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C^*-ALGEBRAS ASSOCIATED WITH ALGEBRAIC CORRESPONDENCES ON THE RIEMANN SPHERE

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ABSTRACT. Let $p(z, w)$ be a polynomial in two variables. We call the solution of the algebraic equation $p(z, w) = 0$ an algebraic correspondence. We regard it as the graph of the multivalued function $z \mapsto w$ defined implicitly by $p(z, w) = 0$. Algebraic correspondences on the Riemann sphere $\widehat{\mathbb{C}}$ generalize both Kleinian groups and rational functions. We introduce C^*-algebras associated with algebraic correspondences on the Riemann sphere. We show that if an algebraic correspondence is free and expansive on a closed p-invariant subset J of $\widehat{\mathbb{C}}$, then the associated C^*-algebra $\mathcal{O}_p(J)$ is simple and purely infinite.

KEYWORDS: *Algebraic correspondence, complex dynamical system, purely infinite C^*-algebra, Hilbert C^*-bimodule.*

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INTRODUCTION

For a branched covering $\pi : M \to M$, Deaconu and Muhly [9] introduced a C^*-algebra $C^*(M, \pi)$ as the C^*-algebra of the r-discrete groupoid constructed by Renault [27]. In order to capture information of the branched points for the complex dynamical system arising from a rational function R, in [14] we introduced slightly different C^*-algebras $\mathcal{O}_R(\widehat{\mathbb{C}})$, $\mathcal{O}_R = \mathcal{O}_R(J_R)$ and $\mathcal{O}_R(F_R)$ associated with a rational function R on the Riemann sphere $\widehat{\mathbb{C}}$, the Julia set J_R and the Fatou set F_R of R. We showed that the C^*-algebras $\mathcal{O}_R(J_R)$ on the Julia set are always simple and purely infinite if the degree of R is at least two. We also studied a relation between branched points and KMS states in [12]. C. Delaroche [1] and M. Laca–J. Spielberg [20] showed that a certain boundary action of a Kleinian group on the limit set yields a simple nuclear purely infinite C^*-algebra as a groupoid C^*-algebra or a crossed product. Dutkay and Jorgensen study a spectral theory on Hilbert spaces built on general finite-to-one maps ([8]).

On the other hand, Sullivan discovered a dictionary between the theory of complex analytic iteration and the theory of Kleinian groups in [29]. Sullivan's dictionary shows a strong analogy between the limit set Λ_Γ of a Kleinian group Γ and the Julia set J_R of a rational function R. Therefore it is natural to generalize both Kleinian groups and rational maps. In fact there exist such objects called algebraic correspondences or holomorphic correspondences. Many works on algebraic correspondences have been done, for example, in Bullet [4], Bullet–Penrose [5], [6] and Münzner–Rasch [24]. Let $p(z,w)$ be a polynomial in two variables. Then the solution of the algebraic equation $p(z,w) = 0$ is called an algebraic correspondence. We regard it as the graph of the multivalued function $z \mapsto w$ defined implicitly by $p(z,w) = 0$.

In this paper, we introduce C^*-algebras associated with algebraic correspondences on the Riemann sphere. We show that if an algebraic correspondence is free and expansive on a closed p-invariant subset J of $\widehat{\mathbb{C}}$, then the associated C^*-algebra $\mathcal{O}_p(J)$ is simple and purely infinite. We shall show some examples and compute the K-groups of the associated C^*-algebras. For example, let $p(z,w) = (w - z^{m_1})(w - z^{m_2}) \cdots (w - z^{m_r})$ such that $m_1, \dots, m_r$ are all different, where r is the number of irreducible components. Then $J := \mathbb{T}$ is a p-invariant set. Let $b = {}^{\#}B(p)$ be the number of the branched points. Then we have

$$K_0(\mathcal{O}_p(\mathbb{T})) = \mathbb{Z}^b, \quad \text{and} \quad K_1(\mathcal{O}_p(\mathbb{T})) = \mathbb{Z}/(r-1)\mathbb{Z}.$$

If $m_1, m_2, \dots, m_r$ are relatively prime, then the associated C^*-algebra $\mathcal{O}_p(\mathbb{T})$ is simple and purely infinite.

Our C^*-algebras $\mathcal{O}_p(J)$ are related with C^*-algebras of irreversible dynamical systems of Exel–Vershik [10], C^*-algebras associated with subshifts of Matsumoto [21], graph C^*-algebras [19] and their generalization for topological relations of Brenken [3], topological graphs of Katsura [16], and topological quivers of Muhly and Solel [22] and of Muhly and Tomforde [23]. Some of our C^*-algebras are isomorphic to C^*-algebras associated with self-similar sets [15] and Mauldin–Williams graphs [13].

1. ALGEBRAIC CORRESPONDENCES

Let $p(z,w) \in \mathbb{C}[z,w]$ be a polynomial in two variables of degree m in z and degree n in w. We shall study an algebraic function implicitly determined by the algebraic equation $p(z,w) = 0$ on the Riemann sphere $\widehat{\mathbb{C}}$. Note that there exist two different ways to compactify the algebraic curve $p(z,w) = 0$. The standard construction in algebraic geometry is to consider the zeros of a homogeneous polynomial $P(z,w,u)$ in the complex projective plane $\mathbb{C}P^2$. But we choose the second way after [5] and introduce four variables z_1, z_2, w_1, w_2 and a polynomial

$$\widetilde{p}(z_1, z_2, w_1, w_2) = z_2^m w_2^n p(z_1/z_2, w_1/w_2),$$

which is separately homogeneous in z_1, z_2 and in w_1, w_2. We identify the Riemann sphere $\widehat{\mathbb{C}}$ with the complex projective line $\mathbb{C}P^1$. We denote by $[z_1, z_2]$ an element of $\mathbb{C}P^1$. Then the *algebraic correspondence* $\mathcal{C}_p$ of $p(z,w)$ on the Riemann sphere is a closed subset of $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ defined by

$$\mathcal{C}_p := \{([z_1, z_2], [w_1, w_2]) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} : \widetilde{p}(z_1, z_2, w_1, w_2) = 0\}.$$

Then $\mathcal{C}_p$ is compact. In fact, it is a continuous image of a compact subset

$$\{(z_1, z_2, w_1, w_2) \in \mathbb{C}^4 : \widetilde{p}(z_1, z_2, w_1, w_2) = 0 \text{ and } |z_1|^2 + |z_2|^2 + |w_1|^2 + |w_2|^2 = 1\}.$$

To simplify notation, we write

$$\mathcal{C}_p = \{(z,w) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} : p(z,w) = 0\}$$

for short if no confusion can arise. It is also convenient to consider change of variables $u = \frac{1}{z}$ or $v = \frac{1}{w}$ instead.

For example, let $R(z) = \frac{P(z)}{Q(z)}$ be the rational function with polynomials $P(z)$, $Q(z)$. Put $p(z,w) = Q(z)w - P(z)$. Then the algebraic correspondence $\mathcal{C}_p$ of $p(z,w)$ on the Riemann sphere is exactly the following graph, of R,

$$\{(z,w) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} : w = R(z),\ z \in \widehat{\mathbb{C}}.\}$$

Therefore we regard the algebraic correspondence $\mathcal{C}_p$ of a general polynomial $p(z,w)$ as the graph of the algebraic function $z \mapsto w$ implicitly defined by the equation $p(z,w) = 0$. Then the iteration of the algebraic function is described naturally by a sequence $z_1, z_2, z_3, \dots$ satisfying $(z_k, z_{k+1}) \in \mathcal{C}_p$ for $k = 1, 2, 3, \dots$.

Any non-zero polynomial $p(z,w) \in \mathbb{C}[z,w]$ has a unique factorization into irreducible polynomials:

$$p(z,w) = g_1(z,w)^{n_1} \cdots g_p(z,w)^{n_p}$$

where each $g_i(z,w)$ is irreducible and g_i and g_j $(i \neq j)$ are prime to each other.

Throughout the paper, we assume that any polynomial $p(z,w)$ we consider is reduced, that is, the above powers $n_i = 1$ for any i. We also assume that any $g_i(z,w)$ is not a function only in z or w. In particular the degree m in z and the degree n in w of $p(z,w)$ are both greater than or equal to one.

We need to recall an elementary fact as follows:

DEFINITION 1.1. Let $p(z,w)$ be a non-zero polynomial in two variables of degree m in z and degree n in w. Then we sometimes rewrite $p(z,w)$ as

$$\begin{aligned} p(z,w) &= a_m(w)z^m + a_{m-1}(w)z^{m-1} + \cdots + a_1(w)z + a_0(w) \\ &= b_n(z)w^n + b_{n-1}(z)w^{n-1} + \cdots + b_1(z)w + b_0(z). \end{aligned}$$

Fix $w = w_0 \in \widehat{\mathbb{C}}$. Then the equation $f(z) := p(z, w_0) = 0$ in $z \in \widehat{\mathbb{C}}$ has m roots with multiplicities. Take any root $z = z_0$. The *branch index* of $p(z,w)$ at (z_0, w_0), denoted by $e_p(z_0, w_0)$ or $e(z_0, w_0)$, is defined to be the multiplicity for the root $z = z_0$ of $f(z) = p(z, w_0) = 0$. For example, let $R(z) = \frac{P(z)}{Q(z)}$ be the rational function with polynomials $P(z)$, $Q(z)$. Put $p(z,w) = Q(z)w - P(z)$. Then the branch index $e_p(z_0, R(z_0))$ coincides with the usual branch index $e_R(z_0)$ of R at $z = z_0$.

2. ASSOCIATED C^*-ALGEBRAS

We recall Cuntz–Pimsner algebras [25]. Let A be a C^*-algebra and X be a Hilbert right A-module. We denote by $L(X)$ the algebra of the adjointable bounded operators on X. For ξ, $\eta \in X$, the "rank one" operator $\theta_{\xi,\eta}$ is defined by $\theta_{\xi,\eta}(\zeta) = \xi(\eta|\zeta)$ for $\zeta \in X$. The closure of the linear span of rank one operators is denoted by $K(X)$. We say that X is a Hilbert C^*-bimodule (or C^*-correspondence) over A if X is a Hilbert right A-module with a homomorphism $\phi : A \to L(X)$. In this note, we assume that X is full and ϕ is injective.

