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LATTICE ISOMORPHISMS BETWEEN SPACES OF INTEGRABLE FUNCTIONS WITH RESPECT TO VECTOR MEASURES

A. FERNÁNDEZ, F. MAYORAL, F. NARANJO, and E.A. SÁNCHEZ-PÉREZ

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ABSTRACT. In this paper we study the relation between different spaces of vector measures $(\Omega_1, \Sigma_1, m_1)$ and $(\Omega_2, \Sigma_2, m_2)$; where (Ω_1, Σ_1) and (Ω_2, Σ_2) are measurable spaces and m_1 and m_2 are countably additive vector measures taking values in real Banach spaces X and Y, respectively, when the corresponding spaces of integrable functions $L^1(m_1)$ and $L^1(m_2)$ are lattice isomorphic. As a consequence, we give a description of the lattice isomorphisms between spaces of integrable functions with respect to a vector measure.

KEYWORDS: *Vector measure, integrable function, lattice isomorphism, multiplication operator, composition operator, Boolean algebra.*

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INTRODUCTION

The representation of Banach lattices as function spaces is a useful technique in Functional Analysis, since it allows to study abstract problems using tools and specific properties coming from the particular properties of the function spaces. For instance, when the reference spaces for this representation are Banach function spaces, Banach ideals of Lebesgue integrable function spaces, it is possible to use powerful tools that comes from measure theory and integration theory. Recently, and using important results that go back to the work of Bartle, Dunford and Schwartz in the fifties and Lewis in the early seventies, it has been established representation theorems of Banach lattices by means of spaces of integrable scalar functions with respect to vector measures; see for example Theorem 8 of [2], Theorem 2.5 of [3], Theorem 4 of [4] or Theorem 2.4 of [8]. Let us show this by means of the following, in a sense canonical, result.

THEOREM 0.1 ([2], Theorem 8). *Let E be an order continuous Banach lattice with weak order unit. Then there exists an E-valued positive vector measure m defined* *on a measurable space* (Ω, Σ) *such that* E *is lattice isomorphic to the space* $L^1(m)$ *of integrable scalar functions with respect to* m.

Some of these representations have been already applied in the context of the operator theory for obtaining Maurey–Rosenthal type factorization theorems and also to analyze the optimal domain of relevant operators defined on Banach function spaces (see for instance [3], [5], [7], [9]). In the case of the factorization theorems obtained with this technique, the factorization depends strongly on the vector measure that is used for the representation (see [7], [9]). The same happens when these arguments are applied for obtaining concrete representations of Banach lattices. However, the same Banach lattice E can be represented using different vector measures, as the next example shows.

EXAMPLE 0.2. Let us consider the following three vector measures defined on $\mathcal{B}$, the Borel σ-algebra of the interval $[0,1]$:

$$m_1 : A \in \mathcal{B} \to m_1(A) := \lambda(A) \in \mathbb{R},$$
$$m_2 : A \in \mathcal{B} \to m_2(A) := \Big(\int_A r_n(t)\mathrm{d}t \Big)_n \in c_0,$$
$$m_3 : A \in \mathcal{B} \to m_3(A) := \chi_A \in L^1[0,1],$$

where λ denotes the Lebesgue measure and $(r_n)_n$ is the sequence of Rademacher functions. They all generate the same space, namely,

$$L^1(m_1) = L^1(m_2) = L^1(m_3) = L^1[0,1].$$

The aim of this paper is to analyze the relation between different *spaces of vector measures* $(\Omega_1, \Sigma_1, m_1)$ and $(\Omega_2, \Sigma_2, m_2)$; (Ω_1, Σ_1) and (Ω_2, Σ_2) being two measurable spaces, and $m_1 : \Sigma_1 \to X$ and $m_2 : \Sigma_2 \to Y$ being two vector measures with values in two Banach spaces X and Y, respectively, under the assumption that the spaces $L^1(m_1)$ and $L^1(m_2)$ are lattice isomorphic.

A quick look at the Example 0.2 above reveals that there are no clear links between the spaces X and Y where the vector measures are defined related to the existence of a lattice isomorphism $T : L^1(m_1) \to L^1(m_2)$.

On the other hand, it seems that it is not possible to relax the condition of T being a *lattice isomorphism* if we want to obtain relevant relationships between the corresponding measurable spaces. For instance, it is well-known that the Fourier transform $\mathcal{F} : L^2[0,1] \to \ell_2(\mathbb{N})$ is an isometry onto. The corresponding measurable spaces are in this case $([0,1], \mathcal{B})$ and $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, respectively, where $\mathcal{B}$ denotes the σ-algebra of Borel sets of the interval $[0,1]$ and $\mathcal{P}(\mathbb{N})$ is the σ-algebra of the subsets of the set of natural numbers $\mathbb{N}$. From the lattice point of view the spaces $L^2[0,1]$ and $\ell_2(\mathbb{N})$ are really different; likewise their related measurable spaces are too. However, both spaces can be represented as $L^1(m)$ spaces for suitable vector measures m, but the first one *has no atoms* and the second one is *purely atomic.*

The structure of the paper is as follows. First, we begin by analyzing the relation, from the Boolean algebras point of view, between the σ-algebras Σ_1 and Σ_2. The starting point is the case of a lattice isomorphism between spaces of bounded functions, which forces to the corresponding σ-algebras to be isomorphic as Boolean algebras. In Section 2 we extend this study to the case of a lattice isomorphism $T : L^1(m_1) \to L^1(m_2)$, assuming that the measures m_1 and m_2 are positive. In Section 3 we analyze which ones of the results obtained in the previous section can be carried out to the general case of non necessarily positive vector measures; actually, we shall show that positivity of the vector measure is a fundamental property in order to obtaining complete characterizations of the relations that hold between the measurable spaces. In Section 4 we show, using results of Sikorski [18], [19] and Kuratowski [12], that for the isomorphic results obtained in previous sections to be true in the case of *Borel spaces* Ω_1 and Ω_2 they must be *essentially equal*, that is, it must be possible to define (almost everywhere) a one-to-one measurable map from Ω_1 onto Ω_2 with measurable inverse. As a consequence, we obtain the structure of *any* lattice isomorphism $T : L^1(m_1) \to L^1(m_2)$ for positive measures. Essentially, T must be the composition of a *multiplication operator* with a *composition operator.* Similar results were obtained by Lamperti for scalar measures and isometries between L_p spaces, see [13] or XV Section 5 of [16]. Finally, we would like to mention that the description and study of a certain order relation between measures that represent the same space of integrable functions have been done in [10].

1. PREMIMINARIES

Let $m : \Sigma \to X$ be a vector measure defined on a σ-algebra Σ of subsets of a nonempty set Ω; this will always mean that m is countably additive on Σ with values in a real Banach space X. We denote by X' its dual space. The semivariation of the measure m is the set function $\|m\| : \Sigma \to [0, \infty)$ defined by

$$\|m\|(A) := \sup\{|\langle m, x'\rangle|(A) : \|x'\| \leqslant 1\}, \quad A \in \Sigma,$$

where $|\langle m, x'\rangle|$ is the total variation measure of the scalar measure $\langle m, x'\rangle$ given by $\langle m, x'\rangle(A) := \langle m(A), x'\rangle$, for all $A \in \Sigma$. A set $A \in \Sigma$ is called m-null if $\|m\|(A) = 0$.

A measurable function $f : \Omega \to \mathbb{R}$ is called *integrable* with respect to m if:

(i) $f \in L^1(|\langle m, x'\rangle|)$ for all $x' \in X'$, and

(ii) for each $A \in \Sigma$ there exists an element $\int_A f \mathrm{d}m \in X$ (called the *integral* of f over A) such that $\left\langle \int_A f \mathrm{d}m, x' \right\rangle = \int_A f \mathrm{d}\langle m, x'\rangle$ for all $x' \in X'$.

