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ISOMORPHISMS OF NON-COMMUTATIVE DOMAIN ALGEBRAS

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ABSTRACT. Noncommutative domain algebras were introduced by Popescu as the non-selfadjoint operator algebras generated by weighted shifts on the Full Fock space. This paper uses results from several complex variables to classify many noncommutative domain algebras, and it uses results from operator theory to obtain new bounded domains in $\mathbb{C}^n$ with non-compact automorphic group.

KEYWORDS: *Non-self-adjoint operator algebras, disk algebra, weighted shifts, biholomorphic domains, circular domains.*

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1. INTRODUCTION

Noncommutative domain algebras, introduced in [14], generalize the noncommutative disk algebras and are defined as norm closures of the algebras generated by a family of weighted shifts on the Full Fock space. This paper investigates the isomorphism problem for this class of algebras. The fundamental tool we use is the theory of functions in several complex variables in $\mathbb{C}^n$, which has been used in this context in [4] as well. Formally, we apply Cartan's Lemma [3] and Sunada's Theorem [17] to domains in $\mathbb{C}^n$ naturally associated with our noncommutative domain algebras to derive a first classification result: if there exists an isometric isomorphism between two noncommutative domain algebras whose dual map takes the zero character to the zero character, then the noncommutative domain algebras are related via a simple linear transformation of their generators. When $n = 2$, Thullen characterization of domains of $\mathbb{C}^2$ with non-compact automorphic group [18] gives more information.

An application of our result is that there are many non-isomorphic noncommutative domain algebras. In addition, we characterize the disk algebras among all noncommutative domain algebras, and we obtain interesting examples of new domains with noncompact automorphic group in $\mathbb{C}^n$.

Our paper is structured as follows. We first present a survey of the field of noncommutative domain algebras, which allows us to introduce our notations as well. We then construct contravariant functors between noncommutative domain algebras with completely contractive unital homomorphisms and domains in $\mathbb{C}^m$ with holomorphic maps and deduce from them a few classification results. Last, we present several applications to important examples of noncommutative domains.

In this paper, the set $\mathbb{N}\backslash\{0\}$ of nonzero natural numbers will be denoted by $\mathbb{N}^*$. The identity map on a Hilbert space $\mathcal{H}$ will be denoted by $1_{\mathcal{H}}$, or simply 1 when no confusion may arise. The algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$.

We also will adopt the convention that if χ is an n-tuple of elements in a set E then for all $i = 1, \ldots, n$ we write χ_i for its i^{th} projection, i.e. $\chi = (\chi_1, \ldots, \chi_n)$.

The domain algebras are represented by formal power series of "free" variables $X_1, \ldots, X_n$. For example, the non-commutative disc algebra with n variable is given by $X_1 + \cdots + X_n$. To illustrate some computations and results of the paper we look at the noncommutative domains with free formal power series

$$f = X_1 + X_2 + X_1 X_2 \quad \text{and} \quad g = X_1 + X_2 + \frac{1}{2} X_1 X_2 + \frac{1}{2} X_2 X_1$$

and we prove that they are not isomorphic. We consider these algebras as test examples. Their power series are close to the power series of the noncommutative disc algebra and they cannot be distinguished by their 1-dimensional representations.

2. BACKGROUND

This first section allows us to introduce the category of noncommutative domain algebras. These algebras were introduced in [14] and much notation is required for a proper description, so this section is also presented as a guide for the reader about these objects.

2.1. THE FULL HILBERT–FOCK SPACE OF A HILBERT SPACE. We start by introducing notations which we will use all throughout this paper. For any Hilbert space $\mathcal{H}$, the full Fock space $\mathcal{F}(\mathcal{H})$ of $\mathcal{H}$ is the completion of:

$$\bigoplus_{k \in \mathbb{N}} \mathcal{H}^{\otimes k} = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots$$

for the Hilbert norm associated to the inner product $\langle \cdot, \cdot \rangle$ defined on elementary tensors by:

$$\langle \xi_0 \otimes \cdots \otimes \xi_m, \zeta_0 \otimes \cdots \otimes \zeta_k \rangle = \begin{cases} 0 & \text{if } m \neq k, \\ \prod_{j=0}^{n} \langle \xi_j, \zeta_j \rangle_{\mathcal{H}} & \text{otherwise,} \end{cases}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the inner product on $\mathcal{H}$. We shall denote $\mathcal{F}(\mathbb{C}^n)$ by $\mathcal{F}_n$ for all $n \in \mathbb{N} \backslash \{0\}$. We now shall exhibit, for any nonzero natural number n, a natural isomorphism of Hilbert space between $\mathcal{F}_n$ and $\ell^2(\mathbb{F}_n^+)$, where $\mathbb{F}_n^+$ is the free semigroup on n generators $g_1, \ldots, g_n$ with identity element e.

Let $n \in \mathbb{N}^*$ be fixed and let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathbb{C}^n$. For any word $\alpha \in \mathbb{F}_n^+$ we define:

$$e_\alpha = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_{|\alpha|}}$$

where $|\alpha|$ is the length of α and $\alpha = g_{i_1} \cdots g_{i_{|\alpha|}}$. Note that this decomposition of α is unique in the semigroup $\mathbb{F}_n^+$. The map which sends, for any $\alpha \in \mathbb{F}_n^+$, the element $e_\alpha \in \mathcal{F}_n$ to the series $\delta_\alpha \in \ell^2(\mathbb{F}_n^+)$ defined by

$$\forall \beta \in \mathbb{F}_n^+ \quad \delta_\alpha(\beta) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

extends to a linear isomorphism from $\mathcal{F}_n$ onto $\ell^2(\mathbb{F}_n^+)$. The easy proof is left to the reader. In this paper, we will identify these two Hilbert spaces.

2.2. LEFT CREATION OPERATORS AND NONCOMMUTATIVE DISK ALGEBRAS.

At the root of the study of disk algebras reside the concept of row contractions.

DEFINITION 2.1. Let $\mathcal{H}$ be a Hilbert space and $n \in \mathbb{N}^*$. An *n-row contraction* on $\mathcal{H}$ is an n-tuple of operators $(T_1, \ldots, T_n)$ on $\mathcal{H}$ such that:

$$\sum_{i=0}^{n} T_i T_i^* \leqslant 1_{\mathcal{H}}.$$

In other words, the operator $T = [T_1 \ T_2 \ \cdots \ T_n]$ acting on $\mathcal{H}^n$ is a contraction since $TT^* \leqslant 1_{\mathcal{H}_n}$. We now construct a row contraction which plays a fundamental role in our study.

DEFINITION 2.2. Let $n \in \mathbb{N}^*$. For $i \in \{1, \ldots, n\}$, the i^{th} left creation operator S_i on $\mathcal{F}_n$ is the unique continuous linear extension of:

$$\forall \alpha \in \mathbb{F}_n^+ \quad S_i(\delta_\alpha) = \delta_{g_i \alpha}.$$

We note that the linear extension of $\delta_\alpha \mapsto \delta_{g_i\alpha}$ is an isometry on the span of $\{\delta_\alpha : \alpha \in \mathbb{F}_n^+\}$ and therefore is uniformly continuous on this span which is dense in $\mathcal{F}_n$ so the continuous extension is well-defined and unique. One can easily check that

PROPOSITION 2.3. *The n-tuple $(S_1, \ldots, S_n)$ is a row contraction of isometries with orthogonal ranges.*

In this paper, we will focus our attention on algebras generated by various shifts operators. We introduce:

DEFINITION 2.4. The *noncommutative disk algebra $\mathcal{A}_n$* is the norm closure of the algebra generated by $\{S_1, \ldots, S_n, 1\}$.

This algebra enjoys the following remarkable property.

THEOREM 2.5 ([11]). *Let $(T_1, \dots, T_n)$ be a n-tuple of operators on a Hilbert space $\mathcal{H}$. Then $(T_1, \dots, T_n)$ is a row contraction if and only if there exists a completely bounded unital morphism $\psi : \mathcal{A}_n \longrightarrow \mathcal{B}(\mathcal{H})$ such that $\psi(S_i) = T_i$ for $i = 1, \dots, n$.*

We will sketch a proof of this result at the end of this section. But first, we notice that Theorem 2.5 formalizes the fundamental idea that $(S_1, \dots, S_n)$ are universal among all row contractions. We also notice the following useful corollary.

COROLLARY 2.6 ([12]). *The spectrum of $\mathcal{A}_n$ (i.e. the set of its characters, or one dimensional representations) is the closed unit ball of $\mathbb{C}^n$.*

Proof. A character of $\mathcal{A}_n$ is fully defined by its image on the generators $S_1, \dots, S_n$. Moreover, any unital morphism from $\mathcal{A}_n$ into the Abelian algebra $\mathbb{C}$ is completely contractive. Let $(z_1, \dots, z_n) \in \mathbb{C}^n$. Then by Theorem 2.5 there exists a morphism ψ from $\mathcal{A}_n$ into $\mathbb{C}$ with $\psi(S_i) = z_i$ for all $i = 1, \dots, n$ if and only if $[z_1 \ \cdots \ z_n]$ is a row contraction, i.e. $1 \geqslant \sum_{i=1}^{n} z_i \bar{z}_i = \sum_{i=1}^{n} |z_i|^2$ as claimed. ∎

We now sketch a proof of Theorem 2.5. The original proof of Theorem 2.5 combined a dilation theorem for row contractions with a Wold decomposition of isometric row contractions. In [13], Popescu found a shorter proof using Poisson transforms, which are explicit dilations for row contractions $(T_1, \dots, T_n)$ with the additional property that, if $\varphi : A \in \mathcal{B}(\mathcal{H}) \mapsto \sum_{i=1}^{n} T_i A T_i^*$ then $\lim_{k\to\infty} \varphi^k(A) = 0$ in the strong operator topology for all $A \in \mathcal{B}(\mathcal{H})$. Such row contractions will be called $C_{\cdot 0}$-row contraction. The proof goes as follows: assume that $T = (T_1, \dots, T_n)$ is a $C_{\cdot 0}$-row contraction on a Hilbert space $\mathcal{H}$ and let:

$$\Delta = \Big(I - \sum_{i \leqslant n} T_i T_i^*\Big)^{1/2}.$$

Define $K : \mathcal{H} \to \ell_2(\mathbb{F}_n^+) \otimes \mathcal{H}$ by

$$\forall h \in \mathcal{H} \quad K(h) = \sum_{\alpha \in \mathbb{F}_n^+} \delta_a \otimes \Delta(T_\alpha)^*(h).$$

It is not hard to check that K is an isometry that satisfies:

$$\forall i \in \{1, \dots, n\} \quad K^*(S_i \otimes 1) = T_i K^*.$$

From this it follows that if we set

$$\Phi : a \in \mathcal{A}_n \longmapsto K^*(a \otimes I)K$$

then Φ is a completely contractive unital homomorphism which satisfies Theorem 2.5. The map K is called the Poisson kernel of $(T_1, \dots, T_n)$, and the map Φ is called the Poisson transform of $(T_1, \dots, T_n)$.