Let $F(X) = \bigoplus_{n=0}^{\infty} X^{\otimes n}$ be the Fock module of X with the convention $X^{\otimes 0} = A$. For $\xi \in X$, the creation operator $T_\xi \in L(F(X))$ is defined by

$$T_\xi(a) = \xi a \quad \text{and} \quad T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

We define $i_{F(X)} : A \to L(F(X))$ by

$$i_{F(X)}(a)(b) = ab \quad \text{and} \quad i_{F(X)}(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \phi(a)\xi_1 \otimes \cdots \otimes \xi_n$$

for $a, b \in A$. The Cuntz–Toeplitz algebra $\mathcal{T}_X$ is the C^*-subalgebra of $L(F(X))$ generated by $i_{F(X)}(a)$ with $a \in A$ and T_ξ with $\xi \in X$. Let $j_K : K(X) \to \mathcal{T}_X$ be the homomorphism defined by $j_K(\theta_{\xi,\eta}) = T_\xi T_\eta^*$. We consider the ideal $I_X := \phi^{-1}(K(X))$ of A. Let $\mathcal{J}_X$ be the ideal of $\mathcal{T}_X$ generated by $\{i_{F(X)}(a) - (j_K \circ \phi)(a); a \in I_X\}$. Then the Cuntz–Pimsner algebra $\mathcal{O}_X$ is the quotient $\mathcal{T}_X / \mathcal{J}_X$. Let $\pi : \mathcal{T}_X \to \mathcal{O}_X$ be the quotient map. Put $S_\xi = \pi(T_\xi)$ and $i(a) = \pi(i_{F(X)}(a))$. Let $i_K : K(X) \to \mathcal{O}_X$ be the homomorphism defined by $i_K(\theta_{\xi,\eta}) = S_\xi S_\eta^*$. Then $\pi((j_K \circ \phi)(a)) = (i_K \circ \phi)(a)$ for $a \in I_X$. We note that the Cuntz–Pimsner algebra $\mathcal{O}_X$ is the universal C^*-algebra generated by $i(a)$ with $a \in A$ and S_ξ with $\xi \in X$ satisfying that $i(a)S_\xi = S_{\phi(a)\xi}$, $S_\xi i(a) = S_{\xi a}$, $S_\xi^* S_\eta = i((\xi|\eta)_A)$ for $a \in A$, $\xi, \eta \in X$ and $i(a) = (i_K \circ \phi)(a)$ for $a \in I_X$. We usually identify $i(a)$ with a in A. We denote by $\mathcal{O}_X^{\mathrm{alg}}$ the $*$-algebra generated algebraically by A and S_ξ with $\xi \in X$. There exists an action $\gamma : \mathbb{R} \to \operatorname{Aut} \mathcal{O}_X$ with $\gamma_t(S_\xi) = e^{it} S_\xi$, which is called the gauge action. Since we assume that $\phi : A \to L(X)$ is isometric, there is an embedding $\phi_n : L(X^{\otimes n}) \to L(X^{\otimes n+1})$ with $\phi_n(T) = T \otimes \mathrm{id}_X$ for $T \in L(X^{\otimes n})$ with the convention $\phi_0 = \phi : A \to L(X)$. We denote by $\mathcal{F}_X$ the C^*-algebra generated by all $K(X^{\otimes n})$, $n \geqslant 0$ in the inductive limit algebra $\varinjlim L(X^{\otimes n})$. Let $\mathcal{F}_n$ be the C^*-subalgebra of $\mathcal{F}_X$ generated by $K(X^{\otimes k})$, $k = 0, 1, \ldots, n$, with the convention $\mathcal{F}_0 = A = K(X^{\otimes 0})$. Then $\mathcal{F}_X = \varinjlim \mathcal{F}_n$.

Let $p(z,w)$ be a non-zero polynomial in two variables and $\mathcal{C}_p$ the algebraic correspondence of $p(z,w)$ on the Riemann sphere. Consider the C^*-algebra

$A = C(\widehat{\mathbb{C}})$ of continuous functions on $\widehat{\mathbb{C}}$. Let $X = C(\mathcal{C}_p)$. Then X is an A-A bimodule by

$$(a \cdot f \cdot b)(z,w) = a(z)f(z,w)b(w)$$

for $a, b \in A$ and $f \in X$. We introduce an A-valued inner product $(\cdot|\cdot)_A$ on X by

$$(f|g)_A(w) = \sum_{\{z\in\widehat{\mathbb{C}}:(z,w)\in\mathcal{C}_p\}} e_p(z,w)\overline{f(z,w)}g(z,w)$$

for $f, g \in X$ and $w \in \widehat{\mathbb{C}}$. We need the branch index $e_p(z,w)$ in the formula of the inner product above. Put $\|f\|_2 = \|(f|f)_A\|_\infty^{1/2}$.

LEMMA 2.1. *The above A-valued inner product is well defined, that is, $\widehat{\mathbb{C}} \ni w \mapsto (f|g)_A(w) \in \mathbb{C}$ is continuous.*

Proof. If we consider $p(z,w) = a_m(w)z^m + a_{m-1}(w)z^{m-1} + \cdots + a_1(w)z + a_0(w)$, as a polynomial in z, then each coefficient $(a_k(w))_k$ is continuous in w. Then the continuity of the map $\widehat{\mathbb{C}} \ni w \mapsto (f|g)_A(w) \in \mathbb{C}$ follows from the definition of the branch index and the continuity of the roots with multiplicities of a polynomial on the Riemann sphere. See, for example, [7]. ∎

The left multiplication of A on X gives the left action $\phi : A \to L(X)$ such that $(\phi(a)f)(z,w) = a(z)f(z,w)$ for $a \in A$ and $f \in X$.

PROPOSITION 2.2. *Let $p(z,w)$ be a non-zero polynomial in two variables. Then $X = C(\mathcal{C}_p)$ is a full Hilbert C^*-bimodule over $A = C(\widehat{\mathbb{C}})$ without completion. The left action $\phi : A \to L(X)$ is unital and faithful.*

Proof. Let m be the degree of $p(z,w)$ in z. For any $f \in X = C(\mathcal{C}_p)$, we have

$$\|f\|_\infty \leqslant \|f\|_2 := \Big(\sup_w \sum_{\{z\in\widehat{\mathbb{C}}:(z,w)\in\mathcal{C}_p\}} e_p(z,w)|f(z,w)|^2\Big)^{1/2} \leqslant \sqrt{m}\|f\|_\infty.$$

Therefore the two norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent. Since $C(\mathcal{C}_p)$ is complete with respect to $\|\cdot\|_\infty$, it is also complete with respect to $\|\cdot\|_2$.

Since $(1|1)_A(w) = \sum_{\{z\in\widehat{\mathbb{C}}:(z,w)\in\mathcal{C}_p\}} e_p(z,w)1 = m$, $(X|X)_A$ contains the identity I_A of A. Therefore X is full. If $a \in A$ is not zero, then there exists $x_0 \in \widehat{\mathbb{C}}$ with $a(x_0) \neq 0$. Since the degree m in z of $p(z,w)$ is greater than or equal to one, there exists $w_0 \in \widehat{\mathbb{C}}$ with $(x_0, w_0) \in \mathcal{C}_p$. Choose $f \in X$ with $f(x_0, w_0) \neq 0$. Then $\phi(a)f \neq 0$. Thus ϕ is faithful. ∎

DEFINITION 2.3. We introduce the C^*-algebra $\mathcal{O}_p(\widehat{\mathbb{C}})$ associated with an algebraic correspondence $\mathcal{C}_p = \{(z,w) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} : p(z,w) = 0\}$ as the Cuntz–Pimsner algebra [25] of the Hilbert C^*-bimodule $X_p = C(\mathcal{C}_p)$ over $A = C(\widehat{\mathbb{C}})$.

A closed subset J in $\widehat{\mathbb{C}}$ is said to be *p-invariant* if the following conditions are satisfied: For $z, w \in \widehat{\mathbb{C}}$,

(i) $z \in J$ and $p(z,w) = 0$ implies $w \in J$,
(ii) $w \in J$ and $p(z,w) = 0$ implies $z \in J$.

Under the condition, we can define $\mathcal{C}_p(J) = \{(z,w) \in J \times J : p(z,w) = 0\}$, $A = C(J)$, $X_p(J) = C(\mathcal{C}_p(J))$ similarly. Then $X_p(J)$ is a full Hilbert C^*-bimodule (C^*-correspondence) over $A = C(J)$ and the left action is unital and faithful.

We also introduce the C^*-algebra $\mathcal{O}_p(J)$ as the Cuntz–Pimsner algebra of the Hilbert C^*-bimodule $X_p(J) = C(\mathcal{C}_p(J))$ over $A = C(J)$.

We define the set $B(p)$ of "branched points" and the set $C(p)$ of "branched values":

$$B(p) := \{z \in \widehat{\mathbb{C}} : \text{there exists } w \in \widehat{\mathbb{C}} \text{ such that } p(z,w) = 0 \text{ and } e(z,w) \geqslant 2\};$$

$$C(p) := \{w \in \widehat{\mathbb{C}} : \text{there exists } z \in \widehat{\mathbb{C}} \text{ such that } p(z,w) = 0 \text{ and } e(z,w) \geqslant 2\}.$$

In the above definitions, we may replace $e(z,w) \geqslant 2$ by $\frac{\partial p}{\partial z}(z,w) = 0$ after appropriate change of variables. Symmetrically we define:

$$\widetilde{B}(p) := \Big\{w \in \widehat{\mathbb{C}} : \text{there exists } z \in \widehat{\mathbb{C}} \text{ such that } p(z,w) = 0 \text{ and } \frac{\partial p}{\partial w}(z,w) = 0\Big\};$$

$$\widetilde{C}(p) := \Big\{z \in \widehat{\mathbb{C}} : \text{there exists } w \in \widehat{\mathbb{C}} \text{ such that } p(z,w) = 0 \text{ and } \frac{\partial p}{\partial w}(z,w) = 0\Big\}.$$

We need some known estimates of the above sets.

LEMMA 2.4. *Let $p(z,w)$ be a non-zero polynomial in two variables of degree m in z and degree n in w. Then $B(p)$, $C(p)$, $\widetilde{B}(p)$ and $\widetilde{C}(p)$ are finite sets. More precisely we have ${}^\#B(p) \leqslant 2m(m-1)n$, ${}^\#C(p) \leqslant 2(m-1)n$, ${}^\#\widetilde{B}(p) \leqslant 2n(n-1)m$ and ${}^\#\widetilde{C}(p) \leqslant 2(n-1)m$.*

Proof. It follows from Proposition 2 in [5] that ${}^\#C(p) \leqslant 2(m-1)n$. Since $p(z,w)$ has degree m in z, we also have ${}^\#B(p) \leqslant 2m(m-1)n$. The rest is symmetrically obtained. ∎

Let $I_X = I_{X_p(J)} = \phi^{-1}(\phi(C(J)) \cap K(X_p(J)))$.

PROPOSITION 2.5. $I_{X_p(J)} = \{a \in C(J) : a|_{B(p)}\} = 0$.

The proof is a direct consequence of Proposition 4.4 in [12] or [23].