Two measurable functions f and g defined on Ω are identified if they are equal m-a.e., that is, if $\{w \in \Omega : f(w) \neq g(w)\}$ is a m-null set. The space $L^1(m)$

of all (equivalence classes of) integrable functions (modulo m-a.e.) becomes an order continuous Banach lattice (with weak order unit) when it is endowed with the pointwise order m-a.e. and the norm

$$\|f\|_{L^1(m)} := \sup\Big\{\int_\Omega |f| \mathrm{d}|\langle m, x'\rangle| : \|x'\| \leqslant 1\Big\}, \quad f \in L^1(m).$$

See [11], [14] and [15] for details. In particular, it is known that every bounded function is integrable ([11], II. Section 3). We denote by $L^\infty(m)$ the space of (equivalence classes of) essentially bounded functions (modulo m-a.e.) equipped with the supremum norm $\|\cdot\|_{L^\infty(m)}$. Moreover, the inclusion $L^\infty(m) \subseteq L^1(m)$ is continuous.

The following lemmas will be useful in the sequel. Their proofs are straightforward. For a function $f \in L^1(m)$ we denote by

$$\mathrm{supp}(f) := \{w \in \Omega : f(w) \neq 0\}.$$

LEMMA 1.1. *Let $f, g \in L^1(m)$. If $\|m\|(\mathrm{supp}(f) \cap \mathrm{supp}(g)) = 0$, then*

$$\mathrm{supp}(f+g) = \mathrm{supp}(f) \cup \mathrm{supp}(g),$$

modulo m-a.e. If f and g are non-negative functions, we have

$$\mathrm{supp}(\inf\{f, g\}) = \mathrm{supp}(f) \cap \mathrm{supp}(g),$$

modulo m-a.e.

Our basic reference for Banach lattices is the book by Aliprantis and Burkinshaw [1]. All the not explained terminology of this paper can be found there.

LEMMA 1.2. *Let E and F be Banach lattices and let $T : E \to F$ be a lattice isomorphism.*

(i) *If $0 < e \in E$ is a (weak) order unit in E, then $Te \in F$ is a (weak) order unit in F.*

(ii) *If $0 < e \in E$, then $T(\mathcal{C}(e)) = \mathcal{C}(Te)$, where $\mathcal{C}(x)$ is the collection of all components of the element $0 < x \in E$.*

LEMMA 1.3. *Given a vector measure space (Ω, Σ, m), the following statements hold:*

(i) *A function $h \in L^1(m)$ is a weak order unit in $L^1(m)$ if and only if $h > 0$. In particular, χ_Ω is a weak order unit in $L^1(m)$.*

(ii) *A function $0 < h \in L^\infty(m)$ is an order unit if and only if $\inf h > 0$. In particular, χ_Ω is an order unit in $L^\infty(m)$.*

(iii) *If $0 < h \in L^1(m)$, then $\mathcal{C}(h) = \{h \cdot \chi_A : A \in \Sigma\}$. In particular,*

$$\mathcal{C}(\chi_\Omega) = \{\chi_A : A \in \Sigma\}.$$

Given a vector measure space, (Ω, Σ, m) we consider the σ-ideal $\mathcal{N}$ of the m-null sets, that is, $\mathcal{N} := \{N \in \Sigma : \|m\|(N) = 0\}$. For a set $A \in \Sigma$, denote by $[A]$ the class of all sets $B \in \Sigma$ for which $\chi_A = \chi_B$ in $L^1(m)$ holds and set $\Sigma[m] := \{[A] : A \in \Sigma\}$. Then $\Sigma[m]$ is a σ-complete Boolean algebra with respect to the usual operations of union, intersection and difference, and the corresponding order $[A] \leqslant [B]$ if and only if $\chi_A \leqslant \chi_B$ in $L^1(m)$, where $A, B \in \Sigma$. Observe that $\Sigma[m]$ is the quotient-algebra Σ modulo the σ-ideal of m-null sets. For the terminology on Boolean algebras we refer to Sikorski [18], [19]. Note also that $\Sigma[m]$ is a complete metric space under the metric d given by

$$d([A],[B]) := \|\chi_A - \chi_B\|_{L^1(m)}, \quad A, B \in \Sigma.$$

Given a mapping $\Phi : \Sigma[m_1] \to \Sigma[m_2]$ and a set $A \in \Sigma_1$, observe that $\Phi([A])$ is a class of sets. As usual we write $\chi_{\Phi(A)}$, for the characteristic function of any element of this class, that must be understood as an element of $L^1(m_2)$. Thus $\Phi(A)$ is a representative of the class $\Phi([A])$.

The lattice structure of spaces of bounded functions is completely determined by the corresponding structure of their associated Boolean algebras. More precisely, if $(\Omega_1, \Sigma_1, m_1)$ and $(\Omega_2, \Sigma_2, m_2)$ are vector measure spaces and we consider the associated Boolean algebras $\Sigma_1[m_1]$ and $\Sigma_2[m_2]$, then the following results are true.

PROPOSITION 1.4. *If $\Phi : \Sigma_1[m_1] \to \Sigma_2[m_2]$ is an isomorphism of Boolean algebras, then there is a unique isometric multiplicative lattice isomorphism*

$$T : L^\infty(m_1) \to L^\infty(m_2)$$

such that $T(\chi_A) = \chi_{\Phi(A)}$ for all $A \in \Sigma_1$. In particular, $T(\chi_{\Omega_1}) = \chi_{\Omega_2}$.

Proof. For each simple function $\varphi = \sum_{k=1}^{N} a_k \chi_{A_k}$ of $L^\infty(m_1)$ we define

$$T(\varphi) := \sum_{k=1}^{N} a_k \chi_{\Phi(A_k)}.$$

It is not difficult to see that it can be extended from $L^\infty(m_1)$ to $L^\infty(m_2)$. ∎

Reciprocally we have the following

PROPOSITION 1.5. *If $T : L^\infty(m_1) \to L^\infty(m_2)$ is a lattice isomorphism, then $\Phi : [A] \in \Sigma_1[m_1] \to \Phi([A]) := [\mathrm{supp}(T\chi_A)] \in \Sigma_2[m_2]$ is an isomorphism of Boolean algebras such that $T(\chi_A) = T(\chi_{\Omega_1}) \cdot \chi_{\Phi(A)}$ for all $A \in \Sigma_1$. If moreover T is multiplicative, then $T(\chi_{\Omega_1}) = \chi_{\Omega_2}$ and $T(\chi_A) = \chi_{\Phi(A)}$ for all $A \in \Sigma_1$.*

Proof. It is straightforward to show that Φ is an isomorphism of Boolean algebras by using Lemma 1.1, Lemma 1.2 and Lemma 1.3. ∎

REMARK 1.6. If $T : L^\infty(m_1) \to L^\infty(m_2)$ is a multiplicative lattice isomorphism, we have that $(T(\chi_A))^2 = T(\chi_A)$ for all $A \in \Sigma_1$. Therefore, for all $A \in \Sigma_1$ there exists $B \in \Sigma_2$ such that $T(\chi_A) = \chi_B$. Note that

$$B := \{w \in \Omega_2 : T(\chi_A)(w) = 1\} = \mathrm{supp}(T(\chi_A)),$$

that justifies the definition of the isomorphism of Boolean algebras Φ.

2. LATTICE ISOMORPHISMS BETWEEN SPACES OF INTEGRABLE FUNCTIONS WITH RESPECT TO POSITIVE VECTOR MEASURES

In this section we consider a couple of positive vector measure spaces, denoted by $(\Omega_1, \Sigma_1, m_1)$ and $(\Omega_2, \Sigma_2, m_2)$, where the measure m_1 takes values in the Banach lattice X and that the measure m_2 takes values in the Banach lattice Y. It is obvious that the existence of an isomorphism of Boolean algebras $\Phi : \Sigma_1[m_1] \to \Sigma_2[m_2]$ does not imply that the spaces $L^1(m_1)$ and $L^1(m_2)$ are lattice isomorphic. In order to obtain a lattice isomorphism between $L^1(m_1)$ and $L^1(m_2)$ coming from a Boolean algebra isomorphism $\Phi : \Sigma_1[m_1] \to \Sigma_2[m_2]$ we need to impose some more extra conditions on Φ. Namely,

(C1) There is a constant $K_1 > 0$ such that for each $0 \leqslant y' \in B_{Y'}$ and each $\pi \in \Pi(\Omega_1)$ there exists $0 \leqslant x' \in B_{X'}$ satisfying that

$$\langle m_2(\Phi(A)), y' \rangle \leqslant K_1 \langle m_1(A), x' \rangle, \quad A \in \pi.$$

(C2) There is a constant $K_2 > 0$ such that for each $0 \leqslant x' \in B_{X'}$ and each $\pi \in \Pi(\Omega_1)$ there exists $0 \leqslant y' \in B_{Y'}$ satisfying that

$$\langle m_1(A), x' \rangle \leqslant K_2 \langle m_2(\Phi(A)), y' \rangle, \quad A \in \pi.$$

As usual, $\Pi(\Omega_1)$ denotes the set of all finite Σ_1-partitions of the set $\Pi(\Omega_1)$. With these conditions in hands we can prove the following result holds.