The general case for row contractions follows from the observation that if $T = (T_1, \ldots, T_n)$ is a row contraction, then $T_r = (rT_1, \ldots, rT_n)$ is a $C_{\cdot 0}$-row contraction for $0 < r < 1$. One then concludes by taking the limit as $r \to 1$.

The converse assertion in Theorem 2.5 is clear.

The Poisson transform is a useful tool to study these algebras. For example in [2] Poisson transforms were used to obtain multivariate non-commutative and commutative Nevanlinna–Pick and Caratheodory interpolation theorems. The same results were obtained in [5] using different methods. Multivariate interpolation problems, particularly commutative ones, have received a lot of attention in recent years.

This paper deals with natural generalizations of $\mathcal{A}_n$ where the left creation operators are replaced by weighted shifts. We present these notions in the next section.

2.3. WEIGHTED SHIFTS AND NONCOMMUTATIVE DOMAIN ALGEBRAS. Popescu and the first author [1] proved that the construction of a Poisson kernel and Poisson transform, as was done previously for row-contractions, could be extended to a class of weighted shifts $(T_1, \ldots, T_n)$ satisfying the condition that

$$\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha T_\alpha T_\alpha^* \leqslant 1$$

for some coefficients $a_\alpha \geqslant 0$ $(\alpha \in \mathbb{F}_n^+)$ in lieu of $\sum\limits_{i=1}^{n} T_i T_i^* \leqslant 1$ and with the notations defined below.

DEFINITION 2.7. Let $\alpha \in \mathbb{F}_n^+$ and $T_1, \ldots, T_n$ be operators on $\mathcal{H}$. Then $T_\alpha = T_{i_1} \cdots T_{i_{|\alpha|}}$ where $\alpha = g_{i_1} \cdots g_{i_{|\alpha|}}$. In other words, $\alpha \mapsto T_\alpha$ is the unique unital multiplicative map such that $g_i \mapsto T_i$ for $i = 1, \ldots, n$.

The coefficients $(a_\alpha)_{\alpha \in \mathbb{F}_n^+}$ were defined in relation with the weight of the shifts $(T_1, \ldots, T_n)$ as in a paper of Quiggin [15] and were known to satisfy $a_{g_i} > 0$ for all $i = 1, \ldots, n$ and $a_\alpha \geqslant 0$ for $|\alpha| \geqslant 1$. They were in general difficult to compute.

In [14], Popescu generalizes further the study of weighted shifts and their associated algebras by starting with a general collection of nonnegative coefficients $(a_\alpha)_{\alpha \in \mathbb{F}_n^+}$ satisfying the following conditions.

DEFINITION 2.8. Let $(a_\alpha)_{\alpha \in \mathbb{F}_n^+}$ be a family of nonnegative numbers and consider the formal power series in n free variables $X_1, \ldots, X_n$ defined by $f = \sum a_\alpha X_\alpha$ with $X_a = X_{i_1} X_{i_2} \cdots X_{i_{|\alpha|}}$ when $\alpha = g_{i_1} \cdots g_{i_{|\alpha|}}$. Then f is called a *positive regular n-free formal power series* when the following conditions are met:

$$a_e = 0; \quad a_{g_i} > 0 \quad \text{for all } i \in \{1, \ldots, n\}; \tag{2.1}$$

$$\sup_{n \in \mathbb{N}^*} \Big(\Big| \sum_{|\alpha| = n} a_\alpha^2 \Big| \Big)^{1/n} = M < \infty.$$

Popescu then produced a universal model for all operator n-tuples satisfying $\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha T_\alpha T_\alpha^* \leqslant 1$ such that $f = \sum a_\alpha X_\alpha$ is a positive regular n-free formal power series, based upon weighted shifts, in a manner similar to the above constructions of the disk algebra. More formally, he proves in [14] the following fundamental result.

THEOREM 2.9 ([14]). *Let us be given a positive regular n-free formal power series $f = \sum a_\alpha X_\alpha$. Then there exists positive real numbers $(b_\alpha)_{\alpha \in \mathbb{F}_n^+}$ such that, if for all $i = 1, \ldots, n$ we define W_i^f as the unique continuous linear extension of*

$$W_i^f : \delta_\alpha \in \mathcal{F}_n \mapsto \sqrt{\frac{b_\alpha}{b_{g_i \alpha}}} \delta_{g_i \alpha};$$

then

$$\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha W_\alpha^f W_\alpha^{f*} \leqslant 1.$$

Moreover, if $(T_1, \ldots, T_n)$ is an n-tuple of operators acting on some Hilbert space $\mathcal{H}$, then $\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha T_\alpha T_\alpha^* \leqslant 1$ if and only if there exists a unique completely contractive morphism of algebra Φ from the algebra generated by $\{W_1^f, \ldots, W_n^f\}$ into $\mathcal{B}(\mathcal{H})$ such that $W_i^f \mapsto T_i$ for all $i = 1, \ldots, n$.

The fundamental role played by the algebra generated by the weighted shifts $W_1^f, \ldots, W_n^f$ in Theorem 2.9 justifies the following definition, where we use the notations of Theorem 2.9.

DEFINITION 2.10. Let $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$ be a positive regular n-free formal power series. The algebra $\mathcal{A}(\mathcal{D}_f)$ is the norm closure of the algebra generated by the weighted shifts $W_1^f, \ldots, W_n^f$ of Theorem 2.9 and the identity. The algebra $\mathcal{A}(\mathcal{D}_f)$ is called the *noncommutative domain algebra associated to f.*

In addition, following Popescu's notations, we define the noncommutative domain associated to a positive regular free formal power series f in a Hilbert space $\mathcal{H}$ as the "preimage" of the unit ball by f i.e.

DEFINITION 2.11. Let $\mathcal{H}$ be a Hilbert space and $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$ be a positive regular n-free formal power series. Then:

$$\mathcal{D}_f(\mathcal{H}) = \Big\{ (T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{H}) : \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha T_\alpha T_\alpha^* \leqslant 1 \Big\}.$$

The following corollary of Theorem 2.9 will play a fundamental role in our paper, and its proof is simply a direct application of Theorem 2.9.

COROLLARY 2.12. *Let $T = (T_1, \dots, T_n) \in M^n_{k\times k}$. Then $T \in \mathcal{D}_f(\mathbb{C}^k)$ if and only if there exists a necessarily unique completely contractive unital algebra morphism $\Phi : \mathcal{A}(\mathcal{D}_f) \longrightarrow M_{k\times k}$ such that $\Phi(W_i^f) = T_i$ for $i \leqslant n$.*

NOTATION 2.13. Sometimes we denote the Φ of Corollary 2.12 by $\langle T, \cdot\rangle^n_f$ and when no confusion may arise, we will denote $\langle T, \cdot\rangle^k_f$ simply as $\langle T, \cdot\rangle$. That is,

$$(\langle T, W_1^f\rangle^k_f, \dots, \langle T, W_n^f\rangle^k_f) = T.$$

We also will write $\varphi(T)$ for $\langle T, \varphi\rangle$ for any $\varphi \in \mathcal{A}(\mathcal{D}_f)$ and $T \in \mathcal{D}_f(\mathbb{C}^k)$ following the notation in [14] that emphasizes the functional calculus of $\mathcal{A}(\mathcal{D}_f)$ (see Lemma 3.6). The notation $\langle \cdot, \cdot\rangle$ is meant to emphasize the role of duality in our present work.

For the reader convenience, and as a mean to fix our notation, we present a sktech of the proof of Theorem 2.9. We refer the reader to [14] for the details of this proof and the development of the general theory of noncommutative domain algebras.

We fix a positive regular n-free formal power series $f = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha X_\alpha$. Let us choose $r > 0$ such that $rM < \frac{1}{2}$. Denote by $\|\cdot\|$ the norm of operators acting on $\mathcal{F}_n$, the full Fock space. We have:

$$\begin{aligned}\Big\|\sum_{|\alpha|=k} a_\alpha r^k S_\alpha\Big\|^2 &= r^{2k}\Big\|\Big(\sum_{|\alpha|=k} a_\alpha r^k S_\alpha\Big)^*\Big(\sum_{|\alpha|=k} a_\alpha r^k S_\alpha\Big)\Big\| \\ &= r^{2k}\Big\|\sum_{|\alpha|=|\beta|=k} a_\alpha a_\beta S_\alpha^* S_\beta\Big\| = r^{2k}\Big\|\sum_{|\alpha|=k} a_\alpha^2\Big\|\end{aligned}$$

so $\Big\|\sum_{|\alpha|=k} a_\alpha r^k S_\alpha\Big\| \leqslant (rM)^k$ and thus $\Big\|\sum_{|\alpha|\geqslant 1} a_\alpha r^{|\alpha|} S_\alpha\Big\| < 1$ and hence that $I - \sum_{|\alpha|\geqslant 1} a_\alpha r^{|\alpha|} S_\alpha$ is invertible in $\mathcal{A}_n$. Therefore there exist coefficients $(b_\alpha)_{\alpha\in\mathbb{F}_n^+}$ such that

$$\Big(I - \sum_{|\alpha|\geqslant 1} a_\alpha r^{|\alpha|} S_\alpha\Big)^{-1} = \sum_{|\alpha|\geqslant 1} b_\alpha r^{|\alpha|} S_\alpha \in \mathcal{A}_n. \tag{2.2}$$

The relation between $(b_\alpha)_{\alpha\in\mathbb{F}_n^+}$ and $(a_\alpha)_{\alpha\in\mathbb{F}_n^+}$ is given by the equalities:

$$\begin{aligned}&\sum_{k=0}^{\infty}\Big(\sum_{|\alpha|\geqslant 1} a_\alpha r^{|\alpha|} S_\alpha\Big)^k = \sum_{|\alpha|\geqslant 1} b_\alpha r^{|\alpha|} S_\alpha, \\ &\Big(I - \sum_{|\alpha|\geqslant 1} a_\alpha r^{|\alpha|} S_\alpha\Big)\Big(\sum_{|\alpha|\geqslant 1} b_\alpha r^{|\alpha|} S_\alpha\Big) = 1, \\ &\Big(\sum_{|\alpha|\geqslant 1} b_\alpha r^{|\alpha|} S_\alpha\Big)\Big(I - \sum_{|\alpha|\geqslant 1} a_\alpha r^{|\alpha|} S_\alpha\Big) = 1.\end{aligned} \tag{2.3}$$

The first equality in (2.3) is valid because in any Banach algebras B, if $t \in B$ and $\|t\| < 1$ then $\sum_{k \geqslant 0} t^k = (1-t)^{-1}$. The other two are clear.

