall x_1, x_2, y_1, y_2 \in \mathcal{M}.$$

It is easy to check that the complete positivity of φ is equivalent to the positivity of $\langle \cdot, \cdot\rangle_\varphi$. Let $\mathcal{H}_\varphi$ be the completion of $\mathcal{H}_0/\sim$, where $\sim$ is the equivalence modulo the null space of $\langle \cdot, \cdot\rangle_\varphi$ in $\mathcal{H}_0$. Then $\mathcal{H}_\varphi$ is a correspondence of $\mathcal{M}$ and the bimodule structure is given by $x(x_1 \otimes y_1)y = xx_1 \otimes y_1 y$ (see [12]). We call $\mathcal{H}_\varphi$ the *correspondence of $\mathcal{M}$ associated to φ.*

The correspondence $\mathcal{H}_{\mathrm{id}}$ associated to the identity operator on $\mathcal{M}$ is called the *identity correspondence* of $\mathcal{M}$. It is easy to see that $\mathcal{H}_{\mathrm{id}}$ and $L^2(\mathcal{M}, \tau)$ are equivalent as correspondences of $\mathcal{M}$. The correspondence $\mathcal{H}_{\mathrm{co}}$ associated to the rank one normal completely positive map $\varphi(x) = \tau(x)1$ is called the *coarse correspondence* of $\mathcal{M}$. If $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and $E_{\mathcal{N}}$ is the unique τ -preserving normal conditional expectation from $\mathcal{M}$ to $\mathcal{N}$, then the correspondence of $\mathcal{M}$ associated to $E_{\mathcal{N}}$ is denoted by $\mathcal{H}_{\mathcal{N}}$ instead of $\mathcal{H}_{E_{\mathcal{N}}}$.

1.6. LEFT τ-BOUNDED VECTORS. Let $\mathcal{N}, \mathcal{M}$ be finite von Neumann algebras with faithful normal traces $\tau_{\mathcal{N}}$ and $\tau_{\mathcal{M}}$, respectively, and $\mathcal{H}$ be a correspondence between $\mathcal{N}$ and $\mathcal{M}$. Let $\xi \in \mathcal{H}$ be a vector. Recall that ξ is a *left (or right) τ-bounded vector* if there is a positive number K such that $\langle \xi, \xi x\rangle_{\mathcal{H}} \leqslant K\tau_{\mathcal{M}}(x)$ (or $\langle x\xi, \xi\rangle_{\mathcal{H}} \leqslant K\tau_{\mathcal{N}}(x)$, respectively) for all $x \in \mathcal{N}_+$ (or $x \in \mathcal{M}_+$, respectively). A vector ξ is called a *τ-bounded vector* if it is both left τ-bounded and right τ-bounded. The set of τ-bounded vectors is a dense vector subspace of $\mathcal{H}$ (see Lemma 1.2.2 of [12]).

1.7. COEFFICIENTS. Let $\mathcal{N}, \mathcal{M}$ be finite von Neumann algebras with faithful normal traces $\tau_{\mathcal{N}}$ and $\tau_{\mathcal{M}}$, respectively, and $\mathcal{H}$ be a correspondence between $\mathcal{N}$ and $\mathcal{M}$. For a left τ-bounded vector ξ, we can define a bounded operator $T : L^2(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{H}$ by $T(y\Omega_{\mathcal{M}}) = \xi y$ for every $y \in \mathcal{M}$. Let $\Phi_\xi(x) = T^*\pi_{\mathcal{N}}(x)T$, where $\pi_{\mathcal{N}}(x)$ is the left action of $x \in \mathcal{N}$ on $\mathcal{H}$. Then Φ_ξ is a normal completely positive map from $\mathcal{N}$ to $\mathcal{M}$ (see 1.2.1 of [12]), and Φ_ξ is called the *coefficient* corresponding to ξ, which is uniquely determined by

$$\langle \Phi_\xi(x) y\Omega_{\mathcal{M}}, z\Omega_{\mathcal{M}}\rangle_{\tau_{\mathcal{M}}} = \langle x\xi y, \xi z\rangle_{\mathcal{H}} \tag{1.1}$$

for all $x \in \mathcal{N}$ and $y, z \in \mathcal{M}$. Therefore,

$$\Phi_\xi(x) = \frac{\mathrm{d}\langle x\xi\cdot, \xi\rangle_{\mathcal{H}}}{\mathrm{d}\tau_{\mathcal{M}}}, \quad \text{i.e., } \tau_{\mathcal{M}}(\Phi_\xi(x)y) = \langle x\xi y, \xi\rangle_{\mathcal{H}}, \quad \forall x \in \mathcal{N}, y \in \mathcal{M}.$$

If $\mathcal{N} = \mathcal{M}$, $\tau_{\mathcal{N}} = \tau_{\mathcal{M}}$, and $x \geqslant 0$,

$$\tau_{\mathcal{M}}(\Phi_\xi(x)) = \langle \Phi_\xi(x)\Omega_{\mathcal{M}}, \Omega_{\mathcal{M}}\rangle_\tau = \langle x\xi, \xi\rangle_{\mathcal{H}} \leqslant K\tau_{\mathcal{M}}(x).$$

By Lemma 1.2.1 of [13], Φ_ξ can be extended to a bounded operator T_{Φ_ξ} from $L^2(\mathcal{M}, \tau)$ to $L^2(\mathcal{M}, \tau)$.

It follows by a maximality argument that $\mathcal{H}$ is a direct sum of cyclic correspondences associated to coefficients as above.

