J. OPERATOR THEORY
67:1(2012), 33–72

SECTORIAL FORMS AND DEGENERATE DIFFERENTIAL OPERATORS

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Communicated by Nikolai K. Nikolski

ABSTRACT. If a is a densely defined sectorial form in a Hilbert space which is possibly not closable, then we associate in a natural way a holomorphic semigroup generator with a. This allows us to remove in several theorems of semigroup theory the assumption that the form is closed or symmetric. Many examples are provided, ranging from complex sectorial differential operators, to Dirichlet-to-Neumann operators and operators with Robin or Wentzell boundary conditions.

KEYWORDS: *Sectorial forms, semigroups, Dirichlet-to-Neumann operator, degenerate operators, m-sectorial operators, boundary conditions.*

MSC (2000): 47A07, 47D06, 35Hxx, 35J70, 47A52.

1. INTRODUCTION

Form methods are most efficient to solve evolution equations in a Hilbert space H. The theory establishes a correspondence between closable sectorial forms and holomorphic semigroups on H which are contractive on a sector (see [17], [24] and [19], for example). The aim of this article is to extend the theory in two directions and apply the new criteria to differential operators. Our first result shows that the condition of closability can be omitted completely. To be more precise, consider a sesquilinear form

$$a\colon D(a) \times D(a) \to \mathbb{C}$$

where $D(a)$ is a dense subspace of a Hilbert space H. The form a is called *sectorial* if there exist a (closed) sector

$$\Sigma_\theta = \{re^{\mathrm{i}\alpha} : r \geqslant 0,\ |\alpha| \leqslant \theta\}$$

with $\theta \in [0, \frac{\pi}{2})$, and $\gamma \in \mathbb{R}$, such that $a(u) - \gamma\|u\|_H^2 \in \Sigma_\theta$ for all $u \in D(a)$, where $a(u) = a(u,u)$. We shall show that there exists an operator A in H such that for all $x, f \in H$ one has $x \in D(A)$ and $Ax = f$ if and only if there exist $u_1, u_2, \ldots \in D(a)$

such that $(\operatorname{Re} a(u_n))_n$ is bounded, $\lim_{n\to\infty} u_n = x$ in H and $\lim_{n\to\infty} a(u_n, v) = (f, v)_H$ for all $v \in D(a)$. It is part of the following theorem that f is independent of the sequence $u_1, u_2, \ldots$.

THEOREM 1.1 (Incomplete case). *The operator A is well-defined and $-A$ generates a holomorphic C_0-semigroup on the interior of $\Sigma_{\pi/2-\theta}$.*

This is a special case of Theorem 3.2 below, but we give a short proof already in Section 2. Recall that the form a is called *closable* if for every Cauchy sequence $u_1, u_2, \ldots$ in $D(a)$ such that $\lim_{n\to\infty} u_n = 0$ in H one has $\lim_{n\to\infty} a(u_n) = 0$. Here $D(a)$ carries the natural norm $\|u\|_a = (\operatorname{Re} a(u) + (1-\gamma)\|u\|_H^2)^{1/2}$. In Theorem 1.1 we do not assume that a is closable. Nonetheless, the operator A is well-defined.

For our second extension of the theory on form methods we consider the *complete case*, where the form a is defined on a Hilbert space V. However, we do not assume that V is embedded in H, but merely that there exists a not necessarily injective operator j from V into H. This case is actually the first we consider in Section 2. It is used for the proof of Theorem 1.1 given in Section 2. In Theorem 3.2 we give a common extension of both Theorem 1.1 and the main theorem of Section 2. It turns out that many examples can be treated by our extended form method and Section 4 is devoted to several applications. Our most substantial results concern degenerate elliptic differential operators of second order with complex measurable coefficients on an open set Ω in $\mathbb{R}^d$. If the coefficients satisfy merely a sectoriality condition (which can be very degenerate including the case where the coefficients are zero on some part of Ω), then Theorem 1.1 shows right away that the corresponding operator generates a holomorphic C_0-semigroup on $L_2(\Omega)$. We prove a Davies–Gaffney type estimate which gives us locality properties and in case of Neumann boundary conditions and real coefficients, the invariance of the constant functions. This extends results for positive symmetric forms on $\mathbb{R}^d$ in [13] and [12]. We also extend the criteria for closed convex sets due to Ouhabaz [21] to our more general situation and show that the semigroup is submarkovian if the coefficients are real (but possibly non-symmetric). As a second application, we present an easy and direct treatment of the Dirichlet-to-Neumann operator on a Lipschitz domain Ω. Here it is essential to allow non injective $j \colon D(a) \to H$. As a result, we obtain submarkovian semigroups on $L_p(\partial\Omega)$. Most interesting are Robin boundary conditions which we consider in Subsection 4.3 on an open bounded set Ω of $\mathbb{R}^d$ with the $(d-1)$-dimensional Hausdorff measure on $\partial\Omega$. Using Theorem 1.1 we obtain directly a holomorphic semigroup on $L_2(\Omega)$. Moreover, for every element in the domain of the generator there is a unique trace in $L_2(\partial\Omega, \sigma)$ realising Robin boundary conditions. Such boundary conditions on rough domains had been considered before in [9] and [8]. We also give a new simple proof for the existence of a trace for such general domains. We use these results on the trace to consider Wentzell boundary conditions in Subsection 4.5. These boundary conditions obtained much attention recently [14],

[25]. By our approach we may allow degenerate coefficients for the elliptic operator and the boundary condition. Our final application in Subsection 4.2 concerns multiplicative perturbation of the Laplacian.

Throughout this paper we use the notation and conventions as in [17]. Moreover, the field is $\mathbb{C}$, except if indicated explicitly. We will only consider univocal operators.

2. GENERATION THEOREMS FOR THE COMPLETE CASE

The first step in the proof of Theorem 1.1 is the following extension of the "French" approach to closed sectorial forms (see Chapter XVIIA, Example 3 of [10], Sections 2.2 and 3.6 of [24], and [18]). It is a generation theorem for forms with a complete form domain. It differs from the usual well-known result for closed forms in the following point. We do not assume that the form domain is a subspace of the given Hilbert space, but that there exists a linear mapping j from the form domain into the Hilbert space. Moreover, we do not assume that the mapping is injective. In the injective case, and also in the general case by restricting j to the orthogonal complement of its kernel, we could reduce our result to the usual case. It seems to us simpler to give a direct proof, though, which is adapted from Section 3.6, Application 2 of [24], treating the usual case.

Let V be a normed space and $a\colon V \times V \to \mathbb{C}$ a sesquilinear form. Then a is continuous if and only if there exists a $c > 0$ such that

$$|a(u,v)| \leqslant c\|u\|_V \|v\|_V \tag{2.1}$$

for all $u, v \in V$. Let H be a Hilbert space and $j\colon V \to H$ a bounded linear operator. The form $a\colon V \times V \to \mathbb{C}$ is called *j-elliptic* if there exist $\omega \in \mathbb{R}$ and $\mu > 0$ such that

$$\operatorname{Re} a(u) + \omega \|j(u)\|_H^2 \geqslant \mu \|u\|_V^2 \tag{2.2}$$

for all $u \in V$. The form a is called *coercive* if (2.2) is valid with $\omega = 0$.

An operator $A\colon D(A) \to H$ with $D(A) \subset H$ is called *sectorial* if there are $\gamma \in \mathbb{R}$, called a *vertex*, and $\theta \in [0, \frac{\pi}{2})$, called a *semi-angle*, such that

$$(Ax, x) - \gamma \|x\|_H^2 \in \Sigma_\theta$$

for all $x \in D(A)$. Moreover, A is called *m-sectorial* if it is sectorial and $\lambda I - A$ is surjective for some $\lambda \in \mathbb{R}$ with $\lambda < \gamma$. Then an operator A on H is m-sectorial if and only if $-A$ generates a holomorphic C_0-semigroup S, which is quasi-contractive on some sector, i.e. there exist $\theta \in (0, \frac{\pi}{2})$ and $\omega \in \mathbb{R}$ such that $\|e^{-\omega z} S_z\|_{\mathcal{L}(H)} \leqslant 1$ for all $z \in \Sigma_\theta$. (See Theorem IX.1.24 of [17] and proof of Theorem 1.58 of [21].)

The main theorem of this section is as follows. We repeat that in our setting an operator is always univocal.

THEOREM 2.1. *Let H, V be Hilbert spaces and $j\colon V \to H$ a bounded linear operator such that $j(V)$ is dense in H. Let $a\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form which is j-elliptic. Then one has the following:*

(i) *There exists an operator A in H such that for all $x, f \in H$ one has $x \in D(A)$ and $Ax = f$ if and only if*

> *there exists a $u \in V$ such that $j(u) = x$ and $a(u, v) = (f, j(v))_H$ for all $v \in V$.*

(ii) *The operator A of statement* (i) *is m-sectorial.*

We call the operator A in statement (i) of Theorem 2.1 the *operator associated with (a, j)*.

In the proof of Theorem 2.1 we need two subspaces of V which we need throughout the paper. Set

$$D_H(a) = \{u \in V : \text{there exists an } f \in H \text{ such that } a(u, v) = (f, j(v))_H \text{ for all } v \in V\}$$

and

$$V(a) = \{u \in V : a(u, v) = 0 \text{ for all } v \in \ker j\}.$$

Clearly $D_H(a) \subset V(a)$ and $V(a)$ is closed in V.

Proof of Theorem 2.1. The proof consists of several steps.

Step 1. First, we prove that the restriction map $j|_{V(a)}\colon V(a) \to H$ is injective. If $u \in V(a)$ and $j(u) = 0$, then $a(u) = 0$. The j-ellipticity (2.2) of a then implies that $\|u\|_V = 0$. So $u = 0$ and $j|_{V(a)}$ is injective.

Step 2. Next we prove statement (i). If $u \in V$, then it follows from the density of $j(V)$ in H that there exists at most one $f \in H$ such that $a(u, v) = (f, j(v))_H$ for all $v \in V$. But $j|D_H(a)$ is injective. Therefore we can define the operator A by $D(A) = j(D_H(a))$ and

$$a(u, v) = (Aj(u), j(v))_H \quad \text{for all } u \in D_H(a) \text{ and } v \in V. \tag{2.3}$$

(We emphasize that (2.3) is restricted to $u \in D_H(a)$ and need not to be valid for all $u \in V$ with $j(u) \in D(A)$. An example will be given in Example 3.14.)

Step 3. Let c, ω and μ be as in (2.1) and (2.2). Let $x \in D(A)$. There exists a $u \in D_H(a)$ such that $x = j(u)$. Then $((\omega I + A)x, x) = a(u) + \omega \|j(u)\|_H^2$ and $\mathrm{Re}((\omega I + A)x, x) \geqslant \mu \|u\|_V^2$. Therefore

$$|\,\mathrm{Im}((\omega I + A)x, x)| = |\,\mathrm{Im}\, a(u)| \leqslant c\|u\|_V^2 \leqslant \frac{c}{\mu} \mathrm{Re}((\omega I + A)x, x).$$

So A is sectorial with vertex $-\omega$.

Finally, set $\lambda = \omega + 1$. We shall show that the range of $\lambda I + A$ equals H. Define the form b on V by $b(u, v) = a(u, v) + \lambda(j(u), j(v))_H$. Then b is continuous and coercive. Let $f \in H$. The Lax–Milgram theorem implies that there exists a unique $u \in V$ such that $b(u, v) = (f, j(v))_H$ for all $v \in V$. Then $j(u) \in D(A)$ and $(\lambda I + A)j(u) = f$. Thus A is m-sectorial. This proves Theorem 2.1. ∎

Although Theorem 1.1 is a special case of Theorem 3.2, a short direct proof can be given at this stage.

Proof of Theorem 1.1. Denote by V the completion of $(D(a), \|\cdot\|_a)$. The injection of $(D(a), \|\cdot\|_a)$ into H is continuous. Hence there exists a $j \in \mathcal{L}(V, H)$ such that $j(u) = u$ for all $u \in D(a)$. Since a is sectorial, there exists a unique continuous extension $\widetilde{a}\colon V \times V \to \mathbb{C}$. This extension is j-elliptic. Let A be the operator associated with $(\widetilde{a}, j)$. If $u_1, u_2, \ldots \in D(a)$ with $u_1, u_2, \ldots$ convergent in H and $\operatorname{Re} a(u_1), \operatorname{Re} a(u_2), \ldots$ bounded, then $u_1, u_2, \ldots$ is bounded in $D(a)$. Therefore it has a weakly convergent subsequence in V. It follows from the density of $D(a)$ in V that A equals the operator from Theorem 1.1. In particular, the operator is well-defined. Now the result follows from Theorem 2.1. ∎

In the definition of j-elliptic the assumption is that (2.2) is valid for all $u \in V$. For a version of the Dirichlet-to-Neumann operator in Subsection 4.4 this condition is too strong. One only needs (2.2) to be valid for all $u \in V(a)$ if in addition $V = V(a) + \ker j$ (cf. Theorem 2.5(i)).

COROLLARY 2.2. *Let H, V be Hilbert spaces and $j\colon V \to H$ a bounded linear operator such that $j(V)$ is dense in H. Let $a\colon V \times V \to \mathbb{C}$ be a continuous sesquilinear form. Suppose that there exist $\omega \in \mathbb{R}$ and $\mu > 0$ such that*

$$\operatorname{Re} a(u) + \omega \|j(u)\|_H^2 \geqslant \mu \|u\|_V^2 \tag{2.4}$$

for all $u \in V(a)$. In addition suppose that $V = V(a) + \ker j$. Then one has the following:

(i) *There exists an operator A in H such that for all $x, f \in H$ one has $x \in D(A)$ and $Ax = f$ if and only if*

there exists a $u \in V$ such that $j(u) = x$ and $a(u, v) = (f, j(v))_H$ for all $v \in V$.

(ii) *The operator A of statement* (i) *is m-sectorial.*

Again we call the operator A in statement (i) of Corollary 2.2 the *operator associated with* (a, j).

Proof of Corollary 2.2. Define the form b by $b = a|_{V(a) \times V(a)}$. Then $(b, j|_{V(a)})$ satisfies the assumptions of Theorem 2.1. Let B be the operator associated with $(b, j|_{V(a)})$.

Clearly $D_H(a) \subset D_H(b)$. Conversely, if $u \in D_H(b)$, then there is an $f \in H$ such that $b(u, v) = (f, j(v))_H$ for all $v \in V(a)$. Then $a(u, v) = (f, j(v))_H$ for all $v \in V(a)$, but also for all $v \in \ker j$ by definition of $V(a)$. Since $V = V(a) + \ker j$ by assumption, it follows that $u \in D_H(a)$. Therefore $D_H(a) = D_H(b)$. But $j|_{D_H(a)}$ is injective. Hence the operator B satisfies the requirements of statement (i). Then statement (ii) is obvious. ∎

We return to the situation of Theorem 2.1. If the form a is j-elliptic and if $\tau \in \mathbb{C}$, then obviously the operator $A + \tau I$ is associated with (b, j), where b is the j-elliptic form $b(u, v) = a(u, v) + \tau (j(u), j(v))_H$ on V.

Although it is very convenient that we do not assume that the operator j is injective, the second statement in the next proposition shows that without loss of generality one might assume that j is injective, by considering a different form. The proposition is a kind of uniqueness result. It determines the dependence of the operator on the choice of V.

PROPOSITION 2.3. *Suppose the form a is j-elliptic and let A be the operator associated with (a, j). Then one has the following:*

(i) *If U is a closed subspace of V such that $D_H(a) \subset U$, then A equals the operator associated with $(a|_{U \times U}, j|_U)$. If, in addition, the restriction $j|_U$ is injective, then $U = \overline{D_H(a)}$, where the closure is taken in V.*

(ii) *$V(a) = \overline{D_H(a)}$. Moreover, $j|_{V(a)}$ is injective and A equals the operator associated with $(a|_{V(a) \times V(a)}, j|_{V(a)})$.*

(iii) *If U is a closed subspace of $V(a)$ such that $j(U)$ is dense in H and A is the operator associated with $(a|_{U \times U}, j|_U)$, then $U = V(a)$.*

Proof. (i) Note that $j(U)$ and $j(V(a))$ both contain $j(D_H(a)) = D(A)$. Therefore $j(U)$ and $j(V(a))$ are dense in H. Let $b_1 = a|_{U \times U}$ and $b_2 = a|_{V(a) \times V(a)}$. Further, let B_1 and B_2 be the operators associated with $(b_1, j|_U)$ and $(b_2, j|_{V(a)})$. Then for all $u \in D_H(a)$ one deduces that $(Aj(u), j(v))_H = a(u, v) = b_1(u, v)$ for all $v \in U$. Therefore $u \in D_H(b_1)$ and $B_1 j(u) = Aj(u)$. So $A \subset B_1$. But both $-A$ and $-B_1$ are semigroup generators. Therefore $B_1 = A$. Similarly, $A = B_2$. Finally, if j is injective on U, then it follows from the inclusion $V(a) \subset U$ and the uniqueness theorem for closed sectorial forms, ([17], Theorem VI.2.7) that $U = V(a)$. This proves statement (i).

(ii) The injectivity of $j|_{V(a)}$ has been proved in Step 1 of the proof of Theorem 2.1. Then (ii) is a special case of (i).

Finally, statement (iii) follows from (i) with a replaced by $a|_{U \times U}$. ∎

It is easy to construct examples with $V(a) \neq V$. Therefore the injectivity condition in Proposition 2.3(i) is necessary.

COROLLARY 2.4. *Assume the notation and conditions of Corollary 2.2. Let A be the operator associated with (a, j). Then one has the following:*

(i) *$V(a) = \overline{D_H(a)}$. Moreover, $j|_{V(a)}$ is injective and A equals the operator associated with $(a|_{V(a) \times V(a)}, j|_{V(a)})$.*

(ii) *Let U be a closed subspace of $V(a)$ such that $j(U)$ is dense in H. Then $a|_{U \times U}$ is $j|_U$-elliptic. Suppose A is the operator associated with $(a|_{U \times U}, j|_U)$. Then $U = V(a)$.*

Proof. Let $b = a|_{V(a) \times V(a)}$. Then it follows from the proof of Corollary 2.2 that b is $j|_{V(a)}$-elliptic and A is the associated operator. Moreover, $D_H(a) = D_H(b)$. Then

$$V(b) = \{u \in V(a) : a(u, v) = 0 \text{ for all } v \in V(a) \cap \ker j\} = V(a).$$

It follows from Proposition 2.3(ii) applied to $(b, j|_{V(a)})$ that $V(b) = \overline{D_H(b)}$. Hence $V(a) = \overline{D_H(a)}$. Moreover, $(j|_{V(a)})|_{V(b)}$ is injective. Therefore $j|_{V(a)}$ is injective.