From the first equality in (2.3), we deduce that $b_0 = 1, b_{g_i} = a_{g_i}$ for $i = 1, \ldots, n, b_{g_i g_j} = a_{g_i g_j} + a_{g_i} a_{g_j}$, and more generally that for $\alpha \in \mathbb{F}_n^+$

$$b_\alpha = \sum_{j=1}^{|\alpha|} \sum_{\gamma_1 \cdots \gamma_j = \alpha;\, |\gamma_1| \geqslant 1, \ldots, |\gamma_j| \geqslant 1} a_{\gamma_1} a_{\gamma_2} \cdots a_{\gamma_j} > 0. \tag{2.4}$$

This implies that

$$b_{\alpha\beta} \geqslant b_\alpha b_\beta. \tag{2.5}$$

From the second and third equalities in (2.3) we deduce that for $|\alpha| \geqslant 1$

$$b_\alpha = \sum_{\beta\gamma = \alpha,\, |\beta| \geqslant 1} a_\beta b_\gamma = \sum_{\beta\gamma = \alpha,\, |\gamma| \geqslant 1} b_\beta a_\gamma. \tag{2.6}$$

For $i \in \{1, \ldots, n\}$ and $\alpha \in \mathbb{F}_n^+$ we define W_i^f as the linear extension of

$$W_i^f \delta_\alpha = \sqrt{\frac{b_\alpha}{b_{g_i \alpha}}} \delta_{g_i \alpha}.$$

It follows from inequality (2.5) that for all $i = 1, \ldots, n$ we have $\sqrt{\frac{b_\alpha}{b_{g_i \alpha}}} \leqslant \sqrt{\frac{b_0}{b_{g_i}}} = \sqrt{\frac{1}{b_{g_i}}}$ and thus $\|W_i^f\| = \sqrt{\frac{1}{b_{g_i}}}$. So W_i^f can be extended uniquely as a linear operator on $\mathcal{F}_n$.

We now check that $(W_1^f, \ldots, W_n^f) \in \mathcal{D}_f(\mathcal{F}_n)$. Let $\alpha, \beta \in \mathbb{F}_n^+$ with $|\beta| \geqslant 1$ we have:

$$W_\alpha^f W_\alpha^{f*} \delta_\beta = \begin{cases} \frac{b_\gamma}{b_\beta} \delta_\beta & \text{if } \beta = \gamma\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

We deduce that $\Big(\sum_{|\alpha| \geqslant 1} a_\alpha W_\alpha^f W_\alpha^{f*} \Big) \delta_\beta = \Big(\sum_{\alpha\gamma = \beta,\, |\alpha| \geqslant 1} \frac{a_\alpha b_\gamma}{b_\beta} \Big) \delta_\beta = \delta_\beta$.

Then if $\beta \in \mathbb{F}_n^+$ and $|\beta| \geqslant 1$ and we have

$$\Big(\sum_{|\alpha| \geqslant 1} a_\alpha W_\alpha^f W_\alpha^{f*} \Big) \delta_\beta = \Big(\sum_{\alpha\gamma = \beta,\, |\alpha| \geqslant 1} \frac{a_\alpha b_\gamma}{b_\beta} \Big) \delta_\beta = \delta_\beta$$

by equality (2.6). Then it follows that $\sum_{|\alpha| \geqslant 1} a_\alpha W_\alpha^f W_\alpha^{f*}$ is the projection orthogonal to $\mathbb{C}\delta_0$ and therefore is less than or equal to the identity as desired.

To conclude, we note that we can use a Poisson kernel again to prove the characterization of completely bounded unital representations of $\mathcal{A}(\mathcal{D}_f)$. For any Hilbert space $\mathcal{H}$, any $(T_1, \ldots, T_n) \in \mathcal{D}_f(\mathcal{H})$ satisfies the $C_{\cdot 0}$-condition when for all $A \in \mathcal{B}(\mathcal{H})$ we have that $\lim_{k \to \infty} \varphi^k(A) = 0$ in the strong operator topology with φ :

$A \in \mathcal{B}(\mathcal{H}) \mapsto \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha T_\alpha A T_\alpha^*$. Given an element $(T_1, \ldots, T_n)$ of $\mathcal{D}_f(\mathcal{H})$ satisfying the $C_{\cdot 0}$-condition, we define $\Delta = \sqrt{1 - \sum_{|\alpha| \geqslant 1} a_\alpha T_\alpha T_\alpha^*}$ and set:

$$K : h \in \mathcal{H} \mapsto \sum_{\alpha \in \mathbb{F}_n^+} \sqrt{b_\alpha} \delta_\alpha \otimes \Delta (T_\alpha)^*(h) \in \mathcal{F}_n \otimes \mathcal{H}.$$

Then K is an isometry such that $K^*(W_i^f \otimes 1)K = T_i$ for $i = 1, \ldots, n$. The computations which remain are along the same line as for Theorem 2.5.

We now set informally the problem which this paper begin to address:

PROBLEM 2.14. *Given positive regular free formal power series f and g, under what conditions is $\mathcal{A}(\mathcal{D}_f)$ isomorphic to $\mathcal{A}(\mathcal{D}_g)$?*

To be more precise, we introduce the proper notion of isomorphism for this paper.

DEFINITION 2.15. Let NCD be the category of noncommutative domains, defined as follows: the objects of NCD are the noncommutative domain algebras $\mathcal{A}(\mathcal{D}_f)$ for all positive regular free formal power series f and the morphisms between these algebras are the completely isometric algebra unital isomorphisms.

Given a Hilbert space $\mathcal{H}$, Definition 2.11 shows how to associate to any object in NCD a set of operators. We will see that when $\mathcal{H}$ is finite dimensional then we can extends this map between objects into a functor of well-chosen categories. This functor will be the invariant we shall use further on to distinguish between many objects in NCD.

3. ISOMORPHISMS BETWEEN NONCOMMUTATIVE DOMAINS

We start by observing that a completely bounded isomorphism between noncommutative domain algebras gives rise to a sequence of multivariate holomorphic maps. We thus construct a functor from NCD to the category of domains in $\mathbb{C}^k$. We start by recalling basic definitions [9] from multivariate analysis to set our notations.

An n-index is an element $k \in \mathbb{N}^n$. Given $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$, we define $x \cdot y = (x_1 y_1, \ldots, x_n y_n)$. In the same spirit, for any n-index $k = (k_1, \ldots, k_n)$, we write z^k for $(z_1^{k_1}, \ldots, z_n^{k_n})$.

DEFINITION 3.1. Let $U \subseteq \mathbb{C}^n$ be a nonempty open set. A function $f : U \longrightarrow \mathbb{C}^n$ is holomorphic on U when, for all $z \in U$, there exists a neighborhood $V \subseteq U$ of z and complex numbers $(\alpha_k)_{k \in \mathbb{N}^{*k}}$ such that for all $y \in V$ we have:

$$f(y) = \sum \alpha_k \cdot (y - z)^k.$$

Note in particular that if we write $f = (f_1, \dots, f_n)$ then f_i is a function from U to $\mathbb{C}$ which can be written as a power series in its n variables. Conversely, if $f_1, \dots, f_n$ all are power series in their n variables then f is holomorphic.

To ease the discussion in this paper, we will allow ourselves the following mild extension of the definition of holomorphy to sets with nonempty interior.

DEFINITION 3.2. Let $F \subseteq \mathbb{C}^n$ be a set of nonempty interior $\overset{\circ}{F}$. Then $f : F \longrightarrow \mathbb{C}^n$ is holomorphic on F if f restricted to $\overset{\circ}{F}$ is holomorphic on $\overset{\circ}{F}$ and f is continuous on F.

We will be interested in the matter of mapping a set onto another one by means of a holomorphic map with a holomorphic inverse.

DEFINITION 3.3. Let $U, V \subseteq \mathbb{C}^n$ be two sets with nonempty interiors. If there exists a holomorphic map $f : U \longrightarrow V$ and a holomorphic map $g : V \longrightarrow U$ such that $f \circ g = \mathrm{Id}_U$ and $g \circ f = \mathrm{Id}_V$ then U and V are biholomorphically equivalent, and f and g are called *biholomorphic maps*.

We refer to [8] for a survey of the problem of holomorphic mapping, which was a great motivation for the development of multivariate complex analysis. Unlike the case of domains of the complex plane, where the Riemann mapping theorem shows that any simply connected proper open subset of $\mathbb{C}$ is biholomorphically equivalent to the open unit disk, there are many biholomorphic equivalence classes of domains in $\mathbb{C}^n$ for $n > 1$; in other words there is great rigidity for biholomorphic maps between domains in $\mathbb{C}^n$ ($n > 1$). We shall exploit this rigidity in this paper to help classify the noncommutative domain algebras.

3.1. HOLOMORPHIC MAPS FROM ISOMORPHISMS.

The fundamental observation of this paper is the following result which creates a bridge between noncommutative domain algebra theory and the theory of holomorphic mapping of domains in $\mathbb{C}^m$. To be formal, we set:

DEFINITION 3.4. Let $k \in \mathbb{N}$. Let HD_k be the category of domains in $\mathbb{C}^k$, i.e. the category whose objects are open connected subsets of $\mathbb{C}^k$ for any $k \in \mathbb{N}$, and whose morphisms are holomorphic maps.

We shall now construct a contravariant functor from NCD to HD_k. To do so, we shall find the following two lemmas useful. These two lemmas are part of Popescu's construction found in [14]. We include a proof of these results for the reader's convenience, as they can be established directly, and it is again helpful to fix notations for the rest of this paper.

The first result shows that any element in $\mathcal{A}(\mathcal{D}_f)$ admits a form of Fourier series expansion, where convergence is understood in the radial sense. The second lemma proves that the pairings between noncommutative domain algebras and the complex domains introduced in (2.11) define holomorphic functions.

LEMMA 3.5 ([14]). *Let $a \in \mathcal{A}(\mathcal{D}_f)$ for some positive regular free formal series f. Then there exists a unique collection $(c_\alpha)_{\alpha\in\mathbb{F}_n^+}$ of complex numbers such that for all $r \in (0,1)$ the series*

$$\sum_{j=0}^{\infty} \sum_{\alpha\in\mathbb{F}_n^+,\,|\alpha|=j} c_\alpha r^{|\alpha|} W_\alpha$$

converge in norm in $\mathcal{A}(\mathcal{D}_f)$ to a_r defined by $\langle (rW_1,\ldots,rW_n), a\rangle_{\ell^2(\mathbb{F}_n^+)}$. Note that $\lim_{r\to 1^-} a_r = a$ in norm in $\mathcal{A}(\mathcal{D}_f)$.

We will denote $\sum_{\alpha\in\mathbb{F}_n^+,\,|\alpha|=j} c_\alpha W_\alpha$ by $[a]_j$. Then $\|[a]_j\| \leqslant \|a\|$ in $\mathcal{A}(\mathcal{D}_f)$.

Proof. Let $a \in \mathcal{A}(\mathcal{D}_f)$ be fixed. By Definition 2.10, the bounded linear operator a acts on the Fock space $\ell^2(\mathbb{F}_n^+)$. In particular, there exists a collection $(c_\alpha)_{\alpha\in\mathbb{F}_n^+}$ of complex numbers such that

$$a(\delta_0) = \sum_{\alpha\in\mathbb{F}_n^+} c_\alpha \sqrt{b_\alpha}\,\delta_\alpha$$

where $\{\delta_\alpha : \alpha \in \mathbb{F}_n^+\}$ is the usual Hilbert basis of $\ell^2(\mathbb{F}_n^+)$. We now wish to prove that a is the radial limit of $\sum_{j\in\mathbb{N}} [a]_j$ in the sense described above. A standard Fourier-type argument shows that we have for all $j \in \mathbb{N}$

$$\|[a]_j\| \leqslant \|a\|. \tag{3.1}$$

Note that inequality (3.1) also follows from Lemma 4.2 later in this paper, and that lemma is independent from the one we are now proving.