1.8. COMPOSITION OF CORRESPONDENCES. Suppose that $\mathcal{M}, \mathcal{N}, \mathcal{P}$ are finite von Neumann algebras, and $\tau_{\mathcal{P}}$ is a faithful normal trace on $\mathcal{P}$. Let $\mathcal{H}$ be a correspondence between $\mathcal{N}$ and $\mathcal{P}$ and $\mathcal{K}$ be a correspondence between $\mathcal{P}$ and $\mathcal{M}$. Let $\mathcal{H}'$

and $\mathcal{K}'$ be vector subspaces of the τ-bounded vectors in $\mathcal{H}$ and $\mathcal{K}$, respectively. For $\xi_1, \xi_2 \in \mathcal{H}'$ and $\eta_1, \eta_2 \in \mathcal{K}'$,

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1 p, \xi_2 \rangle_{\mathcal{H}} = \langle q\eta_1, \eta_2 \rangle_{\mathcal{K}} = \tau_{\mathcal{P}}(qp)$$

defines an inner product on $\mathcal{H}' \otimes \mathcal{K}'$, where p and q are Radon–Nikodym derivatives of normal linear forms $\mathcal{P} \ni z \to \langle z\eta_1, \eta_2 \rangle_{\mathcal{K}}$ and $\mathcal{P} \ni z \to \langle \xi_1 z, \xi_2 \rangle_{\mathcal{H}}$ with respect to the trace $\tau_{\mathcal{P}}$, respectively (see [12]). The *composition correspondence* (or the *tensor product correspondence*) $\mathcal{H} \otimes_{\mathcal{P}} \mathcal{K}$ is the completion of $\mathcal{H}' \otimes \mathcal{K}' / \sim$, where $\sim$ is the equivalence modulo the null space of $\langle \cdot, \cdot \rangle$ in $\mathcal{H}' \otimes \mathcal{K}'$, and the $\mathcal{N} - \mathcal{M}$ bimodule structure is given by $x(\xi \otimes \eta)y = x\xi \otimes \eta y$. It is easy to verify that the composition of correspondences is associative.

1.9. INDUCED CORRESPONDENCES. We recall Popa's notion of induced correspondences. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ and $\mathcal{N}$ a von Neumann subalgebra. If $\mathcal{H}$ is a correspondence of $\mathcal{N}$, then the *induced correspondence* by $\mathcal{H}$ from $\mathcal{N}$ up to $\mathcal{M}$ is $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H}) = L^2(\mathcal{M}, \tau) \otimes_{\mathcal{N}} \mathcal{H} \otimes_{\mathcal{N}} L^2(\mathcal{M}, \tau)$, where the first $L^2(\mathcal{M})$ is regarded as a left $\mathcal{M}$ and right $\mathcal{N}$ module and the second $L^2(\mathcal{M})$ is regarded as a left $\mathcal{N}$ and right $\mathcal{M}$ module. If $\mathcal{H}$ is the identity correspondence of $\mathcal{N}$, then $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H})$ is the correspondence $\mathcal{H}_{\mathcal{N}}$ of $\mathcal{M}$ (see Proposition 1.3.6 of [12]).

1.10. RELATIVE AMENABILITY. Let $\mathcal{H}, \mathcal{K}$ be two correspondences between $\mathcal{N}$ and $\mathcal{M}$. We say that $\mathcal{H}$ is *weakly contained* in $\mathcal{K}$, if for every $\varepsilon > 0$, and finite subsets $E \subseteq \mathcal{N}$, $F \subseteq \mathcal{M}$, $\{\xi_1, \ldots, \xi_n\} \subseteq \mathcal{H}$, there exists $\{\eta_1, \ldots, \eta_n\} \subseteq \mathcal{K}$ such that

$$|\langle x\xi_i y, \xi_j \rangle_{\mathcal{H}} - \langle x\eta_i y, \eta_j \rangle_{\mathcal{K}}| < \varepsilon,$$

for all $x \in E, y \in F$ and $1 \leqslant i, j \leqslant n$. If $\mathcal{H}$ is weakly contained in $\mathcal{K}$, we will denote this by $\mathcal{H} \prec \mathcal{K}$. We refer the reader to [7], [12] for more details on weak containment and topology on correspondences.

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ, and $\mathcal{N}$ a von Neumann subalgebra. Recall that $\mathcal{M}$ is *relative amenable to* $\mathcal{N}$ if $\mathcal{H}_{\mathrm{id}} \prec \mathcal{H}_{\mathcal{N}}$. The algebra $\mathcal{M}$ is relative amenable to $\mathcal{N}$ if and only if there exists a conditional expectation from the basic construction $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ onto $\mathcal{M}$ (see [12]).

2. REMOVAL OF THE SUBTRACIAL CONDITION

The following definition is given by Popa in [13].

DEFINITION 2.1. Let $\mathcal{M}$ be a finite von Neumann algebra and $\mathcal{N}$ a von Neumann subalgebra. $\mathcal{M}$ has *Haagerup's approximation property relative to* $\mathcal{N}$ if there exists a normal faithful trace τ on $\mathcal{M}$ and a net of normal completely positive $\mathcal{N}$-bimodular maps $\{\varphi_\alpha\}_{\alpha \in \Lambda}$ from $\mathcal{M}$ to $\mathcal{M}$ satisfying the following conditions:

(i) $\tau \circ \varphi_\alpha \leqslant \tau, \forall \alpha \in \Lambda$;

(ii) $T_{\varphi_\alpha} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle), \forall \alpha \in \Lambda$;

(iii) $\lim_{\alpha} \|\varphi_\alpha(x) - x\|_2 = 0, \forall x \in \mathcal{M}$.

In this section we will prove the following theorem.

THEOREM 2.2. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ and $\mathcal{N}$ a von Neumann subalgebra. Suppose $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ is a net of normal completely positive $\mathcal{N}$-bimodular maps from $\mathcal{M}$ to $\mathcal{M}$ satisfying the conditions* (ii) *and* (iii) *as in Definition* 2.1*, i.e.* $\lim_{\alpha} \|\varphi_\alpha(x) - x\|_2 = 0$ *for all* $x \in \mathcal{M}$ *and the map* $x\Omega \to \varphi_\alpha(x)\Omega$ *extends to a bounded operator* T_{φ_α} *in* $\mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}}\rangle)$ *for every* $\alpha \in \Lambda$. *Then there exists a net* $\{\psi_\beta\}_{\beta\in\Gamma}$ *of normal completely positive $\mathcal{N}$-bimodular maps from $\mathcal{M}$ to $\mathcal{M}$ satisfying:*

(i) $\psi_\beta(1) = 1$ *and* $\tau \circ \psi_\beta = \tau$, $\forall \beta \in \Gamma$;

(ii) $T_{\psi_\beta} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}}\rangle), \forall \beta \in \Gamma$;

(iii) $\lim_{\beta} \|\psi_\beta(x) - x\|_2 = 0, \forall x \in \mathcal{M}$.

In particular, $\mathcal{M}$ has Haagerup's approximation property relative to $\mathcal{N}$.