THEOREM 2.1. *Let $\Phi : \Sigma_1[m_1] \to \Sigma_2[m_2]$ be an isomorphism of Boolean algebras such that the conditions* (C1) *and* (C2) *hold. Then there exists a unique lattice isomorphism $T : L^1(m_1) \to L^1(m_2)$ such that $T(L^\infty(m_1)) = L^\infty(m_2)$. Moreover, the restriction $T : L^\infty(m_1) \to L^\infty(m_2)$ is an isometric multiplicative lattice isomorphism satisfying $T(\chi_A) = \chi_{\Phi(A)}$ for all $A \in \Sigma_1$. In particular, $T(\chi_{\Omega_1}) = \chi_{\Omega_2}$.*

Proof. Suppose that $\Phi : \Sigma_1[m_1] \to \Sigma_2[m_2]$ is an isomorphism of Boolean algebras. For each simple function $\varphi = \sum_{k=1}^{N} a_k \chi_{A_k}$ of $L^1(m_1)$ we define

$$T(\varphi) := \sum_{k=1}^{N} a_k \chi_{\Phi(A_k)},$$

which is a simple function in $L^1(m_2)$. As in the L^∞ case we have defined a map $T : S(m_1) \to S(m_2)$ from the set of simple functions of $L^1(m_1)$ into the set of simple functions of $L^1(m_2)$ that is a multiplicative and lattice preserving linear bijection. Moreover, it satisfies $\|T(\varphi)\|_{L^\infty(m_2)} = \|\varphi\|_{L^\infty(m_1)}$ for all $\varphi \in S(m_1)$.

Let us prove now that the requirements (C1) and (C2) in the statement of the theorem imply continuity of the operator $T : S(m_1) \to S(m_2)$ and its inverse with respect to the norms $\|\cdot\|_{L^1(m_1)}$ and $\|\cdot\|_{L^1(m_2)}$, respectively. Since the measure m_2

is positive, for each simple function $\varphi = \sum_{k=1}^{N} a_k \chi_{A_k}$ an application of the Hahn–Banach Theorem gives:

$$\|T(\varphi)\|_{L^1(m_2)} = \Big\| \int_{\Omega_2} |T(\varphi)| \mathrm{d}m_2 \Big\|_Y = \Big\| \sum_{k=1}^{N} |a_k| m_2(\Phi(A_k)) \Big\|_Y \tag{2.1}$$
$$= \Big\langle \sum_{k=1}^{N} |a_k| m_2(\Phi(A_k)), y' \Big\rangle = \sum_{k=1}^{N} |a_k| \langle m_2(\Phi(A_k)), y' \rangle,$$

for a certain $0 \leqslant y' \in B_{Y'}$. From (C1) we obtain that there exists $0 \leqslant x' \in B_{X'}$ such that $\langle m_2(\Phi(A_k)), y' \rangle \leqslant K_1 \langle m_1(A_k), x' \rangle$ for every $k = 1, 2, \ldots, N$. Thus, (2.1) implies

$$\|T(\varphi)\|_{L^1(m_2)} = \sum_{k=1}^{N} |a_k| \langle m_2(\Phi([A_k])), y' \rangle \leqslant K_1 \sum_{k=1}^{N} |a_k| \langle m_1(A_k), x' \rangle$$
$$\leqslant K_1 \Big\| \sum_{k=1}^{N} |a_k| m_1(A_k) \Big\|_X \leqslant K_1 \|\varphi\|_{L^1(m_1)}.$$

A similar argument proves that $\|\varphi\|_{L^1(m_1)} \leqslant K_2 \|T(\varphi)\|_{L^1(m_2)}$ for every $\varphi \in S(m_1)$. Therefore, T can be extended from $L^1(m_1)$ to $L^1(m_2)$ and the extension satisfies the conditions in the statement of the theorem, since $S(m_2)$ is dense in $L^1(m_2)$.

The uniqueness and properties of $T : L^\infty(m_1) \to L^\infty(m_2)$ are clearly seen from Proposition 1.4. ∎

On the opposite way, we have the one that follows, that is the analogous result to Proposition 1.5.

THEOREM 2.2. *Let $T : L^1(m_1) \to L^1(m_2)$ be a lattice isomorphism such that $T(L^\infty(m_1)) = L^\infty(m_2)$. Then*

$$\Phi : [A] \in \Sigma_1[m_1] \to \Phi([A]) := [\mathrm{supp}(T\chi_A)] \in \Sigma_2[m_2]$$

is a Boolean algebra isomorphism that satisfies the conditions (C1) *and* (C2). *Moreover, $T(\chi_A) = T(\chi_{\Omega_1}) \cdot \chi_{\Phi(A)}$ for all $A \in \Sigma_1$ and if T is multiplicative in $L^\infty(m_1)$, then $T(\chi_{\Omega_1}) = \chi_{\Omega_2}$ and so $T(\chi_A) = \chi_{\Phi(A)}$ for all $A \in \Sigma_1$.*

Proof. As we mentioned in the proof of Proposition 1.5 it is not difficult to see that Φ is a Boolean algebra isomorphism. The same holds for the last part of the statement of the theorem. Thus we only prove that (C1) and (C2) hold for the isomorphism Φ. Continuity of T implies that there is a constant $K > 0$ such that $\|T(f)\|_{L^1(m_2)} \leqslant K\|f\|_{L^1(m_1)}$ for all $f \in L^1(m_1)$. On the other hand, since $T(L^\infty(m_1)) = L^\infty(m_2)$, we get that $T : L^\infty(m_1) \to L^\infty(m_2)$ is a lattice isomorphism and therefore the fact that $h := T(\chi_{\Omega_1})$ is an order unit in $L^\infty(m_2)$ implies, according to Lemma 1.3 (ii), that $0 < \inf h$. Moreover, it is easy to see that

$T(\varphi) = h \cdot \sum_{k=1}^{n} a_k \chi_{\Phi(B_k)}$ for each simple function $\varphi = \sum_{k=1}^{n} a_k \chi_{B_k} \in S(m_1)$. Consider the constant $K_1 := K/\inf h$. Now, take an element $0 \leqslant y' \in B_{Y'}$ and a Σ_1-partition $A_1, A_2, \ldots, A_N$ of Ω_1. In what follows we fix y' and the partition. For each positive element $0 \leqslant \mathbf{a} := (a_1, a_2, \ldots, a_N) \in \mathbb{R}^N$ we define the function $\Psi_{\mathbf{a}} : B^+_{X'} \to \mathbb{R}$ as

$$\Psi_{\mathbf{a}}(x') := \sum_{k=1}^{N} a_k(\langle m_2(\Phi(A_k)), y'\rangle - K_1\langle m_1(A_k), x'\rangle).$$

We are going to use a separation argument based on Ky Fan's Lemma (see Lemma 9.10 of [6]).

(i) Each function $\Psi_{\mathbf{a}}$ is convex because it is an affine map. Indeed,

$$\Psi_{\mathbf{a}}(x') = \Big\langle \sum_{k=1}^{N} a_k m_2(\Phi(A_k)), y' \Big\rangle - \Big\langle K_1 \sum_{k=1}^{N} a_k m_1(A_k), x' \Big\rangle.$$

(ii) The class of functions $\{\Psi_{\mathbf{a}} : 0 \leqslant \mathbf{a} \in \mathbb{R}^N\}$ is convex; that is, if $0 \leqslant t \leqslant 1$, $0 \leqslant x' \in B_{X'}$, $0 \leqslant \mathbf{a} \in \mathbb{R}^N$ and $0 \leqslant \mathbf{b} \in \mathbb{R}^N$, then

$$t\Psi_{\mathbf{a}}(x') + (1-t)\Psi_{\mathbf{b}}(x') = \Psi_{t\mathbf{a}+(1-t)\mathbf{b}}(x').$$

(iii) Each function $\Psi_{\mathbf{a}}$ is clearly continuous with respect to the weak* topology of $B^+_{X'}$.