We conclude from inequality (3.1) that if $r \in (0,1)$ then $\sum_{j\in\mathbb{N}} r^j [a]_j$ converges in norm in $\mathcal{A}(\mathcal{D}_f)$, and we denote this limit a_r. Using Theorem 2.9, we also see that we must have:

$$a_r = \langle (rW_1,\ldots,rW_n), a\rangle.$$

In particular, we note that $\|a_r\| \leqslant \|a\|$ for all $r \in (0,1)$. Last, if a is a finite linear combination of the operators W_α^f ($\alpha \in \mathbb{F}_n^+$) then clearly $\lim_{r\to 1^-} a_r = a$. In general, for any a and any $\varepsilon > 0$ we can find a finite linear combination p of the operators W_α^f ($\alpha \in \mathbb{F}_n^+$) such that $\|a-p\| < \frac{1}{3}\varepsilon$. It is then immediate that $\|a_r - p_r\| < \frac{1}{3}\varepsilon$ as well. Moreover, using continuity in r there exists some $R \in (0,1)$ such that $\|p - p_r\| < \frac{1}{3}\varepsilon$ for $r \in (R,1)$. Hence $\lim_{r\to 1^-} \|a - a_r\| = 0$ as desired. ∎

As an observation and using the notation of Lemma 3.5, we note that if T is in $\mathcal{D}_f(\mathbb{C}^k)$ for some $k \in \mathbb{N}^*$ then the convergence of $a(T) = \langle T, a\rangle = \sum_{j=0}^{\infty}\Big(\sum_{\alpha\in\mathbb{F}_n^+,\,|\alpha|=j} c_\alpha T_\alpha\Big)$ is understood in the following sense:

$$a(T_1,\ldots,T_n) = \langle T, a\rangle = \lim_{r\to 1^-} \sum_{j=0}^{\infty} r^j \Big(\sum_{\alpha\in\mathbb{F}_n^+,\,|\alpha|=j} c_\alpha T_\alpha\Big). \tag{3.2}$$

In general, we will write an element $a \in \mathcal{A}(\mathcal{D}_f)$ as the sum of the series $(\sum c_\alpha W_\alpha)_{\alpha\in\mathbb{F}_n^+}$, or simply as $a = \sum_{\alpha\in\mathbb{F}_n^+} c_\alpha W_\alpha$, where convergence is understood as in Lemma 3.5.

LEMMA 3.6 ([14]). *Let $a \in \mathcal{A}(\mathcal{D}_f)$ for some positive regular free formal series f. Then for all $k \in \mathbb{N}^*$ the function*

$$T \in \mathcal{D}_f(\mathbb{C}^k) \mapsto a(T) = \langle T, a\rangle_k \in M_{k\times k}$$

is continuous on $\mathcal{D}_f(\mathbb{C}^k)$ and holomorphic on the interior of $\mathcal{D}_f(\mathbb{C}^k)$.

Proof. Fix $a \in \mathcal{A}(\mathcal{D}_f)$ and $k \in \mathbb{N}^*$, and endow $\mathbb{C}^{nk^2}$ and $\mathbb{C}^{k^2}$ with an arbitrary norm. The continuity of $\langle \cdot, a\rangle_k$ is addressed first. Let $\varepsilon > 0$ and $T \in \mathcal{D}_f(\mathbb{C}^k)$. Let $p \in \mathcal{A}(\mathcal{D}_f)$ be a finite linear combinations of the operators W_α for some finite subset of indices α in $\mathbb{F}_n^+$ such that $\|a - p\| < \frac{1}{3}\varepsilon$. The map $\langle \cdot, p\rangle$, as seen as a function from $\mathbb{C}^{nk^2}$ into $\mathbb{C}^{k^2}$, is a k^2-tuple of polynomials in nk^2 variables, and thus is continuous on $\mathcal{D}_f(\mathbb{C}^k)$. Therefore there exists $\delta > 0$ such that if $T' \in \mathcal{D}_f(\mathbb{C}^k)$ and $\|T - T'\| < \delta$ then $\|p(T) - p(T')\| < \frac{1}{3}\varepsilon$. We conclude:

$$\|a(T)-a(T')\| \leqslant \|a(T)-p(T)\|+\|p(T)-p(T')\|+\|p(T')-a(T')\| < \frac{1}{3}(\varepsilon+\varepsilon+\varepsilon) = \varepsilon.$$

This concludes the proof of continuity of $\langle \cdot, a\rangle_k$ on $\mathcal{D}_f(\mathbb{C}^k)$.

We now turn to the question of holomorphy. Let K be an arbitrary compact subset of the interior of $\mathcal{D}_f(\mathbb{C}^k)$. Let $\tau > 0$ be chosen so that $(1+\tau)K$ is also in the interior of $\mathcal{D}_f(\mathbb{C}^k)$. Then, using Lemma 3.5, we have for every $T \in K$ and $j \in \mathbb{N}$ that since $\langle (1+\tau)T, [a]_j\rangle = (1+\tau)^j \langle T, [a]_j\rangle$ by definition:

$$\begin{aligned}\|\langle T, [a]_j\rangle\| &= \frac{1}{(1+\tau)^j}\|(1+\tau)^j\langle T, [a]_j\rangle\| = \frac{1}{(1+\tau)^j}\|\langle (1+\tau)T, [a]_j\rangle\| \\ &\leqslant \frac{1}{(1+\tau)^j}\|[a]_j\| \leqslant \frac{1}{(1+\tau)^j}\|a\|.\end{aligned} \tag{3.3}$$

Hence for any $N \in \mathbb{N}$:

$$\left\|\langle T, a\rangle - \left\langle T, \sum_{j=0}^{N} [a]_j \right\rangle\right\| \leqslant \|a\| \sum_{j=N+1}^{\infty} \frac{1}{(1+\tau)^j} \xrightarrow[N\to\infty]{} 0.$$

Hence $\langle \cdot, a\rangle$ is the uniform limit of the functions $p_N = \left\langle \cdot, \sum_{j=0}^{N} [a]_j \right\rangle$. These functions are given as k^2 polynomials in nk^2 variables, and are thus holomorphic the interior of $\mathcal{D}_f(\mathbb{C}^k)$. Hence $\langle \cdot, a\rangle$ is holomorphic on the interior of $\mathcal{D}_f(\mathbb{C}^k)$. ∎

Using inequality (3.3) and the notation of Lemma 3.5, we note that if T is in the interior of $\mathcal{D}_f(\mathbb{C}^k)$ for some $k \in \mathbb{N}^*$ then we in fact have the norm convergence in $\mathbb{C}^{k^2}$:

$$a(T_1,\dots,T_n) = \langle T, a\rangle = \sum_{j=0}^{\infty}\Big(\sum_{\alpha\in\mathbb{F}_n^+,\,|\alpha|=j} c_\alpha T_\alpha\Big).$$

We are now ready to deduce our main theorem.

THEOREM 3.7. *Let f, g be two positive regular free formal power series, and let n be the number of indeterminates of f. Let $\Phi : \mathcal{A}(\mathcal{D}_f) \mapsto \mathcal{A}(\mathcal{D}_g)$ be an isomorphism in* NCD. *Then for all $k \in \mathbb{N}^*$ there exists a biholomorphic function $\widehat{\Phi}_k : \mathcal{D}_g(\mathbb{C}^k) \longrightarrow \mathcal{D}_f(\mathbb{C}^k)$ such that for any $T \in \mathcal{D}_g(\mathbb{C}^k)$ we have, for $i = 1,\dots,n$:*

$$\langle T, \Phi(\cdot)\rangle_g^k = \langle \widehat{\Phi}_k(T), \cdot\rangle_f^k.$$

Moreover, for all $k \in \mathbb{N}^*$ the function $\widehat{\Phi}_k$ maps the interior of $\mathcal{D}_g(\mathbb{C}^k)$ onto the interior of $\mathcal{D}_f(\mathbb{C}^k)$, where the interior is computed in the topology of $\mathbb{C}^{nk^2}$.

For $k \in \mathbb{N}^*$ we can thus define a contravariant functor $\mathbb{D}^k$ from NCD *into* HD_{nk^2} *where, for any object $\mathcal{A}(\mathcal{D}_f)$ and isomorphism Φ in* NCD, *$\mathbb{D}^k(\mathcal{A}(\mathcal{D}_f))$ is the interior of $\mathcal{D}_f(\mathbb{C}^k)$ and $\mathbb{D}^k(\Phi) = \widehat{\Phi}_k$.*

Proof. Let $k \in \mathbb{N}^*$ and let $T \in \mathcal{D}_g(\mathbb{C}^k)$. It follows from the proof of Theorem 2.9 that:

$$\widehat{\Phi_k}(T) = \Big(\sum_{\alpha\in\mathbb{F}_n^+} c_{1,\alpha}T_\alpha, \dots, \sum_{\alpha\in\mathbb{F}_n^+} c_{n,\alpha}T_\alpha\Big)$$

where the convergence of each entry is understood in the sense of (3.2), and where $\Phi(W_i^f) = \sum_{\alpha\in\mathbb{F}_n^+} c_{i,\alpha}W_\alpha^g$ for $i \leqslant n$.

By Lemma 3.6, the function $T \in \mathcal{D}_g(\mathbb{C}^k) \mapsto \langle T, \Phi_k(W_i^f)\rangle$ is continuous on $\mathcal{D}_g(\mathbb{C}^k)$ and holomorphic on the interior of $\mathcal{D}_g(\mathbb{C}^k)$ for all $i = 1,\dots,n$. Therefore, the function $\widehat{\Phi_k}$ is continuous on $\mathcal{D}_g(\mathbb{C}^k)$ and holomorphic on the interior of $\mathcal{D}_g(\mathbb{C}^k)$. Moreover, since $\Phi^{-1} : \mathcal{A}(\mathcal{D}_g) \to \mathcal{A}(\mathcal{D}_f)$ is also an isomorphism in NCD, we deduce the same properties for $\widehat{\Phi_k^{-1}} : \mathcal{D}_f(\mathbb{C}^k) \to \mathcal{D}_g(\mathbb{C}^k)$.

Last, let $T \in \mathcal{D}_g(\mathbb{C}^k)$ be an interior point of $\mathcal{D}_g(\mathbb{C}^k)$ in the topology of $\mathbb{C}^{k^2n}$ and let $U \subseteq \mathcal{D}_g(\mathbb{C}^k)$ be an open set in $\mathbb{C}^{nk^2}$ with $T \in U$. By assumption, $\widehat{\Phi}_k$ is a homeomorphism from $\mathcal{D}_g(\mathbb{C}^k)$ onto $\mathcal{D}_f(\mathbb{C}^k)$ (in their relative topology). In particular, identifying $\mathbb{C}^{nk^2}$ with $\mathbb{R}^{2nk^2}$ (via the canonical linear isomorphism), $\widehat{\Phi}_k$ restricted to $U \subseteq \mathbb{R}^{2nk^2}$ is a continuous injective $\mathbb{R}^{2nk^2}$-valued function, so by the invariance of domain principle ([16], Theorem 16, page 199) we conclude that $\widehat{\Phi}_k(U)$ is open in $\mathbb{R}^{2nk^2}$ since U is open in $\mathbb{R}^{2nk^2}$. Hence since $\widehat{\Phi}_k(T) \in \widehat{\Phi}_k(U) \subseteq \mathcal{D}_f(\mathbb{C}^k)$, we conclude that $\widehat{\Phi}_k(T)$ is indeed an interior point of $\mathcal{D}_f(\mathbb{C}^k)$.