We consider Hilbert C^*-bimodules of iteration of the "algebraic function". Put $X_A^{\otimes 2} = X \otimes_A X$, $X_A^{\otimes n} = X^{\otimes n-1} \otimes_A X$.

We define the path space $\mathcal{P}_n = \mathcal{P}_n(J)$ of length n in J by

$$\mathcal{P}_n = \{(z_1, z_2, \ldots, z_{n+1}) \in J^{n+1} : (z_i, z_{i+1}) \in \mathcal{C}_p(J), i = 1, \ldots, n\}.$$

Then $\mathcal{P}_n$ is compact, since it is a continuous image of a compact subset. We extend the branched index for paths of length n as

$$e(z_1, z_2, \ldots, z_{n+1}) = e(z_1, z_2)e(z_2, z_3) \cdots e(z_n, z_{n+1}).$$

Then $C(\mathcal{P}_n)$ is a Hilbert bimodule over A by

$$(a \cdot f \cdot b)(z_1, z_2, \ldots, z_{n+1}) = a(z_1) f(z_1, z_2, \ldots, z_n) b(z_{n+1}),$$
$$(f|g)_A(w) = \sum_{\{(z_1,\ldots,z_n):(z_1,\ldots,z_n,w)\in\mathcal{P}_n\}} e(z_1, \ldots, z_n, w)\overline{f(z_1, \ldots, z_n, w)} g(z_1, \ldots, z_n, w),$$

for $a, b \in A$, $f, g \in C(\mathcal{P}_n)$.

LEMMA 2.6. *The above A-valued inner product is well defined, that is, $\widehat{\mathbb{C}} \ni w \mapsto (f|g)_A(w) \in \mathbb{C}$ is continuous, for any $f, g \in C(\mathcal{P}_n)$.*

Proof. It is enough to assume that $J = \widehat{\mathbb{C}}$. We may and do assume that $n = 2$, because a similar argument holds for general n. We already know that $X \otimes_A X$ has an $A = C(\widehat{\mathbb{C}})$-valued inner product. Therefore for $f_1 \otimes f_2, g_1 \otimes g_2 \in X \otimes_A X$, $\widehat{\mathbb{C}} \ni w \mapsto (f_1 \otimes f_2 | g_1 \otimes g_2)_A(w) \in \mathbb{C}$ is continuous. Define $f, g \in C(\mathcal{P}_2)$ by

$$f(z_1, z_2, w) = f_1(z_1, z_2) f_2(z_2, w), \quad g(z_1, z_2, w) = g_1(z_1, z_2) g_2(z_2, w).$$

Then

$$\begin{aligned}
(f|g)_A(w) &= \sum_{\{(z_1,z_2)\in\mathcal{P}_1:(z_1,z_2,w)\in\mathcal{P}_2\}} e(z_1, z_2, w)\overline{f(z_1, z_2, w)} g(z_1, z_2, w) \\
&= \sum_{\{(z_1,z_2)\in\mathcal{P}_1:(z_1,z_2,w)\in\mathcal{P}_2\}} e(z_1, z_2) e(z_2, w)\overline{f_1(z_1, z_2) f_2(z_2, w)} g_1(z_1, z_2) g_2(z_2, w) \\
&= \sum_{\{z_2\in\widehat{\mathbb{C}}:p(z_2,w)=0\}} e(z_2, w)\overline{f_2(z_2, w)} \Big(\sum_{\{z_1\in\widehat{\mathbb{C}}:p(z_1,z_2)=0\}} e(z_1, z_2)\overline{f_1(z_1, z_2)} g_1(z_1, z_2) g_2(z_2, w) \Big) \\
&= \sum_{\{z_2\in\widehat{\mathbb{C}}:p(z_2,w)=0\}} e(z_2, w)\overline{f_2(z_2, w)} (f_1|g_1)_A(z_2) g_2(z_2, w) \\
&= (f_2|(f_1|g_1)_A g_2)_A(w) = (f_1 \otimes f_2 | g_1 \otimes g_2)_A(w).
\end{aligned}$$

Hence $w \mapsto (f|g)_A(w)$ is continuous. Then for finite sums

$$f(z_1, z_2, w) = \sum_i f_{1,i}(z_1, z_2) f_{2,i}(z_2, w), \quad g(z_1, z_2, w) = \sum_i g_{1,i}(z_1, z_2) g_{2,i}(z_2, w),$$

$w \mapsto (f|g)_A(w)$ is also continuous. Put

$$C(\mathcal{P}_2)^0 = \Big\{ f \in C(\mathcal{P}_2) : f(z_1, z_2, w) = \sum_{\text{finite } i} f_{1,i}(z_1, z_2) f_{2,i}(z_2, w) \text{ for } f_{1,i}, f_{2,i} \in X \Big\}.$$

Since $C(\mathcal{P}_2)^0$ is a $*$-subalgebra of $C(\mathcal{P}_2)$ and separates points, $C(\mathcal{P}_2)^0$ is uniformly dense in $C(\mathcal{P}_2)$. Note that the uniform norm $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent, because $\|\cdot\|_\infty \leqslant \|\cdot\|_2 \leqslant m^{n/2}\|\cdot\|_\infty$. For any $f, g \in C(\mathcal{P}_2)$, there exist sequences $(f_n)_n$ and $(g_n)_n$ in $C(\mathcal{P}_2)^0$ such that $f_n \to f$ and $g_n \to g$ uniformly. Since

$$\begin{aligned}
&|(f|g)_A(w) - (f_n|g_n)_A(w)| \\
&\leqslant \sum_{\{(z_1,z_2):(z_1,z_2,w)\in\mathcal{P}_2\}} e(z_1, z_2, w)|\overline{f(z_1, z_2, w)} g(z_1, z_2, w) - \overline{f_n(z_1, z_2, w)} g_n(z_1, z_2, w)|,
\end{aligned}$$

we see that $(f_n|g_n)_A(w)$ converges to $(f|g)_A(w)$ uniformly in w. Since a uniform limit of continuous functions is continuous, $(f|g)_A(w)$ is continuous in w. ∎

Now it is easy to check the following proposition.

PROPOSITION 2.7. *Let $p(z,w)$ be a non-zero polynomial in two variables. Then $X = C(\mathcal{P}_n)$ is a full Hilbert bimodule over $A = C(J)$ without completion. The left action $\phi : A \to L(X)$ is unital and faithful.*

PROPOSITION 2.8. *There exists an isometric A-A bimodule homomorphism $\varphi : X_A^{\otimes n} \to C(\mathcal{P}_n)$ such that, for $f_1, \dots, f_n \in X$,*

$$\varphi(f_1 \otimes f_2 \otimes \cdots \otimes f_n)(z_1, z_2, \dots, z_{n+1}) = f_1(z_1, z_2) f_2(z_2, z_3) \cdots f_n(z_n, z_{n+1}).$$

Proof. It is easy to check that φ is a well defined A-A bimodule map. The proof of Lemma 2.6 shows that φ is isometric. Since $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent, φ is onto. ∎

We need to define another compact space $\mathcal{G}_n = \mathcal{G}_n(J)$ by

$$\mathcal{G}_n = \{(z_1, z_{n+1}) \in J^2 : \text{ there exists } (z_1, z_2, \dots, z_{n+1}) \in \mathcal{P}_n\}.$$

Then $C(\mathcal{G}_n)$ is a Hilbert C^*-bimodule over A by:

$$(a \cdot f \cdot b)(z_1, z_{n+1}) = a(z_1) f(z_1, z_n) b(z_{n+1}),$$
$$(f|g)_A(w) = \sum_{\{(z_1, z_2, \dots, z_n) : (z_1, z_2, \dots, z_n, w) \in \mathcal{P}_n\}} e(z_1, z_2, \dots, z_n, w) \overline{f(z_1, w)} g(z_1, w),$$

for a, $b \in A$, f, $g \in C(\mathcal{G}_n)$. Define a continuous onto map $\rho : \mathcal{P}_n \to \mathcal{G}_n$ by $\rho((z_1, z_2, \dots, z_{n+1})) = (z_1, z_{n+1})$ for $(z_1, z_2, \dots, z_{n+1}) \in \mathcal{P}_n$. Then it is clear that the induced map $\rho^* : C(\mathcal{G}_n) \to C(\mathcal{P}_n)$ defined by $\rho^*(f) = f \circ \rho$ is an isometric Hilbert bimodule embedding.

3. SIMPLICITY AND PURE INFINITENESS

In this section we consider a sufficient condition for a polynomial so that the associated $\mathcal{O}_p(J)$ is simple and purely infinite.

Let J be a p-invariant subset of $\widehat{\mathbb{C}}$. For any subset U of J and a natural number n, we define a subset $U^{(n)}$ of J by

$$U^{(n)} = \{w \in J : (z_1, z_2, \dots, z_n, w) \in \mathcal{P}_n \text{ for some } z_1 \in U, z_2, \dots, z_n \in J\}.$$

DEFINITION 3.1. Let $p(z,w)$ be a non-zero polynomial in two variables and J a p-invariant subset of $\widehat{\mathbb{C}}$. Then p is said to be *expansive* on J if for any nonempty open set $U \subset J$ in J with the relative topology there exists a natural number n such that $U^{(n)} = J$.

EXAMPLE 3.2. Let $R(z) = \frac{P(z)}{Q(z)}$ be the rational function with polynomials $P(z)$, $Q(z)$ and $\deg R \geqslant 2$. Put $p(z,w) = Q(z)w - P(z)$. Then $U^{(n)}$ is exactly $R^n(U)$. Therefore p is expansive on the Julia set J_R by Theorem 4.2.5 of [2].

EXAMPLE 3.3. Let $p(z,w) = z^2 + w^2 - 1$. Then $J := \{0,1,-1\}$ is a p-invariant set and p is not expansive on J. In fact, let $U = \{0\}$, then $U^{(2n)} = \{0\}$ and $U^{(2n+1)} = \{1,-1\}$. p is not expansive on $\widehat{\mathbb{C}}$, because an open set $U := \widehat{\mathbb{C}} \setminus \{0,1,-1\}$ is p-invariant and $U^{(n)} = U \neq \widehat{\mathbb{C}}$ for any n. In general, for any polynomial, if p has a finite p-invariant set, then p is not expansive on $\widehat{\mathbb{C}}$ similarly.

EXAMPLE 3.4. Let $p(z,w) = z^m - w$, $m \geqslant 2$. Then $J := \mathbb{T}$ is a p-invariant set and p is expansive on J, because $\mathbb{T}$ is a Julia set of $w = R(z) = z^m$.

Let $p(z,w) = z - w^n$, $n \geqslant 2$. Then $J := \mathbb{T}$ is a p-invariant set but p is not expansive on J. In fact, let $U := \widehat{\mathbb{T}} \setminus \{1\}$. Then $U^{(k)} = U \neq \widehat{\mathbb{T}}$ for any k.