If U is a closed subspace of $V(a)$ such that $j(U)$ is dense in H, then $a|_{U\times U}$ is $j|_U$-elliptic. Then statement (ii) follows from Proposition 2.3(iii). ∎

In Subsection 4.4 we give an example that Proposition 2.3(i) cannot be extended to the setting of Corollary 2.2.

One can decompose a form a in its real and imaginary parts as $a = h + \mathrm{i}k$, where $h, k \colon D(a) \times D(a) \to \mathbb{C}$ are symmetric sesquilinear forms. We write $\Re a = h$ and $\Im a = k$.

The next theorem gives a connection between the current forms a together with the map j and the closed sectorial forms in Section VI.2 of [17].

THEOREM 2.5. *Suppose the form a is j-elliptic and let A be the operator associated with (a, j). Then the following holds:*

(i) $\ker j \oplus V(a) = V$ *as vector spaces.*

(ii) *Let a_{c} be the form on H defined by*

$$D(a_{\mathrm{c}}) = j(V) \quad \textit{and} \quad a_{\mathrm{c}}(j(u), j(v)) = a(u, v) \quad (u, v \in V(a)).$$

Then a_{c} is the unique closed, sectorial form such that A is associated with a_{c}.

Proof. (i) Let $\omega \in \mathbb{R}$ and $\mu > 0$ be as in (2.2). We can assume that $\omega = -1$ and the form a is coercive. Otherwise we replace a by the form $(u, v) \mapsto a(u, v) + (\omega + 1)(j(u), j(v))_H$. Let $h = \Re a$ and $k = \Im a$ be the real and imaginary part of a. Then $\langle u, v \rangle := h(u, v)$ defines an equivalent scalar product on V. So we may assume that $\|u\|_V = \|u\|_h$ for all $u \in V$. Let $V_1 = \ker j$ and $V_2 = (\ker j)^\perp$. Moreover, let π_1 and π_2 be the projection from V onto V_1 and V_2, respectively. Then $h(u_1, v_2) = 0$ for all $u_1 \in V_1$ and $v_2 \in V_2$. There exists a unique operator $T \in \mathcal{L}(V)$ such that $k(u, v) = h(Tu, v)$ for all $u, v \in V$. Let $T_{11} = \pi_1 \circ T|_{V_1} \in \mathcal{L}(V_1)$ and $T_{12} = \pi_1 \circ T|_{V_2} \in \mathcal{L}(V_2, V_1)$. If $(u_1, u_2) \in V_1 \times V_2$, then $u_1 + u_2 \in V(a)$ if and only if $0 = a(u_1 + u_2, v_1) = h((I + \mathrm{i}T)(u_1 + u_2), v_1) = h((I + \mathrm{i}T_{11})u_1 + \mathrm{i}T_{12}u_2, v_1)$ for all $v_1 \in V_1$. So

$$V(a) = \{u_1 + u_2 : (u_1, u_2) \in V_1 \times V_2 \text{ and } (I + \mathrm{i}T_{11})u_1 + \mathrm{i}T_{12}u_2 = 0\}.$$

But T is self-adjoint since h and k are symmetric. So $I + \mathrm{i}T_{11}$ is invertible. Thus for all $u_2 \in V_2$ there exists a $u_1 \in V_1$ such that $u_1 + u_2 \in V(a)$. Consequently, $j(V(a)) = j(V_2) = j(V)$. This implies that $\ker j + V(a) = V$. That the sum is direct has been proved in Step 1 of the proof of Theorem 2.1.

(ii) Define on $j(V(a))$ the scalar product carried over from $V(a)$ by j. Then the form a_{c} is clearly continuous and elliptic, which is the same as sectorial and closed (Lemma 3.1). The operator A is clearly the operator associated with a_{c}. ∎

We call the form a_{c} in Theorem 2.5 the *classical form* associated with (a, j). It equals the classical form associated with the m-sectorial operator A. The proof of

Theorem 2.5 also allows to estimate the real part of the classical form of a by the classical form of the real part of a as follows.

PROPOSITION 2.6. *Suppose the form a is j-elliptic and let A be the operator associated with (a, j). Suppose $\omega \leqslant -1$ in* (2.2). *Let h be the real part of a and h_c the classical form associated with (h, j). Then $D(a_c) = D(h_c)$. Moreover, there exists a constant $C > 0$ such that $\operatorname{Re} a_c(x) \leqslant C h_c(x)$ for all $x \in j(V)$.*

Proof. The first statement is obvious since $D(a_c) = j(V) = D(h_c)$. We use the notation introduced in the proof of Theorem 2.5. Moreover, we may assume that the inner product on V is given by $(u, v) \mapsto h(u, v)$. Let $u \in V(a)$. Then $(I + \mathrm{i}T_{11})u_1 + \mathrm{i}T_{12}u_2 = 0$, where $u_1 = \pi_1(u)$ and $u_2 = \pi_2(u)$. Therefore $u_1 = -\mathrm{i}(I + \mathrm{i}T_{11})^{-1}T_{12}u_2$. Moreover, $j(u) = j(u_2)$ and $u_2 \in V(h)$. So $a_c(j(u)) = a(u)$ and $h_c(j(u)) = h_c(j(u_2)) = h(u_2) = \|u_2\|_V^2$. Since the operator $(I + \mathrm{i}T_{11})^{-1}T_{12}$ is bounded one estimates

$$\operatorname{Re} a_c(j(u)) = \operatorname{Re} a(u) = h(u) = \|u_1\|_V^2 + \|u_2\|_V^2 \leqslant C\|u_2\|_V^2 = C h_c(j(u))$$

where $C = \|(I + \mathrm{i}T_{11})^{-1}T_{12}\|^2 + 1$. ∎

The next lemma gives a sufficient condition for the resolvents to be compact.

LEMMA 2.7. *Suppose the form a is j-elliptic and let A be the operator associated with (a, j). If j is compact, then $(\lambda I + A)^{-1}$ is compact for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \omega$, where ω is as in* (2.2).

Proof. By the Lax–Milgram theorem there exists a $B \in \mathcal{L}(H, V)$ such that

$$(f, j(v))_H = a(Bf, v) + \lambda(j(Bf), j(v))_H$$

for all $f \in H$ and $v \in V$. Then $B(H) \subset D_H(a)$ and $(A + \lambda I)j(Bf) = f$ for all $f \in H$. Therefore $(\lambda I + A)^{-1} = j \circ B$ is compact. ∎

REMARK 2.8. If B is the operator associated with (a^*, j) where a^* is the j-elliptic form on V given by $a^*(u, v) = \overline{a(v, u)}$, then A^* is an extension of B. But both $-A^*$ and $-B$ are generators of semigroups. Therefore A^* is the operator associated with (a^*, j).

In Theorem 2.2 of [21] there is a characterization of closed convex subsets which are invariant under the semigroup S. For a background of this theorem we refer to the Notes for Section 2.1 in [21]. Using the two statements of Theorem 2.5, the theorem of Ouhabaz can be reformulated in the current context. Recall that a sesquilinear form b is called *accretive* if $\operatorname{Re} b(u) \geqslant 0$ for all $u \in D(b)$.

PROPOSITION 2.9. *Suppose the form a is j-elliptic, let A be the operator associated with (a, j) and S the semigroup generated by $-A$. Moreover, suppose that a is accretive. Let $C \subset H$ be a non-empty closed convex set and $P\colon H \to C$ the orthogonal projection. Then the following conditions are equivalent:*

(i) *$S_t C \subset C$ for all $t > 0$.*

(ii) *For all* $u \in V$ *there exists a* $w \in V$ *such that*

$$Pj(u) = j(w) \quad \textit{and} \quad \operatorname{Re} a(w, u - w) \geqslant 0.$$

(iii) *For all* $u \in V$ *there exists a* $w \in V$ *such that*

$$Pj(u) = j(w) \quad \textit{and} \quad \operatorname{Re} a(u, u - w) \geqslant 0.$$

(iv) *There exists a dense subset* D *of* V *such that for all* $u \in D$ *there exists a* $w \in V$ *such that*

$$Pj(u) = j(w) \quad \textit{and} \quad \operatorname{Re} a(w, u - w) \geqslant 0.$$

Proof. (i) $\Rightarrow$ (ii) Let $u \in V$. By Theorem 2.5 there exists a $u' \in V(a)$ such that $j(u') = j(u)$. Then $Pj(u') \in D(a_c)$ by Theorem 2.2, 1)$\Rightarrow$2) of [21]. So there exists a $w \in V(a)$ such that $Pj(u') = j(w)$. Then $\operatorname{Re} a(w, u' - w) = \operatorname{Re} a_c(j(w), j(u') - j(w)) = \operatorname{Re} a_c(Pj(u'), j(u') - Pj(u')) \geqslant 0$ again by Theorem 2.2, 1)$\Rightarrow$2) of [21]. But $a(w, u - u') = 0$ since $w \in V(a)$ and $u - u' \in \ker j$. So $\operatorname{Re} a(w, u - w) \geqslant 0$.

(ii) $\Rightarrow$ (iii) Trivial, since $\operatorname{Re} a(u - w, u - w) \geqslant 0$.

(iii) $\Rightarrow$ (i) Let $u \in V(a)$. By assumption there exists a $w \in V$ such that $Pj(u) = j(w)$ and $\operatorname{Re} a(u, u - w) \geqslant 0$. Let $w' \in V(a)$ be such that $j(w) = j(w')$. Then $a(u, w - w') = 0$ since $u \in V(a)$ and $w - w' \in \ker j$. So $\operatorname{Re} a(u, u - w') \geqslant 0$ and $\operatorname{Re} a_c(j(u), j(u) - Pj(u)) \geqslant 0$. Then the implication follows from Theorem 2.2, 3)$\Rightarrow$1) of [21].

(ii) $\Rightarrow$ (iv) Trivial.

(iv) $\Rightarrow$ (ii) Since a is continuous there exists a $c > 0$ such that $|a(u, v)| \leqslant c\|u\|_V \|v\|_V$ for all $u, v \in V$. Let $u \in V$. There exist $u_1, u_2, \ldots \in D$ such that $\lim u_n = u$ in V. For all $n \in \mathbb{N}$ there exists by assumption a $w_n \in V$ such that $Pj(u_n) = j(w_n)$ and $\operatorname{Re} a(w_n, u_n - w_n) \geqslant 0$. Let μ and ω be as in (2.2). Then

$$\begin{aligned} \mu\|w_n\|_V^2 &\leqslant \operatorname{Re} a(w_n) + \omega\|j(w_n)\|_H^2 \\ &= \operatorname{Re} a(w_n, u_n) - \operatorname{Re} a(w_n, u_n - w_n) + \omega\|j(w_n)\|_H^2 \\ &\leqslant \operatorname{Re} a(w_n, u_n) + \omega\|j(w_n)\|_H^2 \leqslant c\|w_n\|_V \|u_n\|_V + \omega\|Pj(u_n)\|_H^2 \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\{u_n : n \in \mathbb{N}\}$ is bounded in V and $\{Pj(u_n) : n \in \mathbb{N}\}$ is bounded in H by continuity of j and contractivity of P, it follows that the set $\{w_n : n \in \mathbb{N}\}$ is bounded in V. So there exist $w \in V$ and a subsequence $w_{n_1}, w_{n_2}, \ldots$ of $w_1, w_2, \ldots$ such that $\lim_{k\to\infty} w_{n_k} = w$ weakly in V. Then $\lim_{k\to\infty} Pj(u_{n_k}) = \lim j(w_{n_k}) = j(w)$ weakly in H. On the other hand, the continuity of j and P gives $\lim_{n\to\infty} Pj(u_n) = Pj(u)$ strongly in H. So $Pj(u) = j(w)$. Next, since $\operatorname{Re} a(w_n, u_n - w_n) \geqslant 0$ one has $\operatorname{Re} a(w_n) \leqslant \operatorname{Re} a(w_n, u_n)$ for all $n \in \mathbb{N}$. Moreover, $\lim_{k\to\infty} \operatorname{Re} a(w_{n_k}, u_{n_k}) = \operatorname{Re} a(w, u)$. In addition, since a is accretive and j-elliptic it follows that $v \mapsto (\operatorname{Re} a(v) + \varepsilon\|j(v)\|_H^2)^{1/2}$ is an equivalent norm associated with an inner product on V for all $\varepsilon > 0$. Therefore $\operatorname{Re} a(w) \leqslant \liminf_{k\to\infty} \operatorname{Re} a(w_{n_k})$. So $\operatorname{Re} a(w) \leqslant \operatorname{Re} a(w, u)$ and $\operatorname{Re} a(w, u - w) \geqslant 0$ as required. ∎

3. GENERATION THEOREMS IN THE INCOMPLETE CASE

In this section we consider forms for which the form domain is not necessarily a Hilbert space. First we reformulate the complete case.

Let $a \colon D(a) \times D(a) \to \mathbb{C}$ be a sesquilinear form, where the domain $D(a)$ of a is a complex vector space, the domain of a. Let H be a Hilbert space and $j \colon D(a) \to H$ a linear map. We say that a is a *j-sectorial form* if there are $\gamma \in \mathbb{R}$, called a *vertex*, and $\theta \in [0, \frac{\pi}{2})$, called a *semi-angle*, such that

$$a(u) - \gamma \|j(u)\|_H^2 \in \Sigma_\theta$$

for all $u \in D(a)$. If a is j-sectorial with vertex γ, then we define the semi-inner product $(\cdot, \cdot)_a$ in the space $D(a)$ by

$$(u, v)_a = (\Re a)(u, v) + (1 - \gamma)(j(u), j(v))_H.$$

Again we do not include the γ in the notation. Then the associated seminorm $\| \cdot \|_a$ is a norm if and only if $\operatorname{Re} a(u) = j(u) = 0$ implies $u = 0$ for all $u \in D(a)$. A j-sectorial form a is called *closed* if $\| \cdot \|_a$ is a norm and $(D(a), \| \cdot \|_a)$ is a Hilbert space. This term coincides with the term for closed forms in Section VI.1.3 of [17] if j is an inclusion map.

The alluded reformulation is as follows.

LEMMA 3.1. *Let V be a vector space, $a \colon V \times V \to \mathbb{C}$ a sesquilinear form, H a Hilbert space and $j \colon V \to H$ a linear map. Then the following are equivalent:*

(i) *The form a is j-sectorial and closed.*

(ii) *There exists a norm $\| \cdot \|_V$ on V such that V is a Banach space, the map j is bounded from $(V, \| \cdot \|_V)$ into H, the form a is j-elliptic and a is continuous.*

Moreover, if condition (ii) *is valid, then the norms $\| \cdot \|_a$ and $\| \cdot \|_V$ are equivalent.*

The easy proof is left to the reader.

In this section we drop the assumption that $(D(a), \| \cdot \|_a)$ is complete. So H is a Hilbert space, $a \colon D(a) \times D(a) \to \mathbb{C}$ is a sesquilinear form, $j \colon D(a) \to H$ is a linear map and we assume that a is merely j-sectorial and $j(D(a))$ is dense in H. We will again associate a sectorially bounded holomorphic semigroup generator with (a, j). The next theorem is an extension of both Theorem 1.1 and Theorem 2.1.

In a natural way one can define the notion of Cauchy sequence in a seminormed vector space.

THEOREM 3.2. *Let a be a sesquilinear form, H a Hilbert space and $j \colon D(a) \to H$ a linear map. Assume that a is j-sectorial and $j(D(a))$ is dense in H. Then one has the following:*

(a) *There exists an operator A in H such that for all $x, f \in H$ one has $x \in D(A)$ and $Ax = f$ if and only if there exist $u_1, u_2, \ldots \in D(a)$ such that:*

(i) $\lim_{n \to \infty} j(u_n) = x$ *weakly in H,*

(ii) $\sup_{n\in\mathbb{N}} \operatorname{Re} a(u_n) < \infty$, *and,*

(iii) $\lim_{n\to\infty} a(u_n, v) = (f, j(v))_H$ *for all* $v \in D(a)$.

(b) *The operator* A *of statement* (a) *is m-sectorial.*

(c) *Let* $x, f \in H$. *Then* $x \in D(A)$ *and* $Ax = f$ *if and only if there exists a Cauchy sequence* $u_1, u_2, \ldots$ *in* $D(a)$ *such that* $\lim_{n\to\infty} j(u_n) = x$ *in* H *and* $\lim_{n\to\infty} a(u_n, v) = (f, j(v))_H$ *for all* $v \in D(a)$.

If V_0 is a vector space with a semi-inner product, then there exist a Hilbert space V and an isometric map $q \colon V_0 \to V$ such that $q(V_0)$ is dense in V. Then V and q are unique, up to unitary equivalence. We call (V, q) the *completion* of V_0. If also V_0' is a vector space with a semi-inner product and (V', q') is its completion, then for every linear map $T_0 \colon V_0 \to V_0'$ and $c \geqslant 0$ such that $\|T_0 u\|_{V_0'} \leqslant c\|u\|_{V_0}$ for all $u \in V_0$, there exists a unique $T \in \mathcal{L}(V, V')$ such that $q' \circ T_0 = T \circ q$. Then $\|T\| \leqslant c$. We call T the *continuous extension* of T_0 to V.