Last, if $(T_1,\dots,T_n) \in \mathcal{D}_f(\mathbb{C}^k)$ then so does $(tT_1,\dots,tT_n)$ for $t \in [0,1]$ so every point in $\mathcal{D}_f(\mathbb{C}^k)$ is path connected to 0. Hence $\mathcal{D}_f(\mathbb{C}^k)$ is connected. Therefore the interior $\mathbb{D}^k(\mathcal{A}(\mathcal{D}_f))$ of $\mathcal{D}_f(\mathbb{C}^k)$ is an object in HD_{nk^2}. It is then easy to check that $\mathbb{D}^k$ defines a contravariant functor. ∎

REMARK 3.8. When $k = 1$, we do not need to assume that Φ is completely isometric. Indeed, if Φ is a continuous unital algebra isomorphism, then for any $\chi \in \mathcal{D}_g(\mathbb{C})$ the map $\langle \chi, \cdot \rangle \circ \Phi$ is continuous and scalar valued, hence completely contractive, so the proof of Theorem 3.7 applies.

REMARK 3.9. It is worth noticing that if the interiors of $\mathcal{D}_f(\mathbb{C})$ and $\mathcal{D}_g(\mathbb{C})$ are not biholomorphically equivalent, then we can already conclude that there is no bounded isomorphism between $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ — not just no completely isometric isomorphism.

We introduce a notation to ease the presentation in this paper. Given an object $\mathcal{A}(\mathcal{D}_f)$ in NCD we denote the domain $\mathbb{D}^k(\mathcal{A}(\mathcal{D}_f))$ simply by $\mathbb{D}^k_f$. It is also worth pointing out a terminology conflict between the literature in complex analysis and the literature on non selfadjoint operator algebras. A domain in complex analysis usually refers to a connected open subset of some Hermitian space, while it is common in operator theory to call $\mathcal{D}_f(\mathcal{H})$ domains for any Hilbert space $\mathcal{H}$, even though the sets $\mathcal{D}_f(\mathcal{H})$ are closed and connected. We hope that the definition of the functors $\mathbb{D}^k$ clarifies this point and establishes the bridge not only formally, but from a lexical point of view as well.

Thus, the functors $\mathbb{D}^k$ $(k \in \mathbb{N}^*)$ define a new family of invariants for the noncommutative domains. We are thus led to study the geometry of the sets $\mathbb{D}^k_f$ (seen as objects in HD_{nk^2}) in order to classify objects in NCD.

3.2. THE INVARIANT $\mathbb{D}^1$. As observed in Theorem 3.7 and Remark 3.8, the biholomorphic equivalence of $\mathbb{D}^1_f$ is an isomorphism invariant for the noncommutative domain $\mathcal{A}(\mathcal{D}_f)$ for any positive regular free formal series f. We thus can provide a first easy partial classification result for the algebras $\mathcal{A}(\mathcal{D}_f)$ by studying the geometry of $\mathbb{D}^1_f$. We start by noticing that they have many symmetries, in the following sense.

DEFINITION 3.10. Let $D \subseteq \mathbb{C}^m$ be an open connected set. Then D is a *Reinhardt domain* if for all $\omega \in \mathbb{T}^m$ and $z \in D$ we have $\omega \cdot z = (\omega_1 z_1, \ldots, \omega_m z_m) \in D$.

PROPOSITION 3.11. *Let f be a positive regular free series. Then $\mathbb{D}^1_f$ is a Reinhardt domain.*

Proof. Let $\chi \in \mathbb{D}^1_f$ and $\omega = \exp(\mathrm{i}\theta) \in \mathbb{T}$ (where $\theta \in [0, 2\pi]$). Then by definition $\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha |\chi_\alpha|^2 < 1$ so we have the following that concludes our proof:

$$\sum_{\alpha \in \mathbb{F}_n^+} a_\alpha |\omega^{|\alpha|} \chi_\alpha|^2 = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha |\chi_\alpha|^2 < 1. \quad ∎$$

Now, two Reinhardt domains containing 0 are biholomorphic only under very rigid conditions, as proven by Sunada in [17].

THEOREM 3.12 ([17]). *Let D and D' be two Reinhardt domains containing 0. Then D and D' are biholomorphic if and only if there exists a permutation σ and scalars $\lambda_1, \ldots, \lambda_n$ such that the following map is a biholomorphic map:*

$$(z_1, \ldots, z_n) \mapsto (\lambda_1 z_{\sigma(1)}, \ldots, \lambda_n z_{\sigma(n)}).$$

We can now conclude by putting Theorem 3.7 and Theorem 3.12 together.

THEOREM 3.13. *Let f, g be positive regular free formal power series. If $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ are isomorphic in* NCD *then:*

(i) *The series f and g have the same number of indeterminates, denoted by n.*

(ii) *There exists a permutation σ of $\{1, \ldots, n\}$ and a function $\lambda : \{1, \ldots, n\} \longrightarrow \mathbb{C}$ such that the following map induces an isomorphism from $\mathbb{D}_f^1$ onto $\mathbb{D}_g^1$:*

$$(z_1, \ldots, z_n) \in \mathbb{C}^n \mapsto (\lambda_1 z_{\sigma(1)}, \ldots, \lambda_n z_{\sigma(n)}).$$

We can already prove that there are many different noncommutative domain algebras (i.e. non isomorphic) using Theorem 3.13.

Moreover, we shall see in a later subsection that it is possible to construct isomorphisms in NCD from simply rescaling weighted shifts in a manner suggested by Theorem 3.13. However, it is not true that isomorphisms in $\mathrm{HD}_{n\cdot 1^2}$ can be lifted to isomorphisms in NCD even when they are presented in as nice a form as in Theorem 3.13. The following example is our main illustration of the results of this paper, as it displays the role of the higher invariants $\mathbb{D}^k$ ($k > 1$) to capture some of the noncommutative aspects of the classification problem in NCD.

EXAMPLE 3.14. Let:

$$f = X_1 + X_2 + X_1 X_2 \quad \text{and} \quad g = X_1 + X_2 + \frac{1}{2}(X_1 X_2 + X_2 X_1).$$

Then by definition $\mathbb{D}_f^1 = \mathbb{D}_g^1 = \{(\lambda_1, \lambda_2) : |\lambda_1|^2 + |\lambda_2|^2 + |\lambda_1 \lambda_2|^2 < 1\}$. Yet we shall see in the next section that $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ are not isomorphic in NCD.

In other words, Theorem 3.13 does not have a converse.

The higher invariants $\mathbb{D}^k$ ($k > 1$) will be used to show that unital completely isometric isomorphisms that map the zero character to the zero character are very simple. To illustrate this suppose that f and g are free formal power series and that $\Phi : \mathcal{A}(\mathcal{D}_f) \to \mathcal{A}(\mathcal{D}_g)$ is a unital isometric isomorphism with $\widehat{\Phi}_1(0) = 0$. Then $\widehat{\Phi}_1 : \mathbb{D}_g^1 \to \mathbb{D}_f^1$ is given by

$$\widehat{\Phi}_1(\lambda_1, \ldots, \lambda_n) = \Big(\sum_{|\alpha| \geqslant 1} c_{1,\alpha} \lambda_\alpha, \ldots, \sum_{|\alpha| \geqslant 1} c_{n,\alpha} \lambda_\alpha \Big),$$

where $\Phi(W_i^f) = \sum_{|\alpha|\geqslant 1} c_{i,\alpha} W_\alpha^g$ for $i \leqslant n$. Now, by Cartan's Lemma, which appears below, $\widehat{\Phi}_1$ is linear and we conclude that for $i \leqslant n$,

$$\sum_{|\alpha|\geqslant 1} c_{i,\alpha}\lambda_\alpha = c_{i,g_1}\lambda_1 + \cdots + c_{i,g_n}\lambda_n.$$

Most terms cancel. Look at the left hand side and notice that the term $(\lambda_1)^2$ appears only once and has coefficient c_{1,g_1g_1}. Hence $c_{1,g_1g_1} = 0$. The term $\lambda_1\lambda_2$ appears twice and has coefficients c_{i,g_1g_2} and c_{i,g_2g_1}. Hence $c_{i,g_1g_2} + c_{i,g_2g_1} = 0$. The main point of the next subsection is that the higher invariants $\mathbb{D}^k$ ($k > 1$) can be used to separate c_{i,g_1g_2} from c_{i,g_2g_1}, and more generally, to conclude that $c_{i,\alpha} = c_{i,\alpha} = 0$ whenever $|\alpha| > 1$.

3.3. THE INVARIANTS $\mathbb{D}^k$. Let f be a positive regular free series. The domains $\mathbb{D}_f^k$ are not usually Reinhardt domains for $k \in \mathbb{N}^*\backslash\{1\}$, yet they still enjoy enough symmetry to make some very useful observations.

DEFINITION 3.15. Let $D \subseteq \mathbb{C}^m$ be an open connected set. Then D is *circular* if for all $\omega \in \mathbb{T}$ and $z \in D$ then $\omega z = (\omega z_1, \ldots, \omega z_m) \in D$.

PROPOSITION 3.16. *For any positive regular free series f and any $k \in \mathbb{N}^*$ the domain $\mathbb{D}_f^k$ is circular.*

A powerful tool to classify circular domains is Cartan's lemma which we now quote for the convenience of the reader.

THEOREM 3.17 (Cartan's Lemma [3]). *Let $m \in \mathbb{N}^*$. Let D, D' be bounded circular domains in $\mathbb{C}^m$ containing 0. Let $\Psi : D \longrightarrow D'$ be biholomorphic such that $\Psi(0) = 0$. Then Ψ is the restriction of a linear map of $\mathbb{C}^m$.*

Therefore, if two noncommutative domains $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ are isomorphic in NCD then for all $k \in \mathbb{N}^*$ the domains $\mathbb{D}_f^k$ and $\mathbb{D}_g^k$ are linearly isomorphic, provided the dual map of the isomorphism maps the zero character to the zero character. Using this, we can conclude the main result of this paper.

THEOREM 3.18. *Let f, g be positive regular free series. Let n be the number of variables of f. Then if there exists an isomorphism $\Phi : \mathcal{A}(\mathcal{D}_f) \longrightarrow \mathcal{A}(\mathcal{D}_g)$ in* NCD *such that $\widehat{\Phi}_1(0) = 0$ then there exists an invertible scalar $n \times n$ matrix M such that*

$$\begin{bmatrix} \Phi(W_1^f) \\ \Phi(W_2^f) \\ \vdots \\ \Phi(W_n^f) \end{bmatrix} = (M \otimes 1_{\ell^2(\mathbb{F}_n^+)}) \begin{bmatrix} W_1^g \\ W_2^g \\ \vdots \\ W_n^g \end{bmatrix},$$

(i.e. for each $i \leqslant n$, $\Phi(W_i^f)$ is a linear combination of the set of weighted shifts $\{W_1^g, \ldots, W_n^g\}$).