The proof of Theorem 2.2 uses Lemma 1.1.2 in [13], Day's trick and Proposition 2.1 of [11]. We begin by recalling 2° and 3° of Lemma 1.1.2 in [13], here labelled Lemma 2.3 and Lemma 2.4, for the sake of completeness.

LEMMA 2.3. *Let φ be a normal completely positive $\mathcal{N}$-bimodular map from $\mathcal{M}$ to $\mathcal{M}$. Let $a = 1 \vee \varphi(1)$ and $\varphi'(\cdot) = a^{-1/2}\varphi(\cdot)a^{-1/2}$. Then φ' is a normal completely positive $\mathcal{N}$-bimodular map from $\mathcal{M}$ to $\mathcal{M}$ and satisfies $\varphi'(1) \leqslant 1$, $\tau \circ \varphi' \leqslant \tau \circ \varphi$ and the estimate:*

$$\|\varphi'(x) - x\|_2 \leqslant \|\varphi(x) - x\|_2 + 2\|\varphi(1) - 1\|_2^{1/2} \cdot \|x\|, \quad \forall x \in \mathcal{M}.$$

LEMMA 2.4. *Let φ be a normal completely positive $\mathcal{N}$-bimodular map from $\mathcal{M}$ to $\mathcal{M}$ such that $\varphi(1) \leqslant 1$. Let $b = 1 \vee (\mathrm{d}\tau \circ \varphi/\mathrm{d}\tau)$ and $\varphi'(\cdot) = \varphi(b^{-1/2} \cdot b^{-1/2})$. Then φ' is a normal completely positive $\mathcal{N}$-bimodular map from $\mathcal{M}$ to $\mathcal{M}$ and satisfies $\varphi'(1) \leqslant \varphi(1) \leqslant 1$, $\tau \circ \varphi' \leqslant \tau$ and the estimate:*

$$\|\varphi'(x) - x\|_2^2 \leqslant 2\|\varphi(x) - x\|_2 + 5\|\tau \circ \varphi - \tau\|^{1/2} \cdot \|x\|, \quad \forall x \in \mathcal{M}.$$

LEMMA 2.5. *Let φ be a normal completely positive $\mathcal{N}$-bimodular map from $\mathcal{M}$ to $\mathcal{M}$ such that $\varphi(1) \leqslant 1$ and $\tau \circ \varphi \leqslant \tau$. Let $h = \varphi(1)$ and $k = \mathrm{d}\tau \circ \varphi/\mathrm{d}\tau$. Then $0 \leqslant h, k \leqslant 1$, $h, k \in \mathcal{N}' \cap \mathcal{M}$, and $E_{\mathcal{N}}(h) = E_{\mathcal{N}}(k)$.*

Proof. It is easy to see that $0 \leqslant h, k \leqslant 1$ and $h, k \in \mathcal{M}$. Since φ is $\mathcal{N}$-bimodular,

$$bh = b\varphi(1) = \varphi(b) = \varphi(1)b = hb, \quad \forall b \in \mathcal{N}.$$

Note that for all $x \in \mathcal{M}$ and $b \in \mathcal{N}$,

$$\begin{aligned} \tau(x(bk - kb)) &= \tau(xbk) - \tau(bxk) = \tau(\varphi(xb)) - \tau(\varphi(bx)) \\ &= \tau(\varphi(x)b) - \tau(b\varphi(x)) = 0; \end{aligned}$$

$$\tau(E_{\mathcal{N}}(h)x) = \tau(hE_{\mathcal{N}}(x)) = \tau(\varphi(1)E_{\mathcal{N}}(x)) = \tau(\varphi(E_{\mathcal{N}}(x)))$$
$$= \tau(E_{\mathcal{N}}(x)k) = \tau(xE_{\mathcal{N}}(k)) = \tau(E_{\mathcal{N}}(k)x).$$

Hence, $bk = kb$ and $E_{\mathcal{N}}(h) = E_{\mathcal{N}}(k)$. ∎

LEMMA 2.6. *Let φ be a normal completely positive $\mathcal{N}$-bimodular map from $\mathcal{M}$ to $\mathcal{M}$ such that $\varphi(1) \leqslant 1 - \varepsilon$ for some $\varepsilon > 0$ and $\tau \circ \varphi \leqslant \tau$. Let $h = \varphi(1)$ and $k = \mathrm{d}\tau \circ \varphi / \mathrm{d}\tau$. Then there exist positive operators $a, b \in \mathcal{N}' \cap \mathcal{M}$ such that*

$$1 - h = aE_{\mathcal{N}}(b) \quad \textit{and} \quad 1 - k = E_{\mathcal{N}}(a)b.$$

Proof. Let $b = 1 - k$. By Lemma 2.5, b is a positive operator in $\mathcal{N}' \cap \mathcal{M}$ and

$$E_{\mathcal{N}}(b) = 1 - E_{\mathcal{N}}(k) = 1 - E_{\mathcal{N}}(h) = E_{\mathcal{N}}(1 - h).$$

Since $h \leqslant 1 - \varepsilon$, $1 - h \geqslant \varepsilon$ and therefore $E_{\mathcal{N}}(1 - h) \geqslant \varepsilon$. Hence $(E_{\mathcal{N}}(1 - h))^{-1}$ exists. For all $b \in \mathcal{N}$, by Lemma 2.5,

$$bE_{\mathcal{N}}(1 - h) = E_{\mathcal{N}}(b(1 - h)) = E_{\mathcal{N}}((1 - h)b) = E_{\mathcal{N}}(1 - h)b.$$

Hence, $E_{\mathcal{N}}(1 - h) \in \mathcal{N} \cap \mathcal{N}'$ and $(E_{\mathcal{N}}(1 - h))^{-1} \in \mathcal{N} \cap \mathcal{N}'$. So $a = (1 - h)(E_{\mathcal{N}}(1 - h))^{-1}$ is a positive operator in $\mathcal{N}' \cap \mathcal{M}$. Since $E_{\mathcal{N}}(b) = E_{\mathcal{N}}(1 - h)$, it is routine to check that $1 - h = aE_{\mathcal{N}}(b)$ and $1 - k = E_{\mathcal{N}}(a)b$. ∎

Proof of Theorem 2.2. Let $a_\alpha = 1 \vee \varphi_\alpha(1)$ and $\varphi'_\alpha(\cdot) = a_\alpha^{-1/2}\varphi_\alpha(\cdot)a_\alpha^{-1/2}$. By Lemma 2.3, $\{\varphi'_\alpha\}_\alpha$ satisfy the condition (iii) in Theorem 2.2 and $\varphi'_\alpha(1) \leqslant 1$ for every $\alpha \in \Lambda$. By Lemma 2.5, $T_{\varphi'_\alpha} = a_\alpha^{-1/2} J a_\alpha^{-1/2} J T_{\varphi_\alpha} \in \langle \mathcal{M}, e_{\mathcal{N}} \rangle$. Since $T_{\varphi_\alpha} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$, $T_{\varphi'_\alpha} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$ for every $\alpha \in \Lambda$ (see Lemma 1.2.1 of [13]).