Proof of Theorem 3.2. Let (V, q) be the completion of $D(a)$. Since $\|j(u)\|_H \leqslant \|u\|_a$ for all $u \in D(a)$ the continuous extension $\widetilde{j} \in \mathcal{L}(V, H)$ of j is a contraction. Note that $\widetilde{j} \circ q = j$ and $\widetilde{j}(V)$ is dense in H. Next, since

$$|a(u, v) - \gamma(j(u), j(v))_H| \leqslant (1 + \tan\theta)\|u\|_a\|v\|_a$$

for all $u, v \in D(a)$, where θ is the semi-angle of a and we used the estimate (1.15) of Subsection VI.1.2 in [17], there exists a unique continuous sesquilinear form $\widetilde{a} \colon V \times V \to \mathbb{C}$ such that $\widetilde{a}(q(u), q(v)) = a(u, v)$ for all $u, v \in D(a)$. Then $\widetilde{a}$ is $\widetilde{j}$-sectorial with vertex γ and semi-angle θ. Moreover, $\widetilde{a}$ is $\widetilde{j}$-elliptic. Now let A be the operator associated with $(\widetilde{a}, \widetilde{j})$.

Let $x, f \in H$. We next show that the following statements are equivalent:

(i) $x \in D(A)$ and $Ax = f$;

(ii) there exists a Cauchy sequence $u_1, u_2, \ldots$ in $(D(a), \|\cdot\|_a)$ such that $\lim_{n\to\infty} j(u_n) = x$ and $\lim_{n\to\infty} a(u_n, v) = (f, j(v))_H$ for all $v \in D(a)$; and

(iii) there exists a bounded sequence $u_1, u_2, \ldots$ in $(D(a), \|\cdot\|_a)$ such that $\lim_{n\to\infty} j(u_n) = x$ weakly in H and $\lim_{n\to\infty} a(u_n, v) = (f, j(v))_H$ for all $v \in D(a)$.

(i) $\Rightarrow$ (ii) It follows from the definition (2.3) that there exists a $\widetilde{u} \in V$ such that $\widetilde{j}(\widetilde{u}) = x$ and $\widetilde{a}(\widetilde{u}, \widetilde{v}) = (f, \widetilde{j}(\widetilde{v}))_H$ for all $\widetilde{v} \in V$. Then there exist $u_1, u_2, \ldots \in D(a)$ such that $\lim q(u_n) = \widetilde{u}$ in V. Hence $u_1, u_2, \ldots$ is a Cauchy sequence in $(D(a), \|\cdot\|_a)$. Moreover,

$$(f, j(v))_H = (f, \widetilde{j}(q(v)))_H = \widetilde{a}(\widetilde{u}, q(v)) = \lim \widetilde{a}(q(u_n), q(v)) = \lim a(u_n, v)$$

for all $v \in D(a)$ and $\lim j(u_n) = \lim \widetilde{j}(q(u_n)) = \widetilde{j}(\widetilde{u}) = x$ in H.

(ii) $\Rightarrow$ (iii) Trivial.

(iii) $\Rightarrow$ (i) Since $q(u_1), q(u_2), \ldots$ is a bounded sequence in V_0 the weak limit $\widetilde{u} = \lim q(u_n)$ exists in V after passing to a subsequence, if necessary. Then $\widetilde{j}(\widetilde{u}) =$

$\lim \widetilde{j}(q(u_n)) = \lim j(u_n) = x$ weakly in H. Moreover,

$$\widetilde{a}(\widetilde{u}, q(v)) = \lim \widetilde{a}(q(u_n), q(v)) = \lim a(u_n, v) = (f, j(v))_H = (f, \widetilde{j}(q(v)))_H$$

for all $v \in D(a)$. Since $q(D(a))$ is dense in V one deduces that $\widetilde{a}(\widetilde{u}, \widetilde{v}) = (f, \widetilde{j}(\widetilde{v}))_H$ for all $\widetilde{v} \in V$. So $x \in D(A)$ and $Ax = f$ as required.

We have proved the existence of the operator A in statement (a) of the theorem, together with the characterization (c). Now statement (b) follows from Theorem 2.1. ∎

We call the operator A in statement (a) of Theorem 3.2 the *operator associated with* (a, j). Note that this is the same operator as in Theorem 2.1 if $D(a)$ was provided with a Hilbert space structure such that j is continuous, a is continuous and a is j-elliptic.

In the proof of Theorem 3.2 we also proved the following fact.

PROPOSITION 3.3. *Let a be a sesquilinear form, H a Hilbert space and $j \colon D(a) \to H$ a linear map. Assume that a is j-sectorial and $j(D(a))$ is dense in H. Let (V, q) be the completion of $D(a)$. Then there exists a unique continuous sesquilinear form $\widetilde{a} \colon V \times V \to \mathbb{C}$ such that $\widetilde{a}(q(u), q(v)) = a(u, v)$ for all $u, v \in D(a)$. Moreover, $\widetilde{a}$ is $\widetilde{j}$-elliptic, where $\widetilde{j}$ is the continuous extension of j to V and the operator associated with (a, j) equals the operator associated with $(\widetilde{a}, \widetilde{j})$.*

REMARK 3.4. Let a be a sesquilinear form, H a Hilbert space and $j \colon D(a) \to H$ a linear map. Suppose that a is j-sectorial. Let D be a *core* for a, i.e. a dense subspace of $D(a)$. Then $j(D)$ is dense in H and the operator associated with (a, j) equals the operator associated with $(a|_{D \times D}, j|_D)$. This follows immediately from Theorem 3.2(c)

REMARK 3.5. Let a be a sesquilinear form, H a Hilbert space and $j \colon D(a) \to H$ a linear map. Assume that a is j-sectorial and $j(D(a))$ is dense in H. Then a^* is j-sectorial. Moreover, if B is the operator associated with (a^*, j) and A is the operator associated with (a, j), then $B = A^*$. Indeed, using the notation as in the proof of Theorem 3.2 it follows that A is the operator associated with $(\widetilde{a}, \widetilde{j})$. Moreover, $(u, v)_{a^*} = (u, v)_a$ for all $u, v \in D(a) = D(a^*)$. Therefore (V, q) is also the completion of $D(a^*)$. Then $\widetilde{a^*} = (\widetilde{a})^*$. By construction the operator B is the operator associated with $(\widetilde{a^*}, \widetilde{j}) = ((\widetilde{a})^*, \widetilde{j})$. Hence $B = A^*$ by Remark 2.8. In particular, if a is symmetric, then A is self-adjoint.

REMARK 3.6. It follows from the construction that the operator $\lambda I + A$ is invertible for all $\lambda > -\gamma$ if A is the operator associated with a j-sectorial form a with vertex γ.

The next theorem is of the nature of Theorem VIII.3.6 of [17]. If $F_1, F_2, \ldots$ are subsets of a set F, then define $\liminf\limits_{n \to \infty} F_n = \bigcup\limits_{n=1}^{\infty} \bigcap\limits_{k=n}^{\infty} F_k$.

THEOREM 3.7. *Let a be a sesquilinear form, H a Hilbert space and $j\colon D(a) \to H$ a linear map. Assume that a is j-sectorial with vertex γ. For all $n \in \mathbb{N}$ let a_n be a sesquilinear form with $D(a_n) \subset D(a)$. Suppose that there exist $\theta \in [0, \frac{\pi}{2})$ and for all $n \in \mathbb{N}$ a $\gamma_n \in \mathbb{R}$ such that*

$$a_n(u) - a(u) - \gamma_n \|j(u)\|_H^2 \in \Sigma_\theta \tag{3.1}$$

for all $u \in D(a_n)$. Assume that $\lim_{n\to\infty} \gamma_n = 0$. Moreover, suppose that there exists a core D for a such that $D \subset \liminf_{n\to\infty} D(a_n)$ and $\lim_{n\to\infty} a_n(u) = a(u)$ for all $u \in D$. Finally, suppose that $j(D(a_n))$ is dense in H for all $n \in \mathbb{N}$. Let A be the operator associated with (a, j) and for all $n \in \mathbb{N}$ let A_n be the operator associated with $(a_n, j|_{D(a_n)})$. Fix $\lambda > -\gamma$. Then, for all $f \in H$,

$$\lim_{n\to\infty} (\lambda I + A_n)^{-1} f = (\lambda I + A)^{-1} f.$$

Proof. Without loss of generality we may assume that $\gamma = 0$ and $\gamma_n < 1$ for all $n \in \mathbb{N}$. Then a_n is j-sectorial with vertex γ_n and $D(a_n)$ has the norm $\|u\|_{a_n}^2 = \operatorname{Re} a_n(u) + (1 - \gamma_n)\|j(u)\|_H^2$. We use the construction as in the proof of Theorem 3.2. For the form a we construct V, q, $\widetilde{j}$, $\widetilde{a}$ and for the form a_n we construct V_n, q_n, $\widetilde{j}_n$, $\widetilde{a}_n$.

Let $n \in \mathbb{N}$. It follows from (3.1) that $\|u\|_a^2 \leqslant \|u\|_{a_n}^2$ for all $u \in D(a_n)$. Let Φ_n be the continuous extension of the inclusion map $D(a_n) \subset D(a)$. So $\Phi_n \in \mathcal{L}(V_n, V)$ and $\Phi_n \circ q_n = q$. Then $\widetilde{j}_n(q_n(u)) = j(u) = \widetilde{j}(q(u)) = \widetilde{j}(\Phi_n(q_n(u)))$ for all $u \in D(a_n)$ and by density $\widetilde{j}_n = \widetilde{j} \circ \Phi_n$. Define the sectorial form $b_n\colon D(a_n) \times D(a_n) \to \mathbb{C}$ by

$$b_n(u, v) = a_n(u, v) - a(u, v) - \gamma_n (j(u), j(v))_H.$$

Then $|b_n(u)| \leqslant \|u\|_{a_n}^2$, so there exists a unique continuous accretive sectorial form $\widetilde{b}_n\colon V_n \times V_n \to \mathbb{C}$ such that $\widetilde{b}_n(q_n(u), q_n(v)) = b_n(u, v)$ for all $u, v \in D(a_n)$. Then

$$\widetilde{a}_n(u, v) = \widetilde{a}(\Phi_n(u), \Phi_n(v)) + \widetilde{b}_n(u, v) + \gamma_n (\widetilde{j}_n(u), \widetilde{j}_n(v))_H \tag{3.2}$$

first for all $u, v \in q_n(D(a_n))$ and then by density for all $u, v \in V_n$.

In order not to duplicate too much of the proof for the current theorem for the proof of Theorem 3.8 we first prove a little bit more. Let $f, f_1, f_2, \ldots \in H$ and suppose that $\lim f_n = f$ weakly in H. For all $n \in \mathbb{N}$ there exists a unique $\widetilde{u}_n \in D_H(\widetilde{a}_n)$ such that $\widetilde{j}_n(\widetilde{u}_n) = (\lambda I + A_n)^{-1} f_n$. Set $u_n = \Phi_n(\widetilde{u}_n) \in V$. Then $\widetilde{j}(u_n) = \widetilde{j}_n(\widetilde{u}_n)$. We shall show that there exists a subsequence (u_{n_k}) of (u_n) and a $u \in D_H(\widetilde{a})$ such that $\lim u_{n_k} = u$ weakly in V and $\widetilde{j}(u) = (\lambda I + A)^{-1} f$.

Since $\widetilde{u}_n \in D_H(\widetilde{a}_n)$ and $\lambda \widetilde{j}_n(\widetilde{u}_n) + A_n \widetilde{j}_n(\widetilde{u}_n) = f_n$ it follows from (2.3) that

$$\lambda(\widetilde{j}_n(\widetilde{u}_n), \widetilde{j}_n(v))_H + \widetilde{a}_n(\widetilde{u}_n, v) = (f_n, \widetilde{j}_n(v))_H \tag{3.3}$$

for all $v \in V_n$. Taking $v = \widetilde{u}_n$ in (3.3) and using (3.2) we obtain

$$(3.4) \qquad (\lambda + \gamma_n) \|\widetilde{j}_n(\widetilde{u}_n)\|_H^2 + \operatorname{Re} \widetilde{a}(u_n) + \operatorname{Re} \widetilde{b}_n(\widetilde{u}_n)$$
$$= \operatorname{Re}(f_n, \widetilde{j}_n(\widetilde{u}_n))_H \leqslant \|f_n\|_H \|\widetilde{j}_n(\widetilde{u}_n)\|_H \leqslant \frac{\lambda}{2} \|\widetilde{j}_n(\widetilde{u}_n)\|_H^2 + \frac{2}{\lambda} \|f_n\|_H^2.$$

Since $\frac{\lambda}{4} + \gamma_n \geqslant 0$ for large n this implies that the set $\{\widetilde{j}(u_n) : n \in \mathbb{N}\} = \{\widetilde{j}_n(\widetilde{u}_n) : n \in \mathbb{N}\}$ is bounded in H, and that the two sets $\{\operatorname{Re} \widetilde{a}(u_n) : n \in \mathbb{N}\}$ and $\{\operatorname{Re} \widetilde{b}_n(\widetilde{u}_n) : n \in \mathbb{N}\}$ are bounded. In particular the sequence $u_1, u_2, \ldots$ is bounded in V. Passing to a subsequence, if necessary, it follows that there exists a $u \in V$ such that $\lim u_n = u$ weakly in V. Then $\lim \widetilde{j}(u_n) = \widetilde{j}(u)$ weakly in H.

Let $n \in \mathbb{N}$. Then $\widetilde{b}_n$ is $\widetilde{j}_n$-sectorial with vertex 0 and semi-angle θ. Therefore

$$|\widetilde{b}_n(\widetilde{u}_n, v)| \leqslant (1 + \tan\theta)(\operatorname{Re} \widetilde{b}_n(\widetilde{u}_n))^{1/2} (\operatorname{Re} \widetilde{b}_n(v))^{1/2}$$

for all $v \in V_n$. Now let $v \in D$. Then $\lim_{n\to\infty} \operatorname{Re} b_n(v) = 0$ by assumption. Hence $\lim_{n\to\infty} \widetilde{b}_n(\widetilde{u}_n, q_n(v)) = 0$. It follows from (3.2) and (3.3) that

$$\lambda(\widetilde{j}(u_n), j(v))_H + \widetilde{a}(u_n, q(v)) + \widetilde{b}_n(\widetilde{u}_n, q_n(v)) + \gamma_n(\widetilde{j}(u_n), j(v))_H = (f_n, j(v))_H.$$

Taking the limit $n \to \infty$ gives

$$(3.5) \qquad \lambda(\widetilde{j}(u), j(v))_H + \widetilde{a}(u, q(v)) = (f, j(v))_H$$

for all $v \in D$. Since D is a core for a one deduces that (3.5) is valid for all $v \in D(a)$ and then again by density one establishes that

$$(3.6) \qquad \lambda(\widetilde{j}(u), \widetilde{j}(v))_H + \widetilde{a}(u, v) = (f, \widetilde{j}(v))_H$$

for all $v \in V$. Thus $u \in D_H(\widetilde{a})$, and by definition of A it follows that $\widetilde{j}(u) = (\lambda I + A)^{-1} f$.

Now we prove the theorem. Let $f \in H$ and apply the above with $f_n = f$ for all $n \in \mathbb{N}$. In order to deduce that $\lim \widetilde{j}(u_n) = \widetilde{j}(u)$ strongly in H, by Proposition 3.6 in [16] it suffices to show that $\limsup \|\widetilde{j}(u_n)\|_H \leqslant \|\widetilde{j}(u)\|_H$.

Substituting $v = u_n$ in (3.6) gives

$$\lambda(\widetilde{j}(u), \widetilde{j}(u_n))_H + \widetilde{a}(u, u_n) = (f, \widetilde{j}(u_n))_H$$

for all $n \in \mathbb{N}$. Hence by (3.4) one deduces that

$$\begin{aligned}
\lambda \|\widetilde{j}(u_n)\|_H^2 &\leqslant \lambda \|\widetilde{j}_n(\widetilde{u}_n)\|_H^2 + \operatorname{Re} \widetilde{b}_n(\widetilde{u}_n) \\
&= \operatorname{Re}((f, \widetilde{j}(\widetilde{u}_n))_H - \widetilde{a}(u_n)) - \gamma_n \|\widetilde{j}(u_n)\|_H^2 \\
&= \operatorname{Re}(\lambda(\widetilde{j}(u), \widetilde{j}(u_n))_H + \widetilde{a}(u, u_n) - \widetilde{a}(u_n)) - \gamma_n \|\widetilde{j}(u_n)\|_H^2
\end{aligned}$$

for all $n \in \mathbb{N}$. But $\operatorname{Re} \widetilde{a}(u) \leqslant \liminf \operatorname{Re} \widetilde{a}(u_n)$ by Lemma VIII.3.14a of [17]. Therefore $\limsup \lambda \|\widetilde{j}(u_n)\|_H^2 \leqslant \operatorname{Re} \lambda \|\widetilde{j}(u)\|_H^2 = \lambda \|\widetilde{j}(u)\|_H^2$ and the strong convergence follows.

We have shown that there exists a subsequence $n_1, n_2, \ldots$ of the sequence $1, 2, \ldots$ such that $\lim_{k\to\infty} (\lambda I + A_{n_k})^{-1} f = (\lambda I + A)^{-1} f$. But this implies that

$$\lim_{n\to\infty} (\lambda I + A_n)^{-1} f = (\lambda I + A)^{-1} f$$

and the proof of the theorem is complete. ∎

For compact maps one obtains a stronger convergence in Theorem 3.7.