Proof. For $i \in \{1, \dots, n\}$ we write (see the proof of Theorem 3.7):

$$\Phi(W_i^f) = \sum_{\alpha \in \mathbb{F}_n^+} c_{i,\alpha} W_\alpha^g.$$

Let $k \in \mathbb{N}^*$ be fixed. By Theorem 3.7, the maps $\widehat{\Phi}_k = \mathbb{D}_k(\Phi)$ are biholomorphic maps from $\mathbb{D}_g^k$ onto $\mathbb{D}_f^k$, both being circular domains containing 0. Hence $\mathbb{D}_g^1$ is an open subset of $\mathbb{C}^n$ and thus g has as many variables as f. Since for $\chi \in \mathbb{D}_g^1$, $(\chi_1, \dots, \chi_n) \in \mathbb{C}^n$ we have

$$\widehat{\Phi}_1(\chi) = \Big(\sum_{\alpha \in \mathbb{F}_n^+} c_{1,\alpha} \chi_\alpha, \dots, \sum_{\alpha \in \mathbb{F}_n^+} c_{n,\alpha} \chi_\alpha \Big)$$

we conclude that $\widehat{\Phi}_1(0) = 0$ imposes that $c_{1,0} = \cdots = c_{n,0} = 0$. Consequently, by construction $\widehat{\Phi}_k(0) = 0$. Since for all $k \in \mathbb{N}^*$ the bounded domains $\mathbb{D}_f^k$ and $\mathbb{D}_g^k$ are circular, contain 0 and the maps $\widehat{\Phi}_k$ are biholomorphic, we conclude by Cartan's Lemma 3.17 that $\widehat{\Phi}_k$ is the restriction of a linear map. We now show that this implies that Φ is induced by a linear map as stated in the theorem.

Fix $k \geqslant 2$ and $i \in \{1, \dots, n\}$. We shall exploit the fact that two power series agree on an open set if and only if their coefficients agree. Let $\beta \in \mathbb{F}_n^+$ be fixed and of length k; we write $\beta = g_{\beta_1} \cdots g_{\beta_k}$ where $\beta_1, \dots, \beta_k \in \{1, \dots, n\}$.

Our goal is to prove that $c_{i,\beta} = 0$.

For any $j, l \in \{1, \dots, k\}$ and any matrix $N \in M_{k\times k}(\mathbb{C})$ we write $N_{j,l}$ for the (j, l) entry of N. Let $T = (T_1, \dots, T_n) \in \mathbb{D}_g^k$, then viewing $\widehat{\Phi}_k(T)$ as an n-tuple of matrices in $M_{k\times k}(\mathbb{C})$, we let $\widehat{\Phi}_{k,i}(T)$ be the i^{th} component of this n-tuple. We then set:

$$\eta = (\widehat{\Phi}_{k,i})_{1,k} : \mathbb{D}_g^k \to \mathbb{C}.$$

Then η is a scalar valued holomorphic function, and in fact it has expression

$$\eta(T_1, \dots, T_n) = \sum_{\alpha \in \mathbb{F}_n^+} c_{i,\alpha} (T_\alpha)_{1,k}.$$

To ease notation, for $T = (T_1, \dots, T_n) \in M_{k\times k}(\mathbb{C})$, $m \in \{1, \dots, n\}$, and $j, l \in \{1, \dots, k\}$, will denote $(T_m)_{j,l}$ by t_{jl}^m . Then η is a holomorphic function on the variables t_{jl}^m.

For a fixed $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = s$, the function $(T_1, \dots, T_n) \mapsto (T_\alpha)_{1,k}$ is the sum of all homogeneous polynomials in the entries of the matrices $T_1, \dots, T_n$ of the form:

$$(T_1, \dots, T_n) \mapsto (T_{\alpha_1})_{1,i_1} (T_{\alpha_2})_{i_1,i_2} \cdots (T_{\alpha_s})_{i_{s-1},k} = t_{1i_1}^{\alpha_1} \cdots t_{i_{s-1}k}^{\alpha_s}.$$

In particular, we note that η is the sum of homogeneous polynomials, and $(T_\alpha)_{1,k}$ is homogeneous of degree the length of α for all $\alpha \in \mathbb{F}_n^+$. However, since $\widehat{\Phi}_k$ is linear, η is also linear and therefore the sum of all homogeneous polynomials of a degree greater than or equal to 2 vanish.

We now wish to identify the coefficient of the polynomial $t_{11}^{\beta_1}t_{12}^{\beta_2}t_{23}^{\beta_3}\cdots t_{(k-1)k}^{\beta_k}$ in η. Let

$$\Sigma_\beta = \{\alpha \in \mathbb{F}_n^+ : (T_\alpha)_{1,k} \text{ contains the term } t_{11}^{\beta_1}t_{12}^{\beta_2}t_{23}^{\beta_3}\cdots t_{(k-1)k}^{\beta_k}\}.$$

The coefficient of $t_{11}^{\beta_1}t_{12}^{\beta_2}t_{23}^{\beta_3}\cdots t_{(k-1)k}^{\beta_k}$ in η is equal to $\sum_{\alpha\in\Sigma_\beta} c_{i,\alpha}$ and since η is linear and $k \geqslant 2$ it follows that $\sum_{\alpha\in\Sigma_\beta} c_{i,\alpha} = 0$. To finish the proof we will check that Σ_β contains only one element, which has to be β. Then it will follow that $c_{i,\beta} = 0$ as desired.

First, if $\alpha \in \Sigma_\beta$ then the length of α is the length of β i.e. k. Let us write $\alpha = g_{\alpha_1}\cdots g_{\alpha_k}$. Now, $(T_\alpha)_{1,k}$ will be the sum of the terms $t_{1i_1}^{\alpha_1}\cdots t_{i_{k-1}k}^{\alpha_k}$ for $i_1,\ldots,i_{k-1} \in \{1,\ldots,n\}$. Hence there exists $i_1,\ldots,i_{k-1} \in \{1,\ldots,k\}$ such that on $\mathbb{D}_g^k$ we have:

$$t_{1i_1}^{\alpha_1}\cdots t_{i_{k-1}k}^{\alpha_k} = t_{11}^{\beta_1}t_{12}^{\beta_2}t_{23}^{\beta_3}\cdots t_{(k-1)k}^{\beta_k}. \tag{3.4}$$

Let us view the map t_{jl}^m as defined from $\mathbb{C}^{nk^2}$ and mapping the coordinate $mk^2 + jk + l$, i.e. we linearly identify $M_{k\times k}(\mathbb{C})$ with $\mathbb{C}^{k^2}$ using canonical bases. Since $\mathbb{D}_g^k$ is open in $\mathbb{C}^{nk^2}$ and contains 0, identity (3.4) must hold on some open hypercube in $\mathbb{C}^{nk^2}$ around 0. Hence the factors in identity (3.4) are identical up to some permutation. Now, there is a unique factor of the form t_{jk}^m on the right hand side, so there must be a unique one on the left hand side. Moreover these two must agree. We deduce that $i_{k-1} = k-1$ and $\alpha_k = \beta_k$. Now, there is a unique factor of the form $t_{j(k-1)}^m$ on the right hand side, so once again since $i_{k-2} = k-2$ and $\beta_{k-1} = \alpha_{k-1}$. By an easy induction, we conclude that $\alpha_m = \beta_m$ for $m \in \{1,\ldots,k\}$. Consequently, $\Sigma_\beta = \{\beta\}$.

Since β is arbitrary of length at least 2 we conclude that $c_{i,\beta} = 0$ for all β of length at least 2 and all $i \in \{1,\ldots,n\}$. Hence if we set

$$M = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n} \end{bmatrix}$$

it is now clear that $\Phi = (M \otimes 1_{\ell^2(\mathbb{F}_n^+)})$. ∎

We now turn to some important examples of applications of Theorem 3.18, including the important Example 3.14.

4. APPLICATIONS AND EXAMPLES

We now use Theorem 3.18 to prove that various important examples of pure formal power series give rise to non isomorphic algebras in NCD. First, we show that combining Thullen's classification of two dimensional domains with Theorem 3.18, we can distinguish in NCD the noncommutative domain algebras associated to $X_1 + X_2 + X_1X_2$ and $X_1 + X_2 + \frac{1}{2}(X_1X_2 + X_2X_1)$ which we encountered in Example 3.14. Note that the invariant $\mathbb{D}^1$ was not enough to do so, thus showing the need to consider higher dimensional characters of noncommutative domain algebras to capture the noncommutativity of the symbol defining them.

We then prove that the noncommutative disk algebras are exactly described by degree one positive regular free series, thus allowing for a very simple characterization of these algebras among all objects in NCD. In particular, the algebras of Example 3.14 are not disk algebras — and are, informally, the simplest such examples. In the process of our investigation, we will also point out new examples of domains in $\mathbb{C}^{nk^2}$ with noncompact automorphism groups which occur naturally within our framework.

4.1. AN APPLICATION OF THULLEN CHARACTERIZATION.

In 1931, Thullen [18] proved that if a bounded Reinhardt domain in $\mathbb{C}^2$ has a biholomorphic map that does not map zero to zero, the domain is linearly equivalent to one of the following:

(i) polydisc $\{(z, w) \in \mathbb{C}^2 : |z| < 1, |w| < 1\}$;
(ii) unit ball $\{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}$;
(iii) Thullen domain $\{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^{2/p} < 1\}$, $p > 0, p \neq 1$.

We will use Thullen's Theorem to prove that the algebras in Example 3.14 are non isomorphic in NCD. Indeed, an isomorphism $\Phi : \mathcal{A}(\mathcal{D}_f) \to \mathcal{A}(\mathcal{D}_g)$ induces a biholomorphic map $\widehat{\Phi} : \mathbb{D}^1_g \to \mathbb{D}^1_f$. Since

$$\mathbb{D}^1_f = \mathbb{D}^1_g = \{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : |\lambda_1|^2 + |\lambda_2|^2 + |\lambda_1\lambda_2|^2 < 1\}$$

is not linearly equivalent to the three examples listed above, we conclude that $\widehat{\Phi}(0) = 0$.

To proceed with the proof we list two useful lemmas, using the notations established in Section 2.