More generally we have the following criterion.

PROPOSITION 3.5. *Let $p(z,w) = z^m - w^n$ for natural numbers m and n. Then $J := \mathbb{T}$ is a p-invariant set, and p is expansive on $\mathbb{T}$ if and only if n is not divisible by m.*

Proof. Suppose that n is divisible by m, so that $n = mj$ for some $j \in \mathbb{N}$. Let $U = \{z \in \mathbb{T} : z^m \neq 1\}$. Then for any $k \in \mathbb{N}$, 1 is not in $U^{(k)}$. In fact, if 1 were in $U^{(k)}$, then there exists $(z_1, z_2, \dots, z_k, 1) \in \mathcal{P}_k$ such that $z_1 \in U$. Hence $z_k^m = 1$ and $z_{k-1}^m = z_k^n = z_k^{mj} = 1$. We continue this argument to obtain $z_1^m = 1$. This contradicts the fact that $z_1 \in U$. Therefore p is not expansive on $\mathbb{T}$.

Next, suppose that n is not divisible by m. Let d be the greatest common divisor of m and n. Then $m = m_0 d$ and $n = n_0 d$ for some natural numbers m_0 and n_0. Since n is not divisible by m, m_0 is greater than or equal to 2. We identify $\mathbb{T}$ with $\mathbb{R}$ (mod $\mathbb{Z}$) by $z = e^{2\pi i\alpha}$ and $w = e^{2\pi i\beta}$. Then $z^m - w^n = 0$ means that $m\alpha = n\beta - k$ for some integer k. Hence

$$\mathcal{C}_p(J) \cong \Big\{([\alpha],[\beta]) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} : \beta = \frac{m}{n}\alpha + \frac{k}{n} \text{ for some integer } k\Big\}.$$

Then $\mathcal{C}_p(J)$ has d connected components, because $m\mathbb{Z} = d\mathbb{Z}$ (mod n).

For $([\alpha],[\beta]) \in \mathcal{G}_2(J)$, there exist $k_1, k_2 \in \mathbb{Z}$ such that

$$\beta = \frac{m}{n}\Big(\frac{m}{n}\alpha + \frac{k_1}{n}\Big) + \frac{k_2}{n} = \frac{m^2}{n^2}\alpha + \frac{mk_1 + nk_2}{n^2}.$$

Since $m\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z}$, $([\alpha],[\beta]) \in \mathcal{G}_2(J)$ if and only if there exists $k \in \mathbb{Z}$ such that $\beta = \frac{m^2}{n^2}\alpha + \frac{dk}{n^2}$. We continue in this way to get that $([\alpha],[\beta]) \in \mathcal{G}_r(J)$ if and only if there exists $k \in \mathbb{Z}$ such that $\beta = \frac{m^r}{n^r}\alpha + \frac{d^{r-1}k}{n^r}$. Since $m^r\mathbb{Z} + n^r\mathbb{Z} = d^r\mathbb{Z}$, $\mathcal{G}_r(J)$ has $\frac{d^n}{d^{n-1}} = d$ connected components. To avoid overlapping, we consider only one connected component. Hence we need to cover an interval $[0, \frac{d^{r-1}}{n^r}d] = [0, \frac{d^r}{n^r}]$.

Take an open interval $I = (0, \frac{1}{m_0^r} + \varepsilon)$. Since $\frac{m^r}{n^r}\frac{1}{m_0} = \frac{d^r}{n^r}$, $I^{(r)}$ contains

$$(0, \frac{d^r}{n^r}] \cup (\frac{d^r}{n^r}, \frac{2d^r}{n^r}] \cup \cdots \cup (\frac{(n_0^r - 1)d^r}{n^r}, \frac{n_0^r d^r}{n^r}] \cup \{0\} = [0,1] \ (\mathrm{mod}\ \mathbb{Z}).$$

It is also true if we replace I by a translation of I. Now for any open set $U \subset J$, there exists an open interval (a,b) with $(a,b) \subset U$. Choose a natural number r such that $|b-a| > \frac{1}{m_0^r}$. Then by the preceding argument we see that $(a,b)^{(r)} = [0,1]$. Hence $U^{(r)} = [0,1] (\mathrm{mod}\ \mathbb{Z}) = J$. This shows that p is expansive on $\mathbb{T}$. ∎

DEFINITION 3.6. Let N be a natural number. We define the set $\mathrm{GP}(N)$ of *N-generalized periodic points* by

$$\begin{aligned}\mathrm{GP}(N) = \{ & w \in J : \exists z \in J\ \exists m,n\ 0 \leqslant m \neq n \leqslant N, \exists (z, z_2, z_3, \dots, z_n, w) \in \mathcal{P}_n, \\ & \exists (z, u_2, u_3, \dots, u_m, w) \in \mathcal{P}_m \}.\end{aligned}$$

Let $R(z) = \frac{P(z)}{Q(z)}$ be the rational function with polynomials $P(z), Q(z)$. Put $p(z,w) = Q(z)w - P(z)$. Then

$$\mathrm{GP}(N) = \bigcup_{n=1}^{N} \{ w \in \widehat{\mathbb{C}} : R^n(w) = w \}.$$

In fact, if $R^n(w) = w$ for some $n \leqslant N$, then it is clear that $w \in \mathrm{GP}(N)$. Conversely, let $w \in \mathrm{GP}(N)$. Then there exists z such that $w = R^n(z) = R^m(z)$ for some $0 \leqslant m < n \leqslant N$. Then $R^{n-m}(w) = w$.

DEFINITION 3.7. A polynomial p in two variables is said to be *free* on J if for any natural number N, $\mathrm{GP}(N)$ is a finite set.

For example, let $R(z) = \frac{P(z)}{Q(z)}$ be the rational function with polynomials $P(z)$, $Q(z)$. Put $p(z,w) = Q(z)w - P(z)$. If $\deg R \geqslant 2$, then p is free on any p-invariant set J.

LEMMA 3.8. *Let $p(z,w) = z^m - w^n$. Then p is free on $J = \mathbb{T}$ if and only if $m \neq n$.*

Proof. Assume that $m \neq n$. We identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$ by $z = e^{2\pi i \alpha}$ and $w = e^{2\pi i \beta}$. For any natural number N, $[\beta] \in \mathrm{GP}(N)$ if and only if there exist $[\alpha] \in \mathbb{R}/\mathbb{Z}$ and $r,s \quad 0 \leqslant r \neq s \leqslant N$ such that $([\alpha],[\beta]) \in \mathcal{G}_r(J)$ and $([\alpha],[\beta]) \in \mathcal{G}_s(J)$. Therefore there exist $k_1, k_2 \in \mathbb{Z}$ with $0 \leqslant k_1 \leqslant n^r - 1$ and $0 \leqslant k_2 \leqslant n^s - 1$ such that

$$\beta = \frac{m^r}{n^r}\alpha + \frac{d^{r-1}k_1}{n^r} = \frac{m^s}{n^s}\alpha + \frac{d^{s-1}k_2}{n^s}.$$

Since two lines with different slopes meet at at most one point, $^{\#}\mathrm{GP}(N) \leqslant n^{3N}$. Hence p is free on $\mathbb{T}$.

Conversely assume that $m = n$. Then any $(z, z, \dots, z) \in J^{k+1}$ is in $\mathcal{P}_k(J)$. Hence for any natural number N, $\mathrm{GP}(N) = \mathbb{T}$ is an infinite set. Thus p is not free on $\mathbb{T}$. ∎

REMARK 3.9. The above example is related with an example of Katsura in Section 4 of [17] . If m and n are relatively prime, then his example coincides with our example. If m and n are not relatively prime, then his example is different from ours. In fact our $\mathcal{P}_n(\mathbb{T})$ is not connected if m and n are not relatively prime. But they are isomorphic as bimodules.

PROPOSITION 3.10. *Let* $R_i(z) = P_i(z)/Q_i(z)$ $i = 1, \ldots, r$ *be rational functions with polynomials* $P_i(z)$, $Q_i(z)$. *Put* $p(z,w) = (Q_1(z)w - P_1(z)) \cdots (Q_r(z)w - P_r(z))$. *Let* $J \subset \widehat{\mathbb{C}}$ *be a p-invariant closed subset. Assume that each* $\deg R_i \geqslant 2$ *and* $\deg R_1, \ldots, \deg R_r$ *are relatively prime. Then p is free on J. Furthermore, if J is the Julia set for some* R_i, *then p is expansive on J.*

Proof. Let N be a natural number and m, n integers with $0 \leqslant n < m \leqslant N$. For $i_1, \ldots, i_m, j_1, \ldots, j_n = 1, 2, \ldots, r$, we shall show that

$$M :=^{\#} \{z \in \widehat{\mathbb{C}} \ : \ R_{i_m} \circ \cdots \circ R_{i_1}(z) = R_{j_n} \circ \cdots \circ R_{j_1}(z)\} < \infty.$$

On the contrary, assume that $M = \infty$. Then covering degrees of both sides coincide. Count the covering degrees and rearrange them. Then we have

$$(\deg R_1)^{s_1} \cdots (\deg R_r)^{s_r} = (\deg R_1)^{t_1} \cdots (\deg R_r)^{t_r}$$

with $s_1 + \cdots + s_r = m$ and $t_1 + \cdots + t_r = n$. Since $\deg R_1, \ldots, \deg R_r$ are relatively prime, $s_i = t_i$ for $i = 1, \ldots, r$. Then $m = n$. This contradicts the fact that $n < m$. Hence $M < \infty$. Therefore $Q(m,n) := \{z \in \widehat{\mathbb{C}} :$ there exists $w \in \widehat{\mathbb{C}}$ such that $(z,w) \in \mathcal{G}_m$, $(z,w) \in \mathcal{G}_n\}$ is a finite set. Hence

$$\mathrm{GP}(N) = \{w \in J : \exists z \in J \ \exists m, n \quad 0 \leqslant n < m \leqslant N, \exists (z,w) \in \mathcal{G}_m, \exists (z,w) \in \mathcal{G}_n\}$$

is also a finite set. This shows that p is free on J.

It is evident that, if J is a Julia set for some R_i, then p is expansive on J. ∎

EXAMPLE 3.11. Let m and n be natural numbers and relatively prime. Consider $p(z,w) = (w - z^m)(w - z^n)$. We note that $J = \mathbb{T}$ is the common Julia set of $w = z^m$ and $w = z^n$. Then p is free on J and expansive on J. We note that there appears a new branched point $(1,1)$ in $\mathcal{C}_p$.