THEOREM 3.8. *Assume the notation and conditions of Theorem* 3.7. *Suppose in addition that the map* $j\colon D(a) \to H$ *is compact. Then, for all* $\lambda > -\gamma$,

$$\lim_{n\to\infty} \|(\lambda I + A_n)^{-1} - (\lambda I + A)^{-1}\| = 0.$$

Proof. Suppose not. Then there exist $\varepsilon > 0$, $n_1, n_2, \ldots \in \mathbb{N}$ and $f_1, f_2, \ldots \in H$ such that $n_k < n_{k+1}$, $\|f_k\|_H \leqslant 1$ and $\|(\lambda I + A_{n_k})^{-1} f_k - (\lambda I + A)^{-1} f_k\| \geqslant \varepsilon$ for all $k \in \mathbb{N}$. Passing to a subsequence, if necessary, there exists an $f \in H$ such that $\lim_{k\to\infty} f_{n_k} = f$ weakly in H. For all $k \in \mathbb{N}$ there exists a $\widetilde{u}_k \in D_H(\widetilde{a}_{n_k})$ such that $\widetilde{j}_{n_k}(\widetilde{u}_k) = (\lambda I + A_{n_k})^{-1} f_k$. Let $u_k = \Phi_{n_k}(\widetilde{u}_k)$, where we use the notation as in the proof of Theorem 3.7. Then it follows from the first part of the proof of Theorem 3.7 that there exists a $u \in D_H(\widetilde{a})$ such that, after passing to a subsequence if necessary, $\lim_{k\to\infty} u_k = u$ weakly in V and $\widetilde{j}(u) = (\lambda I + A)^{-1} f$. Since j is compact, the map $\widetilde{j}$ is compact. Therefore

$$\lim_{k\to\infty} (\lambda I + A_{n_k})^{-1} f_k = \lim \widetilde{j}(u_k) = \widetilde{j}(u) = (\lambda I + A)^{-1} f$$

strongly in H. Moreover, $\lim_{k\to\infty} (\lambda I + A)^{-1} f_k = (\lambda I + A)^{-1} f$ by Lemma 2.7. So $\lim_{k\to\infty} \|(\lambda I + A_{n_k})^{-1} f_k - (\lambda I + A)^{-1} f_k\| = 0$. This is a contradiction. ∎

If a is symmetric and j is the identity map, then Theorem 3.7 is a generalization of Corollary 3.9 in [11], which followed from Theorem VIII.3.11 of [17]. Note that Theorem VIII.3.11 in [17] is a special case of Theorem 3.7. The point in the following corollary is that the form a is merely j-sectorial, but not necessarily j-elliptic. It allows one to describe the associated operator also by a limit of suitable perturbations. This also underlines that the associated operator as we define it is the natural object.

COROLLARY 3.9. *Let* V, H *be Hilbert spaces and* $j \in \mathcal{L}(V, H)$ *with* $j(V)$ *dense in* H. *Let* $a\colon V \times V \to \mathbb{C}$ *be a continuous* j*-sectorial form with vertex* γ. *Let* $b\colon V \times V \to \mathbb{C}$ *be a* j*-elliptic continuous form. Suppose that there exists a* $\theta \in [0, \frac{\pi}{2})$ *such that* $b(u) \in \Sigma_\theta$ *for all* $u \in V$. *For all* $n \in \mathbb{N}$ *define* $a_n = a + \frac{1}{n} b$. *Then* a_n *is* j*-elliptic. Let* A_n *be the operator associated with* (a_n, j) *and* A *the operator associated with* (a, j). *Then, for all* $\lambda > -\gamma$ *and* $f \in H$,

$$\lim_{n\to\infty} (\lambda I + A_n)^{-1} f = (\lambda I + A)^{-1} f.$$

We next consider the classical form associated with the m-sectorial operator A.

PROPOSITION 3.10. *Let a be a sesquilinear form, H a Hilbert space and $j\colon D(a) \to H$ a linear map. Suppose the form a is j-sectorial and $j(D(a))$ is dense in H. Let A be the operator associated with (a, j). Then one has the following:*

(i) *There exists a unique closable sectorial form a_{r} with form domain $j(D(a))$ such that A is associated with $\overline{a_{\mathrm{r}}}$.*

(ii) $D(\overline{a_{\mathrm{r}}}) = \{x \in H :$ *there exists a bounded sequence $u_1, u_2, \ldots$ in $D(a)$ such that* $x = \lim_{n\to\infty} j(u_n)$ *in* $H\}$.

(iii) *There exists a $c > 0$ such that $\|j(u)\|_{a_{\mathrm{r}}} \leqslant c\|u\|_a$ for all $u \in D(a)$. In particular, if D is a core for a, then $j(D)$ is a core for $\overline{a_{\mathrm{r}}}$.*

(iv) *Let h be the real part of a and let h_{r} be defined similarly as in statement* (i). *Then* $D(\overline{a_{\mathrm{r}}}) = D(\overline{h_{\mathrm{r}}})$.

Proof. (i) We use the notation as in the proof of Theorem 3.2. Let b be the closed sectorial form associated with A, i.e. the classical form associated with $(\widetilde{a}, \widetilde{j})$ given in Theorem 2.5(ii). So $D(b) = \widetilde{j}(V) = \widetilde{j}(V(\widetilde{a}))$ and $b(\widetilde{j}(u), \widetilde{j}(v)) = \widetilde{a}(u, v)$ for all $u, v \in V(\widetilde{a})$. Then $j(D(a)) = \widetilde{j}(q(D(a))) \subset \widetilde{j}(V) = D(b)$. We show that $j(D(a))$ is a core for b. Let $x \in D(b)$. There exists a unique $u \in V(\widetilde{a})$ such that $\widetilde{j}(u) = x$. There exist $u_1, u_2, \ldots \in D(a)$ such that $\lim q(u_n) = u$ in V. Let π_2 be the projection of V onto $V(\widetilde{a})$ along the decomposition $V = \ker \widetilde{j} \oplus V(\widetilde{a})$. Clearly $\pi_2(u) = u$. In addition, π_2 is continuous and $j(u_n) = \widetilde{j}(q(u_n)) = \widetilde{j}(\pi_2(q(u_n)))$ for all $n \in \mathbb{N}$. Therefore

$$\|x - j(u_n)\|_{D(b)} = \|\pi_2(u) - \pi_2(q(u_n))\|_{V(\widetilde{a})} \leqslant \|\pi_2\| \|u - q(u_n)\|_V$$

for all $n \in \mathbb{N}$, from which one deduces that $\lim j(u_n) = x$ in $D(b)$. We have shown that $j(D(a))$ is a core for b. Let $a_{\mathrm{r}} = b|_{j(D(a)) \times j(D(a))}$. Then $b = \overline{a_{\mathrm{r}}}$. This proves existence of a_{r}. The uniqueness is obvious from Theorem VI.2.7 of [17].

(ii) "$\subset$" Let $x \in D(\overline{a_{\mathrm{r}}}) = D(b)$. Let $u_1, u_2, \ldots \in D(a)$ and $u \in V(\widetilde{a})$ be as in the proof of statement (i). Then $\lim j(u_n) = x$ in $D(b)$, therefore also in H. Moreover, $\lim q(u_n) = u$ in V. So the sequence $q(u_1), q(u_2), \ldots$ is bounded in V. But $\|u_n\|_a = \|q(u_n)\|_V$ for all $n \in \mathbb{N}$. Thus the sequence $u_1, u_2, \ldots$ satisfies the requirements.

"$\supset$" Let $u_1, u_2, \ldots$ be a bounded sequence in $D(a)$, $x \in H$ and suppose that $\lim j(u_n) = x$ in H. Then $q(u_1), q(u_2), \ldots$ is a bounded sequence in V. So passing to a subsequence if necessary, there exists a $v \in V$ such that $\lim q(u_n) = v$ weakly in V. Then $\widetilde{j}(v) = \lim j(u_n)$ weakly in H. Hence $x = \widetilde{j}(v) \in \widetilde{j}(V) = D(\overline{a_{\mathrm{r}}})$.

(iii) The construction in the proof of Theorem 3.2 with h instead of a leads to the same closed space W, then the same normed space V_0 and the same Banach space V. Let $\widetilde{h}\colon V \times V \to \mathbb{C}$ be the unique continuous form on V such that $\widetilde{h}(q(u), q(v)) = h(u, v)$ for all $u, v \in V$. Then $\widetilde{h} = \Re \widetilde{a}$, the real part of $\widetilde{a}$. Let h_{c} be the classical form associated with $(\widetilde{h}, \widetilde{j})$. Then $h_c = \overline{h_r}$ and $b = \overline{a_r}$ by part (i). Then statement (iv) follows from Proposition 2.6.

(iii) Again by Proposition 2.6 there exists a $c \geqslant 1$ such that $\operatorname{Re} b(x) \leqslant c h_c(x)$ for all $x \in \widetilde{j}(V)$. But $h_c(\widetilde{j}(u)) \leqslant \widetilde{h}(u) = \operatorname{Re} \widetilde{a}(u)$ for all $u \in V$. So $\|\widetilde{j}(u)\|_b \leqslant c\|u\|_{\widetilde{a}}$ for all $u \in V$. Then $\|j(u)\|_{a_r} \leqslant c\|q(u)\|_{\widetilde{a}} = c\|u\|_a$ for all $u \in D(a)$. The last assertion in statement (iii) is an immediate consequence. ∎

We call a_r the *regular part* and $\overline{a_r}$ the *relaxed form* of the j-sectorial form a. If $D(a) \subset H$ and j is the identity map, then it follows from the proof of Proposition 3.10(i) that $a_r(x) = \widetilde{a}(\pi_2(x))$ for all $x \in D(a)$, with the notation introduced there. So if in addition a is symmetric and *positive*, i.e. if the numerical range $\{a(u) : u \in D(a)\}$ is contained in $[0, \infty)$, then this terminology coincides with the one employed by Simon ([23], Section 2). Under these assumptions Simon characterized the regular part of a as the largest closable form lying below a for the order relation $b_1 \leqslant b_2$ if and only if $D(b_2) \subset D(b_1)$ and $b_1(u) \leqslant b_2(u)$ for all $u \in D(b_2)$. Of course, such an order relation does not exist for sectorial forms. It seems to us, though, that the direct formula in Theorem 1.1 expressing the generator directly in terms of the form a, is frequently more useful than the computation of a_r. For positive a Simon proved Proposition 3.10(ii) in Theorem 3 of [22]. Note that for general a (but still j the inclusion), the form a is closable if and only if a_r coincides with a on $D(a)$.

Let a be a densely defined sectorial form and A its associated operator, as above. If the form a is symmetric, then the associated operator A is self-adjoint. But the converse is not true if the form a is not closable. In order to see this, it suffices to consider the form $(1 + \mathrm{i})a$ where a is the form as in Example 3.14 below.

For general j-sectorial forms we also consider invariance of closed convex subsets.

PROPOSITION 3.11. *Let a be a sesquilinear form, H a Hilbert space and $j\colon D(a) \to H$ a linear map. Suppose the form a is accretive, j-sectorial and $j(D(a))$ is dense in H. Let A be the operator associated with (a, j) and S the semigroup generated by $-A$. Let $C \subset H$ be a non-empty closed convex set and $P\colon H \to C$ the orthogonal projection. Then the following are equivalent:*

(i) *$S_t C \subset C$ for all $t > 0$.*

(ii) *For all $u \in D(a)$ there exists a Cauchy sequence $w_1, w_2, \ldots$ in $(D(a), \|\cdot\|_a)$ such that $\lim_{n\to\infty} j(w_n) = Pj(u)$ in H and $\lim_{n\to\infty} \operatorname{Re} a(w_n, u - w_n) \geqslant 0$.*

(iii) *For all $u \in D(a)$ there exists a bounded sequence $w_1, w_2, \ldots$ in $(D(a), \|\cdot\|_a)$ such that $\lim_{n\to\infty} j(w_n) = Pj(u)$ in H and $\limsup_{n\to\infty} \operatorname{Re} a(w_n, u - w_n) \geqslant 0$.*

Proof. We use the notation as in the proof of Theorem 3.2. Clearly the form $\widetilde{a}$ is accretive by continuity and density of V_0. We shall prove the equivalence with condition (iv) in Proposition 2.9 for $D = V_0 = q(D(a))$, $\widetilde{a}$ and $\widetilde{j}$.

(i) $\Rightarrow$ (ii) Let $u \in D(a)$. By Proposition 2.9, (i) $\Rightarrow$ (iv) there exists a $w \in V$ such that $\widetilde{j}(w) = Pj(u)$ and $\mathrm{Re}\, \widetilde{a}(w, q(u) - w) \geqslant 0$. There are $w_1, w_2, \ldots \in D(a)$ such that $\lim q(w_n) = w$ in V. Then the sequence $w_1, w_2, \ldots$ satisfies the requirements.

(ii) $\Rightarrow$ (iii) Trivial.

(iii) $\Rightarrow$ (i) Let $u \in D(a)$. By assumption there exists a bounded sequence $w_1, w_2, \ldots$ in the space $(D(a), \| \cdot \|_a)$ such that $\lim\limits_{n\to\infty} j(w_n) = Pj(u)$ in H and $\limsup\limits_{n\to\infty} a(w_n, u - w_n) \geqslant 0$. Then $q(w_1), q(w_2), \ldots$ is a bounded sequence in V, so passing to a subsequence if necessary, it follows that it is weakly convergent. Let $w = \lim\limits_{n\to\infty} q(w_n)$ weakly in V. Then $\widetilde{j}(w) = \lim j(w_n)$ weakly in H, so $\widetilde{j}(w) = Pj(u) = P\widetilde{j}(q(u))$. Moreover, $\widetilde{a}(w, q(u)) = \lim \widetilde{a}(q(w_n), q(u))$ and $\mathrm{Re}\, \widetilde{a}(w, w) = \Re \widetilde{a}(w) \leqslant \liminf \Re \widetilde{a}(q(w_n))$ by Lemma VIII.3.14a of [17]. So $\mathrm{Re}\, \widetilde{a}(w, q(u) - w) \geqslant \limsup\limits_{n\to\infty} \mathrm{Re}\, a(w_n, u - w_n) \geqslant 0$. Then condition (i) follows from Proposition 2.9, (iv) $\Leftrightarrow$ (i). ∎

REMARK 3.12. Clearly condition (ii) in Proposition 3.11 is valid if for all $u \in D(a)$ there exists a $w \in D(a)$ such that $j(w) = Pj(u)$ and $\mathrm{Re}\, a(w, u - w) \geqslant 0$.

Proposition 3.11 has several consequences which will be useful for differential operators in the next section. If $(X, \mathcal{B}, m)$ is a measure space and a is a sesquilinear form in $L_2(X)$, then we call *a real* if $\mathrm{Re}\, u \in D(a)$ and $a(\mathrm{Re}\, u, \mathrm{Im}\, u) \in \mathbb{R}$ for all $u \in D(a)$.

COROLLARY 3.13. *Let $(X, \mathcal{B}, m)$ be a measure space and a a densely defined sectorial form in $L_2(X)$. Let A be the operator associated with a as in Theorem* 1.1 *and let S be the semigroup generated by the operator $-A$.*

(i) *If a is real, then $S_t L_2(X, \mathbb{R}) \subset L_2(X, \mathbb{R})$ for all $t > 0$.*

(ii) *If a is real, $u^+ \in D(a)$ and $a(u^+, u^-) \leqslant 0$ for all $u \in D(a) \cap L_2(X, \mathbb{R})$, then S is positive. In particular, $|S_t u| \leqslant S_t |u|$ for all $t > 0$ and $u \in L_2(X)$.*

(iii) *If a is accretive, real, $u \wedge \mathbf{1} \in D(a)$ and $a(u \wedge \mathbf{1}, (u - \mathbf{1})^+) \geqslant 0$ for all $u \in D(a) \cap L_2(X, \mathbb{R})$, then S is submarkovian, i.e., $\|S_t u\|_\infty \leqslant \|u\|_\infty$ for all $u \in L_2(X) \cap L_\infty(X)$ and $t > 0$.*

(iv) *If a is accretive, real, $u \wedge \mathbf{1} \in D(a)$ and $a((u - \mathbf{1})^+, u \wedge \mathbf{1}) \geqslant 0$ for all $u \in D(a) \cap L_2(X, \mathbb{R})$, then $\|S_t u\|_1 \leqslant \|u\|_1$ for all $u \in L_1(X) \cap L_2(X)$ and $t > 0$.*

Proof. (i) Replacing a by $(u, v) \mapsto a(u, v) + \gamma (u, v)_H$ we may assume that a is accretive. Let $u \in D(a)$. Set $w = \mathrm{Re}\, u$. Then $w \in D(a)$ and $\mathrm{Re}\, a(w, u - w) = \mathrm{Re}\, a(\mathrm{Re}\, u, \mathrm{i}\, \mathrm{Im}\, u) = \mathrm{Im}\, a(\mathrm{Re}\, u, \mathrm{Im}\, u) = 0$. So by Proposition 3.11 the set $L_2(X, \mathbb{R})$ is invariant under S. (See also Remark 3.12.)

(ii) Again we may assume that a is accretive. Let $C = \{u \in L_2(X, \mathbb{R}) : u \geqslant 0\}$. Then C is closed and convex. Let P be the orthogonal projection of $L_2(X)$ onto C. Let $u \in D(a)$. Then $Pu = (\mathrm{Re}\, u)^+ \in D(a)$. Moreover, $\mathrm{Re}\, a(Pu, u - Pu)$

$= \operatorname{Re} a((\operatorname{Re} u)^+, -(\operatorname{Re} u)^- + \mathrm{i} \operatorname{Im} u) = -a((\operatorname{Re} u)^+, (\operatorname{Re} u)^-) \geqslant 0$. So by Proposition 3.11 the set C is invariant under S.

(iii) Let $C = \{u \in L_2(X, \mathbb{R}) : u \leqslant \mathbf{1}\}$. Then C is closed and convex in $L_2(X)$. The orthogonal projection $P\colon L_2(X) \to C$ is given by $Pu = (\operatorname{Re} u) \wedge \mathbf{1}$. It follows by assumption and Proposition 3.11 that the set C is invariant under S. Hence S is submarkovian.

(iv) This follows by duality from statement (iii) and Remark 3.5. ∎

We end this section with an example which shows that in general (2.3) is restricted to $u \in D_H(\widetilde{a})$.

EXAMPLE 3.14. Let $H = L_2(0,1)$, $D(a) = C[0,1]$ and

$$a(u,v) = \sum_{n=1}^{\infty} 2^{-n} u(q_n) \overline{v(q_n)}$$

where $\{q_n : n \in \mathbb{N}\} = [0,1] \cap \mathbb{Q}$ with $q_n \neq q_m$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Moreover, let j be the inclusion map. Then it is not hard to characterize the completion of $D(a)$ and to show that the operator A associated with a is the zero operator.

4. APPLICATIONS

We illustrate the theorems of the previous sections by several examples.

4.1. SECTORIAL DIFFERENTIAL OPERATORS. First we consider differential operators on open sets in $\mathbb{R}^d$. We emphasize that the operators do not have to be symmetric and may have complex coefficients. The next lemma, whose proof is trivial, provides an efficient way to construct sectorial operators.