LEMMA 4.1. *For every $\alpha \in \mathbb{F}^+_n$ we have $\|W_\alpha\| = \sqrt{\frac{1}{b_\alpha}}$.*

Proof. Let $\alpha \in \mathbb{F}^+_n$. Then the operator W_α maps the orthonormal basis $\{\delta_\beta : \beta \in \mathbb{F}^+_n\}$ into an orthogonal family. Therefore, we have the following and the lemma is proven:

$$\|W_\alpha\| = \sup_{\beta \in \mathbb{F}^+_n} \|W_\alpha \delta_\beta\| = \sup_{\beta \in \mathbb{F}^+_n} \sqrt{\frac{b_\beta}{b_{\alpha\beta}}} \leqslant \sqrt{\frac{b_0}{b_\alpha}} = \|W_\alpha \delta_0\|. \quad \blacksquare$$

LEMMA 4.2. *For every $k \in \mathbb{N}$ and scalar family $(x_\alpha)_{\alpha\in\mathbb{F}_n^+}$ we have:*

$$\Big\| \sum_{|\alpha|=k} x_\alpha W_\alpha \Big\| = \sqrt{\sum_{|\alpha|=k} \frac{|x_\alpha|^2}{b_\alpha}}.$$

Proof. Similarly to the proof of Lemma 4.1, the operator $\sum_{|\alpha|=k} x_\alpha W_\alpha$ maps the orthonormal family $\{\delta_\beta : \beta \in \mathbb{F}_n^+\}$ onto an orthogonal family. Hence, we have the following that completes our proof:

$$\begin{aligned}\Big\| \sum_{|\alpha|=k} x_\alpha W_\alpha \Big\| &= \sup_{\beta\in\mathbb{F}_n^+} \Big\| \sum_{|\alpha|=k} x_\alpha W_\alpha \delta_\beta \Big\| = \sup_{\beta\in\mathbb{F}_n^+} \Big\| \sum_{|\alpha|=k} x_\alpha \sqrt{\frac{b_\beta}{b_{\alpha\beta}}} \delta_{\alpha\beta} \Big\| \\ &= \sqrt{\sum_{|\alpha|=k} |x_\alpha|^2 \frac{b_\beta}{b_{\alpha\beta}}} \leqslant \sqrt{\sum_{|\alpha|=k} |x_\alpha|^2 \frac{b_0}{b_\alpha}} = \Big\| \sum_{|\alpha|=k} x_\alpha W_\alpha \delta_0 \Big\|. \qquad \blacksquare\end{aligned}$$

We now can prove:

THEOREM 4.3. *Let:*

$$f = X_1 + X_2 + X_1 X_2, \quad g = X_1 + X_2 + \frac{1}{2} X_1 X_2 + \frac{1}{2} X_2 X_1.$$

Then the noncommutative domain algebras $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ are not isomorphic in NCD.

Proof. Remaining consistent with the notations we used in this paper, we will write $f = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^f X_\alpha$ and $g = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^g X_\alpha$ (so $a_{g_1}^f = a_{g_2}^f = a_{g_1}^g = a_{g_2}^g = 1 = a_{g_1 g_2}^f$ yet $a_{g_1 g_2}^g = a_{g_2 g_1}^g = \frac{1}{2}$ and all other coefficients are null). We also denote by $(b_\alpha^f)_{\alpha\in\mathbb{F}_n^+}$ (respectively $(b_\alpha^g)_{\alpha\in\mathbb{F}_n^+}$) the weight given by equality (2.4) for the coefficients $(a_\alpha^f)_{\alpha\in\mathbb{F}_n^+}$ (respectively $(a_\alpha^g)_{\alpha\in\mathbb{F}_n^+}$). The weighted shifts associated to f (respectively g) are denoted by W_α^f (respectively W_α^g) for all $\alpha \in \mathbb{F}_n^+$.

Since, from equality (2.4), we have $a_{g_i} = b_{g_i}$ and $b_{g_i g_j} = b_{g_i} a_{g_i} + a_{g_i g_j}$ ($i \neq j \in \{1,2\}$) so we easily compute:

$$\begin{aligned} &b_{g_1}^f = b_{g_2}^f = b_{g_1}^g = b_{g_2}^g = 1, \\ &b_{g_1 g_1}^f = b_{g_2 g_1}^f = b_{g_2 g_2}^f = 1 \quad \text{and} \quad b_{g_1 g_2}^f = 2, \qquad (4.1) \\ &b_{g_1 g_1}^g = b_{g_2 g_2}^g = 1 \quad \text{and} \quad b_{g_1 g_2}^g = b_{g_2 g_1}^g = \frac{3}{2}. \end{aligned}$$

We conclude from Lemma 4.1 and equalities (4.1) that:

$$\|W^f_{g_1}\| = \|W^f_{g_2}\| = \|W^g_{g_1}\| = \|W^g_{g_2}\| = 1,$$
$$\|W^f_{g_1}W^f_{g_1}\| = \|W^f_{g_2}W^f_{g_1}\| = \|W^f_{g_2}W^f_{g_2}\| = 1 \quad \text{and} \quad \|W^f_{g_1}W^f_{g_2}\| = \frac{1}{\sqrt{2}},$$
$$\|W^g_{g_1}W^g_{g_1}\| = \|W^g_{g_2}W^g_{g_2}\| = 1 \text{ and } \|W^g_{g_2g_1}\| = \|W^g_{g_1g_2}\| = \sqrt{\frac{2}{3}}.$$

Assume now that there exists an Φ isomorphism in NCD from $\mathcal{A}(\mathcal{D}_f)$ onto $\mathcal{A}(\mathcal{D}_g)$. By Thullen, we have $\widehat{\Phi}_1(0) = 0$. Now, by Theorem 3.18, there exist $t_{11}, t_{12}, t_{21}, t_{22} \in \mathbb{C}$ such that:

$$\Phi(W^f_{g_1}) = t_{11}W^g_{g_1} + t_{12}W^g_{g_2}, \quad \Phi(W^f_{g_2}) = t_{21}W^g_{g_1} + t_{22}W^g_{g_2}.$$

Moreover, Lemma 4.2 implies that $T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ is unitary. Now:

$$\begin{aligned}\Phi(W^f_{g_1}W^f_{g_1}) &= (t_{11}W^g_{g_1} + t_{12}W^g_{g_2})(t_{11}W^g_{g_1} + t_{12}W^g_{g_2}) \\ &= t_{11}^2 W^g_{g_1g_1} + t_{11}t_{12}(W^g_{g_1g_2} + W^g_{g_2g_1}) + t_{12}^2 W^g_{g_2g_2}.\end{aligned}$$

From Lemma 4.2 and since Φ is an isometry we conclude that:

$$1 = \|W^f_{g_1}W^f_{g_1}\| = \|\Phi(W^f_{g_1}W^f_{g_1})\| = \sqrt{|t_{11}|^2 + \frac{4}{3}|t_{11}t_{12}|^2 + |t_{12}|^2}.$$

Since $|t_{11}|^2 + |t_{12}|^2 = 1$ we conclude that $|t_{11}t_{12}| = 0$. Hence either $t_{11} = 0$ or $t_{12} = 0$.

If $t_{11} = 0$ and since T is unitary, we conclude that then $|t_{12}| = |t_{21}| = 1$ and $|t_{22}| = 0$. We then have

$$1 = \|W^f_{g_2g_1}\| = \|\Phi(W^f_{g_2g_1})\| = \|W^g_{g_1g_2}\| = \sqrt{\frac{2}{3}}$$

which is obviously a contradiction! A similar computation shows that if instead $t_{12} = 0$ then we are led to a similar contradiction. Hence, Φ does not exist and our theorem is proven. ∎

We wish to point out that this proof is an example where, while $\mathbb{D}^1$ was not able to distinguish between these two examples, results from multivariate complex analysis together with our invariant $\mathbb{D}^k$ allows us to prove that they are not isomorphic. We chose to use Thullen's result in our proof, yet other routes would have been possible. For instance, we could have used the results of [17] and [7] as well. Since this example only requires $\mathbb{D}^1$ and $\mathbb{D}^2$, Thullen's classification is sufficient. Yet, for many examples, one may need to use $\mathbb{D}^k$ for $k \geqslant 2$, in which case [17] and [7] are appropriate.

4.2. CHARACTERIZATION OF NONCOMMUTATIVE DISK ALGEBRAS. In this section we characterize the noncommutative disk algebras $\mathcal{A}(\mathcal{D}_f)$ in terms of their symbol f. We also note that the domains of $\mathbb{C}^n$ obtained from our algebras appear to be new and interesting examples.

The following simple result will prove useful.

LEMMA 4.4. *Let $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$ be a positive regular free series and $W_1, \ldots, W_n$ be the weighted shifts associated with f. If $c_1, \ldots, c_n$ are non-negative numbers, then $c_1 W_1, c_2 W_2, \ldots, c_n W_n$ are the weighted shifts spaces associated to some positive regular free series $g = \sum_{\alpha \in \mathbb{F}_n^+} a'_\alpha X_\alpha$. Therefore $\mathcal{A}_n(\mathcal{D}_f) \cong \mathcal{A}_n(\mathcal{D}_g)$, and $\mathcal{D}_g(\mathbb{C})$ is obtained from $\mathcal{D}_f(\mathbb{C})$ by scaling the coordinates in $\mathbb{C}^n$ (i.e., by a diagonal linear map with respect to the canonical basis of $\mathbb{C}^n$).*

Proof. It is clearly enough to scale only one of the W_i's. Assume that $W'_1 = c_1 W_1$, and that $W'_j = W_j$ for $j \in \{2, \ldots, n\}$. We claim that there exist scalars $(a'_\alpha)_{|\alpha| \geqslant 1}$ satisfying the conditions (2.1) such that $W'_1, W'_2, \ldots, W'_n$ are the weighted shifts associated to the positive regular free series $g = \sum_{\alpha \in \mathbb{F}_n^+} a'_\alpha X_\alpha$.

Let $r_1 : \mathbb{F}_n^+ \to \mathbb{N}$ be the function that counts the number of g_1's in a word of $\mathbb{F}_n^+$. For example, $r_1(g_2 g_1 g_3 g_1) = 2$ and $r_1(g_2 g_4) = 0$. For $\alpha \in \mathbb{F}_n^+$, define

$$a'_\alpha = \frac{1}{c_1^{2r_1(\alpha)}} a_\alpha.$$

Notice that $(a'_\alpha)_{|\alpha| \geqslant 1}$ satisfies the conditions (2.1). Then from Theorem 2.9 there exist positive real numbers b'_α and n weighted shifts $V_1, \ldots, V_n$ associated to $g = \sum_{\alpha \in \mathbb{F}_n^+} a'_\alpha X_\alpha$. The b'_α's must satisfy equation (2.3) and we can easily check that these are given by

$$b'_\alpha = \frac{1}{c_1^{2r_1(\alpha)}} b_\alpha.$$

Therefore, the weights derived from $(a'_\alpha)_{|\alpha| \geqslant 1}$ are given by the follwing formulas, and the proof is complete:

$$V_1 \delta_\alpha = \sqrt{\frac{b'_\alpha}{b'_{g_1 \alpha}}} \delta_{g_1 \alpha} = c_1 \sqrt{\frac{b_\alpha}{b_{g_1 \alpha}}} \delta_{g_1 \alpha} = c_1 W_1 \delta_a; \quad V_j \delta_\alpha = \sqrt{\frac{b'_\alpha}{b'_{g_j \alpha}}} \delta_{g_j \alpha} = W_j \delta_a \quad \text{for } j \geqslant 2.$$