(iv) Let us see now that for each function $\Psi_{\mathbf{a}}$, $\mathbf{a} := (a_1, a_2, \ldots, a_N)$, there exists an element $0 \leqslant x' \in B^+_{X'}$ such that $\Psi_{\mathbf{a}}(x') \leqslant 0$. Let us write φ for the function $\varphi := \sum_{k=1}^{N} a_k \chi_{A_k}$. Note that φ takes non-negative values. From the continuity of T together with the Hahn–Banach Theorem, we find an element $0 \leqslant x' \in B_{X'}$ such that

$$\begin{aligned}\|T(\varphi)\|_{L^1(m_2)} &\leqslant K\|\varphi\|_{L^1(m_1)} = K\Big\| \int_{\Omega_1} \varphi \, \mathrm{d}m_1 \Big\|_X \\ &= K\Big\langle \int_{\Omega_1} \varphi \mathrm{d}m_1, x' \Big\rangle = K \sum_{k=1}^{N} a_k \langle m_1(A_k), x'\rangle = \sum_{k=1}^{N} a_k K \langle m_1(A_k), x'\rangle.\end{aligned}$$

Note that for the element $0 \leqslant y' \in B_{Y'}$ we obtain

$$\begin{aligned}\inf h \cdot \Big(\sum_{k=1}^{N} a_k \langle m_2(\Phi(A_k)), y'\rangle\Big) &\leqslant \Big\langle \int_{\Omega_2} (h \sum_{k=1}^{N} a_k \chi_{\Phi(A_k)}) \mathrm{d}m_2, y' \Big\rangle = \Big\langle \int_{\Omega_2} T(\varphi) \mathrm{d}m_2, y' \Big\rangle \\ &\leqslant \Big\| \int_{\Omega_2} T(\varphi) \mathrm{d}m_2 \Big\|_Y = \|T(\varphi)\|_{L^1(m_2)}.\end{aligned}$$

Thus, $\inf h \cdot \Big(\sum_{k=1}^{N} a_k \langle m_2(\Phi(A_k)), y' \rangle \Big) \leqslant \sum_{k=1}^{N} a_k K \langle m_1(A_k), x' \rangle$. So,

$$\sum_{k=1}^{N} a_k (\langle m_2(\Phi(A_k)), y' \rangle - \frac{K}{\inf h} \langle m_1(A_k), x' \rangle) \leqslant 0$$

and then $\Psi_{\mathbf{a}}(x') \leqslant 0$.

Now we can apply Ky Fan's Lemma, since $B^+_{X'}$ is weak* compact. Then there is an element $0 \leqslant x'_0 \in B^+_{X'}$ such that $\Psi_{\mathbf{a}}(x'_0) \leqslant 0$ for all

$$0 \leqslant \mathbf{a} := (a_1, a_2, \ldots, a_N) \in \mathbb{R}^N.$$

In particular, taking the elements $\mathbf{a} = \mathbf{e}_k := (0, \ldots, 1^k, \ldots, 0) \in \mathbb{R}^N$ we get

$$\langle m_2(\Phi(A_k)), y' \rangle \leqslant K_1 \langle m_1(A_k), x'_0 \rangle$$

for each $k = 1, 2, \ldots, N$.

The proof of the condition (C2) can be obtained just by using the same argument and the continuity of the inverse operator T^{-1}. ∎

REMARK 2.3 (See Section 4 of [9]). If the Banach lattice X is a generalized AL-space, namely, the topology of X can be defined by a lattice norm $\|\cdot\|$ that is additive on the positive cone, that is, $\|x + y\| = \|x\| + \|y\|$ for all $0 \leqslant x, y \in X$, then the condition (C1) is equivalent to each one of the following two (equivalent) conditions:

(i) There is a constant $K_1 > 0$ such that for each $0 \leqslant y' \in B_{Y'}$ and each $A \in \Sigma_1$ there is $0 \leqslant x' \in B_{X'}$ satisfying $\langle m_2(\Phi(A)), y' \rangle \leqslant K_1 \langle m_1(A), x' \rangle$.

(ii) There exists a constant $K_1 > 0$ such that

$$\|m_2\|(\Phi(A)) \leqslant K_1 \|m_1\|(A), \quad A \in \Sigma_1.$$

Analogously if the Banach lattice Y is a generalized AL-space, similar equivalences are true for the condition (C2). Summing up all this comments, if the measures m_1 and m_2 take values on generalized AL-spaces, the conditions (C1) and (C2) can be rewritten in the following way: there is a constant $K > 0$ such that

$$\frac{1}{K}\|m_1\|(A) \leqslant \|m_2\|(\Phi(A)) \leqslant K \|m_1\|(A), \quad A \in \Sigma_1. \tag{2.2}$$

Note that (2.2) is equivalent to say that $\Phi : \Sigma[m_1] \to \Sigma[m_2]$ and its inverse Φ^{-1} are Lipschitz maps.

In the rest of this section we study the case when the lattice isomorphism

$$T : L^1(m_1) \to L^1(m_2)$$

does not necessarily satisfy the equality $T(L^\infty(m_1)) = L^\infty(m_2)$ that is required in Theorem 2.2. To do that consider a positive vector measure space (Ω, Σ, m), where m takes values in the Banach lattice X. If $0 < h \in L^1(m)$, we can define another positive vector measure as

$$m_h : A \in \Sigma \to m_h(A) := \int_A h \, \mathrm{d}m \in X.$$

It is well-known that the equality $\|m_h\|(A) = \|h\chi_A\|_{L^1(m)}$ holds for each $A \in \Sigma$. Therefore $\|m_h\|(A) = 0$ if and only if $\|m\|(A) = 0$, and the classes of null sets are the same for both measures. Moreover, for each $x' \in X'$ and each $A \in \Sigma$, we have $|\langle m_h, x'\rangle|(A) = \int_A h \, \mathrm{d}|\langle m, x'\rangle|$. Actually, it can be easily seen that the following result holds.

LEMMA 2.4. $L^1(m_h) = \{f : \Omega \to \mathbb{R} \text{ measurable} : h \cdot f \in L^1(m)\}$.

On the other hand, we have the following result for the multiplication operator defined by the function h.

LEMMA 2.5. *The multiplication operator $M_h : L^1(m_h) \to L^1(m)$, defined as $M_h(f) := h \cdot f$, is an isometric lattice isomorphism. Moreover, its inverse is given by $D_h : f \in L^1(m) \to D_h(f) := f/h \in L^1(m_h)$, which is an isometric lattice isomorphism that satisfies $D_h M_h = \mathrm{Id}_{L^1(m_h)}$ and $M_h D_h = \mathrm{Id}_{L^1(m)}$.*

Proof. As we have noted before, the operator M_h is well-defined and it is clearly a lattice homomorphism. On the other hand, if φ is a Σ-simple function, then $\int_\Omega \varphi \, \mathrm{d}m_h = \int_\Omega \varphi \cdot h \, \mathrm{d}m$ and so we have

$$\|M_h(\varphi)\|_{L^1(m)} = \|h \cdot \varphi\|_{L^1(m)} = \Big\| \int_\Omega h \cdot |\varphi| \mathrm{d}m \Big\|_X = \Big\| \int_\Omega |\varphi| \mathrm{d}m_h \Big\|_X = \|\varphi\|_{L^1(m_h)}.$$

Taking into account that the set of simple functions is dense, we obtain that M_h is an isometry. To see that M_h is onto, recall that if $g \in L^1(m)$, then $g/h \in L^1(m_h)$. The proof of the last part is straightforward. ∎

REMARK 2.6. Note that $[\mathrm{supp}(M_h(\chi_A))] = [A]$ for each $A \in \Sigma$. Thus, the isomorphism of Boolean algebras $\Phi : \Sigma[m_h] \to \Sigma[m]$ associated with the multiplication M_h (or with D_h) is just the identity map. However, Φ does not satisfy in general the conditions (C1) or (C2) for the measures m and m_h.