Let $f_\alpha = \tau \circ \varphi'_\alpha$. Then $\{f_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{M}_*$. Since $\lim\limits_\alpha \|\varphi'_\alpha(x) - x\|_{2,\tau} = 0$ for every x in $\mathcal{M}$, $\lim\limits_\alpha f_\alpha(x) = \tau(x)$ for every $x \in \mathcal{M}$. Since $\mathcal{M}$ is the dual space of $\mathcal{M}_*$, this implies that $\lim\limits_\alpha f_\alpha = \tau$ in the weak topology on $\mathcal{M}_*$. Since the weak closure and the strong closure of a convex set in $\mathcal{M}_*$ are the same, τ is in the norm closure of the convex hull of $\{f_\alpha\}_{\alpha \in \Lambda}$. Note that $\tau \circ \left(\sum\limits_{i=1}^{n} \lambda_{\alpha_i} \varphi'_{\alpha_i} \right) = \sum\limits_{i=1}^{n} \lambda_{\alpha_i} f_{\alpha_i}$. By taking finitely many convex combinations of $\{\varphi'_\alpha\}_{\alpha \in \Lambda}$, we can see that there exists a net $\{\psi'_\beta\}_{\beta \in \Gamma}$ of completely positive $\mathcal{N}$-bimodular maps from $\mathcal{M}$ to $\mathcal{M}$ satisfying the conditions (ii) and (iii) in Theorem 2.2, $\psi'_\beta(1) \leqslant 1$ for all $\beta \in \Gamma$ and the following condition: $\lim\limits_\beta \|g'_\beta - \tau\|_1 = 0$ for $g'_\beta = \tau \circ \psi'_\beta$.

Let $b'_\beta = 1 \vee (dg'_\beta / d\tau)$ and $\psi''_\beta(\cdot) = \psi'_\beta((b'_\beta)^{-1/2} \cdot (b'_\beta)^{-1/2})$. By Lemma 2.4, $\{\psi''_\beta\}_{\beta \in \Gamma}$ is a net of completely positive $\mathcal{N}$-bimodular maps from $\mathcal{M}$ to $\mathcal{M}$, and satisfies (iii) in Theorem 2.2, $\psi''_\beta(1) \leqslant 1$ and $\tau \circ \psi''_\beta \leqslant \tau$ for all $\beta \in \Gamma$. By Lemma 2.5, $T_{\varphi''_\beta} = T_{\psi'_\beta} {b'_\beta}^{-1/2} J {b'_\beta}^{-1/2} J \in \langle \mathcal{M}, e_{\mathcal{N}} \rangle$. Since $T_{\psi'_\beta} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$, $T_{\psi''_\beta} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$ for every $\beta \in \Gamma$.

We may further assume that $\psi''_\beta(1) \leqslant 1 - \varepsilon_\beta$, $\varepsilon_\beta > 0$. Otherwise we can choose a net of positive numbers λ_β with $0 < \lambda_\beta < 1$ and $\lim_\beta \lambda_\beta = 1$ and consider $\lambda_\beta \cdot \psi''_\beta$. Let $h_\beta = \psi''_\beta(1)$ and $k_\beta = \mathrm{d}\tau \circ \psi''_\beta / \mathrm{d}\tau$. By Lemma 2.6, there exist positive operators a_β, b_β in $\mathcal{N}' \cap \mathcal{M}$ such that $1 - h_\beta = a_\beta E_{\mathcal{N}}(b_\beta)$ and $1 - k_\beta = E_{\mathcal{N}}(a_\beta) b_\beta$.

The rest is essentially Jolissaint's trick from the proof of Proposition 2.1 in [11]. For every $\beta \in \Gamma$, define $\psi_\beta : \mathcal{M} \to \mathcal{M}$ by

$$\psi_\beta(x) = \psi''_\beta(x) + a_\beta E_{\mathcal{N}}(b_\beta^{1/2} x b_\beta^{1/2}).$$

Clearly, every ψ_β is a normal completely positive $\mathcal{N}$-bimodular map. We have

$$\begin{aligned}
\psi_\beta(1) &= \psi''_\beta(1) + a_\beta E_{\mathcal{N}}(b_\beta) = h_\beta + 1 - h_\beta = 1;\\
\tau(\psi_\beta(x)) &= \tau(\psi''_\beta(x)) + \tau(a_\beta E_{\mathcal{N}}(b_\beta^{1/2} x b_\beta^{1/2}))\\
&= \tau(x k_\beta) + \tau(E_{\mathcal{N}}(a_\beta) b_\beta^{1/2} x b_\beta^{1/2})\\
&= \tau(x k_\beta) + \tau(E_{\mathcal{N}}(a_\beta) b_\beta x) = \tau(k_\beta x) + \tau((1 - k_\beta)x) = \tau(x).
\end{aligned}$$

This proves that $\{\psi_\beta\}_{\beta \in \Gamma}$ satisfies the condition (i) of Theorem 2.2.

Note that $T_{\psi_\beta} = T_{\psi''_\beta} + a_\beta e_{\mathcal{N}} b_\beta^{1/2} J b_\beta^{1/2} J$. Since $e_{\mathcal{N}} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$, $T_{\psi_\beta} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$ for every $\beta \in \Gamma$. This proves that $\{\psi_\beta\}_{\beta \in \Gamma}$ satisfies the condition (ii) of Theorem 2.2.