LEMMA 4.1. *Let $\Omega \subset \mathbb{R}^d$ be open. For all $i, j \in \{1, \ldots, d\}$ let $a_{ij} \in L_{1,\mathrm{loc}}(\Omega)$. Let $D(a)$ be a subspace of $L_2(\Omega)$ with $C_c^\infty(\Omega) \subset D(a)$. Assume that $\partial_i u \in L_{1,\mathrm{loc}}(\Omega)$ as a distribution and*

$$\int_\Omega |(\partial_i u) a_{ij} \partial_j v| < \infty$$

for all $u, v \in D(a)$ and $i, j \in \{1, \ldots, d\}$. Define the form $a\colon D(a) \times D(a) \to \mathbb{C}$ by

$$a(u,v) = \sum_{i,j=1}^{d} \int_\Omega (\partial_i u) a_{ij} \overline{\partial_j v}.$$

Let $\theta \in [0, \frac{\pi}{2})$ and assume that $\sum_{i,j=1}^{d} a_{ij}(x) \xi_i \overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$. Then the form a is sectorial with vertex 0 and semi-angle θ.

We call an operator A associated with a form a which satisfies the assumptions of Lemma 4.1 a *sectorial differential operator* and a a *sectorial differential form*. Then $-A$ generates a holomorphic semigroup.

The assumptions on the domain $D(a)$ and the coefficients a_{ij} are very general. For example one can choose $D(a) = C_c^\infty(\Omega)$ together with the condition $a_{ij} \in L_{1,\mathrm{loc}}(\Omega)$, or alternatively if $a_{ij} \in L_\infty(\Omega)$ one can choose for $D(a)$ any subspace of $H^1(\Omega)$ with $C_c^\infty(\Omega) \subset D(a)$.

In order to avoid too many cases we will not consider unbounded coefficients in this paper. Let $\Omega \subset \mathbb{R}^d$ be open. For all $i, j \in \{1, \ldots, d\}$ let $a_{ij} \in L_\infty(\Omega)$. Define the form $a \colon D(a) \times D(a) \to \mathbb{C}$ by

$$a(u,v) = \sum_{i,j=1}^{d} \int_\Omega (\partial_i u) a_{ij} \overline{\partial_j v},$$

where $D(a)$ is a subspace of $H^1(\Omega)$ with $C_c^\infty(\Omega) \subset D(a)$.

We call (a_{ij}) *strongly elliptic* if there exists a $\mu > 0$ such that

$$\mathrm{Re} \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \overline{\xi_j} \geqslant \mu |\xi|^2$$

for all $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$. Clearly if (a_{ij}) is strongly elliptic, then there exists a $\theta \in [0, \frac{\pi}{2})$ such that $\sum\limits_{i,j=1}^{d} a_{ij}(x) \xi_i \overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$. We then also say that the form a and associated operator are strongly elliptic.

Let $\theta \in [0, \frac{\pi}{2})$ and suppose that $\sum\limits_{i,j=1}^{d} a_{ij}(x) \xi_i \overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$. Let $l \colon D(a) \times D(a) \to \mathbb{C}$ be defined by $l(u,v) = \sum\limits_{i=1}^{d} \int\limits_\Omega (\partial_i u) \overline{\partial_i v}$. For all $n \in \mathbb{N}$ let $a^{(n)} = a + \frac{1}{n} l$. Although a is not strongly elliptic in general, the form $a^{(n)}$ is strongly elliptic for all $n \in \mathbb{N}$. If A, A_n, S and $S^{(n)}$ are the associated operators and semigroups, then the conditions of Theorem 3.7 are satisfied. In particular the A_n converge to A strongly in the resolvent sense and therefore $S_t^{(n)}$ converges strongly to S_t for all $t > 0$.

We next show that under a mild condition on the form domain $D(a)$ the semigroup associated with a sectorial differential operator satisfies certain Davies–Gaffney bounds. If F and G are two non-empty subsets of $\mathbb{R}^d$, then $d(F, G)$ denotes the Euclidean distance. The value of M can be improved significantly if the coefficients are real. (See Proposition 3.1 of [13].) In this paper the following version for complex coefficients suffices.

THEOREM 4.2. *Let $\Omega \subset \mathbb{R}^d$ be open. For all $i, j \in \{1, \ldots, d\}$ let $a_{ij} \in L_\infty(\Omega)$. Let $\theta \in [0, \frac{\pi}{2})$. Suppose $\sum\limits_{i,j=1}^{d} a_{ij}(x) \xi_i \overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$. Define the* *form* $a\colon D(a)\times D(a)\to\mathbb{C}$ *by*

$$a(u,v)=\sum_{i,j=1}^{d}\int_{\Omega}(\partial_i u)a_{ij}\overline{\partial_j v},$$

where $D(a)$ *is a subspace of* $H^1(\Omega)$ *with* $C_c^\infty(\Omega)\subset D(a)$. *Suppose* $e^{\rho\psi}u\in D(a)$ *for all* $u\in D(a)$, $\rho\in\mathbb{R}$ *and* $\psi\in C_b^\infty(\mathbb{R}^d,\mathbb{R})$. *Let* S *be the semigroup associated with* a. *Then*

$$|(S_t u,v)|\leqslant e^{-d(\Omega_1,\Omega_2)^2(4Mt)^{-1}}\|u\|_2\|v\|_2 \tag{4.1}$$

for all non-empty open $\Omega_1,\Omega_2\subset\Omega$, $u\in L_2(\Omega_1)$, $v\in L_2(\Omega_2)$ *and* $t>0$, *where* $M=3(1+\tan\theta)^2\Big(1+\sum_{i,j=1}^{d}\|a_{ij}\|_\infty\Big)$.

Proof. We first suppose that (a_{ij}) is strongly elliptic. Let $\rho>0$ and $\psi\in C_b^\infty(\mathbb{R}^d,\mathbb{R})$ with $\|\nabla\psi\|_\infty\leqslant 1$. Define the form $a_\rho\colon D(a)\times D(a)\to\mathbb{C}$ by

$$a_\rho(u,v)=\sum_{i,j=1}^{d}\int_{\Omega}(\partial_i u+\rho\psi_i u)a_{ij}\overline{\partial_j v-\rho\psi_j v},$$

where $\psi_i=\partial_i\psi$ for all $i\in\{1,\ldots,d\}$. Then

$$\begin{aligned}\operatorname{Re}a_\rho(u)=\operatorname{Re}a(u)+\rho\operatorname{Re}\int_{\Omega}\sum_{i,j=1}^{d}\psi_i u a_{ij}\overline{\partial_j u}-\rho\operatorname{Re}\int_{\Omega}\sum_{i,j=1}^{d}(\partial_i u)a_{ij}\psi_j\overline{u}\\ -\rho^2\operatorname{Re}\int_{\Omega}\sum_{i,j=1}^{d}\psi_i a_{ij}\psi_j|u|^2\end{aligned}\tag{4.2}$$

for all $u\in D(a)$. It follows from the estimate (1.15) of Subsection VI.1.2 in [17] that

$$\begin{aligned}\Big|\sum_{i,j=1}^{d}a_{ij}(x)\xi_i\overline{\eta_j}\Big|&\leqslant(1+\tan\theta)\Big(\operatorname{Re}\sum_{i,j=1}^{d}a_{ij}(x)\xi_i\overline{\xi_j}\Big)^{1/2}\Big(\operatorname{Re}\sum_{i,j=1}^{d}a_{ij}(x)\eta_i\overline{\eta_j}\Big)^{1/2}\\&\leqslant\varepsilon\operatorname{Re}\sum_{i,j=1}^{d}a_{ij}(x)\xi_i\overline{\xi_j}+\frac{(1+\tan\theta)^2}{4\varepsilon}\operatorname{Re}\sum_{i,j=1}^{d}a_{ij}(x)\eta_i\overline{\eta_j}\end{aligned}$$

for all $\xi,\eta\in\mathbb{C}^d$, $\varepsilon>0$ and a.e. $x\in\Omega$. Choosing $\xi_i=(\partial_i u)(x)$, $\eta_i=(\psi_i u)(x)$ and $\varepsilon=\frac{1}{4\rho}$ it follows that

$$\rho\Big|\int_{\Omega}\sum_{i,j=1}^{d}(\partial_i u)a_{ij}\psi_j\overline{u}\Big|\leqslant\frac{1}{4}\operatorname{Re}a(u)+(1+\tan\theta)^2\rho^2\operatorname{Re}\int_{\Omega}\sum_{i,j=1}^{d}\psi_i a_{ij}\psi_j|u|^2.$$

Similarly the second term in (4.2) can be estimated. Hence

$$\operatorname{Re} a_\rho(u) \geqslant \frac{1}{2}\operatorname{Re} a(u) - (1+2(1+\tan\theta)^2)\rho^2 \operatorname{Re}\int_\Omega \sum_{i,j=1}^d \psi_i a_{ij}\psi_j |u|^2$$

$$\geqslant \frac{1}{2}\operatorname{Re} a(u) - M\rho^2\|u\|_2^2. \tag{4.3}$$

Define $U_{\pm\rho}\colon L_2(\Omega) \to L_2(\Omega)$ by $U_{\pm\rho}v = \mathrm{e}^{\pm\rho\psi}v$. Then $U_{\pm\rho}D(a) \subset D(a)$. Moreover, $a_\rho(u,v) = a(U_\rho u, U_{-\rho}v)$ for all $u,v \in D(a)$. Since (a_{ij}) is strongly elliptic, the forms a and a_ρ are sectorial (cf. Lemmas 3.6 and 3.7 in [4]). Let A and A_ρ be the associated operators and let $S^{(\rho)}$ be the semigroup generated by $-A_\rho$. Then $A_\rho = U_{-\rho}AU_\rho$ and $S_t^{(\rho)} = U_{-\rho}S_tU_\rho$ for all $t > 0$. It follows from (4.3) that

$$\|S_t^{(\rho)}\|_{2\to 2} \leqslant \mathrm{e}^{M\rho^2 t} \tag{4.4}$$

for all $t > 0$. Then

$$|(S_t u, v)| = |(S_t^{(\rho)}U_{-\rho}u, U_\rho v)| \leqslant \|S_t^{(\rho)}\|_{2\to 2}\|U_{-\rho}u\|_2\|U_\rho v\|_2$$
$$\leqslant \mathrm{e}^{M\rho^2 t}\mathrm{e}^{-\rho d_\psi(\Omega_1,\Omega_2)}\|u\|_2\|v\|_2$$

for all $u \in L_2(\Omega_1)$ and $v \in L_2(\Omega_2)$, where

$$d_\psi(\Omega_1,\Omega_2) = \inf_{x\in\Omega_1}\psi(x) - \sup_{x\in\Omega_2}\psi(x).$$

Minimizing over all $\psi \in C_{\mathrm{b}}^\infty(\mathbb{R}^d)$ with $\|\nabla\psi\|_\infty \leqslant 1$ one deduces that

$$|(S_t u, v)| \leqslant \mathrm{e}^{M\rho^2 t}\mathrm{e}^{-\rho d(\Omega_1,\Omega_2)}\|u\|_2\|v\|_2$$

and choosing $\rho = \frac{d(\Omega_1,\Omega_2)}{2Mt}$ gives, for all $u \in L_2(\Omega_1)$, $v \in L_2(\Omega_2)$ and $t > 0$,

$$|(S_t u, v)| \leqslant \mathrm{e}^{-d(\Omega_1,\Omega_2)^2(4Mt)^{-1}}\|u\|_2\|v\|_2.$$

Finally we drop the assumption that (a_{ij}) is strongly elliptic. For all $n \in \mathbb{N}$ define $a_{ij}^{(n)} = a_{ij} + \frac{1}{n}\delta_{ij}$. Then $(a_{ij}^{(n)})$ is strongly elliptic. If $S^{(n)}$ is the associated semigroup, then $\lim_{n\to\infty} S_t^{(n)} = S_t$ strongly for all $t > 0$ by Theorem 3.7. Hence the theorem follows. ∎

We next consider locality properties of the relaxed form $\overline{a_{\mathrm{r}}}$ of the sectorial form a.

COROLLARY 4.3. *Assume the notation and assumptions of Theorem* 4.2. *Then* $\overline{a_{\mathrm{r}}}(u,v) = 0$ *for all* $u,v \in D(\overline{a_{\mathrm{r}}})$ *with disjoint compact supports.*

Proof. There exist open non-empty $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ such that $\operatorname{supp} u \subset \Omega_1$, $\operatorname{supp} v \subset \Omega_2$ and $d(\Omega_1,\Omega_2) > 0$. Then it follows from Theorem 4.2 that there exists a $b > 0$ such that

$$|((I - S_t)u, v)| = |(S_t u, v)| \leqslant \mathrm{e}^{-bt^{-1}}\|u\|_2\|v\|_2$$

for all $t > 0$. Hence by Lemma 1.56 of [21] one deduces the following, as required:

$$|\overline{a_{\mathrm{r}}}(u,v)| = \lim_{t\downarrow 0} t^{-1}|((I - S_t)u, v)| \leqslant \lim_{t\downarrow 0} t^{-1}e^{-bt^{-1}}\|u\|_2\|v\|_2 = 0. \quad \blacksquare$$

If $\Omega \subset \mathbb{R}^d$ define

$$L_{2,c}(\Omega) = \{u \in L_2(\Omega) : \operatorname{supp} u \text{ is compact}\}.$$

Another corollary of Theorem 4.2 is that S_t maps $L_{2,c}(\Omega)$ into $L_1(\Omega)$. This is a special case of the following lemma.

For all $R > 0$ let B_R denote the open ball in $\mathbb{R}^d$ with centre 0 and radius R. Set $\chi_R = \mathbf{1}_{B_R}$.

LEMMA 4.4. *Let $d \in \mathbb{N}$. There exists a constant $c_d > 0$ such that the following holds. Let $\Omega \subset \mathbb{R}^d$ be open and $T \in \mathcal{L}(L_2(\Omega))$. Let $N > 0$ and suppose that*

$$|(Tu, v)| \leqslant e^{-d(\Omega_1,\Omega_2)^2/N}\|u\|_2\|v\|_2$$

for all non-empty open $\Omega_1, \Omega_2 \subset \Omega$, $u \in L_2(\Omega_1)$ and $v \in L_2(\Omega_2)$. Then $TL_{2,c}(\Omega) \subset L_1(\Omega)$ and

$$\|(\mathbf{1} - \chi_{2R})Tu\|_1 \leqslant c_d R^{-1} N^{(d+2)/4} e^{-R^2/(2N)}\|u\|_2$$

for all $R > 0$ and $u \in L_2(\Omega)$ with $\operatorname{supp} u \subset B_R$.

Proof. Since $\chi_{2R}Tu \in L_2(\Omega \cap B_{2R}) \subset L_1(\Omega)$ it suffices to show the estimate. Let $\varphi \in C_c(\Omega)$. Then

$$\begin{aligned}
|((\mathbf{1} - \chi_{2R})Tu, \varphi)| = |(Tu, (\mathbf{1} - \chi_{2R})\varphi)| &\leqslant \sum_{n=1}^{\infty} |(Tu, (\chi_{(n+2)R} - \chi_{(n+1)R})\varphi)| \\
&\leqslant \sum_{n=1}^{\infty} e^{-n^2R^2/N}\|u\|_2\|(\chi_{(n+2)R} - \chi_{(n+1)R})\varphi\|_2 \\
&\leqslant \sum_{n=1}^{\infty} e^{-n^2R^2/N}((n+2)R)^{d/2}|B_1|^{1/2}\|u\|_2\|\varphi\|_\infty \\
&\leqslant 3^{d/2}|B_1|^{1/2}e^{-R^2/(2N)}\|u\|_2\|\varphi\|_\infty \sum_{n=1}^{\infty} e^{-n^2R^2/(2N)}(nR)^{d/2}.
\end{aligned}$$

Let $c' > 0$ be such that $x^{d/4} \leqslant c'e^x$ uniformly for all $x > 0$. Then c' can be chosen to depend only on d. Note that $\sum_{n=1}^{\infty} e^{-an^2} \leqslant \int_0^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{4a}}$ for all $a > 0$. Therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} e^{-n^2R^2/(2N)}(nR)^{d/2} &= (4N)^{d/4}\sum_{n=1}^{\infty} e^{-n^2R^2/(2N)}\Big(\frac{n^2R^2}{4N}\Big)^{d/4} \\
&\leqslant c'(4N)^{d/4}\sum_{n=1}^{\infty} e^{-n^2R^2/(4N)} \leqslant c'(4N)^{d/4}\Big(\frac{\pi N}{R^2}\Big)^{1/2}.
\end{aligned}$$

Then the lemma follows by taking the supremum over all φ with $\|\varphi\|_\infty \leqslant 1$. $\blacksquare$

As a consequence one deduces L_1-convergence of the approximate semigroups on $L_{2,c}(\Omega)$. Recall that the coefficients in Theorem 4.2 are complex.

LEMMA 4.5. *Assume the notation and assumptions of Theorem* 4.2. *For all $n \in \mathbb{N}$ let $a^{(n)} = a + \frac{1}{n} l$, where l is the form with $D(l) = D(a)$ and $l(u,v) = \sum_{i=1}^{d} \int_\Omega (\partial_i u) \overline{\partial_i v}$. Let $S^{(n)}$ be the semigroup associated with $a^{(n)}$. Then $\lim_{n\to\infty} S_t^{(n)} u = S_t u$ in $L_1(\Omega)$ for all $t > 0$ and $u \in L_{2,c}(\Omega)$.*

Proof. It follows from Theorem 4.2 that there exists an $M > 0$ such that

$$|(S_t u, v)| \vee |(S_t^{(n)} u, v)| \leqslant \mathrm{e}^{-d(\Omega_1,\Omega_2)^2 (4Mt)^{-1}} \|u\|_2 \|v\|_2$$

for all $n \in \mathbb{N}$, non-empty open $\Omega_1, \Omega_2 \subset \Omega$, $u \in L_2(\Omega_1)$, $v \in L_2(\Omega_2)$ and $t > 0$, Let $c_d > 0$ be as in Lemma 4.4. Let $u \in L_{2,c}(\Omega)$ and $t > 0$. Then

$$\|(\mathbf{1} - \chi_{2R}) S_t^{(n)} u\|_1 \leqslant c_d R^{-1} (4Mt)^{(d+2)/4} \mathrm{e}^{-R^2 (8Mt)^{-1}} \|u\|_2$$

for all $n \in \mathbb{N}$ and $R > 0$ with $\operatorname{supp} u \subset B_R$. So $\lim_{R\to\infty} (\mathbf{1} - \chi_{2R}) S_t^{(n)} u = 0$ in $L_1(\Omega)$ uniformly in $n \in \mathbb{N}$. Similarly, $\lim_{R\to\infty} (\mathbf{1} - \chi_{2R}) S_t u = 0$ in $L_1(\Omega)$. So it suffices to prove that $\lim_{n\to\infty} \chi_{2R} (S_t^{(n)} u - S_t u) = 0$ for large $R > 0$. Since $\|\chi_{2R} (S_t^{(n)} u - S_t u)\|_1 \leqslant |B_{2R}|^{1/2} \|S_t^{(n)} u - S_t u\|_2$ for all $n \in \mathbb{N}$ and $R > 0$, it follows from Theorem 3.7 that $\lim_{n\to\infty} \chi_{2R} (S_t^{(n)} u - S_t u) = 0$ in $L_1(\Omega)$ for all $R > 0$. ∎

For strongly elliptic operators one can strengthen the conclusions of Theorem 4.2.