LEMMA 2.7. *The identity map $\Phi : \Sigma[m_h] \to \Sigma[m]$, the isomorphism of Boolean algebras associated with the multiplication M_h, satisfies the condition* (C1) *if and only if $1/h \in L^\infty(m)$; and it satisfies the condition* (C2) *if and only if $h \in L^\infty(m)$.*

Proof. We only prove the second equivalence. The proof of the first one is similar. For the identity map $\Phi : \Sigma[m_h] \to \Sigma[m]$, the condition (C2) reads as: there is a constant $K_2 > 0$ such that for every $0 \leqslant x' \in B_{X'}$ and each $\pi \in \Pi(\Omega)$ there exists $0 \leqslant y' \in B_{X'}$ such that $\int_A h \, \mathrm{d}\langle m, x'\rangle \leqslant K_2 \langle m(A), y'\rangle$ for all $A \in \pi$.

It is clear that the condition (C2) holds if $h \in L^\infty(m)$, since for each $0 \leqslant x' \in B_{X'}$ and $A \in \Sigma$ we have that $\int_A h \, \mathrm{d}\langle m, x' \rangle \leqslant \|h\|_{L^\infty(m)} \langle m(A), x' \rangle$; it is enough to take $K_2 := \|h\|_{L^\infty(m)}$ and $y' = x'$. Reciprocally, assume now the condition (C2) and let us prove that $h \in L^\infty(m)$. For each $N = 1, 2, \ldots$ consider the measurable sets $A_N := \{\omega \in \Omega : N \leqslant h(\omega)\}$. Note that the function $h \in L^\infty(m)$ if and only if there exists $N \geqslant 1$ such that $\|m\|(A_N) = 0$. For each natural number N consider the partition $\{A_N, \Omega - A_N\} \in \Pi(\Omega)$, and let $0 \leqslant x'_N \in B_{X'}$ such that

$$\langle m(A_N), x'_N \rangle = \|m(A_N)\| = \|m\|(A_N).$$

Then by assumption there is an element $0 \leqslant y'_N \in B_{X'}$ such that

$$\int_{A_N} h \, \mathrm{d}\langle m, x'_N \rangle \leqslant K_2 \langle m(A_N), y'_N \rangle$$

for all $N = 1, 2, \ldots$ On the other hand

$$\int_{A_N} h \, \mathrm{d}\langle m, x'_N \rangle \geqslant N \langle m(A_N), x'_N \rangle = N \|m\|(A_N).$$

Thus we have obtained that

$$N \|m\|(A_N) \leqslant K_2 \langle m(A_N), y'_N \rangle \leqslant K_2 \|m\|(A_N)$$

for all $N = 1, 2, \ldots$ But this can only happen if there exists an $N \geqslant 1$ such that $\|m\|(A_N) = 0$. Consequently, $h \in L^\infty(m)$. ∎

REMARK 2.8. Observe that $M_h(L^\infty(m_h)) \subseteq L^\infty(m)$ if and only if $h \in L^\infty(m)$. This shows that it is possible to find examples of isometric lattice isomorphisms between spaces $L^1(m)$ and $L^1(m_h)$ such that the corresponding bounded functions subspaces are not fixed by the isometry M_h; this only happens if the multiplication is defined by a bounded function h with bounded inverse $1/h$.

Finally, we obtain the following general result.

THEOREM 2.9. *Let $T : L^1(m_1) \to L^1(m_2)$ be a lattice isomorphism. Then $\Phi : [A] \in \Sigma_1[m_1] \to \Phi([A]) := [\mathrm{supp}(T\chi_A)] \in \Sigma_2[m_2]$ is a Boolean algebra isomorphism that satisfies:*

(C1′) *There exists a constant $K_1 > 0$ such that for each $0 \leqslant y' \in B_{Y'}$ and each $\pi \in \Pi(\Omega_1)$ there exists $0 \leqslant x' \in B_{X'}$ satisfying that*

$$\int_{\Phi(A)} T(\chi_{\Omega_1}) \, \mathrm{d}\langle m_2, y' \rangle \leqslant K_1 \langle m_1(A), x' \rangle, \quad A \in \pi.$$

(C2′) *There exists a constant $K_2 > 0$ such that for each $0 \leqslant x' \in B_{X'}$ and each $\pi \in \Pi(\Omega_1)$ there exists $0 \leqslant y' \in B_{Y'}$ satisfying*

$$\langle m_1(A), x' \rangle \leqslant K_2 \int_{\Phi(A)} T(\chi_{\Omega_1}) \, \mathrm{d}\langle m_2, y' \rangle, \quad A \in \pi.$$

Moreover, $T(\chi_A) = T(\chi_{\Omega_1}) \cdot \chi_{\Phi(A)}$ *for all* $A \in \Sigma_1$.

Proof. For the proof of this result we use Theorem 2.2. Let us denote by h the function $h := T(\chi_{\Omega_1})$ and observe that $0 < h \in L^1(m_2)$. Now consider the measure $m_{2h} : A \in \Sigma_2 \to m_{2h}(A) := \int_A h \, \mathrm{d}m_2 \in Y$ and the corresponding space $L^1(m_{2h})$. We know by Lemma 2.5 that the multiplication

$$D_h : f \in L^1(m_2) \to D_h(f) := \frac{f}{h} \in L^1(m_{2h})$$

is an isometric lattice isomorphism such that $D_h(h) = \chi_{\Omega_2}$. Thus, the composition operator $\widetilde{T} := D_h T$, that is,

$$L^1(m_1) \xrightarrow{\ T\ } L^1(m_2) \xrightarrow{\ D_h\ } L^1(m_{2h})$$

is a lattice isomorphism that satisfies $\widetilde{T}(\chi_{\Omega_1}) = D_h T(\chi_{\Omega_1}) = D_h(h) = \chi_{\Omega_2}$. Thus, $\widetilde{T}(L^\infty(m_1)) = L^\infty(m_{2h})$. An application of Theorem 2.2 gives that

$$\Phi : [A] \in \Sigma_1[m_1] \to \Phi([A]) := [\operatorname{supp}(\widetilde{T}\chi_A)] \in \Sigma_2[m_2]$$

is a Boolean algebra isomorphism that satisfies the conditions (C1) and (C2) for the measures m_1 and m_{2h}. Moreover, for each $A \in \Sigma_1$ it holds that

$$\widetilde{T}(\chi_A) = \widetilde{T}(\chi_{\Omega_1}) \cdot \chi_{\Phi(A)} = \chi_{\Omega_2} \cdot \chi_{\Phi(A)} = \chi_{\Phi(A)},$$

that is, $D_h T(\chi_A) = \chi_{\Phi(A)}$. So we can conclude that

$$T(\chi_A) = h \cdot \chi_{\Phi(A)} = T(\chi_{\Omega_1}) \cdot \chi_{\Phi(A)}$$

for all $A \in \Sigma_1$. Now let us note that for all $A \in \Sigma_1$ we have

$$[\operatorname{supp}(\widetilde{T}(\chi_A))] = [\operatorname{supp}(D_h T(\chi_A))] = \Big[\operatorname{supp}\Big(\frac{1}{h} T(\chi_A)\Big)\Big] = [\operatorname{supp}(T(\chi_A))]$$

and also for each $0 \leqslant y' \in B_{Y'}$ and each $\pi \in \Pi(\Omega_1)$ it is satisfied that

$$\langle m_{2h}(\Phi(A)), y' \rangle = \Big\langle \int_{\Phi(A)} h \, \mathrm{d}m_2, y' \Big\rangle = \int_{\Phi(A)} h \, \mathrm{d}\langle m_2, y' \rangle = \int_{\Phi(A)} T(\chi_{\Omega_1}) \, \mathrm{d}\langle m_2, y' \rangle$$

for all $A \in \pi$. This implies the conditions (C1′) and (C2′). ∎

3. LATTICE ISOMORPHISMS BETWEEN SPACES OF INTEGRABLE FUNCTIONS WITH RESPECT TO GENERAL VECTOR MEASURES

In this section we shall consider any couple of spaces of vector measures $(\Omega_1, \Sigma_1, m_1)$ and $(\Omega_2, \Sigma_2, m_2)$. Assume that the measure m_1 takes values in the Banach space X and m_2 takes values in the Banach space Y. With the appropriate changes in the conditions (C1) and (C2) we can prove an analogous to Theorem 2.1 for a general couple of vector measures. It is the following result.