Finally, for every positive operator x in $\mathcal{M}$,

$$\begin{aligned}
\psi_\beta(x) - \psi''_\beta(x) &= a_\beta E_{\mathcal{N}}(b_\beta^{1/2} x b_\beta^{1/2}) \leqslant \|x\| a_\beta E_{\mathcal{N}}(b_\beta)\\
&= \|x\|(1 - h_\beta) = \|x\|(1 - \psi''_\beta(1)),
\end{aligned}$$

which shows that $\{\psi_\beta\}_{\beta \in \Gamma}$ satisfies the condition (iii) of Theorem 2.2. ∎

Paul Jolissaint has pointed out that a modification of the proof of Proposition 2.4 of [11] using Proposition 2.4.2 of [13] yields that the relative Haagerup property considered by Popa in Definition 2.1 of [13] is independent of the choice of faithful normal trace on $\mathcal{M}$. This observation and the proof of the argument above establish that the notion of "relative Haagerup property" considered by Boca in [3] is equivalent to the notion of relative Haagerup property considered by Popa in Definition 2.1 of [13].

3. C_0-CORRESPONDENCES

We now show that Theorem 2.2 enables us to interpret Haagerup's approximation property in the framework of Connes's theory of correspondences. Throughout this section $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal trace τ and $\mathcal{H}$ is a correspondence of $\mathcal{M}$.

DEFINITION 3.1. We say that $\mathcal{H}$ is a C_0-correspondence of $\mathcal{M}$ with respect to τ if $\mathcal{H} \cong \bigoplus_{\alpha\in\Lambda} \mathcal{H}_{\varphi_\alpha}$, where each $\mathcal{H}_{\varphi_\alpha}$ is the correspondence of $\mathcal{M}$ associated to a completely positive map $\varphi_\alpha : \mathcal{M} \to \mathcal{M}$ such that the extension operator T_{φ_α} of φ_α is a compact operator in $\mathcal{B}(L^2(\mathcal{M},\tau))$.

REMARK 3.2. Paul Jolissaint has pointed out that C_0-correspondences indeed depend on the choice of trace. In particular, given a c.p. map φ such that $T_{\varphi,\tau} : x\Omega_\tau \to \varphi(x)\Omega_\tau$ is bounded, the map $T_{\varphi,\tau'}$ taken with respect to another trace τ' may not be bounded. For instance, let $\phi \in M_*$ be a normal state, and define $\varphi(x) = \phi(x)1$. Then it is straightforward to show that $T_{\varphi,\tau}$ is bounded if and only if $\tau((\mathrm{d}\phi/\mathrm{d}\tau)^2) < \infty$. For this reason, only if the trace in question is clear we will write "H is a C_0-correspondence of M" instead of "H is a C_0-correspondence of M with respect to τ".

Note that with a given trace τ, the coarse correspondence $\mathcal{H}_{\mathrm{co}}$ of $\mathcal{M}$ is an example of a C_0-correspondence with respect to τ.

REMARK 3.3. Let G be a discrete group, and π be a unitary representation of G on a Hilbert space $\mathcal{H}$. Then π is unitarily equivalent to a direct sum of cyclic representations π_{f_α} of G, where each π_{f_α} is the representation associated to a positive definite function f_α on G. Recall that the representation π is a C_0-representation if all matrix coefficients $\omega_{\xi,\eta}(g) = \langle \pi(g)\xi, \eta\rangle$ belong to $C_0(G)$. It is easy to check that π is a C_0-representation if and only if every $f_\alpha \in C_0(G)$. By [9], [5], for every f_α, there is a unique normal completely positive map φ_{f_α} from the group von Neumann algebra $L(G)$ to itself satisfying $\varphi_{f_\alpha}(L_g) = f_\alpha(g)L_g$, where L_g is the unitary operator associated to g. By Lemma 1 and Lemma 2 of [5], f_α is in $C_0(G)$ if and only if the extension operator $T_{\varphi_{f_\alpha}}$ of φ_{f_α} is a compact operator in $\mathcal{B}(L^2(G))$. Hence, the correspondence $\mathcal{H}_{\varphi_{f_\alpha}}$ of $L(G)$ associated to φ_{f_α} is a C_0-correspondence of $\mathcal{M}$. So our definition of C_0-correspondences of finite von Neumann algebras is a natural analogue of the notion of C_0-representations of groups.

The following theorem is the main result of this section.

THEOREM 3.4. *A finite von Neumann algebra $\mathcal{M}$ has Haagerup's approximation property if and only if the identity correspondence of $\mathcal{M}$ is weakly contained in some C_0-correspondence of $\mathcal{M}$ with respect to some trace τ.*

To prove the above theorem, we need the following lemmas.

LEMMA 3.5. *Let φ be a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$ such that the extension operator T_φ of φ is a compact operator in $\mathcal{B}(L^2(\mathcal{M},\tau))$. Let $\xi = \sum_{i=1}^{n} a_i \otimes b_i$ be a vector in the correspondence $\mathcal{H}_\varphi$ of $\mathcal{M}$ associated to φ . Then ξ is a left τ-bounded vector and the coefficient Φ_ξ corresponding to ξ is a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$ such that T_{Φ_ξ} is a compact operator in $\mathcal{B}(L^2(\mathcal{M},\tau))$.*

Proof. To see ξ is a left τ-bounded vector, we may assume that $\xi = a \otimes b$. Then

$$\begin{aligned}\|\xi x\|_\varphi^2 &= \langle \xi x, \xi x\rangle_\varphi = \langle a\otimes (bx), a\otimes (bx)\rangle_\varphi = \tau(\varphi(a^*a)bxx^*b^*)\\ &= \tau(x^*(b^*\varphi(a^*a)b)x) \leqslant \|b^*\varphi(a^*a)b\|\tau(x^*x) = \|b^*\varphi(a^*a)b\|\|x\|_2^2.\end{aligned}$$

Hence Φ_ξ is a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$. For every $x, y, z \in \mathcal{M}$, by equation (1.1) in Section 1.7,

$$\begin{aligned}\langle \Phi_\xi(x)y\Omega, z\Omega\rangle_\tau &= \langle x\xi y, \xi z\rangle_\varphi = \sum_{i,j=1}^n \langle xa_j\otimes b_jy, a_i\otimes b_iz\rangle_\varphi\\ &= \sum_{i,j=1}^n \tau(\varphi(a_i^*xa_j)b_jyz^*b_i^*) = \Big\langle \sum_{i,j=1}^n b_i^*\varphi(a_i^*xa_j)b_jy\Omega, z\Omega\Big\rangle_\tau.\end{aligned}$$