LEMMA 4.6. *Assume the notation and assumptions of Theorem* 4.2. *In addition suppose that the operator is strongly elliptic. Then one has the following:*

(i) *$S_t L_2(\Omega) \subset H^1(\Omega)$ for all $t > 0$.*

(ii) *There exist $c, M' > 0$ such that*

$$|(\partial_i S_t u, v)| \leqslant c t^{-1/2} \mathrm{e}^{-d(\Omega_1,\Omega_2)^2 (M't)^{-1}} \|u\|_2 \|v\|_2$$

for all non-empty open $\Omega_1, \Omega_2 \subset \Omega$, $u \in L_2(\Omega_1)$, $v \in L_2(\Omega_2)$ and $t > 0$.

(iii) *If $u \in L_{2,c}(\Omega)$, then $S_t u, \partial_i S_t u \in L_1(\Omega)$ for all $t > 0$ and $i \in \{1, \ldots, d\}$. Moreover, $t \mapsto \|\partial_i S_t u\|_1$ is locally bounded.*

Proof. (i) Let b be the sectorial differential form with form domain $D(b) = H^1(\Omega)$ and coefficients a_{ij}. Since (a_{ij}) is strongly elliptic if follows that b is closed. Clearly b is an extension of a. So a is closable. Let A be the operator associated with a. Then A is the operator associated with $\overline{a}$. Since S is holomorphic one deduces that $S_t L_2(\Omega) \subset D(A) \subset D(\overline{a}) \subset D(b) = H^1(\Omega)$ for all $t > 0$.

(iii) This is a consequence of Lemma 4.4 and the estimates of Theorem 4.2 and statement (ii), which we next prove.

(ii) We use the notation as in the proof of Theorem 4.2. Fix $\theta' \in (\theta, \frac{\pi}{2})$. For all $\varphi \in \mathbb{R}$ with $|\varphi| < \theta' - \theta$ define $a^{[\varphi]}_{ij} = e^{i\varphi} a_{ij}$ for all $i, j \in \{1, \ldots, d\}$. Then $\sum\limits_{i,j=1}^{d} a^{[\varphi]}_{ij}(x)\xi_i\overline{\xi_j} \in \Sigma_{\theta'}$ for all $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$. Let $a^{[\varphi]}$ be the corresponding form with form domain $D(a)$. For all $\rho > 0$ let $a^{[\varphi]}_\rho$, $A^{[\varphi]}$, $A^{[\varphi]}_\rho$, $S^{[\varphi]}$ and $S^{[\varphi]\rho}$ be the form, operators and semigroups defined naturally as in the proof of Theorem 4.2. Then it follows from (4.4) that

$$\|S^{[\varphi]\rho}_t\|_{2\to 2} \leqslant e^{M_1\rho^2 t}$$

for all $\rho, t > 0$ and $|\varphi| < \theta' - \theta$, where $M_1 = 3(1 + \tan\theta')^2\Big(1 + \sum\limits_{i,j=1}^{d} \|a_{ij}\|_\infty\Big)$. But $S^{[\varphi]\rho}_t = e^{-te^{i\varphi}A_\rho} = S^\rho_{te^{i\varphi}}$. So $\|S^\rho_{te^{i\varphi}}\|_{2\to 2} \leqslant e^{M_1\rho^2 t}$ for all $t, \rho > 0$ and $|\varphi| < \theta' - \theta$. Since S^ρ is a holomorphic semigroup on the interior of $\Sigma_{\pi/2-\theta'}$ it follows that

$$S^\rho_t = \frac{1}{2\pi i} \int\limits_{\Gamma_r(t)} \frac{1}{z - t} S^\rho_z \, dz$$

for all $t > 0$, where $\Gamma_r(t)$ is the circle centred at t and radius $r = ct$ and $c = \sin\frac{1}{2}(\frac{\pi}{2} - \theta')$. Therefore

$$\|A_\rho S^\rho_t\|_{2\to 2} \leqslant \frac{1}{2\pi} \int\limits_{\Gamma_r(t)} \frac{1}{|z - t|^2} \|S^\rho_z\|_{2\to 2} \, d|z| \leqslant \frac{1}{ct} e^{M_2\rho^2 t}$$

for all $\rho, t > 0$, where $M_2 = M_1(1 + c)$. It then follows from (4.3) that

$$\begin{aligned}\frac{1}{2}\mu \sum_{i=1}^{d} \|\partial_i S^\rho_t u\|_2^2 &\leqslant \operatorname{Re} a_\rho(S^\rho_t u, S^\rho_t u) + M\rho^2 \|S^\rho_t u\|_2^2 \leqslant \|A_\rho S^\rho_t u\|_2 \|S^\rho_t u\|_2 + M\rho^2 \|S^\rho_t u\|_2^2 \\ &\leqslant \frac{1}{ct} e^{M_2\rho^2 t} e^{M\rho^2 t} \|u\|_2^2 + M\rho^2 e^{2M\rho^2 t} \|u\|_2^2.\end{aligned}$$

Hence there exist $c_3, M_3 > 0$ such that

$$\|\partial_i S^\rho_t\|_{2\to 2} \leqslant c_3 t^{-1/2} e^{M_3\rho^2 t}$$

for all $i \in \{1, \ldots, d\}$ and $\rho, t > 0$. Since

$$\|U_{-\rho}\partial_i S_t U_\rho\|_{2\to 2} = \|(\partial_i + \rho\psi_i)S^\rho_t\|_{2\to 2} \leqslant \|\partial_i S^\rho_t\|_{2\to 2} + |\rho\psi_i| \|S^\rho_t\|_{2\to 2} \leqslant c_4 t^{-1/2} e^{M_4\rho^2 t}$$

for suitable $c_4, M_4 > 0$, statement (ii) follows as at the end of the proof of Theorem 4.2. ∎

The conditions on the form domain in Theorem 4.2 are satisfied in case of Neumann boundary conditions, i.e. if $D(a) = H^1(\Omega)$. We next show that if $D(a) = H^1(\Omega)$, then a strong locality property is valid. We start with a lemma for (complex) strongly elliptic operators.

LEMMA 4.7. *Let $\Omega \subset \mathbb{R}^d$ be open. For all $i,j \in \{1,\ldots,d\}$ let $a_{ij} \in L_\infty(\Omega)$. Suppose (a_{ij}) is strongly elliptic. Define $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by*

$$a(u,v) = \sum_{i,j=1}^{d} \int_{\Omega} (\partial_i u) a_{ij} \overline{\partial_j v}.$$

Let S be the semigroup associated with a. Then $(S_t u, \mathbf{1}) = (u, \mathbf{1})$ for all $u \in L_{2,c}(\Omega)$ and $t > 0$.

Proof. Fix $\tau \in C_c^\infty(\mathbb{R}^d)$ such that $\tau|_{B_1} = \mathbf{1}$. For all $n \in \mathbb{N}$ define $\tau_n \in C_c^\infty(\mathbb{R}^d)$ by $\tau_n(x) = \tau(n^{-1}x)$. For all $n \in \mathbb{N}$ define $f_n\colon (0,\infty) \to \mathbb{C}$ by $f_n(t) = (S_t u, \tau_n \mathbf{1}_\Omega)$. Note that $\tau_n \mathbf{1}_\Omega \in H^1(\Omega) = D(a)$ for all $n \in \mathbb{N}$. Therefore

$$f_n'(t) = -a(S_t u, \tau_n \mathbf{1}_\Omega) = -\sum_{i,j=1}^{d} (\partial_i S_t u, a_{ij} \partial_j(\tau_n \mathbf{1}_\Omega)) = -\sum_{i,j=1}^{d} (\partial_i S_t u, a_{ij} (\partial_j \tau_n) \mathbf{1}_\Omega)$$

and

$$|f_n'(t)| \leqslant \sum_{i,j=1}^{d} \|\partial_i S_t u\|_1 \|a_{ij}\|_\infty n^{-1} \|\partial_j \tau\|_\infty$$

for all $n \in \mathbb{N}$ and $t > 0$, where we used that $\partial_i S_t u \in L_1(\Omega)$ by Lemma 4.6(iii). So $\lim_{n\to\infty} f_n'(t) = 0$ locally uniform on $(0,\infty)$. In addition, $\lim_{n\to\infty} f_n(t) = (S_t u, \mathbf{1})$ for all $t \in (0,\infty)$. Therefore $t \mapsto (S_t u, \mathbf{1})$ is constant. Since $\lim_{t\downarrow 0}(S_t u, \mathbf{1}) = (u, \mathbf{1})$ the lemma follows. ∎

We are now able to prove strong locality for Neumann sectorial differential operators. Note that our conditions allow that the coefficients are 0 on part or even the entire domain.

PROPOSITION 4.8. *Let $\Omega \subset \mathbb{R}^d$ be open. For all $i,j \in \{1,\ldots,d\}$ let $a_{ij} \in L_\infty(\Omega)$. Let $\theta \in [0, \frac{\pi}{2})$. Suppose $\sum_{i,j=1}^{d} a_{ij}(x) \xi_i \overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$. Define the form a with form domain $D(a) = H^1(\Omega)$ by*

$$a(u,v) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} (\partial_i u) a_{ij} \overline{\partial_j v}.$$

Then one has the following:

(i) *$\overline{a_{\mathrm{r}}}(u,v) = 0$ for all $u,v \in D(\overline{a_{\mathrm{r}}})$ with compact support such that v is constant on a neighbourhood of the support of u.*

(ii) *If S is the semigroup associated with a, then $(S_t u, \mathbf{1}) = (u, \mathbf{1})$ for all $t > 0$ and $u \in L_{2,c}(\Omega)$.*

Proof. We first prove statement (ii). For all $n \in \mathbb{N}$ let $a^{(n)} = a + \frac{1}{n} l$, where l is the form with $D(l) = H^1(\Omega)$ and $l(u,v) = \sum_{i=1}^{d} \int_\Omega (\partial_i u) \overline{\partial_i v}$. Let $S^{(n)}$ be the semigroup associated with $a^{(n)}$. Then $(S_t u, \mathbf{1}) = \lim_{n\to\infty} (S_t^{(n)} u, \mathbf{1}) = (u, \mathbf{1})$ for all $t > 0$ and $u \in L_{2,c}(\Omega)$ by Lemmas 4.5 and 4.7.

Next let $u, v \in D(\overline{a_{\mathrm{r}}})$ with compact support such that v is constant on a neighbourhood of the support of u. Then there exist an open set U and a $\lambda \in \mathbb{C}$ such that $\operatorname{supp} u \subset U$ and $v(x) = \lambda$ for all $x \in U$. Therefore $(u, v) = \lambda(u, \mathbf{1}) = \lambda(S_t u, \mathbf{1})$ for all $t > 0$.

Let $c_d > 0$ be the constant in Lemma 4.4, which depends only on d. Moreover, set

$$M = 3(1 + \tan\theta)^2 \Big(1 + \sum_{i,j=1}^{d} \|a_{ij}\|_\infty\Big).$$

Fix $R > 0$ such that $\operatorname{supp} u \subset B_R$.

Now let $t > 0$. Then

$$((I - S_t)u, v) = \lambda(S_t u, \mathbf{1}) - (S_t u, v) = \lambda(S_t u, \mathbf{1} - \chi_{2R}) + (S_t u, \lambda\chi_{2R} - v).$$

We estimate the terms separately. First, S satisfies the Davies–Gaffney bounds (4.1) of Theorem 4.2. So one estimates

$$|(S_t u, \mathbf{1} - \chi_{2R})| \leqslant \|(\mathbf{1} - \chi_{2R}) S_t u\|_1 \leqslant c_d R^{-1} (4Mt)^{(d+2)/4} \mathrm{e}^{-R^2 (8Mt)^{-1}} \|u\|_2$$

by Lemma 4.4. Next, let $D > 0$ be the distance between $\operatorname{supp} u$ and U^{c}. Then it follows from Theorem 4.2 that

$$\begin{aligned} |(S_t u, \lambda\chi_{2R} - v)| &\leqslant \mathrm{e}^{-D^2 (4Mt)^{-1}} \|u\|_2 \|\lambda\chi_{2R} - v\|_2 \\ &\leqslant (|\lambda| (2R)^{d/2} |B_1|^{1/2} + \|v\|_2) \mathrm{e}^{-D^2 (4Mt)^{-1}} \|u\|_2. \end{aligned}$$

Therefore

$$\begin{aligned} t^{-1} |((I - S_t)u, v)| &\leqslant |\lambda| c_d R^{-1} t^{-1} (4Mt)^{(d+2)/4} \mathrm{e}^{-R^2 (8Mt)^{-1}} \|u\|_2 \\ &\quad + (|\lambda| (2R)^{d/2} |B_1|^{1/2} + \|v\|_2) t^{-1} \mathrm{e}^{-D^2 (4Mt)^{-1}} \|u\|_2 \end{aligned}$$

for all $t > 0$. Since $\overline{a_{\mathrm{r}}}(u, v) = \lim_{t\downarrow 0} t^{-1} ((I - S_t)u, v)$ the proposition follows. ∎

Up to now the coefficients were allowed to be complex in this section. If the coefficients are real, but possibly not symmetric, then one has the following application of Corollary 3.13 and Proposition 4.8.

COROLLARY 4.9. *Let $\Omega \subset \mathbb{R}^d$ be open. For all $i, j \in \{1, \ldots, d\}$ let $a_{ij} \in L_\infty(\Omega, \mathbb{R})$. Let $\theta \in [0, \frac{\pi}{2})$. Suppose $\sum_{i,j=1}^{d} a_{ij}(x) \xi_i \overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$. Define the form $a \colon D(a) \times D(a) \to \mathbb{C}$ by*

$$a(u, v) = \sum_{i,j=1}^{d} \int_\Omega (\partial_i u) a_{ij} \overline{\partial_j v},$$

where $D(a) = H^1(\Omega)$ or $D(a) = H^1_0(\Omega)$. Let S be the semigroup associated with a. Then S is real, positive and S extends consistently to a contraction semigroup on $L_p(\Omega)$ for all $p \in [1, \infty]$, which is a C_0-semigroup if $p \in [1, \infty)$ and the adjoint of a C_0-semigroup if $p = \infty$. Moreover, if $D(a) = H^1(\Omega)$, then $S_t \mathbf{1}_\Omega = \mathbf{1}_\Omega$ for all $t > 0$.

Proof. Only the last statement needs comments. Since $L_{2,c}(\Omega)$ is dense in $L_1(\Omega)$ one deduces from Proposition 4.8(ii) that $(S_t u, \mathbf{1}) = (u, \mathbf{1})$ for all $u \in L_1(\Omega)$. Then the claim follows by duality and Remark 3.5. ∎

Thus for real coefficients and Neumann boundary conditions the semigroup S is stochastic on L_1.

4.2. MULTIPLICATIVE PERTURBATION. Let $\Omega \subset \mathbb{R}^d$ be open and bounded. We perturb the Dirichlet Laplacian by choosing a special function j. Then we obtain a possibly degenerate operator as follows.

PROPOSITION 4.10. *Let $m \colon \Omega \to (0, \infty)$ be such that $\frac{1}{m} \in L_{2,\mathrm{loc}}(\Omega)$. Define the operator, formally denoted by $(m\Delta m)$ on $L_2(\Omega)$ by the following. Let $w, f \in L_2(\Omega)$. Then we define $w \in D((m\Delta m))$ and $(m\Delta m)w = f$ if and only if $mw \in H^1_0(\Omega)$ and $\Delta(mw) = \frac{f}{m}$ in $\mathcal{D}(\Omega)'$.*

Then the operator $(m\Delta m)$ is self-adjoint and $(m\Delta m)$ generates a positive semigroup S. Moreover, the following set is invariant under S:

$$C = \{f \in L_2(\Omega, \mathbb{R}) : f \leqslant \frac{1}{m}\}.$$

Proof. Let $V = H^1_0(\Omega) \cap L_2(\Omega, \frac{1}{m^2} dx)$ and define $j \in \mathcal{L}(V, L_2(\Omega))$ by $j(u) = \frac{u}{m}$. Define $a \colon V \times V \to \mathbb{C}$ by $a(u, v) = \int_\Omega \nabla u \overline{\nabla v}$. Then a is continuous and symmetric. Since Ω is bounded it follows from the (Dirichlet type) Poincaré inequality that the norm

$$u \mapsto \Big(\int_\Omega |\nabla u|^2 + \int_\Omega \frac{|u|^2}{m^2} \Big)^{1/2}$$

is an equivalent norm on V. Therefore the form a is j-elliptic. Let A be the operator associated with (a, j). We shall show that $A = -(m\Delta m)$.

Let $w \in D(A)$ and write $f = Aw$. Then there exists a $u \in V$ such that $w = j(u) = \frac{u}{m}$ and $\int_\Omega \nabla u \overline{\nabla v} = \int_\Omega f \frac{\overline{v}}{m}$ for all $v \in V$. Observe that $\frac{f}{m} \in L_{1,\mathrm{loc}}(\Omega)$. Taking $v \in \mathcal{D}(\Omega)$ one deduces that $-\Delta u = \frac{f}{m}$ in $\mathcal{D}(\Omega)'$. Thus $w \in D((m\Delta m))$ and $-(m\Delta m)w = f$.