EXAMPLE 3.12. Let $R_1(z) = \frac{(z^2+1)^2}{4z(z^2-1)}$ be the rational function given by Lattes. Then the Julia set $J_{R_1} = \widehat{\mathbb{C}}$. Let $R_2(z) = \frac{P_2(z)}{Q_2(z)}$ be any rational function with odd degree. Put $p(z,w) = ((4z(z^2-1))w - (z^2+1)^2)(Q_2(z)w - P_2(z))$. Let $J = \widehat{\mathbb{C}}$. Then p is expansive on J and free on J.

EXAMPLE 3.13. Let $i_1, \ldots, i_n$, $j_1, \ldots, j_n$ be natural numbers. Assume that $i_k \neq 1$ or $j_k \neq 1$ for each k. Suppose that those which are not equal to 1 are relatively prime. Put $J = \mathbb{T}$. Let

$$p(z,w) = (z^{i_1} - w^{j_1})(z^{i_2} - w^{j_2}) \cdots (z^{i_n} - w^{j_n}).$$

Then p is free on J.

EXAMPLE 3.14. Let m be a natural number with $m \geqslant 2$. Put $p(z,w) = (w - z^m)(w^m - z)$. Then p is not free on $\mathbb{T}$. In fact, there exist different paths $(z, z^m, z, z^m, z) \in \mathcal{P}_4(\mathbb{T})$ and $(z, z^m, z) \in \mathcal{P}_2(\mathbb{T})$. Hence $\mathrm{GP}(4) = \mathbb{T}$.

EXAMPLE 3.15. Let $p(z,w) = z^2 + w^2 - 1$. Then p is not free on $J = \widehat{\mathbb{C}}$. In fact, choose any $(z,w) \in \mathcal{C}_p$. Then there exist different paths $(z,w,z,w,z) \in \mathcal{P}_4(\widehat{\mathbb{C}})$ and $(z,w,z) \in \mathcal{P}_2(\widehat{\mathbb{C}})$. Hence $\mathrm{GP}(4) = \widehat{\mathbb{C}}$.

LEMMA 3.16. *Suppose that p is expansive on a p-invariant subset J. Then for any non-zero positive element $a \in A$ and for any $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $f \in X^{\otimes n}$ with $(f|f)_A = I$ such that*

$$\|a\| - \varepsilon \leqslant S_f^* a S_f \leqslant \|a\|.$$

Proof. Let x_0 be a point in J with $|a(x_0)| = \|a\|$. For any $\varepsilon > 0$ there exist an open neighbourhood U of x_0 in J such that for any $x \in U$ we have $\|a\| - \varepsilon \leqslant a(x) \leqslant \|a\|$. Choose another open neighbourhood V of x_0 in J and a compact subset $K \subset J$ satisfying $V \subset K \subset U$. Since p is expansive on J, there exists $n \in \mathbb{N}$ such that $V^{(n)} = J$. We identify $X^{\otimes n}$ with $C(\mathcal{P}_n) \supset \rho^*(C(\mathcal{G}_n))$ as in the paragraph after Proposition 2.8. Define closed subsets F_1 and F_2 of $J \times J$ by

$$F_1 = \{(z,w) \in J \times J : (z,w) \in \mathcal{G}_n, z \in K\},$$
$$F_2 = \{(z,w) \in J \times J : (z,w) \in \mathcal{G}_n, z \in U^c\}.$$

Since $F_1 \cap F_2 = \phi$, there exists $g \in C(\mathcal{G}_n)$ such that $0 \leqslant g(z,w) \leqslant 1$ and

$$g(z,w) = \begin{cases} 1 & (z,w) \in F_1, \\ 0 & (z,w) \in F_2. \end{cases}$$

Since $V^{(n)} = J$, for any $w \in J$ there exists $z_1 \in V$ such that $(z_1, w) \in \mathcal{G}_n$. Then $(z_1, w) \in F_1$ and $g(z_1, w) = 1$. Therefore

$$\begin{aligned}(\rho^*(g)|\rho^*(g))_A(w) &= \sum_{\{(z_1,\dots,z_n)\in\mathcal{P}_{n-1}:(z_1,\dots,z_n,w)\in\mathcal{P}_n\}} e(z_1,\dots,z_n,w)|g(z_1,w)|^2 \\ &\geqslant |g(z_1,w)|^2 = 1.\end{aligned}$$

Let $b := (\rho^*(g)|\rho^*(g))_A$. Then $b(y) = (\rho^*(g)|\rho^*(g))_A(y) \geqslant 1$. Thus $b \in A$ is positive and invertible. We put $f := \rho^*(g) b^{-1/2} \in X^{\otimes n}$. Then

$$(f|f)_A = b^{-1/2}(g|g)_A b^{-1/2} = I.$$

For any $w \in J$ and $(z_1, w) \in \mathcal{G}_n$, if $z \in U$, then $\|a\| - \varepsilon \leqslant a(z)$, and if $z \in U^c$, then $f(z_1, \dots, z_n, w) = g(x,y) b^{-1/2}(w) = 0$. Therefore

$$\begin{aligned}\|a\| - \varepsilon &= (\|a\| - \varepsilon)(f|f)_A(y) \\ &= (\|a\| - \varepsilon) \sum_{\{(z_1,\dots,z_n)\in\mathcal{P}_{n-1}:(z_1,\dots,z_n,w)\in\mathcal{P}_n\}} e(z_1,\dots,z_n,w)|f(z_1,\dots,z_n,w)|^2\end{aligned}$$

$$\leqslant \sum_{\{(z_1,\ldots,z_n)\in\mathcal{P}_{n-1}:(z_1,\ldots,z_n,w)\in\mathcal{P}_n\}} e(z_1,\ldots,z_n,w)a(z_1)|f(z_1,\ldots,z_n,w)|^2$$
$$= (f|af)_A(w) = S_f^*aS_f(w).$$

It is clear that $S_f^*aS_f = (f|af)_A \leqslant \|a\|(f|f)_A = \|a\|$. ∎

LEMMA 3.17. *Suppose that p is expansive on a p-invariant subset J. Then for any non-zero positive element $a \in A$ and for any $\varepsilon > 0$ with $0 < \varepsilon < \|a\|$, there exist $n \in \mathbb{N}$ and $u \in X^{\otimes n}$ such that*

$$\|u\|_2 \leqslant (\|a\| - \varepsilon)^{-1/2} \quad \textit{and} \quad S_u^*aS_u = I.$$

The proof is exactly as same as Lemma 3.5 of [14].

A step in the proof of the main theorem is to show that a certain element S in a Cuntz–Pimsner algebra is 0. It is enough to show that the corresponding element T in the Toeplitz algebra is 0. Since the Toeplitz algebra acts on the Fock module and the Fock module is realized as a function space, we can calculate $Tx = 0$ concretely.

We write $A = X^{\otimes 0}$. If $a \in A$, then T_a means $\phi(a) \otimes I_n$ on $X^{\otimes n}$. The following lemma is a key of the proof of our main theorem.

LEMMA 3.18. *Let i and j be integers with $i, j \geqslant 0$ and $i \neq j$. Take $x \in X^{\otimes i}$ and $y \in X^{\otimes j}$. Suppose that $a \in A = C(J)$ satisfies the following condition: For any $(z_1, z_2, \ldots, z_i, w) \in \mathcal{P}_i$, $(u_1, u_2, \ldots, u_j, w) \in \mathcal{P}_j$, we have $a(z_1)a(u_1) = 0$. Then we have $aT_xT_y^*a^* = 0$.*

Proof. It is enough to show $T_{ax}T_{ay}^*f = 0$ for any $f \in X^{\otimes r}$, $r = 0, 1, 2, \ldots$. If $r < j$, then $T_{ax}T_{ay}^*f = T_{ax}0 = 0$. Hence we may assume that $r \geqslant j$ and $f = f_1 \otimes f_2$ for $f_1 \in X^{\otimes j}$, $f_2 \in X^{\otimes(r-j)}$.

$$\begin{aligned}
&(T_{ax}T_{ay}^*)(f_1 \otimes f_2)(z_1, z_2, \ldots, z_i, z_{i+1}, \ldots, z_{i+r-j+1}) \\
&= (T_{ax}(ay|f_1)_A f_2)(z_1, z_2, \ldots, z_i, z_{i+1}, \ldots, z_{i+r-j+1}) \\
&= (ax \otimes (ay|f_1)_A f_2)(z_1, z_2, \ldots, z_i, z_{i+1}, \ldots, z_{i+r-j+1}) \\
&= a(z_1)x(z_1, \ldots, z_{i+1})(ay|f_1)_A(z_{i+1})f_2(z_{i+1}, \ldots, z_{i+r-j+1}) \\
&= a(z_1)x(z_1, \ldots, z_{i+1}) \cdot \\
&\qquad \Big(\sum_{(u_1,\ldots,u_j,z_{i+1})\in\mathcal{P}_j} e(u_1, \ldots, u_j, z_{i+1})\overline{a(u_1)y(u_1, \ldots, u_j, z_{i+1})}f_1(u_1, \ldots, u_j, z_{i+1}) \Big) \cdot \\
&\qquad f_2(z_{i+1}, \ldots, z_{i+r-j+1}) \\
&= a(z_1)\overline{a(u_1)}x(z_1, \ldots, z_{i+1}) \cdot \\
&\qquad \Big(\sum_{(u_1,\ldots,u_j,z_{i+1})\in\mathcal{P}_j} e(u_1, \ldots, u_j, z_{i+1})\overline{y(u_1, \ldots, u_j, z_{i+1})}f_1(u_1, \ldots, u_j, z_{i+1}) \Big) \cdot \\
&\qquad f_2(z_{i+1}, \ldots, z_{i+r-j+1}) = 0. \quad ∎
\end{aligned}$$

We need to prepare the following elementary fact:

LEMMA 3.19. *Suppose that* $p(z_0, w_0) = 0$, $\frac{\partial p}{\partial z}(z_0, w_0) \neq 0$ *and* $\frac{\partial p}{\partial w}(z_0, w_0) \neq 0$. *Then there exist an open neighbourhood* U *of* z_0, *an open neighbourhood* V *of* w_0 *and a homeomorphism* $\varphi : U \to V$ *such that for any* $z \in U$, $w \in V$, $p(z, w) = 0$ *if and only if* $w = \varphi(z)$.

LEMMA 3.20. *Assume that* p *is free on* J. *Suppose that* J *has no isolated points. Let* N *be a natural number. Then for any non-empty open set* U *in* J, *there exist points* $w_0 \in U, z_i \in J$ $(i = 1, \ldots, m^N)$, *an open neighbourhood* V *of* w_0 *with* $V \subset U$, *open neighbourhoods* W_i *of* z_i *in* J *and homeomorphisms* $\Phi_i : W_i \to V$ *for* $i = 1, \ldots, m^N$ *satisfying the following:*

(i) $W_i \cap W_j =$ *for* $i \neq j$.