This implies that $\Phi_\xi(x) = \sum_{i,j=1}^n b_i^*\varphi(a_i^*xa_j)b_j$. Hence, Φ_ξ can be extended to a bounded operator from $L^2(\mathcal{M},\tau)$ to $L^2(\mathcal{M},\tau)$ such that

$$T_{\Phi_\xi} = \sum_{i,j=1}^n b_i^* J b_j^* J T_\varphi a_i^* J a_j^* J.$$

Since T_φ is a compact operator, T_{Φ_ξ} is also a compact operator. ∎

LEMMA 3.6. *Suppose $\mathcal{H} = \bigoplus_{\alpha\in\Lambda} \mathcal{H}_{\varphi_\alpha}$ is a correspondence of $\mathcal{M}$ such that each T_{φ_α} is a compact operator in $\mathcal{B}(L^2(\mathcal{M},\tau))$. Let $\mathcal{F}$ be the convex hull of the set of coefficients $\left\{\Phi_\xi : \xi = \sum_{i=1}^n a_i\otimes b_i \in \mathcal{H}_{\varphi_\alpha} \text{ for some } \alpha\in\Lambda\right\}$. Then $\mathcal{F}$ is a convex cone and for every $b\in\mathcal{M}$ and $\Phi\in\mathcal{F}$, the completely positive map $b^*\Phi(\cdot)b$ belongs to $\mathcal{F}$. Furthermore, T_Φ is a compact operator in $\mathcal{B}(L^2(\mathcal{M},\tau))$ for all $\Phi\in\mathcal{F}$.*

Proof. It is obvious that $\mathcal{F}$ is a convex cone. To prove the rest, we may assume that $\Phi = \Phi_\xi$ is the coefficient corresponding to $\xi\in\mathcal{H}_\varphi$ as in Lemma 3.5. Let $\eta = \xi b = \sum_{i=1}^n a_i\otimes b_ib \in \mathcal{H}$. By Lemma 3.5, η is a left τ-bounded vector. Let Φ_η be the coefficient corresponding to η. By equation (1.1) in Section 1.7,

$$\langle \Phi_\eta(x)y\Omega, z\Omega\rangle_\tau = \langle x\xi by, \xi bz\rangle_\simeq = \langle \Phi(x)by\Omega, bz\Omega\rangle_\tau = \langle b^*\Phi(x)by\Omega, z\Omega\rangle_\tau.$$

This implies that $\Phi_\eta = b^*\Phi b$. Hence $b^*\Phi b\in\mathcal{F}$. By Lemma 3.5, T_{Φ_η} is compact. ∎

LEMMA 3.7. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ, and $\mathcal{H}$, $\mathcal{K}$ be two correspondences of $\mathcal{M}$. Suppose $\xi\in\mathcal{H}$ and $\eta\in\mathcal{K}$ are two left τ-bounded vectors, and Φ_ξ, Φ_η are the coefficients corresponding to ξ, η, respectively. Then $\xi\oplus\eta$ is also a left τ-bounded vector and $\Phi_\xi+\Phi_\eta$ is the coefficient corresponding to $\xi\oplus\eta\in\mathcal{H}\oplus\mathcal{K}$.*

Proof. It is clear that $\xi \oplus \eta$ is a left τ-bounded vector. By equation (1.1) in Section 1.7,

$$\begin{aligned}\langle(\Phi_\xi + \Phi_\eta)(x)y\Omega, z\Omega\rangle_\tau &= \langle x\xi y, \xi z\rangle_{\mathcal{H}} + \langle x\eta y, \eta z\rangle_{\mathcal{K}} = \langle x(\xi \oplus \eta)y, (\xi \oplus \eta)z\rangle_{\mathcal{H}\oplus\mathcal{K}} \\ &= \langle \Phi_{\xi\oplus\eta}(x)y\Omega, z\Omega\rangle_\tau.\end{aligned}$$

Hence $\Phi_{\xi\oplus\eta} = \Phi_\xi + \Phi_\eta$. ∎

Note that in the proof of Lemma 2.2 of [2], if we replace the arbitrary positive normal form ϕ (on line 10 of page 418) by an arbitrary weak operator topology continuous positive form, then the following lemma follows.

LEMMA 3.8. *Let Ψ be a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$. If Ψ is in the closure of $\mathcal{F}$ in the pointwise weak operator topology, then there exists a net $\{\Phi_\alpha\}_{\alpha\in\Lambda}$ in $\mathcal{F}$ such that $\Phi_\alpha(1) \leqslant \Psi(1)$ for all $\alpha \in \Lambda$ which converges to Ψ in the pointwise weak operator topology.*

Proof of Theorem 3.4. By [11], Haagerup's approximation property is independent of the choice of trace on a finite von Neumann algebra $\mathcal{M}$. Suppose first that $\mathcal{M}$ has Haagerup's approximation property with respect to a trace τ. By Theorem 2.2, there is a net $(\varphi_\alpha)_{\alpha\in\Lambda}$ of unital normal completely positive maps satisfying conditions (i)-(iii) in Theorem 2.2. It immediately follows that the correspondence $\mathcal{H} = \bigoplus_{\alpha\in\Lambda} \mathcal{H}_{\varphi_\alpha}$ is a C_0-correspondence of $\mathcal{M}$ with respect to τ which weakly contains the identity correspondence of $\mathcal{M}$.

Conversely, suppose that $\mathcal{H}$ is a C_0-correspondence of $\mathcal{M}$ with respect to a trace τ which weakly contains the identity correspondence of $\mathcal{M}$. We may assume $\mathcal{H} = \bigoplus_{\beta\in\Gamma} \mathcal{H}_{\varphi_\beta}$, with each $\varphi_\beta : \mathcal{M} \to \mathcal{M}$ is a normal completely positive map such that the extension operator T_{φ_β} of φ_β is a compact operator in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. Since the identity correspondence of $\mathcal{M}$ is weakly contained in $\mathcal{H}$, for every $\varepsilon > 0$ and every finite subset E of $\mathcal{M}$, there exists a $\xi \in \mathcal{H}$ such that

$$|\langle x\xi y, \xi z\rangle_{\mathcal{H}} - \langle x\Omega y, \Omega z\rangle_\tau| < \varepsilon, \quad \forall x, y, z \in E.$$

We may assume that $\xi = \xi_1 \oplus \cdots \oplus \xi_n$, where $\xi_i = \sum_{j=1}^{n_i} a_{ij} \otimes b_{ij} \in \mathcal{H}_{\varphi_{\beta_i}}$. Let Φ_ξ be the coefficient corresponding to ξ. By Lemma 3.6 and Lemma 3.7, $\Phi_\xi \in \mathcal{F}$. By equation (1.1) in Section 1.7,

$$|\langle \Phi_\xi(x)y\Omega, z\Omega\rangle_\tau - \langle x\Omega y, \Omega z\rangle_\tau| = |\langle x\xi y, \xi z\rangle_{\mathcal{H}} - \langle x\Omega y, \Omega z\rangle_\tau| < \varepsilon, \quad \forall x, y, z \in E.$$

This implies that there exists a net $(\Phi_{\alpha'})_{\alpha'\in\Lambda'}$ of completely positive maps in $\mathcal{F}$ such that $\lim_{\alpha'} \Phi_{\alpha'}(x) = x$ in the weak operator topology for every $x \in \mathcal{M}$.