THEOREM 3.1. *Let $\Phi : \Sigma_1[m_1] \to \Sigma_2[m_2]$ be an isomorphism of Boolean algebras such that:*

(i) *There exists a constant $K_1 > 0$ such that for each $y' \in B_{Y'}$ and each $\pi \in \Pi(\Omega_1)$ there exists an element $x' \in B_{X'}$ satisfying that*

$$|\langle m_2, y'\rangle|(\Phi(A)) \leqslant K_1 |\langle m_1, x'\rangle|(A), \quad A \in \pi.$$

(ii) *There exists a constant $K_2 > 0$ such that for each $x' \in B_{X'}$ and each $\pi \in \Pi(\Omega_1)$ there exists an element $y' \in B_{Y'}$ satisfying that*

$$|\langle m_1, x'\rangle|(A) \leqslant K_2 |\langle m_2, y'\rangle|(\Phi(A)), \quad A \in \pi.$$

Then there is a unique lattice isomorphism $T : L^1(m_1) \to L^1(m_2)$ such that

$$T(L^\infty(m_1)) = L^\infty(m_2).$$

Moreover, the restriction $T : L^\infty(m_1) \to L^\infty(m_2)$ is an isometric multiplicative lattice isomorphism that satisfies $T(\chi_A) = \chi_{\Phi(A)}$ for all $A \in \Sigma_1$. In particular, $T(\chi_{\Omega_1}) = \chi_{\Omega_2}$.

The proof is similar to that of Theorem 2.1.

Now we study if it is possible to obtain a sort of converse of Theorem 3.1; this would provide an analogous result to Theorem 2.2 for *any* vector measure, without the restriction of being positive. The following proposition is probably well-known but we include here a simple proof for the sake of completeness.

PROPOSITION 3.2. *Suppose that (Ω, Σ, m) is any vector measure space with the measure m taking values in the Banach space X. Then $L^1(m)$ is lattice isometric to $L^1(\widehat{m})$, where $\widehat{m}$ is the positive vector measure given by $\widehat{m} : A \in \Sigma \to \widehat{m}(A) := \chi_A \in L^1(m)$.*

Proof. Observe that we obtain

$$\|\widehat{m}\|(A) = \|\widehat{m}(A)\|_{L^1(m)} = \|\chi_A\|_{L^1(m)} = \|m\|(A), \quad A \in \Sigma,$$

since the measure $\widehat{m}$ is positive. Thus, null sets coincide for both measures. On the other hand, for each simple function $\varphi := \sum_{k=1}^{n} \alpha_k \chi_{A_k}$,

$$\begin{aligned}\|\varphi\|_{L^1(\widehat{m})} &= \Big\|\sum_{k=1}^{n} \alpha_k \chi_{A_k}\Big\|_{L^1(\widehat{m})} = \Big\|\int_\Omega \Big(\sum_{k=1}^{n} |\alpha_k| \chi_{A_k}\Big) \mathrm{d}\widehat{m}\Big\|_{L^1(m)} = \Big\|\sum_{k=1}^{n} |\alpha_k| \widehat{m}(A_k)\Big\|_{L^1(m)} \\ &= \Big\|\sum_{k=1}^{n} |\alpha_k| \chi_{A_k}\Big\|_{L^1(m)} = \Big\|\sum_{k=1}^{n} \alpha_k \chi_{A_k}\Big\|_{L^1(m)} = \|\varphi\|_{L^1(m)}.\end{aligned}$$

This means that the integration map

$$\mathfrak{I} : f \in L^1(\widehat{m}) \to \mathfrak{I}(f) := \int_\Omega f \mathrm{d}\widehat{m} \in L^1(m),$$

which is always linear and continuous, is, in fact, an isometry from $L^1(\widehat{m})$ onto $L^1(m)$. Moreover, $\mathfrak{I}(f) = \int_\Omega f \mathrm{d}\widehat{m} = f$ for all $f \in L^1(\widehat{m})$, and then $L^1(\widehat{m})$ and $L^1(m)$ are lattice isometric. ∎

THEOREM 3.3. *Let* $T : L^1(m_1) \to L^1(m_2)$ *be a lattice isomorphism. Then* $\Phi : [A] \in \Sigma_1[m_1] \to \Phi([A]) := [\mathrm{supp}(T\chi_A)] \in \Sigma_2[m_2]$ *is a Boolean algebra isomorphism. Moreover,* $T(\chi_A) = T(\chi_{\Omega_1}) \cdot \chi_{\Phi(A)}$ *for all* $A \in \Sigma_1$.

Proof. Consider the corresponding positive measures $\widehat{m}_1$ and $\widehat{m}_2$ and the corresponding spaces $L^1(\widehat{m}_1)$ and $L^1(\widehat{m}_2)$. After Proposition 3.2 we know that the following diagram commutes:

$$\begin{array}{ccc} L^1(m_1) & \xrightarrow{\ T\ } & L^1(m_2) \\ {\scriptstyle \mathfrak{I}_1}\big\uparrow & & \big\uparrow{\scriptstyle \mathfrak{I}_2} \\ L^1(\widehat{m}_1) & \xrightarrow{\ \widehat{T}\ } & L^1(\widehat{m}_2), \end{array}$$

where $\widehat{T} : L^1(\widehat{m}_1) \to L^1(\widehat{m}_2)$ is the lattice isomorphism that coincides with T, that is, $\widehat{T}(f) = T(f)$ for all $f \in L^1(\widehat{m}_1) = L^1(m_1)$. Then, since the measures $\widehat{m}_1$ and $\widehat{m}_2$ are both positive, we obtain as a consequence of Theorem 2.9 that

$$\Phi : [A] \in \Sigma_1[m_1] \to \Phi([A]) := [\mathrm{supp}(T\chi_A)] \in \Sigma_2[m_2]$$

is a Boolean algebra isomorphism satisfying $T(\chi_A) = T(\chi_{\Omega_1}) \cdot \chi_{\Phi(A)}$ for all $A \in \Sigma_1$. ∎

REMARK 3.4. It seems that it is not possible to obtain more accurate results on the Boolean algebra isomorphism

$$\Phi : [A] \in \Sigma_1[m_1] \to \Phi([A]) := [\mathrm{supp}(T\chi_A)] \in \Sigma_2[m_2].$$

However, using the notation above, we also obtain from Theorem 2.9 that the conditions (C1′) and (C2′) hold for the measures $\widehat{m}_1$ and $\widehat{m}_2$. Let us show what the condition (C1′) means in terms of the measures m_1 and m_2. If $y' \in B_{Y'}$, the functional

$$\lambda : f \in L^1(m_2) \to \langle f, \lambda \rangle := \int_{\Omega_2} f \mathrm{d}|\langle m_2, y' \rangle| \in \mathbb{R},$$

satisfies $0 \leqslant \lambda \in B_{(L^1(m_2))'}$. Moreover, observe that

$$\langle \widehat{m}_2, \lambda \rangle (B) = \langle \chi_B, \lambda \rangle = \int_{\Omega_2} \chi_B \, \mathrm{d}|\langle m_2, y' \rangle| = |\langle m_2, y' \rangle|(B),$$

for all $B \in \Sigma_2$, that is, the measures $\langle \widehat{m}_2, \lambda \rangle$ and $|\langle m_2, y' \rangle|$ coincide on Σ_2. Then, (C1′) implies in particular that for each $A \in \Sigma_1$, there is an element $0 \leqslant \mu \in B_{(L^1(m_1))'}$ such that

$$\int_{\Phi(A)} T(\chi_{\Omega_1}) \mathrm{d}|\langle m_2, y' \rangle| \leqslant K_1 \langle \widehat{m}_1(A), \mu \rangle = K_1 \langle \chi_A, \mu \rangle \leqslant K_1 \|\chi_A\|_{L^1(m_1)} = K_1 \|m_1\|(A).$$

This means that $\|T(\chi_{\Omega_1}) \cdot \chi_{\Phi(A)}\|_{L^1(m_2)} \leqslant K_1 \|m_1\|(A)$ for all $A \in \Sigma_1$.