(ii) *For any* $z \in W_i$, $w \in V$, *we have* $(z, w) \in \mathcal{G}_N$ *if and only if* $w = \Phi_i(z)$, *in particular* $w_0 = \Phi_i(z_i)$.

(iii) *For any* $s \in J$ *with* $(z_i, s) \in \mathcal{G}_k$ *for some* k $(1 \leqslant k \leqslant N)$, *there exist an open neighbourhood* $W_{i,s}$ *of* s *and homeomorphisms* $\Phi_{i,s} : W_i \to W_{i,s}$ *satisfying the following: for any* $z \in W_i$, $w \in W_{i,s}$, *we have* $(z, w) \in \mathcal{G}_k$ *if and only if* $w = \Phi_{i,s}(z)$.

(iv) *These open neighbourhoods* W_i *and* $W_{i,s}$ *for* i, s *have empty intersection each other.*

Proof. Let D_1 be the set of $w \in J$ satisfying that there exist $u \in J$, $z \in \mathrm{GP}(N)$ such that $(u, w) \in \mathcal{G}_N$, $(u, z) \in \mathcal{G}_k$ for some $k = 0, 1, \ldots, N$.

Since p is free on J, $\mathrm{GP}(N)$ is a finite set. Hence D_1 is also a finite set. Consider the set D_2 of $w \in J$ satisfying that there exist $u, z \in J$ such that $(u, w) \in \mathcal{G}_N$, $(u, z) \in \mathcal{G}_k$ for some $k = 0, 1, \ldots, N$ and z is in $B(p)$, $C(p)$, $\widetilde{B}(p)$ or $\widetilde{C}(p)$. Then D_2 is a finite set. Since $D_1 \cup D_2$ is a finite set and J has no isolated points, there exist a non-empty open set $V_0 \subset U$ such that $V_0 \subset U \setminus (D_1 \cup D_2)$. Choose $w_0 \in V_0 \subset U \setminus (D_1 \cup D_2)$. There exist distinct $z_i \in J$ for $i = 1, \ldots, m^N$ such that $(z_i, w_0) \in \mathcal{G}_N$. By the Lemma 3.19 , we can choose a sufficiently small non-empty open set $V \subset V_0$, non-empty open neighbourhoods W_i of z_i and homeomorphisms $\Phi_i : W_i \to V$ for $i = 1, \ldots, m^N$ satisfying all the above requirements. ∎

PROPOSITION 3.21. *Let* J *be a* p*-invariant set with no isolated points. Suppose that* p *is expansive and free on* J. *For* $N \in \mathbb{N}$, *for any* $T \in L(X^{\otimes N})$, *for any* $\varepsilon > 0$, *there exists* $a \in A^+ = C(J)^+$ *with* $\|a\| = 1$ *such that*

$$\|\phi(a)T\|^2 \geqslant \|T\|^2 - \varepsilon,$$
$$aS_x S_y^* a = 0 \quad \text{for any } x \in X^{\otimes i}, \text{for any } y \in X^{\otimes j}, 0 \leqslant i, j \leqslant N, i \neq j.$$

Proof. For $N \in \mathbb{N}$, for any $T \in L(X^{\otimes N})$, for any $\varepsilon > 0$, there exists $f \in X^{\otimes N}$ with $\|f\|_2 = 1$ such that $\|T\|^2 \geqslant \|Tf\|_2^2 > \|T\|^2 - \varepsilon$. Hence there exists $w_1 \in J$ such that

$$\|Tf\|_2^2 = \sum_{\{(z_1,\ldots,z_N):(z_1,\ldots,z_N,w_1)\in\mathcal{P}_N\}} e(z_1, \ldots, z_N, w_1)|(Tf)(z_1, \ldots, z_N, w_1)|^2.$$

Since the function

$$w \mapsto \sum_{\{(z_1,z_2,\dots,z_N):(z_1,z_2,\dots,z_N,w)\in\mathcal{P}_N\}} e(z_1,z_2,\dots,z_N,w)|(Tf)(z_1,\dots,w)|^2$$

is continuous, there exists an open neighbourhood U of w_1 such that for any $w \in U$

$$\sum_{\{(z_1,\dots,z_N):(z_1,\dots,z_N,w)\in\mathcal{P}_N\}} e(z_1,\dots,z_N,w)|(Tf)(z_1,\dots,z_N,w)|^2 > \|T\|^2 - \varepsilon.$$

By Propostion 3.21, there exist points $w_0 \in U, z_i \in J$ $(i = 1,\dots,m^N)$, an open neighbourhood V of w_0 with $V \subset U$, open neighbourhoods W_i of z_i in J and homeomorphisms $\Phi_i : W_i \to V$ for $i = 1,\dots,m^N$ satisfying the conditions in the lemma. Choose $b \in A = C(J)$ satisfying

$$b(w_0) = 1, \quad 0 \leqslant b(w) \leqslant 1, \quad \operatorname{supp} b \subset V.$$

Define $a \in C(J)$ by

$$a(z) = \begin{cases} b(\Phi_i(z)) & z \in W_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then this function a satisfies the condition in Lemma 3.18. Therefore for any $x \in X^{\otimes i}, y \in X^{\otimes j}, 0 \leqslant i,j \leqslant N, i \neq j$ we have $aS_xS_y^*a^* = 0$.

Moreover we have

$$\begin{aligned}
\|\phi(a)Tf\|_2^2 &= \sup_w \sum_{\{(z_1,\dots,z_N):(z_1,\dots,z_N,w)\in\mathcal{P}_N\}} e(z_1,\dots,z_N,w)|a(z)(Tf)(z_1,\dots,w)|^2 \\
&\geqslant \sup_w \sum_{\{(z_1,\dots,z_N):(z_1,\dots,z_N,w)\in\mathcal{P}_N\}} e(z_1,\dots,z_N,w)|(Tf)(z_1,\dots,w)b(w)|^2 \\
&\geqslant \sum_{\{(z_1,\dots,z_N):(z_1,\dots,z_N,w_0)\in\mathcal{P}_N\}} e(z_1,\dots,z_N,w_0)|(Tf)(z_1,\dots,w_0)b(w_0)|^2 \\
&> \|T\|^2 - \varepsilon. \quad \blacksquare
\end{aligned}$$

It is important to recall the fact that there exists an isometric $*$-homomorhism

$$\varphi : L(X^{\otimes N}) \supset A \otimes I^N + K(X) \otimes I^{N-1} + \dots + K(X^{\otimes N}) \to \mathcal{O}_p(J)^{\mathbf{T}}$$

as in Pimsner ([25], Proposition 3.11) and Fowler–Muhly–Raeburn ([11], Proposition 4.6) such that

$$\varphi(a + \theta_{x_1\otimes\dots\otimes x_k, y_1\otimes\dots\otimes y_k}) = a + S_{x_1}\cdots S_{x_k}S_{y_k}^*\cdots S_{y_1}^*.$$

To simplify notation, we put $S_x = S_{x_1}\cdots S_{x_k}$ for $x = x_1 \otimes \dots \otimes x_k \in X^{\otimes k}$.

LEMMA 3.22. *Let J be a p-invariant set with no isolated points. Suppose that p is expansive and free on J. Let $b = c^*c$ for some $c \in \mathcal{O}_X^{\mathrm{alg}}$. We decompose $b = \sum_j b_j$ with $\gamma_t(b_j) = \mathrm{e}^{\mathrm{i}jt}b_j$. For any $\varepsilon > 0$ there exists $P \in A$ with $0 \leqslant P \leqslant I$ satisfying the following:*

(i) $Pb_jP = 0$ $(j \neq 0)$;

(ii) $\|Pb_0P\| \geqslant \|b_0\| - \varepsilon$.

Proof. For $x \in X^{\otimes n}$, we define $\text{length}(x) = n$ with the convention $\text{length}(a) = 0$ for $a \in A$. We write c as a finite sum $c = a + \sum_i S_{x_i} S_{y_i}^*$. Put

$$n = 2 \max\{\text{length}(x_i), \text{length}(y_i); i\}.$$

For $j > 0$, each b_j is a finite sum of terms in the form such that

$$S_x S_y^* \quad x \in X^{\otimes(k+j)}, \quad y \in X^{\otimes k} \quad 0 \leqslant k+j \leqslant n.$$

In the case when $j < 0$, b_j is a finite sum of terms in the form such that

$$S_x S_y^* \quad x \in X^{\otimes k}, \quad y \in X^{\otimes(k+|j|)} \quad 0 \leqslant k+|j| \leqslant n.$$

We shall identify b_0 with an element in $L(X^{\otimes n})$. Apply Proposition 3.21 for $T = (b_0)^{1/2}$. Then there exists $a \in A^+ = C(J)^+$ with $\|a\| = 1$ such that

$$\|\phi(a)T\|^2 \geqslant \|T\|^2 - \varepsilon,$$
$$aS_xS_y^*a = 0 \quad \text{for any } x \in X^{\otimes i}, \text{ for any } y \in X^{\otimes j}, 0 \leqslant i,j \leqslant N, i \neq j.$$

Define a positive operator $P = a \in A$. Then

$$\|Pb_0P\| = \|Pb_0^{1/2}\|^2 \geqslant \|b_0^{1/2}\|^2 - \varepsilon = \|b_0\| - \varepsilon.$$

It is evident that $Pb_jP = 0$ for $j \neq 0$. ∎

Since we have prepared technical lemmas adapted to our particular situation, the rest of the proof of our main theorem is a standard one.

THEOREM 3.23. *Let $p(z,w)$ be a reduced non-zero polynomial in two variables with a unique factorization into irreducible polynomials:*

$$p(z,w) = g_1(z,w) \cdots g_p(z,w),$$

where each $g_i(z,w)$ is irreducible and g_i and g_j $(i \neq j)$ are prime to each other. We assume that any $g_i(z,w)$ is not a function only in z or w. Let J be a p-invariant set with no isolated points. Suppose that p is expansive and free on J. Then the associated C^*-algebra $\mathcal{O}_p(J)$ is simple and purely infinite.

Proof. Let $w \in \mathcal{O}_p(J)$ be any non-zero positive element. We shall show that there exist $z_1, z_2 \in \mathcal{O}_p(J)$ such that $z_1 w z_2 = I$. We may assume that $\|w\| = 1$. Let $E : \mathcal{O}_p(J) \to \mathcal{O}_p(J)^\alpha$ be the canonical conditional expectation onto the fixed point algebra by the gauge action α. Since E is faithful, $E(w) \neq 0$. Choose ε such that

$$0 < \varepsilon < \frac{\|E(w)\|}{4} \quad \text{and} \quad \varepsilon(\|E(w)\| - 3\varepsilon)^{-1} \leqslant 1.$$

There exists an element $c \in \mathcal{O}_p(J)^{\text{alg}}$ such that $\|w - c^*c\| < \varepsilon$ and $\|c\| \leqslant 1$. Let $b = c^*c$. Then b is decomposed as a finite sum $b = \sum_j b_j$ with $\gamma_t(b_j) = e^{ijt} b_j$.