By Lemma 3.8, there is a net $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ in $\mathcal{F}$ such that $\lim_\alpha \varphi_\alpha(x) = x$ in the weak operator topology for every $x \in \mathcal{M}$ and $\varphi_\alpha(1) \leqslant 1$ for every $\alpha \in \Lambda$. Now given $x \in \mathcal{M}$:

$$\begin{aligned}\|\varphi_\alpha(x) - x\|_2 &= \tau(\varphi_\alpha(x)^*\varphi_\alpha(x)) + \tau(x^*x) - 2\mathrm{Re}\,\tau(x^*\varphi_\alpha(x)) \\ &\leqslant \|\varphi_\alpha(1)\|\tau(\varphi_\alpha(x^*x)) + \tau(x^*x) - 2\mathrm{Re}\,\tau(x^*\varphi_\alpha(x)) \\ &\leqslant \tau(\varphi_\alpha(x^*x)) + \tau(x^*x) - 2\mathrm{Re}\,\tau(x^*\varphi_\alpha(x)).\end{aligned}$$

Since $\lim_\alpha \varphi_\alpha(x) = x$ in the weak operator topology for every $x \in \mathcal{M}$ it follows that $\lim_\alpha \tau(\varphi_\alpha(x^*x)) = \tau(x^*x)$ and $\lim_\alpha \tau(x^*\varphi_\alpha(x)) = \tau(x^*x)$. Therefore $\lim_\alpha \|\varphi_\alpha(x) - x\|_2 = 0$. This proves that $(\varphi_\alpha)_\alpha$ is a net of completely positive maps that approximate the identity pointwise in the trace-norm. Since $\varphi_\alpha \in \mathcal{F}$, it follows that T_{φ_α} is a compact operator on $L^2(\mathcal{M}, \tau)$. By Theorem 2.2, $\mathcal{M}$ has Haagerup's approximation property with respect to τ. ∎

As an application of Theorem 3.4, we prove the following result.

COROLLARY 3.9. *If a finite von Neumann algebra $\mathcal{M}$ has Haagerup's approximation property, then for any trace τ on $\mathcal{M}$ the class of C_0-correspondences of $\mathcal{M}$ with respect to τ is dense in* $\mathrm{Corr}(\mathcal{M})$.

Proof. By Section 2.6, it is clear that we need only to prove that every cyclic correspondence $\mathcal{H}_{\Phi_\xi}$ of $\mathcal{M}$ associated to the coefficient Φ_ξ of a left τ-bounded vector belongs to the closure of the set of C_0-correspondences of $\mathcal{M}$. Let ξ be a given left τ-bounded vector. Since $\mathcal{M}$ has Haagerup's approximation property, there is a net $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ of normal completely positive maps of $\mathcal{M}$, such that:

(i) $\varphi_\alpha(1) = 1$, $\forall \alpha \in \Lambda$,
(ii) T_{φ_α} is compact, $\forall \alpha \in \Lambda$,
(iii) $\lim_\alpha \|\varphi_\alpha(x) - x\|_2 = 0$, $\forall x \in \mathcal{M}$.

Hence, each $T_{\Phi_\xi\circ\varphi_\alpha} = T_{\Phi_\xi}T_{\varphi_\alpha}$ is compact and $\lim_\alpha \|\Phi_\xi \circ \varphi_\alpha(x) - \Phi_\xi(x)\|_2 = 0$ for every $x \in \mathcal{M}$. By Remark 2.1.4 of [12], $\mathcal{H}_{\Phi_\xi\circ\varphi_\alpha} \to \mathcal{H}_{\Phi_\xi}$. ∎

4. RELATIVE AMENABILITY AND HAAGERUP'S APPROXIMATION PROPERTY

In this section we prove the following result:

THEOREM 4.1. *If $\mathcal{N} \subset \mathcal{M}$ is an amenable inclusion of finite von Neumann algebras and $\mathcal{N}$ has Haagerup's approximation property then $\mathcal{M}$ also has Haagerup's approximation property.*

To prove Theorem 4.1, we need the following lemmas.

LEMMA 4.2. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of finite von Neumann algebras, and $E_\mathcal{N}$ be the normal τ-preserving conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$. If $\mathcal{H}_\varphi$ is the correspondence of $\mathcal{N}$ associated to a normal completely positive map φ from $\mathcal{N}$ to $\mathcal{N}$*

and $\mathcal{H}_{\varphi\circ E_{\mathcal{N}}}$ is the correspondence of $\mathcal{M}$ associated to the normal completely positive map $\varphi \circ E_{\mathcal{N}}$ from $\mathcal{M}$ to $\mathcal{M}$, then $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H}_\varphi) \cong \mathcal{H}_{\varphi\circ E_{\mathcal{N}}}$.