Using a similar argument we can deduce from the condition (C2′) that, for all $A \in \Sigma_1$,

$$\|m_1\|(A) \leqslant K_2 \|T(\chi_{\Omega_1}) \cdot \chi_{\Phi(A)}\|_{L^1(m_2)}.$$

Summing up the comments above we deduce from the conditions (C1′) and (C2′) jointly that there are two constants $M_1 > 0$ and $M_2 > 0$ such that, for all $A \in \Sigma_1$,

$$M_2 \|m_1\|(A) \leqslant \|T(\chi_{\Omega_1}) \cdot \chi_{\Phi(A)}\|_{L^1(m_2)} \leqslant M_1 \|m_1\|(A). \tag{3.1}$$

In the particular case that the lattice isomorphism

$$T : L^1(m_1) \to L^1(m_2)$$

also satisfies $T(L^\infty(m_1)) = L^\infty(m_2)$, the inequalities (3.1) are simply, for all $A \in \Sigma_1$,

$$M_2 \|m_1\|(A) \leqslant \|m_2\|(\Phi(A)) \leqslant M_1 \|m_1\|(A). \tag{3.2}$$

However, as we show in the following example the conditions given by (3.1) or (3.2) for the Boolean algebra isomorphism Φ do not mean at all that we can define a lattice isomorphism between the spaces $L^1(m_1)$ and $L^1(m_2)$ through the isomorphism Φ, as it has been done in Theorem 3.1. And what is more, we don't know if the associated Boolean algebra isomorphism Φ of a lattice isomorphism $T : L^1(m_1) \to L^1(m_2)$ satisfies the conditions (i) and (ii).

EXAMPLE 3.5 (See Example 16 of [9]). Let $1 \leqslant q < p < \infty$ be real numbers. Consider the Lebesgue space L^p and the Lorentz space $L^{p,q}$, both of them over the interval $[0, 1]$. Both of them are Banach lattices with order continuous norm and weak order unit. Now consider the positive vector measures defined over the Borel σ-algebra $\mathcal{B}$ on $[0, 1]$ given by

$$m_1 : A \in \mathcal{B} \to m_1(A) := \chi_A \in L^{p,q}, \quad m_2 : A \in \mathcal{B} \to m_2(A) := \chi_A \in L^p.$$

It is known ([2], Theorem 8) that $L^1(m_1) = L^{p,q}$, and $L^1(m_2) = L^p$. Then the spaces $L^1(m_1)$ and $L^1(m_2)$ cannot be isomorphic if $p < q$. On the other hand, a simple calculation shows that

$$\|m_1\|(A) = \|\chi_A\|_{L^{p,q}} = \left[\frac{p}{q}\right]^{1/q} \|\chi_A\|_{L^p} = \left[\frac{p}{q}\right]^{1/q} \|m_2\|(A),$$

for all $A \in \mathcal{B}$. So the measures m_1 and m_2 satisfy the inequalities (3.2) taking the identity as the isomorphism of Boolean algebras.

4. THE STRUCTURE OF LATTICE ISOMORPHISMS BETWEEN SPACES OF INTEGRABLE FUNCTIONS

In this section we consider two vector measure spaces $(\Omega_1, \mathcal{B}_1, m_1)$ and $(\Omega_2, \mathcal{B}_2, m_2)$, where Ω_1 and Ω_2 are Borel spaces and $\mathcal{B}_1$ and $\mathcal{B}_2$ are the corresponding σ-algebras of the Borel sets of Ω_1 and Ω_2, respectively. We assume that the measure m_1 takes values in the Banach space X and the measure m_2 takes values in the Banach space Y. If we suppose that the measures m_1 and m_2 are positive we assume in addition that X and Y are Banach lattices. Recall that a topological space Ω is a *Borel space* if it is homeomorphic to a Borel subset of the Hilbert cube $\mathcal{H} := \prod_{n\geqslant 1} [0,1]$. It is well-known (see Section 35.III of [12] or Corollary of Theorem II.1.1 in [17]) that any separable complete metric space Ω is a Borel space. Observe that almost all measurable spaces that are usually considered for defining vector (or scalar) measures are of the form $(\Omega, \mathcal{B})$, where Ω is a Borel space and $\mathcal{B}$ is the σ-algebra of the Borel sets of Ω.

The aim here is to describe relationships between Ω_1 and Ω_2 starting from a lattice isomorphism $T : L^1(m_1) \to L^1(m_2)$. Also we present a result on the structure of the lattice isomorphisms between spaces of integrable functions. In a sense, our result is similar to the one of Lamperti (see Theorem 3.1 of [13] or Theorem 15.5.16 of [16]) on the structure of the isometries between spaces of integrable functions with respect to scalar measures.

Sikorski study in [18] Boolean algebra isomorphisms between quotient algebras. In particular, the following result holds (see 6.1 of [18] or Theorem 15.4.12 of [16] for a simple proof).

THEOREM 4.1 (Sikorski). *Let $(\Omega_1, \mathcal{B}_1, m_1)$ and $(\Omega_1, \mathcal{B}_1, m_1)$ be two vector measure spaces, where Ω_1 and Ω_2 are Borel spaces. For each Boolean algebra isomorphism $\Phi : \mathcal{B}_1[m_1] \to \mathcal{B}_2[m_2]$ there are two sets $N_1 \in \mathcal{N}_1$ and $N_2 \in \mathcal{N}_2$ and a map $\alpha : \Omega_2 \setminus N_2 \to \Omega_1 \setminus N_1$ that is a measurable bijection with measurable inverse such that*

$$\Phi([A]) := [\alpha^{-1}(A)], \quad A \in \Sigma_1.$$

In this case it is said that Φ is induced by the measurable map α. Also note that

$$\chi_{\Phi(A)} = \chi_{\alpha^{-1}(A)} = \chi_A \circ \alpha, \quad A \in \Sigma_1. \tag{4.1}$$

THEOREM 4.2. *Let $T : L^1(m_1) \to L^1(m_2)$ be a lattice isomorphism. Then there is a function $0 < h \in L^1(m_2)$ and a map $\alpha : \Omega_2 \to \Omega_1$, defined almost everywhere, bijective, measurable with measurable inverse, such that $T(f) = h \cdot f \circ \alpha$ for all $f \in L^1(m_1)$. If moreover the measures m_1 and m_2 are positive the following holds:*

(i) *There exists a constant $K_1 > 0$ such that for each $0 \leqslant y' \in B_{Y'}$ and each $\pi \in \Pi(\Omega_1)$ there is an element $0 \leqslant x' \in B_{X'}$ satisfying*

$$\int_{\alpha^{-1}(A)} h \mathrm{d}\langle m_2, y' \rangle \leqslant K_1 \langle m_1(A), x' \rangle, \quad A \in \pi.$$

(ii) *There is a constant $K_2 > 0$ such that for each $0 \leqslant x' \in B_{X'}$ and each $\pi \in \Pi(\Omega_1)$ there exists an element $0 \leqslant y' \in B_{Y'}$ satisfying*

$$\langle m_1(A), x' \rangle \leqslant K_2 \int_{\alpha^{-1}(A)} h \mathrm{d}\langle m_2, y' \rangle, \quad A \in \pi.$$

Proof. Let us denote by h the function $T(\chi_{\Omega_1})$. By Theorem 3.3, we know that $\Phi : [A] \in \mathcal{B}_1[m_1] \to \Phi([A]) := [\operatorname{supp}(T\chi_A)] \in \mathcal{B}_2[m_2]$ is a Boolean algebra isomorphism. Moreover it is satisfied that $T(\chi_A) = h \cdot \chi_{\Phi(A)}$ for all $A \in \Sigma_1$. By Sikorski's Theorem, there are two sets $N_1 \in \mathcal{N}_1$ and $N_2 \in \mathcal{N}_2$ and a map $\alpha : \Omega_2 \setminus N_2 \to \Omega_1 \setminus N_1$ that is a measurable bijection with measurable inverse such that $\Phi([A]) := [\alpha^{-1}(A)]$ for all $A \in \mathcal{B}_1$. From (4.1) we obtain that

$$T(\chi_A) = h \cdot \chi_A \circ \alpha$$

for all $A \in \mathcal{B}_1$, and then $T(\varphi) = h \cdot \varphi \circ \alpha$ for every simple function $\varphi \in S(m_1)$. Since the operator T is continuous and taking into account that for each function $0 < f \in L^1(m_1)$ there exists a sequence $(\varphi_n)_n \subseteq S(m_1)$ such that $0 < \varphi_n \uparrow f$ pointwise and in $L^1(m_1)$, we obtain $T(f) = h \cdot f \circ \alpha$, for all $f \in L^1(m_1)$.