Since $\|b\| \leqslant 1$, $\|b_0\| = \|E(b)\| \leqslant 1$. By Lemma 3.22, there exists $P \in A$ with $0 \leqslant P \leqslant I$ satisfying $Pb_jP = 0 \ (j \neq 0)$ and $\|Pb_0P\| \geqslant \|b_0\| - \varepsilon$. Then we have

$$\|Pb_0P\| \geqslant \|b_0\| - \varepsilon = \|E(b)\| - \varepsilon \geqslant \|E(w)\| - \|E(w) - E(b)\| - \varepsilon \geqslant \|E(w)\| - 2\varepsilon.$$

For $T := Pb_0P \in L(X^{\otimes m})$, there exists $f \in X^{\otimes m}$ with $\|f\| = 1$ such that

$$\|T^{1/2}f\|_2^2 = \|(f|Tf)_A\| \geqslant \|T\| - \varepsilon.$$

Hence we have $\|T^{1/2}f\|_2^2 \geqslant \|E(w)\| - 3\varepsilon$. Define $a = S_f^*TS_f = (f|Tf)_A \in A$. Then $\|a\| \geqslant \|E(w)\| - 3\varepsilon > \varepsilon$. By Lemma 3.17, there exist $n \in \mathbb{N}$ and $u \in X^{\otimes n}$ such that

$$\|u\|_2 \leqslant (\|a\| - \varepsilon)^{-1/2} \quad \text{and} \quad S_u^*aS_u = I.$$

Then $\|u\| \leqslant (\|E(w)\| - 3\varepsilon)^{-1/2}$. We have

$$\|S_f^*PwPS_f - a\| \leqslant \|S_f\|^2\|P\|^2\|w - b\| < \varepsilon.$$

Therefore

$$\|S_u^*S_f^*PwPS_fS_u - I\| < \|u\|^2\varepsilon \leqslant \varepsilon(\|E(w)\| - 3\varepsilon)^{-1} \leqslant 1.$$

Hence $S_u^*S_f^*PwPS_fS_u$ is invertible. Thus there exists $v \in \mathcal{O}_X$ with

$$S_u^*S_f^*PwPS_fS_uv = I.$$

Put $z_1 = S_u^*S_f^*P$ and $z_2 = PS_fS_uv$. Then $z_1wz_2 = I$. ∎

REMARK 3.24. Schweizer's theorem in [28] also implies that $\mathcal{O}_p(J)$ is simple. Our theorem gives simplicity and pure infiniteness with a direct proof.

The C^*-algebra $\mathcal{O}_p(J)$ is separable and nuclear, and satisfies the Universal Coefficient Theorem. Hence the isomorphism class of C^*-algebra $\mathcal{O}_p(J)$ is completely determined by the K-group together with the class of the unit by the classification theorem by Kirchberg–Phillips [18], [26].

EXAMPLE 3.25. Let m and n be natural numbers. Consider $p(z,w) = z^m - w^n$ and $J = \mathbb{T}$. If n is not divisible by m, then $\mathcal{O}_p(J)$ is simple and purely infinite.

EXAMPLE 3.26. Let m and n be natural numbers and relatively prime. Consider

$$p(z,w) = (w - z^m)(w - z^n).$$

Let $J = \mathbb{T}$. Then $\mathcal{O}_p(J)$ is simple and purely infinite.

EXAMPLE 3.27. Let $R_1(z) = \frac{(z^2+1)^2}{4z(z^2-1)}$ be the rational function given by Lattes. Let $R_2(z) = \frac{P_2(z)}{Q_2(z)}$ be any rational function with odd degree. Consider

$$p(z,w) = ((4z(z^2-1))w - (z^2+1)^2)(Q_2(z)w - P_2(z)).$$

Let $J = \widehat{\mathbb{C}}$. Then $\mathcal{O}_p(J)$ is simple and purely infinite.

EXAMPLE 3.28. Let $i_1, \dots, i_n, j_1, \dots, j_n$ be natural numbers. Assume that $i_k \neq 1$ or $j_k \neq 1$ for each k. Suppose that those which are not equal to 1 are relatively prime. Put $J = \mathbb{T}$. Let

$$p(z,w) = (z^{i_1} - w^{j_1})(z^{i_2} - w^{j_2}) \cdots (z^{i_n} - w^{j_n}).$$

Then $\mathcal{O}_p(J)$ is simple and purely infinite.

4. K-GROUPS

We shall compute K-groups for several examples.

EXAMPLE 4.1. Let $p(z,w) = z^m - w^n$. Then $J := \mathbb{T}$ is a p-invariant set. Consider the Hilbert C^*-bimodule $X_p = C(\mathcal{C}_p)$ over $A = C(\mathbb{T})$. Then X_p is isomorphic to A^m as a right A-module. In fact, let $u_i(z,w) = \frac{1}{\sqrt{m}} z^i$ for $i = 0, 1, \dots, m-1$. Then $(u_i|u_j)_A = \delta_{i,j} I$ and $\{u_0, u_1, \dots, u_{m-1}\}$ is a basis of X_p, in the sense that $f = \sum_{i=0}^{m-1} u_i (u_i|f)_A$ for any $f \in X_p = C(\mathcal{C}_p)$. Let $a_1(z) = z$ for $z \in \mathbb{T}$. Then

$$(\phi(a_1)u_i)(z,w) = \frac{1}{\sqrt{m}} z^{i+1} = u_{i+1}(z,w)$$

for $i = 0, 1, \dots, m-2$. And

$$(\phi(a_1)u_{m-1})(z,w) = \frac{1}{\sqrt{m}} z^m = \frac{1}{\sqrt{m}} w^n = (u_0 \cdot a_1^n)(z,w).$$

Therefore, if we identify $L(X_p)$ with $M_n(A)$, then $\phi(a_1)_{i,j} = I$ for $i = j+1, j = 1, \dots, m-2$, $\phi(a_1)_{0,m-1} = a_1^n$ and $\phi(a_1)_{i,j} = 0$ for others. Let $\phi_1^* : K_1(A) = \mathbb{Z} \to K_1(A) = \mathbb{Z}$. Since $[a_1]$ is the generator of $K_1(A) = \mathbb{Z}$ and $\phi_1([a_1]) = [a_1^n]$, $\phi_1(k) = nk$ for $k \in \mathbb{Z}$.

Since $\phi(I_A) = I_{M_m(A)}$, $\phi_0^* : K_0(A) = \mathbb{Z} \to K_0(A) = \mathbb{Z}$ is given by $\phi_1^*(k) = mk$ for $k \in \mathbb{Z}$. By the six-term exact sequence due to Pimsner [25], we have

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\mathrm{id}-m\cdot"] & \mathbb{Z} \arrow[r, "i_*"] & K_0(\mathcal{O}_p(\mathbb{T})) \arrow[d, "\delta_0"] \\
K_1(\mathcal{O}_p(\mathbb{T})) \arrow[u, "\delta_1"] & 0 \arrow[l, "i_*"] & \mathbb{Z} \arrow[l, "\mathrm{id}-n\cdot"]
\end{tikzcd}$$

Therefore:

(i) for $n = 1$ and $m = 1$: $K_0(\mathcal{O}_p(\mathbb{T})) \cong \mathbb{Z} \oplus \mathbb{Z}$, $K_1(\mathcal{O}_p(\mathbb{T})) \cong \mathbb{Z} \oplus \mathbb{Z}$;
(ii) for $n = 1$ and $m \neq 1$: $K_0(\mathcal{O}_p(\mathbb{T})) \cong \mathbb{Z} \oplus \mathbb{Z}/(m-1)\mathbb{Z}$, $K_1(\mathcal{O}_p(\mathbb{T})) \cong \mathbb{Z}$;
(iii) for $n \neq 1$ and $m = 1$: $K_0(\mathcal{O}_p(\mathbb{T})) \cong \mathbb{Z}$, $K_1(\mathcal{O}_p(\mathbb{T})) \cong \mathbb{Z} \oplus \mathbb{Z}/(n-1)\mathbb{Z}$;
(iv) for $n \neq 1$ and $m \neq 1$: $K_0(\mathcal{O}_p(\mathbb{T})) \cong \mathbb{Z}/(m-1)\mathbb{Z}$, $K_1(\mathcal{O}_p(\mathbb{T})) \cong \mathbb{Z}/(n-1)\mathbb{Z}$.

EXAMPLE 4.2. Let $p(z,w) = (w - z^m)(w - z^n)$ with $(2 \leqslant m < n)$. Then $J := \mathbb{T}$ is a p-invariant set. Since the set $B(p)$ of branched points in $\mathcal{C}_p$ is non-empty, we need to be careful to compute the K-groups $K_0(\mathcal{O}_p(\mathbb{T}))$ and $K_1(\mathcal{O}_p(\mathbb{T}))$.

If z is a branched point, then $w = z^m = z^n$, so that $z^{n-m} = 1$. Hence $B(p) = \{1, \alpha, \alpha^2, \ldots, \alpha^{n-m-1}\}$, where $\alpha = e^{2\pi i/(n-m)}$ is a primitive $(n-m)$-th root of unity. Put

$$D(p) = \{(z,w) \in \mathcal{C}_p : e(z,w) \geqslant 2\} = \{(1,1), (\alpha, \alpha^m), \ldots, (\alpha^{n-m-1}, \alpha^{m(n-m-1)})\}$$

and any branch index $e(\alpha^k, \alpha^{mk}) = 2$. Let $p_1(z,w) = (w - z^m)$ and $p_2(z,w) = (w - z^n)$. Let $X = C(\mathcal{C}_p)$ and

$$Y = C(\mathcal{C}_{p_1}) \oplus C(\mathcal{C}_{p_2}) = C(\{(1,z,w) : p_1(z,w) = 0\} \cup \{(2,z,w) : p_2(z,w) = 0\}).$$

We shall embed $X = C(\mathcal{C}_p)$ into Y as a bimodule over $A = C(\mathbb{T})$ by identifying the points corresponding to the branched points of p. Let

$$\begin{aligned} Z := &\{f \in Y : f(1,z,w) = f(2,z,w), (z,w) \in D(p)\} \\ = &\{f \in Y : f(1,\alpha^r,\alpha^m r) = f(2,\alpha^r,\alpha^m r),\ \ r = 0,1,\ldots,n-m-1\}. \end{aligned}$$

Then Z is a closed submodule of Y and we can identify X with Z as bimodules.