Proof. Denote by $\mathcal{K} = \mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H}) = L^2(\mathcal{M})\otimes_{\mathcal{N}}\mathcal{H}_\varphi\otimes_{\mathcal{N}}L^2(\mathcal{M})$, where the first $L^2(\mathcal{M})$ is regarded as a left $\mathcal{M}$ and right $\mathcal{N}$ module and the second $L^2(\mathcal{M})$ is regarded as a left $\mathcal{N}$ and right $\mathcal{M}$ module. Let $\xi \in \mathcal{H}_\varphi$ be the vector corresponding to $\Omega \otimes \Omega$, which is a cyclic vector of $\mathcal{H}_\varphi$. Given $x_1, x_2, y_1, y_2 \in \mathcal{M}$, we have

$$\langle x_1 \otimes (\xi \otimes y_1),\, x_2 \otimes (\xi \otimes y_2)\rangle_{\mathcal{K}} = \langle q(\xi \otimes y_1),\, \xi \otimes y_2\rangle_{\mathcal{H}_\varphi\otimes_{\mathcal{N}}L^2(\mathcal{M})},$$

where $q \in \mathcal{N}$ is the Radon–Nikodym derivative of $\mathcal{N} \ni z \mapsto \langle x_1 z,\, x_2\rangle_{L^2(\mathcal{M})}$ with respect to $\tau_{\mathcal{N}}$. Note that

$$\langle x_1 z,\, x_2\rangle_{L^2(\mathcal{M})} = \tau(z x_2^* x_1) = \tau(z E_{\mathcal{N}}(x_2^* x_1)).$$

Hence $q = E_{\mathcal{N}}(x_2^* x_1)$ and

$$\begin{aligned}\langle x_1 \otimes (\xi \otimes y_1), x_2 \otimes (\xi \otimes y_2)\rangle_{\mathcal{K}} &= \langle E_{\mathcal{N}}(x_2^* x_1)\xi \otimes y_1, \xi \otimes y_2\rangle_{\mathcal{H}_\varphi\otimes_{\mathcal{N}}L^2(\mathcal{M})}\\ &= \langle E_{\mathcal{N}}(x_2^* x_1)\xi p, \xi\rangle_{\mathcal{H}_\varphi},\end{aligned}$$

where $p \in \mathcal{N}$ is the Radon–Nikodym derivative of $\mathcal{N} \ni z \mapsto \langle z y_1,\, y_2\rangle_{L^2(\mathcal{M})}$ with respect to $\tau_{\mathcal{N}}$. Note that $\langle z y_1,\, y_2\rangle_{L^2(\mathcal{M})} = \tau(z y_1 y_2^*) = \tau(z E_{\mathcal{N}}(y_1 y_2^*))$. Hence $p = E_{\mathcal{N}}(y_1 y_2^*)$ and

$$\begin{aligned}\langle x_1 \otimes (\xi \otimes y_1), x_2 \otimes (\xi \otimes y_2)\rangle_{\mathcal{K}} &= \langle E_{\mathcal{N}}(x_2^* x_1)\xi p, \xi\rangle_{\mathcal{H}_\varphi}\\ &= \langle E_{\mathcal{N}}(x_2^* x_1)\xi E_{\mathcal{N}}(y_1 y_2^*), \xi\rangle_{\mathcal{H}_\varphi}\\ &= \tau(\varphi(E_{\mathcal{N}}(x_2^* x_1))E_{\mathcal{N}}(y_1 y_2^*))\\ &= \tau(\varphi(E_{\mathcal{N}}(x_2^* x_1))y_1 y_2^*)\\ &= \langle x_1\xi y_1, x_2\xi y_2\rangle_{\mathcal{H}_{\varphi\circ E_{\mathcal{N}}}}.\end{aligned}$$

Therefore the map defined on simple tensors by $(x_1 \otimes \xi) \otimes x_2 \mapsto x_1\xi x_2$ extends to an $\mathcal{M}$-linear isometry from $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H}_\varphi)$ onto $\mathcal{H}_{\varphi\circ E_{\mathcal{N}}}$. ∎

The proof of the following lemma is an easy exercise.

LEMMA 4.3. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of finite von Neumann algebras, and $E_{\mathcal{N}}$ be the normal τ-preserving conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$. Suppose for $\alpha \in \Lambda$, $\mathcal{H}_{\varphi_\alpha}$ is the correspondence of $\mathcal{N}$ associated to a normal completely positive map φ_α from $\mathcal{N}$ to $\mathcal{N}$. Then* $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\bigoplus_{\alpha\in\Lambda}\mathcal{H}_{\varphi_\alpha}) \cong \bigoplus_{\alpha\in\Lambda}\mathcal{H}_{\varphi_\alpha\circ E_{\mathcal{N}}}$.

LEMMA 4.4. *If $\mathcal{N} \subset \mathcal{M}$ is an inclusion of finite von Neumann algebras and $\mathcal{H}$ is a C_0-correspondence of $\mathcal{N}$, then $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H})$ is a C_0-correspondence of $\mathcal{M}$.*

Proof. Let $E_{\mathcal{N}}$ be the τ-preserving normal conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$. Suppose $\mathcal{H} = \bigoplus_{\alpha\in\Lambda}\mathcal{H}_{\varphi_\alpha}$ such that T_{φ_α} is a compact operator in $\mathcal{B}(L^2(\mathcal{N}, \tau))$.

By Lemma 4.2 and Lemma 4.3 we have that $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\bigoplus_{\alpha\in\Lambda}\mathcal{H}_{\varphi_\alpha}) \cong \bigoplus_{\alpha\in\Lambda}\mathcal{H}_{\varphi_\alpha\circ E_{\mathcal{N}}}$. Since $T_{\varphi_\alpha\circ E_{\mathcal{N}}} = T_{\varphi_\alpha}e_{\mathcal{N}}$, the operator $T_{\varphi_\alpha\circ E_{\mathcal{N}}}$ is a compact operator in $\mathcal{B}(L^2(\mathcal{M},\tau))$. So $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H})$ is a C_0-correspondence of $\mathcal{M}$. ∎

Proof of Theorem 4.1. Let $\mathcal{H}$ be a C_0-correspondence of $\mathcal{N}$ that weakly contains the identity correspondence $L^2(\mathcal{N},\tau)$ of $\mathcal{N}$. By Lemma 4.4, $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H})$ is a C_0-correspondence of $\mathcal{M}$. Note that $L^2(\mathcal{N},\tau) \prec \mathcal{H}$. By the continuity of induction operation (see Proposition 2.2.1 of [12]), we see that

$$\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(L^2(\mathcal{N},\tau)) \prec \mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H}).$$

Since $\mathcal{N}\subset\mathcal{M}$ is an amenable inclusion, we have $L^2(\mathcal{M},\tau) \prec \mathcal{H}_{\mathcal{N}} = \mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(L^2(\mathcal{N},\tau))$. By the transitivity of $\prec$ we obtain

$$L^2(\mathcal{M},\tau) \prec \mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H}).$$

By Theorem 3.4, $\mathcal{M}$ has Haagerup's approximation property. ∎

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