We will now see that $\mathcal{D}_g(\mathbb{C})$ can be obtained from $\mathcal{D}_f(\mathbb{C})$ by scaling the canonical coordinates. For $\alpha \in \mathbb{F}_n^+$, notice that

$$|\lambda_\alpha|^2 = |\lambda_1|^{2r_1(\alpha)} |\lambda_2|^{2r_2(\alpha)} \cdots |\lambda_n|^{2r_n(\alpha)},$$

where for $i \leqslant n$, $r_i(\alpha)$ is the number of g_i's in the word α. Then we check easily that $a'_a|\lambda_\alpha|^2 = a_\alpha|\lambda'_\alpha|$, where

$$\lambda' = \Big(\frac{\lambda_1}{c_1}, \lambda_2, \dots, \lambda_n\Big) \in \mathbb{C}^n.$$

Then it follows that $\sum\limits_{|\alpha|\geqslant 1} a'_\alpha|\lambda_\alpha|^2 \leqslant 1$ if and only if $\sum\limits_{|\alpha|\geqslant 1} a_\alpha|\lambda'_\alpha|^2 \leqslant 1$, which proves the result. ∎

Let f, g be positive regular free series, and assume that there exists an isomorphism $\Phi : \mathcal{A}(\mathcal{D}_f) \longrightarrow \mathcal{A}(\mathcal{D}_g)$ in NCD such that $\widehat{\Phi}_1(0) = 0$. After rescailing we can assume that for $i \leqslant n$, $a_i^f = a_i^g = 1$ and $b_i^f = b_i^g = 1$. Then by Lemma 4.2, $\Big\| \sum\limits_{i=1}^n c_i W_i^f \Big\| = \Big\| \sum\limits_{i=1}^n c_i W_i^g \Big\| = \sqrt{\sum\limits_{i=1}^n |c_i|^2}$ for all $c_1, \dots, c_n \in \mathbb{C}$. Since $\sqrt{\sum\limits_{i=1}^n |c_i|^2} = \Big\| \sum\limits_{i=1}^n c_i W_i^f \Big\| = \Big\| \Phi\Big(\sum\limits_{i=1}^n c_i W_i^f \Big) \Big\| = \Big\| \sum\limits_{i=1}^n c_i \sum\limits_{j=1}^n m_{ij} W_j^g \Big\| = \Big\| \sum\limits_{j=1}^n \Big(\sum\limits_{i=1}^n c_i m_{ij} \Big) W_j^g \Big\| = \sqrt{\sum\limits_{j=1}^m \Big| \sum\limits_{i=1}^n c_i m_{ij} \Big|^2}$, we conclude that M is unitary.

COROLLARY 4.5. *After rescaling, we can assume that the M of Theorem* 3.18 *is unitary.*

We denote by $S_1, \dots, S_n$ the unilateral shifts on $\ell^2(\mathbb{F}_n^+)$ generating $\mathcal{A}_n$ as in Section 2. We need the following.

THEOREM 4.6 ([6]). *For every $\chi \in \mathbb{D}^1(\mathcal{A}_n)$ (the open unit ball of $\mathbb{C}^n$) there exists an automorphism $\Phi : \mathcal{A}_n \to \mathcal{A}_n$ in* NCD *such that $\widehat{\Phi}_1(\chi) = 0$.*

We are now ready to characterize noncommutative disk algebras by their symbol.

THEOREM 4.7. *Let f be a positive regular n-free series. Then $\mathcal{A}(\mathcal{D}_f)$ is isomorphic to $\mathcal{A}_n$ in* NCD *if and only if $f = \sum\limits_{i=1}^n c_i X_i$ with $(c_1, \dots, c_n) \in \mathbb{C}^n$.*

Proof. By Lemma 4.4 we can assume, without loss of generality, that $a_{g_1} = \cdots = a_{g_n} = 1$. Let Φ be an isomorphism from $\mathcal{A}_n$ onto $\mathcal{A}(\mathcal{D}_f)$ in NCD. Let $\chi = \widehat{\Phi}_1(0)$, let φ be an automorphism of $\mathcal{A}_n$ such that $\widehat{\varphi}_1(\chi) = 0$ given by Theorem 3.18. We set $\eta = \Phi \circ \varphi$. By functoriality, $\widehat{\eta}_1(0) = 0$ and thus η satisfies the hypotheses of Theorem 3.18. Therefore, there exists a matrix $U = (u_{ij})_{i,j\in\{1,\dots,n\}} \in M_{n\times n}$ such that

$$\eta(S_i) = \sum_{j=1}^n u_{ij} W_{g_j}^f.$$

As in Corollary 4.5, the matrix U is unitary. We compute:

$$1 = \|S_i S_j\| = \|\Phi(S_i S_j)\| = \left\| \sum_{k,l=1}^{n} u_{ik} u_{jl} W^f_{g_k} W^f_{g_l} \right\| = \sqrt{\sum_{k,j=1}^{n} |u_{ik} u_{jl}|^2 \frac{1}{b_{kl}}}.$$

To simplify notation in this proof, we write $a_{i_1 \cdots i_k}$ for $a_{g_{i_1} \cdots g_{i_k}}$ and similarly $b_{i_1 \cdots i_k}$ for $b_{g_{i_1} \cdots g_{i_k}}$. By identity (2.4), we have $b_{kl} = b_k a_l + a_{kl} = 1 + a_{kl} \geqslant 1$ and notice that:

$$\sqrt{\sum_{k,l=1}^{n} |u_{ik} u_{jl}|^2} = 1.$$

Fix $k_0, l_0 \leqslant n$. Let $i, j \leqslant n$ such that $u_{ik_0} \neq 0$ and $u_{jl_0} \neq 0$. Then since $\sum_{k,j=1}^{n} |u_{ik} u_{jl}|^2 \frac{1}{b_{kl}} = 1$ and $\sum_{k,j=1}^{n} |u_{ik} u_{jl}|^2 = 1$, and $u_{ik_0} u_{jl_0} \neq 0$ by assumption, we must have $b_{k_0 l_0} = 1$ (otherwise, since $b_{kl} > 1$ for all k, l, both sum of squares could not be one at the same time). Hence $a_{k_0 l_0} = 0$. As k_0, l_0 are arbitrary, we conclude that $a_\alpha = 0$ for all $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = 2$.

We proceed identically by induction. ∎

4.3. EXAMPLES OF DOMAINS WITH NON COMPACT AUTOMORPHIC GROUP. The Riemann mapping theorem tells us that a bounded simply connected domain $D \subset \mathbb{C}$ is homogeneous, (i.e., if $z \in D$ is fixed, for every $w \in D$ there exists a biholomorphic map from D to D that maps z to w), and hence the group of automorphisms of D is non compact. The situation is much more rigid in higher dimensions. In 1927, Kritikos [10] proved that every automorphism of $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2| < 1\}$ fixes the origin and is linear. In 1931, Thullen [18] characterized all bounded proper Reinhardt domains of $\mathbb{C}^2$ that contain the origin and that have non compact automorphic group (i.e., there exists a biholomorphic map that does not fix the origin). Domains of $\mathbb{C}^n$ with non-compact automorphism group have been studied extensively, leading in particular to many deep characterizations of the unit ball among domains in $\mathbb{C}^n$, as well as many families of domains of $\mathbb{C}^n$ with interesting geometries. We refer to [8] for a survey and references.

We conclude this paper with a remark that Theorem 3.7 combined with the result of Davidson and Pitts [6] mentioned in Theorem 4.6 can be used to obtain new bounded domains in $\mathbb{C}^{nk^2}$ with non-compact automorphic group.

PROPOSITION 4.8. *For every $n, k \in \mathbb{N}$, the following domain does not have a compact automorphic group:*

$$\mathbb{D}^k(\mathcal{A}_n) = \{(T_1, \ldots, T_n) \in (M_{k \times k})^n : T_1 T_1^* + \cdots + T_n T_n^* \leqslant I\}^\circ \subset \mathbb{C}^{nk^2}.$$

Proof. By [6], for any $\chi \in \mathbb{D}^1(\mathcal{A}_n)$ there exists an automorphism Φ of $\mathcal{A}_n$ such that $\widehat{\Phi}_1(0) = \chi$. Now, using the expression of $\widehat{\Phi}_k$ in term of the expansion in series of $\Phi(W_1), \ldots, \Phi(W_n)$ given by Lemma 3.5, we conclude easily that

$\widehat{\Phi}_k(0) = (\chi_1 1_k, \ldots, \chi_n 1_k)$. Hence, the orbit of the point $0 \in \mathbb{D}^k(\mathcal{A}_n)$ by the action of the automorphism group of $\mathcal{A}_n$ admits a boundary point of $\mathbb{D}^k(\mathcal{A}_n)$ as a limit. Therefore, the orbit of 0 by the automorphism group of $\mathbb{D}^k(\mathcal{A}_n)$ is not compact. Hence the group of automorphism of $\mathbb{D}^k(\mathcal{A}_n)$ is not compact, since it acts strongly continuously on $\mathbb{D}^k(\mathcal{A}_n)$. ∎

If $n = k = 2$, $\mathbb{D}^2(\mathcal{A}_2)$ consists of the interior of the set of the $\lambda \in \mathbb{C}^8$ such that if $T_1 = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_5 & \lambda_6 \end{bmatrix}$ and $T_2 = \begin{bmatrix} \lambda_3 & \lambda_4 \\ \lambda_7 & \lambda_8 \end{bmatrix}$, then $T_1 T_1^* + T_2 T_2^* \leqslant I$. Notice that

$$T_1T_1^* + T_2T_2^* = \begin{bmatrix} |\lambda_1|^2+|\lambda_2|^2 & \lambda_1\overline{\lambda}_5 + \lambda_2\overline{\lambda}_6 \\ \lambda_5\overline{\lambda}_1 + \lambda_6\overline{\lambda}_2 & |\lambda_5|^2 + |\lambda_6|^2 \end{bmatrix} + \begin{bmatrix} |\lambda_3|^2 + |\lambda_4|^2 & \lambda_3\overline{\lambda}_7 + \lambda_4\overline{\lambda}_8 \\ \lambda_7\overline{\lambda}_3 + \lambda_8\overline{\lambda}_4 & |\lambda_7|^2 + |\lambda_8|^2 \end{bmatrix}.$$

Now, $T_1T_1^* + T_2T \leqslant 1$ if and only if $\det(1 - T_1T_1^* - T_2T_2^*) \geqslant 0$ and the $(1,1)$ entry of $1 - T_1T_1^* - T_2T_2^*$ is nonnegative. Hence $\mathbb{D}^2(\mathcal{A}_2)$ consists of the interior of the set of the $\lambda \in \mathbb{C}^8$ such that:

$$|\lambda_1|^2 + |\lambda_2|^2 + |\lambda_3|^2 + |\lambda_4|^2 \leqslant 1, \text{ and}$$

$$\sum_{i=1}^{8} |\lambda_i|^2 - \sum_{i=1}^{7}\sum_{j=i+1}^{8} |\lambda_i\lambda_j|^2 + \sum_{i=1}^{3}\sum_{j=i+1}^{4} |\lambda_i\overline{\lambda}_j + \overline{\lambda}_{i+4}\lambda_{j+4}|^2 \leqslant 1.$$

This domain has a noncompact automorphism group but appears not to fit in the families of domains in $\mathbb{C}^n$ with non compact automorphic group encountered in the literature on several complex variables. Thus, distinguishing noncommutative domain algebras may also involve the study of new interesting domains and their biholomorphic equivalence.

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