We introduce a basis $\{u_1, \ldots, u_{m+n}\}$ of Y as follows:
For $i = 1, \ldots, m$,

$$u_i(1,z,w) = \frac{1}{\sqrt{m}} z^{i-1}, \quad u_i(2,z,w) = 0,$$

and for $i = m+1, \ldots, m+n$,

$$u_i(1,z,w) = 0, \quad u_i(2,z,w) = \frac{1}{\sqrt{n}} z^{i-m-1}.$$

Then $(u_i|u_j)_A = \delta_{i,j} I$. Therefore we can identify $f \in Y$ with $(f_i)_i \in A^{m+n}$ by

$$f_i = (u_i|f)_A, \quad f(k,z,w) = \sum_{i=1}^{m+n} u_i(k,z,w) f_i(w), \quad k = 1,2.$$

We claim that $f(1,\alpha^r,\alpha^{mr}) = f(2,\alpha^r,\alpha^{mr})$ if and only if

$$\sum_{i=1}^{m} u_i(1,\alpha^r,\alpha^{mr}) f_i(\alpha^{mr}) = \sum_{i=m+1}^{m+n} u_i(2,\alpha^r,\alpha^{mr}) f_i(\alpha^{mr})$$

if and only if

$$\sum_{i=1}^{m} \frac{1}{\sqrt{m}} \alpha^{r(i-1)} f_i(\alpha^{mr}) = \sum_{i=m+1}^{m+n} \frac{1}{\sqrt{n}} \alpha^{r(i-m-1)} f_i(\alpha^{mr})$$

if and only if the corresponding vector $(f_1(\alpha^{mr}), \ldots, f_{m+n}(\alpha^{mr})) \in \mathbb{C}^{m+n}$ is orthogonal to a vector

$$n_r := \Big(\frac{1}{\sqrt{m}} 1, \frac{1}{\sqrt{m}} \alpha^r, \ldots, \frac{1}{\sqrt{m}} \alpha^{r(m-1)}, -\frac{1}{\sqrt{n}} 1, -\frac{1}{\sqrt{n}} \alpha^r, \ldots, -\frac{1}{\sqrt{n}} \alpha^{r(n-1)}\Big) \in \mathbb{C}^{m+n}.$$

Let $C := \{\alpha^{mr} : r = 0, 1, \ldots, n-m-1\} = \{c_1, c_2, \ldots, c_v\}$ and $c_i \neq c_j$, $(i \neq j)$. For $k = 1, 2, \ldots, v$, put $C(k) = \{r \in \{0, 1, \ldots, n-m-1\} : \alpha^{mr} = c_k\}$. If we identify $Y = A^{m+n} = C(\mathbb{T})^{m+n}$, then

$$Z = \{f = (f_i)_i \in A^{m+n} : (f_i(\alpha^{mr}))_i \text{ is orthogonal to } n_r, \text{ for } r = 0, \ldots, n-m-1\}$$
$$= \bigcap_{k=1}^{v} \{f = (f_i)_i \in A^{m+n} : (f_i(c_k))_i \text{ is orthogonal to } n_r \text{ in } \mathbb{C}^{m+n} \text{ for } \forall r \in B(k)\}.$$

We see that for fixed k, the vectors n_r $(r \in B(k))$ are linearly independent. Therefore the subspace

$$H(k) := \{x = (x_i)_i \in \mathbb{C}^{m+n} : x \text{ is orthogonal to } n_r,\ r \in B(k)\}$$

has dimension $m + n - {}^{\#}C(k) \geqslant m + n - (n-m) = 2m \geqslant 2$. Let

$$L(k) := \text{ span } \{T \in B(\mathbb{C}^{m+n}) : T = \theta_{x,y} \text{ for some } x, y \in H(k)\}.$$

Therefore we have an identification

$$\mathcal{K}(Z) = \{f \in C(\mathbb{T}, M_{m+n}(\mathbb{C})) : f(c_k) \in L(k),\ k = 1, \ldots, v\}.$$

We shall show that the canonical inclusion $i : \mathcal{K}(Z) \to \mathcal{K}(Y) \cong M_{m+n}(C(\mathbb{T}))$ induces the isomorphism

$$i_* : K_r(\mathcal{K}(Z)) \cong \mathbb{Z} \to K_r(\mathcal{K}(Y)) \cong \mathbb{Z}, \quad r = 0, 1.$$

Let

$$J = \{f \in C(\mathbb{T}, M_{m+n}(\mathbb{C})) : f(c_k) = 0,\ k = 1, \ldots, v\}$$

and a finite dimensional algebra $Q = \bigoplus_{k=1}^{v} L(k)$. Then we have an exact sequence

$$0 \to J \to \mathcal{K}(Z) \xrightarrow{\pi} Q \to 0.$$

Consider the six-term exact sequence

$$\begin{array}{ccccc} K_0(J) = 0 & \longrightarrow & K_0(\mathcal{K}(Z)) & \xrightarrow{\pi^*} & K_0(Q) = \mathbb{Z}^v \\ {\scriptstyle\delta_1}\uparrow & & & & \downarrow{\scriptstyle\delta_0} \\ K_1(Q) = 0 & \xleftarrow[\pi^*]{} & K_1(\mathcal{K}(Z)) & \xleftarrow[i^*]{} & K_1(J) = \mathbb{Z}^v \end{array}$$

For $k = 1, \ldots, v$, let $q_k \in L(k)$ be a minimal projection and we consider the projection $r_k = (0, \ldots, 0, q_k, 0, \ldots, 0) \in Q$. Let $f_k \in \mathcal{K}(Z)$ be a lift of r_k defined as a "piecewise linear" map with $f_k(c_j) = \delta_{k,j}$. Since $\delta_0([r_k]) = -[e^{2\pi i f_k}]$, we obtain that

$$\delta_0(n_1, \ldots, n_v) = (n_v - n_1, n_1 - n_2, n_2 - n_3, \ldots, n_{v-1} - n_v).$$

Since $\mathrm{Im}\pi^* = \mathrm{Ker}\delta_0 = \{(n, \ldots, n) \in \mathbb{Z}^r : n \in \mathbb{Z}\} \cong \mathbb{Z}$ and π^* is one to one, we see that $\pi^* : K_0(\mathcal{K}(Z)) \cong \mathbb{Z} \to \mathbb{Z}^r$ is given by $\pi^*(n) = (n, \ldots, n)$. Since $\mathrm{Im}\ \delta_0 = \mathrm{Ker}\ i^*$ and i^* is onto, $i^* : K_1(J) \cong \mathbb{Z}^r \to K_1(\mathcal{K}(Z)) \cong \mathbb{Z}$ is given by $i^*(n_1, \ldots, n_v) = n_1 + \cdots + n_v$. Let $p \in C(\mathbb{T}, M_{m+n}(\mathbb{C}))$ be a projection such that $p(t)$ is a rank one projection for any $t \in \mathbb{T}$ and $p(c_k) \in L(k)$ for $k = 1, \ldots, v$. Then

$[p]$ is a generator of $K_0(\mathcal{K}(Z)) \cong \mathbb{Z}$ and also a generator of $K_0(\mathcal{K}(Y)) \cong \mathbb{Z}$. Let $c_k = \mathrm{e}^{2\pi \mathrm{i}\theta_k}$ with $0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_v$. Let $u \in C(\mathbb{T}, M_{m+n}(\mathbb{C}))$ be a unitary such that $u(\mathrm{e}^{2\pi \mathrm{i}t}) = \mathrm{e}^{2\pi \mathrm{i}t/\theta_1}$ for $0 \leqslant t \leqslant \theta_1$ and $u(\mathrm{e}^{2\pi \mathrm{i}t}) = 1$ for others. Then $[u]$ is a generator of $K_1(\mathcal{K}(Z)) \cong \mathbb{Z}$ and also a generator of $K_1(\mathcal{K}(Y))$. Therefore we conclude that $i_* : K_r(\mathcal{K}(Z)) \cong \mathbb{Z} \to K_r(\mathcal{K}(Y)) \cong \mathbb{Z}$, $r = 0, 1$ is an isomorphism.

Since $I_X = \{f \in C(\mathbb{T}) : f(\alpha^k) = 0 \text{ for } k = 0, 1, m-n-1\}$, we have $K_0(I_X) = 0$ and $K_1(I_X) = \mathbb{Z}^{m-n}$.

Therefore we can identify the left action $\phi : I_X \to K(X) = K(Z)$ with $\phi_1 \oplus \phi_2 : I_X \to K(Y) = K(C(\mathcal{C}_{p_1}) \oplus C(\mathcal{C}_{p_2}))$ on the level of K-groups. Hence $\phi^* : K_1(I_X) = \mathbb{Z}^{m-n} \to K_1(A) = \mathbb{Z}$ is given by $\phi^*(x_1, \ldots, x_{m-n}) = \sum\limits_{i=1}^{m-n} 2x_i$.

By a six-term exact sequence, we have

$$\begin{array}{ccccc}
K_0(I_X) = 0 & \xrightarrow{j^* - \phi^*} & K_0(A) = \mathbb{Z} & \xrightarrow{i_*} & K_0(\mathcal{O}_p(\mathbb{T})) \\
{\scriptstyle \delta_1}\uparrow & & & & \downarrow{\scriptstyle \delta_0} \\
K_1(\mathcal{O}_p(\mathbb{T})) & \xleftarrow[i_*]{} & K_1(A) = \mathbb{Z} & \xleftarrow[j^* - \phi^*]{} & K_1(I_X) = \mathbb{Z}^{m-n}
\end{array}$$

The canonical inclusion map $j : I_X \to A = C(\mathbb{T})$ induces $j^* : K_1(I_X) = \mathbb{Z}^{m-n} \to K_1(A) = \mathbb{Z}$ with $j^*(x_1, \ldots, x_{m-n}) = \sum\limits_{i=1}^{m-n} x_i$. Therefore we have

$$K_0(\mathcal{O}_p(\mathbb{T})) = \mathbb{Z}^{m-n}, \quad \text{and} \quad K_1(\mathcal{O}_p(\mathbb{T})) = 0.$$

EXAMPLE 4.3. Let $p(z, w) = (w - z^{m_1})(w - z^{m_2}) \cdots (w - z^{m_r})$ and $m_1, \ldots, m_r$ are all different, where r is the number of irreducible components. Then $J := \mathbb{T}$ is a p-invariant set. Let $b = {}^{\#}B(p)$ be the number of the branched points. By a similar calculation, we have

$$K_0(\mathcal{O}_p(\mathbb{T})) = \mathbb{Z}^b, \quad \text{and} \quad K_1(\mathcal{O}_p(\mathbb{T})) = \mathbb{Z}/(r-1)\mathbb{Z}.$$

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