Conversely, let $w \in D((m\Delta m))$ and write $f = -(m\Delta m)w$. Set $u = mw \in H^1_0(\Omega)$. Then

$$a(u, v) = \int_\Omega \nabla u \overline{\nabla v} = -\langle \Delta u, \overline{v} \rangle = \Big\langle \frac{f}{m}, \overline{v} \Big\rangle = \int_\Omega f \frac{\overline{v}}{m} = \int_\Omega f \overline{j(v)}$$

for all $v \in \mathcal{D}(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in V by Proposition 3.2 of [3], it follows that $a(u,v) = \int_\Omega f\overline{j(v)}$ for all $v \in V$. Thus $w = j(u) \in D(A)$. This proves that $A = -(m\Delta m)$.

The operator A is self-adjoint since a is symmetric. We next show the invariance of the set C. The set C is closed and convex in $L_2(\Omega)$. Define $P\colon L_2(\Omega) \to C$ by $Pf = (\operatorname{Re} f) \wedge \frac{1}{m}$. The P is the orthogonal projection onto C. Let $u \in V$. Define $w = (\operatorname{Re} u) \wedge 1 \in V$. Then $Pj(u) = j(w)$ and $\operatorname{Re} a(w, u-w) = 0$. Hence it follows from Proposition 2.9 that the set C is invariant under S. Since $f \leqslant 0$ if and only if $nf \in C$ for all $n \in \mathbb{N}$ the invariance of C also implies that the semigroup is positive. ∎

By a similarity transformation we obtain two further kinds of multiplicative perturbations. We leave the proofs to the reader.

PROPOSITION 4.11. *Let $\rho\colon \Omega \to (0,\infty)$ be such that $\frac{1}{\rho} \in L_{1,\mathrm{loc}}(\Omega)$. Define the operator, formally denoted by $(\rho\Delta)$ on $L_2(\Omega, \frac{1}{\rho}\mathrm{d}x)$ by the following. Let $w, f \in L_2(\Omega, \frac{1}{\rho}\mathrm{d}x)$. Then we define $w \in D((\rho\Delta))$ and $(\rho\Delta)w = f$ if and only if $w \in L_2(\Omega, \frac{1}{\rho}\mathrm{d}x) \cap H^1_0(\Omega)$ and $\Delta w = \frac{f}{\rho}$ in $\mathcal{D}(\Omega)'$.*

Then the operator $(\rho\Delta)$ is self-adjoint and generates a submarkovian semigroup.

PROPOSITION 4.12. *Let $\rho\colon \Omega \to (0,\infty)$ be such that $\frac{1}{\rho} \in L_{1,\mathrm{loc}}(\Omega)$. Define the operator, formally denoted by $(\Delta\rho)$ on $L_2(\Omega, \rho\mathrm{d}x)$ by the following. Let $w, f \in L_2(\Omega, \rho\mathrm{d}x)$. Then $w \in D((\Delta\rho))$ and $(\Delta\rho)w = f$ if and only if $\rho w \in H^1_0(\Omega)$ and $\Delta(\rho w) = f$ in $\mathcal{D}(\Omega)'$.*

Then the operator $(\Delta\rho)$ is self-adjoint and generates a submarkovian semigroup.

4.3. ROBIN BOUNDARY CONDITIONS. Let $\Omega \subset \mathbb{R}^d$ be an open set with arbitrary boundary Γ. At first we consider an arbitrary Borel measure on Γ and then specialize to the $(d-1)$-dimensional Hausdorff measure.

For all $i,j \in \{1,\dots,d\}$ let $a_{ij} \in L_\infty(\Omega,\mathbb{C})$. Let $\theta \in [0, \frac{\pi}{2})$. Suppose that $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$. Let μ be a (positive) Borel measure on Γ such that $\mu(K) < \infty$ for every compact $K \subset \Gamma$. Define the form a by

$$D(a) = \Big\{ u \in H^1(\Omega) \cap C(\overline{\Omega}) : \int_\Gamma |u|^2 \mathrm{d}\mu < \infty \Big\}$$

and

$$a(u,v) = \sum_{i,j=1}^{d} \int_\Omega (\partial_i u) a_{ij} \overline{\partial_j v} + \int_\Gamma u\overline{v}\,\mathrm{d}\mu.$$

Then $C_c^\infty(\Omega) \subset D(a) \subset L_2(\Omega)$ and a is sectorial. In order to characterize the associated operator A we need to introduce two concepts and one more condition.

First, define the Neumann form a_N by $D(a_N) = H^1(\Omega)$ and

$$a_N(u,v) = \sum_{i,j=1}^{d} \int_\Omega (\partial_i u) a_{ij} \overline{\partial_j v}.$$

Throughout this subsection we suppose the form a_N is closable. Here we are more interested in the degeneracy caused by μ. If $u \in D(\overline{a_N})$ and $f \in L_2(\Omega)$, then we say that $\mathcal{A}u = f$ *weakly on* Ω if

$$\overline{a_N}(u,v) = \int_\Omega f\overline{v}$$

for all $v \in C_c^\infty(\Omega)$. If $u \in D(\overline{a_N})$, then we say that $\mathcal{A}u \in L_2(\Omega)$ *weakly on* Ω if there exists an $f \in L_2(\Omega)$ such that $\mathcal{A}u = f$ weakly on Ω. Clearly such a function f is unique, if it exists. Secondly, if $u \in D(\overline{a_N})$ and $\varphi \in L_2(\Gamma,\mu)$, then we say that φ is *an* (a,μ)*-trace of* u, or shortly, *a trace of* u, if there exist $u_1, u_2, \ldots \in D(a)$ such that $\lim u_n = u$ in $D(\overline{a_N})$ and $\lim u_n|_\Gamma = \varphi$ in $L_2(\Gamma,\mu)$. Moreover, let $H^1_{a,\mu}(\Omega)$ be the set of all $u \in D(\overline{a_N})$ for which there exists a $\varphi \in L_2(\Gamma,\mu)$ such that φ is a trace of u. We emphasize that φ is not unique (almost everywhere) in general. Clearly $D(a) \subset H^1_{a,\mu}(\Omega)$. With the help of these definitions we can describe the operator A as follows.

PROPOSITION 4.13. *Let* $u, f \in L_2(\Omega)$. *Then* $u \in D(A)$ *and* $Au = f$ *if and only if* $u \in H^1_{a,\mu}(\Omega)$, $\mathcal{A}u = f$ *weakly on* Ω *and there exists a* $\varphi \in L_2(\Gamma,\mu)$ *such that* φ *is a trace of* u *and, for all* $v \in D(a)$,

$$\overline{a_N}(u,v) - \int_\Omega (\mathcal{A}u)\overline{v} = -\int_\Gamma \varphi\overline{v}\,\mathrm{d}\mu. \tag{4.5}$$

If the conditions are valid, then the function φ *is unique.*

Proof. "$\Rightarrow$" There exists a Cauchy sequence $u_1, u_2, \ldots$ in $D(a)$ such that $\lim u_n = u$ in $L_2(\Omega)$ and $\lim a(u_n, v) = (f,v)_H$ for all $v \in D(a)$. Then $u_1, u_2, \ldots$ is a Cauchy sequence in $D(\overline{a_N})$. Therefore $u \in D(\overline{a_N})$ and $\lim u_n = u$ in $D(\overline{a_N})$. Moreover, $u_1|_\Gamma, u_2|_\Gamma, \ldots$ is a Cauchy sequence in $L_2(\Gamma,\mu)$. Therefore $\varphi := \lim u_n|_\Gamma$ exists in $L_2(\Gamma,\mu)$. Then φ is a trace of u. Let $v \in D(a)$. Then

$$\overline{a_N}(u,v) + \int_\Gamma \varphi\overline{v}\,\mathrm{d}\mu = \lim a(u_n,v) = (f,v)_H = \int_\Omega f\overline{v}.$$

Therefore if $v \in C_c^\infty(\Omega)$, then

$$\overline{a_N}(u,v) = \int_\Omega f\overline{v},$$

so $\mathcal{A}u = f$ weakly on Ω. Moreover, for all $v \in D(a)$,

$$\overline{a_N}(u,v) + \int_\Gamma \varphi\overline{v}\,\mathrm{d}\mu = \int_\Omega (\mathcal{A}u)\overline{v}.$$

If also $\varphi' \in L_2(\Gamma, \mu)$ satisfies (4.5), then $\int_\Gamma (\varphi - \varphi')\overline{v}\,\mathrm{d}\mu = 0$ for all $v \in D(a)$. But the space $\{v|_\Gamma : v \in H^1(\Omega) \cap C_c(\overline{\Omega})\}$ is a $*$-algebra which separates the points of Γ. Therefore it is dense in $C_0(\Gamma)$. Let $\psi \in C_c(\Gamma)$. Then there exists a $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi|_{\mathrm{supp}\,\psi} = \mathbb{1}$. By the above there exist $v_1, v_2, \ldots \in H^1(\Omega) \cap C_c(\overline{\Omega})$ such that $\lim v_n|_\Gamma = \psi$ in $C_0(\Gamma)$. Then $\lim (\chi v_n)|_\Gamma = \psi$ in $C_0(\Gamma)$. Moreover, $\mu(\mathrm{supp}(\chi|_\Gamma)) < \infty$. Therefore $\lim(\chi v_n)|_\Gamma = \psi$ in $L_2(\Gamma, \mu)$ and the space $\{v|_\Gamma : v \in H^1(\Omega) \cap C_c(\overline{\Omega})\}$ is dense in $L_2(\Gamma, \mu)$. Thus $\varphi' = \varphi$.

"$\Leftarrow$" There exist $\varphi \in L_2(\Gamma, \mu)$ and $u_1, u_2, \ldots \in D(a)$ such that $\lim u_n = u$ in $D(\overline{a_N})$, $\lim u_n|_\Gamma = \varphi$ in $L_2(\Gamma, \mu)$ and (4.5) is valid for all $v \in D(a)$. Then $u_1, u_2, \ldots$ is a Cauchy sequence in $D(a)$ and

$$\lim_{n\to\infty} a(u_n, v) = \overline{a_N}(u, v) + \int_\Gamma \varphi \overline{v}\,\mathrm{d}\mu = \int_\Omega (\mathcal{A}u)\overline{v} = \int_\Omega f\overline{v}$$

for all $v \in D(a)$. So $u \in D(A)$ and $Au = f$. ∎

This proposition shows how our general results can be easily applied. It is worthwhile to consider more closely the associated closed form since this is intimately related to the problem to define a trace in $L_2(\Gamma, \mu)$ of suitable functions in $H^1(\Omega)$.

Let

$$W = \overline{\{(u, u|_\Gamma) : u \in D(a)\}},$$

where the closure is in $D(\overline{a_N}) \oplus L_2(\Gamma, \mu)$. Then the map $u \mapsto (u, u|_\Gamma)$ from $D(a)$ into W is an isometry and therefore it extends to a unitary map from the completion of $D(a)$ onto W. The form a is closable if and only if the map $j \colon W \to L_2(\Omega)$ defined by $j(u, \varphi) = u$ is injective. Note that if $\varphi \in L_2(\Gamma, \mu)$, then $(0, \varphi) \in W$ if and only if φ is a trace of 0.

The following lemma is due to Daners ([9], Proposition 3.3) in the strongly elliptic case, but our proof is different.

LEMMA 4.14. *There exists a Borel set $\Gamma_{a,\mu} \subset \Gamma$ such that*

$$\{\varphi \in L_2(\Gamma, \mu) : \varphi \text{ is a trace of } 0\} = L_2(\Gamma \setminus \Gamma_{a,\mu}, \mu).$$

Proof. Set $F = \{\varphi \in L_2(\Gamma, \mu) : (0, \varphi) \in W\}$. Then F is a closed subspace of $L_2(\Gamma, \mu)$.

First we show that $u\psi \in F$ for all $\psi \in F$ and $u \in D(a) \cap W^1_\infty(\mathbb{R}^d)$. Since $\psi \in F$ there exist $u_1, u_2, \ldots \in D(a)$ such that $\lim u_n = 0$ in $D(\overline{a_N})$ and $\lim u_n|_\Gamma = \psi$ in $L_2(\Gamma, \mu)$. Then $uu_n \in D(a)$ for all $n \in \mathbb{N}$ and $\lim(uu_n)|_\Gamma = u\psi$ in $L_2(\Gamma, \mu)$. By the Leibniz rule one deduces that

$$(\Re \overline{a_N})(uu_n)^{1/2} \leqslant \|u_n\|_2 \Big(\sum \Big\| \Big| \frac{a_{ij} + \overline{a_{ji}}}{2} \Big| |\partial_i u| |\partial_j u| \Big\|_\infty \Big)^{1/2} + \|u\|_\infty (\Re \overline{a_N})(u_n)^{1/2}$$

for all $n \in \mathbb{N}$ and $\lim uu_n = 0$ in $D(\overline{a_N})$. So $u\psi \in F$.

Secondly, let $P \colon L_2(\Gamma, \mu) \to F$ be the orthogonal projection. Let $\varphi \in L_2(\Gamma, \mu)$ and suppose that $\mu([\varphi \neq 0]) < \infty$. We shall prove that $P\varphi = 0$ a.e. on $[\varphi = 0]$.

Let $A = [\varphi \neq 0]$. Since $\{u|_\Gamma : u \in H^1(\Omega) \cap C_c^\infty(\mathbb{R}^d)\}$ is dense in $L_2(\Gamma, \mu)$ there exist $u_1, u_2, \ldots \in H^1(\Omega) \cap C_c^\infty(\mathbb{R}^d)$ such that $\lim u_n|_\Gamma = \mathbf{1}_A$ in $L_2(\Gamma, \mu)$. Then also $\lim (0 \vee \mathrm{Re}\, u_n \wedge \mathbf{1})|_\Gamma = \mathbf{1}_A$ in $L_2(\Gamma, \mu)$, so we may assume that $u_n \in D(a) \cap W^1_\infty(\mathbb{R}^d)$ and $0 \leqslant u_n \leqslant \mathbf{1}$ for all $n \in \mathbb{N}$. Passing to a subsequence if necessary we may assume that $\lim u_n|_\Gamma = \mathbf{1}_A$ a.e. Therefore $\lim u_n P\varphi = \mathbf{1}_A P\varphi$ in $L_2(\Gamma, \mu)$. Since $u_n P\varphi \in F$ for all $n \in \mathbb{N}$ one deduces that $\mathbf{1}_A P\varphi \in F$. Then

$$\|\varphi - \mathbf{1}_A P\varphi\|_{L_2(\Gamma,\mu)} = \|\mathbf{1}_A(\varphi - P\varphi)\|_{L_2(\Gamma,\mu)} \leqslant \|\varphi - P\varphi\|_{L_2(\Gamma,\mu)}.$$

So $\mathbf{1}_A P\varphi = P\varphi$ and $P\varphi = 0$ a.e. on $A^c = [\varphi = 0]$. Now the lemma easily follows from Zaanen's theorem ([7], Proposition 1.7). ∎

Obviously the set $\Gamma_{a,\mu}$ in Lemma 4.14 is unique in the sense that $\mu(\Gamma_{a,\mu} \Delta \Gamma') = 0$ whenever $\Gamma' \subset \Gamma$ is another Borel set with this property. It follows from the last paragraph of Section 3 in [8] that the set $\Gamma_{a,\mu}$ coincides with the set S in Proposition 3.6 of [8]. In Example 4.2 of [8] there is an example of an open set Ω and μ the $(d-1)$-dimensional Hausdorff measure such that $\mu(\Gamma) < \infty$ and $\mu(\Gamma \setminus \Gamma_{a,\mu}) > 0$, where a is the form of the Laplacian.

It is clear from the construction of $\Gamma_{a,\mu}$ and the definition of $H^1_{a,\mu}(\Omega)$ that there exists a unique map $\mathrm{Tr}_{\,a,\mu} \colon H^1_{a,\mu}(\Omega) \to L_2(\Gamma_{a,\mu}, \mu)$ in a natural way, which we call *trace*. Note that if $u \in H^1_{a,\mu}(\Omega)$, then $\mathrm{Tr}_{\,a,\mu} u$ is the unique $\varphi \in L_2(\Gamma_{a,\mu}, \mu)$ such that φ is an (a, μ)-trace of u. In general, however, the map $\mathrm{Tr}_{\,a,\mu}$ is not continuous from $(H^1_{a,\mu}(\Omega), \|\cdot\|_{\overline{a_N}})$ into $L_2(\Gamma_{a,\mu}, \mu)$. A counter-example is in Remark 3.5(f) of [9].

The map $u \mapsto (u, \mathrm{Tr}_{\,a,\mu} u)$ from $H^1_{a,\mu}(\Omega)$ into $D(\overline{a_N}) \oplus L_2(\Gamma_{a,\mu}, \mu)$ is injective. Therefore one can define a norm on $H^1_{a,\mu}(\Omega)$ by

$$\|u\|^2_{H^1_{a,\mu}(\Omega)} = \|u\|^2_{D(\overline{a_N})} + \|\mathrm{Tr}_{\,a,\mu} u\|^2_{L_2(\Gamma_{a,\mu},\mu)}.$$

It is easy to verify that $H^1_{a,\mu}(\Omega)$ is a Hilbert space. In addition, the operator $\mathrm{Tr}_{\,a,\mu} \colon H^1_{a,\mu}(\Omega) \to L_2(\Gamma_{a,\mu}, \mu)$ is a continuous linear operator with dense range.

It is now possible to reconsider the element $\varphi \in L_2(\Gamma, \mu)$ in Proposition 4.13.