Moreover, if the measures m_1 and m_2 are also positive, the conditions (i) and (ii) can be directly deduced from Theorem 2.9. ∎

In the rest of the section we analyze the converse of the theorem above, concluding the necessity of its hypothesis. We start presenting the following result, that is relevant in our context since it provides a necessary and sufficient condition for a composition operator between spaces of integrable functions (modelled over general vector measure spaces, not necessarily Borel spaces) to be a lattice isomorphism (in particular continuous). Let $\alpha : \Omega_2 \to \Omega_1$ be a bijective map that is defined almost everywhere, measurable with measurable inverse such that $\|m_2\|(\alpha^{-1}(A)) = 0$ if and only if $\|m_1\|(A) = 0$ for all $A \in \Sigma_1$.

PROPOSITION 4.3. *The composition map C_α defines a lattice isomorphism from $L^1(m_1)$ into $L^1(m_2)$ if and only if the following conditions hold:*

(i) *There exists a constant $K_1 > 0$ such that for each $0 \leqslant y' \in B_{Y'}$ and each $\pi \in \Pi(\Omega_1)$ there is an element $0 \leqslant x' \in B_{X'}$ satisfying that*

$$\langle m_2(\alpha^{-1}(A)), y' \rangle \leqslant K_1 \langle m_1(A), x' \rangle, \quad A \in \pi.$$

(ii) *There exists a constant $K_2 > 0$ such that for each $0 \leqslant x' \in B_{X'}$ and each $\pi \in \Pi(\Omega_1)$ there exists $0 \leqslant y' \in B_{Y'}$ satisfying that*

$$\langle m_1(A), x' \rangle \leqslant K_2 \langle m_2(\alpha^{-1}(A)), y' \rangle, \quad A \in \pi.$$

Proof. Under conditions we have noted before referring to the map α it is not difficult to see that

$$\Phi : [A] \in \Sigma_1[m_1] \to \Phi([A]) := [\alpha^{-1}(A)] \in \Sigma_2[m_2]$$

is a Boolean algebra isomorphism. For this map the conditions (C1) and (C2) are exactly (i) and (ii), respectively. So, under these conditions, we know from Theorem 2.1 that there is a lattice isomorphism $T : L^1(m_1) \to L^1(m_2)$ such that $T(\chi_A) = \chi_A \circ \alpha$ for every $A \in \Sigma_1$. This clearly implies that $T(f) = f \circ \alpha$ for all $f \in L^1(m_1)$, that is, T coincides with the *composition operator* C_α defined as the composition with the function α.

Reciprocally, assume that $C_\alpha : L^1(m_1) \to L^1(m_2)$ is a lattice isomorphism. In particular, it is a bijection. Moreover, C_α and its inverse are continuous, that is, there are two constants $K_1 > 0$ and $K_2 > 0$ such that

$$\frac{1}{K_2}\|f\|_{L^1(m_1)} \leqslant \|f \circ \alpha\|_{L^1(m_2)} \leqslant K_1\|f\|_{L^1(m_1)} \tag{4.2}$$

for all $f \in L^1(m_1)$. So, using (4.2) and following the steps of the proof of Theorem 2.2 it can be proved that the conditions (i) and (ii) are satisfied. For instance, for proving the condition (i), take an element $0 \leqslant y' \in B_{Y'}$ and a partition $\{A_1, A_2, \ldots, A_N\} \in \Pi(\Omega_1)$. For each vector $0 \leqslant \mathbf{a} := (a_1, a_2, \ldots, a_N) \in \mathbb{R}^N$ we define the function $\Psi_{\mathbf{a}} : B^+_{X'} \to \mathbb{R}$ by

$$\Psi_{\mathbf{a}}(x') := \sum_{k=1}^{N} a_k(\langle m_2(\alpha^{-1}(A_k)), y'\rangle - K_1\langle m_1(A_k), x'\rangle).$$

Using, as in the previous case, the separation argument based on Ky Fan's Lemma ([6], Lemma 9.10) we find an element $0 \leqslant x'_0 \in B^+_{X'}$ such that $\Psi_{\mathbf{a}}(x'_0) \leqslant 0$ for all $0 \leqslant \mathbf{a} := (a_1, a_2, \ldots, a_N) \in \mathbb{R}^N$. In particular, if $\mathbf{a} = \mathbf{e}_k := (0, \ldots, 1^k, \ldots, 0) \in \mathbb{R}^N$ we obtain

$$\langle m_2(\alpha^{-1}(A_k)), y'\rangle \leqslant K_1\langle m_1(A_k), x'_0\rangle,$$

for each $k = 1, 2, \ldots, N$. That is exactly (i). ∎

REMARK 4.4. Observe that the conditions (i) and (ii) of Theorem 4.2 are exactly the conditions (i) and (ii) of Proposition 4.3 for the measures m_1 and m_{2h}, where the second one, as we know, is defined by

$$m_{2h} : A \in \Sigma \to m_{2h}(A) := \int_A h \, \mathrm{d}m \in X,$$

where $0 < h \in L^1(m)$. After Proposition 4.3, it is easy to see that Theorem 4.2 states that every lattice isomorphism $T : L^1(m_1) \to L^1(m_2)$ factorizes through a multiplication operator $M_h : L^1(m_{2h}) \to L^1(m_2)$, where $0 < h \in L^1(m_2)$, and a composition operator $C_\alpha : L^1(m_1) \to L^1(m_{2h})$, as the following commutative diagram shows:

$$\begin{tikzcd} L^1(m_1) \arrow[rr, "T"] \arrow[dr, "C_\alpha"'] & & L^1(m_2) \\ & L^1(m_{2h}) \arrow[ur, "M_h"'] & \end{tikzcd}$$

Now we get the converse result of Theorem 4.2 as a corollary of Proposition 4.3 and the commutative diagram above.

COROLLARY 4.5. *Let $\alpha : \Omega_2 \to \Omega_1$ be a bijective map defined almost everywhere, measurable and with measurable inverse between two Borel spaces, and let $0 < h \in L^1(m_2)$ satisfying the conditions* (i) *and* (ii) *of Theorem* 4.2. *Then*

$$T : f \in L^1(m_1) \to T(f) := h \cdot f \circ \alpha \in L^1(m_2)$$

is a lattice isomorphism.

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A. FERNÁNDEZ, DPTO. MATEMÁTICA APLICADA II, ESCUELA TÉCNICA SUPERIOR DE INGENIEROS, CAMINO DE LOS DESCUBRIMIENTOS, S/N, 41092-SEVILLA, SPAIN
E-mail address: afernandez@esi.us.es

F. MAYORAL, DPTO. MATEMÁTICA APLICADA II, ESCUELA TÉCNICA SUPERIOR DE INGENIEROS, CAMINO DE LOS DESCUBRIMIENTOS, S/N, 41092-SEVILLA, SPAIN
E-mail address: mayoral@us.es

F. NARANJO, DPTO. MATEMÁTICA APLICADA II, ESCUELA TÉCNICA SUPERIOR DE INGENIEROS, CAMINO DE LOS DESCUBRIMIENTOS, S/N, 41092-SEVILLA, SPAIN
E-mail address: naranjo@.us.es

E.A. SÁNCHEZ-PÉREZ, INSTITUTO UNIVERSITARIO DE MATEMÁTICA PURA Y APLICADA (I.U.M.P.A.), UNIVERSIDAD POLITÉCNICA DE VALENCIA, CAMINO DE VERA, S/N, 46022-VALENCIA, SPAIN
E-mail address: easancpe@mat.upv.es

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