PROPOSITION 4.15. *Let $u, f \in L_2(\Omega)$. Then $u \in D(A)$ and $Au = f$ if and only if $u \in H^1_{a,\mu}(\Omega)$, $\mathcal{A}u = f$ weakly on Ω and, for all $v \in D(a)$,*

$$\overline{a_N}(u, v) - \int_\Omega (\mathcal{A}u)\overline{v} = -\int_\Gamma (\mathrm{Tr}_{\,a,\mu} u)\overline{v}\, d\mu.$$

Proof. Let $u \in D(A)$ and $\varphi \in L_2(\Gamma, \mu)$ be the corresponding unique element as in Proposition 4.13. If $\psi \in L_2(\Gamma \setminus \Gamma_{a,\mu}, \mu) = F$, then there exist $v_1, v_2, \ldots \in D(a)$ such that $\lim v_n = 0$ in $D(\overline{a_N})$ and $\lim v_n|_\Gamma = \psi$ in $L_2(\Gamma, \mu)$. Substituting $v = v_n$ in (4.5) and taking the limit $n \to \infty$ one deduces that $\int_\Gamma \varphi\overline{\psi}\, d\mu = 0$. So $\varphi \in L_2(\Gamma_{a,\mu}, \mu)$ and $\varphi = \mathrm{Tr}_{\,a,\mu} u$. ∎

We now consider the case where μ is the $(d-1)$-dimensional Hausdorff measure, which we denote by σ. In particular, we assume that $\sigma(K) < \infty$ for every compact $K \subset \Gamma$. Moreover, we write $\Gamma_a = \Gamma_{a,\sigma}$ and $\operatorname{Tr}_a = \operatorname{Tr}_{a,\sigma}$. The measure σ coincides with the usual surface measure if Ω is C^1. We continue to consider, however, the case where Ω is an arbitrary bounded open set. If Ω has a Lipschitz continuous boundary and the form a equals the classical Dirichlet form l, then $\Gamma_l = \Gamma_a = \Omega$ by the trace theorem (see Théorème 2.4.2 of [20]). By Proposition 5.5 of [8] it follows that $\sigma(\Gamma_l) > 0$ if Ω is bounded, without any regularity condition on the boundary. (Note, however, that there exists an open connected subset $\Omega \subset \mathbb{R}^3$ such that $\sigma(\Gamma \setminus \Gamma_l) > 0$; see Example 4.3 of [8].) The embedding of $H^1_{l,\sigma}(\Omega)$ into $L_2(\Omega)$ is compact if Ω has finite measure, by Corollary 5.2 of [8]. This surprising phenomenon is a consequence of Maz'ya's inequality. It was Daners [9] who was the first to exploit this inequality to establish results for Robin boundary conditions on rough domains. Further results were given in Section 5 of [8].

We conclude our remarks by considering $\mu = \beta\sigma$, where $\beta \in L_\infty(\Gamma, \mathbb{R})$ and $\beta \geqslant 0$ a.e. We define the weak normal derivative with respect to the matrix (a_{ij}). Let $\varphi \in L_2(\Gamma, \mu)$, $u \in D(\overline{a_N})$ and suppose that $\mathcal{A}u \in L_2(\Omega)$ weakly on Ω. Then we say that φ is the *(a_{ij})-normal derivative of u* if

$$\overline{a_N}(u, v) - \int_\Omega (\mathcal{A}u)\overline{v} = \int_\Gamma \varphi \overline{v} \, d\sigma$$

for all $v \in D(a)$. If Ω is of class C^1, μ is the $(d-1)$-dimensional Hausdorff measure and $u \in C^1(\overline{\Omega})$, then our weak definition coincides with the classical definition by Green's theorem. We reformulate Proposition 4.13.

PROPOSITION 4.16. *Let $u, f \in L_2(\Omega)$. Then $u \in D(A)$ and $Au = f$ if and only if $u \in H^1_{a,\beta\sigma}(\Omega)$, $\mathcal{A}u = f$ weakly on Ω and $-\beta \operatorname{Tr}_{a,\beta\sigma} u$ is the (a_{ij})-normal derivative of u.*

Note that if (a_{ij}) is strongly elliptic and if $u \in D(A)$ and $Au = f$, then $u \in H^1(\Omega)$, $\mathcal{A}u = f$ weakly on Ω, u has a trace $\operatorname{Tr} u$ and $\nu \cdot a\nabla u = -\beta \operatorname{Tr} u$ weakly. Thus one recovers the classical statement.

4.4. THE DIRICHLET-TO-NEUMANN OPERATOR. Let Ω be a bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary Γ, provided with the $(d-1)$-dimensional Hausdorff measure. Let $\operatorname{Tr} : H^1(\Omega) \to L_2(\Gamma)$ be the trace map. We denote by Δ_D the Dirichlet Laplacian on Ω. If $\varphi \in L_2(\Gamma)$, $u \in H^1(\Omega)$ and $\Delta u \in L_2(\Omega)$ weakly on Ω, then we say that $\frac{\partial u}{\partial \nu} = \varphi$ weakly if φ is the (a_{ij})-normal derivative of u, where $a_{ij} = \delta_{ij}$.

Let $\lambda \in \mathbb{R}$ and suppose that $\lambda \notin \sigma(-\Delta_D)$. The *Dirichlet-to-Neumann operator* D_λ on $L_2(\Gamma)$ is defined as follows. Let $\varphi, \psi \in L_2(\Gamma)$. Then we define $\varphi \in D(D_\lambda)$ and $D_\lambda \varphi = \psi$ if there exists a $u \in H^1(\Omega)$ such that $-\Delta u = \lambda u$ weakly on Ω, $\operatorname{Tr} u = \varphi$ and $\frac{\partial u}{\partial \nu} = \psi$ weakly. We next show that the Dirichlet-to-Neumann operator is an example of the m-sectorial operators obtained in Corollary 2.2.

Define the sesquilinear form $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$a(u,v) = \int_\Omega \nabla u \overline{\nabla v} - \lambda \int_\Omega u\overline{v}.$$

Moreover, define $j\colon H^1(\Omega) \to L_2(\Gamma)$ by $j(u) = \operatorname{Tr} u$. Then $\ker j = H^1_0(\Omega)$ by Lemma A.6.10 of [1]. Clearly the form a is continuous, the map j is bounded and $j(H^1(\Omega))$ is dense in $L_2(\Gamma)$. Using the definitions one deduces that

$$V(a) = \{u \in H^1(\Omega) : -\Delta u = \lambda u \text{ weakly on } \Omega\}.$$

It follows from Step 1 in the proof of Proposition 3.3 in [5] that there exist $\omega \in \mathbb{R}$ and $\mu > 0$ such that

$$\operatorname{Re} a(u) + \omega \|j(u)\|^2_{L_2(\Gamma)} \geqslant \mu \|u\|^2_V$$

for all $u \in V(a)$. Moreover, $H^1(\Omega) = V(a) + \ker j$ by Lemma 3.2 in [5]. So the conditions of Corollary 2.2 are satisfied.

Note that if λ_1 is the lowest eigenvalue of the operator $-\Delta_D$ on Ω and $u \in H^1_0(\Omega)$ is an eigenfunction with eigenvalue λ_1, then (2.2) is not valid if $\lambda > \lambda_1$. Therefore Theorem 2.1 is not applicable and this example is the reason why we used the space $V(a)$ in Corollary 2.2. If still $\lambda > \lambda_1$ then there exists a $\widetilde{u} \in H^1_0(\Omega)$ such that $\widetilde{u} \neq 0$ and $a(\widetilde{u}) = 0$. If $U = V(a) + \operatorname{span}\{\widetilde{u}\}$ then U is a closed subspace of $H^1(\Omega)$ such that $D_H(a) \subset U$. But $(a|_{U\times U}, j|_U)$ does not satisfy (2.4). Hence Proposition 2.3(i) cannot be extended to the setting of Corollary 2.2.

Let A be the operator associated with (a,j). We next show that $A = D_\lambda$. Let $\varphi, \psi \in L_2(\Gamma)$. Suppose $\varphi \in D(A)$ and $A\varphi = \psi$. Then there is a $u \in H^1(\Omega)$ such that $\operatorname{Tr} u = \varphi$ and $a(u,v) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in H^1(\Omega)$. For all $v \in H^1_0(\Omega)$ one has

$$\int_\Omega \nabla u \overline{\nabla v} - \lambda \int_\Omega u\overline{v} = a(u,v) = 0,$$

so $-\Delta u = \lambda u$ weakly on Ω. Then

$$\int_\Omega \nabla u \overline{\nabla v} + \int_\Omega (\Delta u)\overline{v} = a(u,v) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$$

for all $v \in H^1(\Omega)$. So $\frac{\partial u}{\partial \nu} = \psi$ weakly. Therefore $\varphi \in D(D_\lambda)$ and $D_\lambda \varphi = \psi$. Conversely, suppose $\varphi \in D(D_\lambda)$ and $D_\lambda \varphi = \psi$. By definition there exists a $u \in H^1(\Omega)$ such that $-\Delta u = \lambda u$ weakly on Ω, $\operatorname{Tr} u = \varphi$ and $\frac{\partial u}{\partial \nu} = \psi$ weakly. Then

$$a(u,v) = \int_\Omega \nabla u \overline{\nabla v} - \lambda \int_\Omega u\overline{v} = \int_\Omega \nabla u \overline{\nabla v} + \int_\Omega (\Delta u)\overline{v} = \int_\Gamma \frac{\partial u}{\partial \nu} \overline{\operatorname{Tr} v} = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$$

for all $v \in H^1(\Omega)$. So $\varphi = j(u) \in D(A)$ and $\psi = Aj(u) = A\varphi$. Thus $D_\lambda = A$ is the operator associated with (a,j).

If S is the semigroup generated by $-D_\lambda$, then it follows as in the proof of Corollary 3.13 that S is real and positive. Moreover, if $\lambda \leqslant 0$, then S extends consistently to a continuous contraction semigroup on $L_p(\Omega)$ for all $p \in [1, \infty]$.

4.5. WENTZELL BOUNDARY CONDITIONS. Let again Ω be an open subset of $\mathbb{R}^d$ with arbitrary boundary Γ and let σ be the $(d-1)$-dimensional Hausdorff measure on Γ. We assume that $\sigma(K) < \infty$ for every compact $K \subset \Gamma$. All L_p spaces on Γ are with respect to the measure σ, except if written different explicitly. For all $i, j \in \{1, \ldots, d\}$ let $a_{ij} \in L_\infty(\Omega)$. Let $\theta \in [0, \frac{\pi}{2})$. Suppose $\sum\limits_{i,j=1}^{d} a_{ij}(x)\xi_i\overline{\xi_j} \in \Sigma_\theta$ for all $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$. Define the form b by

$$D(b) = \Big\{u \in H^1(\Omega) \cap C(\overline{\Omega}) : \int_\Gamma |u|^2 \mathrm{d}\sigma < \infty\Big\}$$

and

$$b(u, v) = \sum_{i,j=1}^{d} \int_\Omega (\partial_i u) a_{ij} \overline{\partial_j v} + \int_\Gamma u\overline{v} \mathrm{d}\sigma.$$

As in Subsection 4.3 we define the Neumann form b_N by $D(b_N) = H^1(\Omega)$ and

$$b_N(u, v) = \sum_{i,j=1}^{d} \int_\Omega (\partial_i u) a_{ij} \overline{\partial_j v}.$$

Throughout this subsection we assume that the form b_N is closable. Set $\widetilde{\Gamma} = \Gamma_{b,\sigma}$ and $\mathrm{Tr} = \mathrm{Tr}_{\,b,\sigma}$. Moreover, we assume that the map $\mathrm{Tr} : (H^1_{b,\sigma}(\Omega), \|\cdot\|_{\overline{b_N}}) \to L_2(\widetilde{\Gamma})$ is continuous.

Fix $\alpha \in L_\infty(\widetilde{\Gamma})$ and $B \in \mathcal{L}(L_2(\widetilde{\Gamma}))$. Throughout this subsection we assume that there exists an $\omega > 0$ such that

$$\omega\|B\varphi\|^2_{L_2(\widetilde{\Gamma})} + \int_{\widetilde{\Gamma}} \mathrm{Re}\,\alpha|\varphi|^2 \geqslant 0 \tag{4.6}$$

for all $\varphi \in L_2(\widetilde{\Gamma})$. As an example, if $\beta \in L_\infty(\Gamma), \omega > 0$ and $\omega|\beta|^2 + \mathrm{Re}\,\alpha \geqslant 0$, then the multiplication operator B, defined by the function β, satisfies (4.6).

Define the form a by

$$D(a) = H^1_{b,\sigma}(\Omega) \quad \text{and} \quad a(u, v) = \sum_{i,j=1}^{d} \int_\Omega (\partial_i u) a_{ij} \overline{\partial_j v} + \int_{\widetilde{\Gamma}} \mathrm{Tr}\, u \overline{\mathrm{Tr}\, v} \alpha \mathrm{d}\sigma.$$

Then a is continuous. Let H be the closure of the space $\{(u, B(\mathrm{Tr}\, u)) : u \in H^1_{b,\sigma}(\Omega)\}$ in the space $L_2(\Omega) \oplus L_2(\widetilde{\Gamma})$ with induced norm. Define the injective map $j \colon H^1_{b,\sigma}(\Omega) \to H$ by

$$j(u) = (u, B(\mathrm{Tr}\, u)).$$

If B has dense range, then $H = L_2(\Omega) \oplus L_2(\widetilde{\Gamma})$ since the space $\{(u, \operatorname{Tr} u) : u \in H^1_{b,\sigma}(\Omega)\}$ is dense in $L_2(\Omega) \oplus L_2(\widetilde{\Gamma})$ by Step a) in the proof of Theorem 2.3 in [6]. Then the claim follows by the range condition on B. Note that the condition (4.6) together with the assumed continuity of $\operatorname{Tr} : (H^1_{b,\sigma}(\Omega), \|\cdot\|_{\overline{b_N}}) \to L_2(\widetilde{\Gamma})$ imply that a is j-elliptic. Let A be the operator associated with (a, j).

PROPOSITION 4.17. *Let $x, y \in H$. Then $x \in D(A)$ and $Ax = y$ if and only if there exist $u \in H^1_{b,\sigma}(\Omega)$ and $\psi \in L_2(\widetilde{\Gamma})$ such that $x = (u, B(\operatorname{Tr} u))$, $\mathcal{A}u \in L_2(\Omega)$ weakly on Ω, $(B^*\psi - \alpha \operatorname{Tr} u)$ is the (a_{ij})-normal derivative of u and $y = (\mathcal{A}u, \psi)$.*

Proof. "$\Rightarrow$" Write $y = (f, \psi) \in H$. There exists a $u \in H^1_{b,\sigma}(\Omega)$ such that $x = j(u)$ and

$$\overline{b_N}(u,v) + \int_{\widetilde{\Gamma}} \operatorname{Tr} u \overline{\operatorname{Tr} v} \alpha \, d\sigma = (y, j(v))_H = \int_{\Omega} f \overline{v} + \int_{\widetilde{\Gamma}} \psi \overline{B(\operatorname{Tr} v)} \, d\sigma$$

for all $v \in H^1_{b,\sigma}(\Omega)$. Taking only $v \in C_c^\infty(\Omega)$ one deduces that $\mathcal{A}u = f$ weakly on Ω. In particular, $y = (f, \psi) = (\mathcal{A}u, \psi)$. Moreover,

$$\overline{b_N}(u,v) - \int_{\Omega} (\mathcal{A}u)\overline{v} = \int_{\widetilde{\Gamma}} (B^*\psi - \alpha \operatorname{Tr} u) \overline{\operatorname{Tr} v} \, d\sigma$$

for all $v \in H^1_{b,\sigma}(\Omega)$, which implies that $(B^*\psi - \alpha \operatorname{Tr} u)$ is the (a_{ij})-normal derivative of u.

"$\Leftarrow$" Let $u \in H^1_{b,\sigma}(\Omega)$ and $\psi \in L_2(\widetilde{\Gamma})$ be such that $x = (u, B(\operatorname{Tr} u))$, $\mathcal{A}u \in L_2(\Omega)$ weakly on Ω, $(B^*\psi - \alpha \operatorname{Tr} u)$ is the (a_{ij})-normal derivative of u and $y = (\mathcal{A}u, \psi)$. Then $x = j(u)$. Since $(B^*\psi - \alpha \operatorname{Tr} u)$ is the (a_{ij})-normal derivative of u one deduces that, for all $v \in H^1_{b,\sigma}(\Omega)$:

$$\overline{b_N}(u,v) - \int_{\widetilde{\Gamma}} (\mathcal{A}u)\overline{v} \, d\sigma = \int_{\widetilde{\Gamma}} (B^*\psi - \alpha \operatorname{Tr} u) \overline{\operatorname{Tr} v} \, d\sigma.$$

So

$$a(u,v) = \overline{b_N}(u,v) + \int_{\widetilde{\Gamma}} \operatorname{Tr} u \overline{\operatorname{Tr} v} \alpha \, d\sigma = \int_{\Omega} (\mathcal{A}u)\overline{v} + \int_{\widetilde{\Gamma}} \psi \overline{B(\operatorname{Tr} v)} \, d\sigma = (y, j(v))_H$$

for all $v \in H^1_{b,\sigma}(\Omega)$. Therefore $x \in D(A)$ and $Ax = y$. ∎

Suppose that B has dense range. Then H is isomorphic with $L_2(\Omega \sqcup \widetilde{\Gamma})$ in a natural way, where $\sqcup$ denotes the disjoint union of the measure spaces. We use this isomorphism to identify H with $L_2(\Omega \sqcup \widetilde{\Gamma})$. It is easy to verify as in the proof of Corollary 3.13(i) that S leaves $L_2(\Omega, \mathbb{R}) \oplus L_2(\widetilde{\Gamma}, \mathbb{R})$ invariant if the form b_N is real, α is real valued and B maps $L_2(\widetilde{\Gamma}; \mathbb{R})$ into itself. We next characterize positivity of S.

PROPOSITION 4.18. *Suppose the form b_N is real, α is real valued and B maps $L_2(\widetilde{\Gamma};\mathbb{R})$ densely into itself.*

(i) *The map B is a lattice homomorphism if and only if the semigroup S is positive.*

(ii) *If $\sigma(\widetilde{\Gamma}) < \infty$, the map B is a lattice homomorphism, $\alpha \geqslant 0$ and there exists a $c \geqslant 1$ such that $\frac{1}{c}\mathbf{1} \leqslant B\mathbf{1} \leqslant c\mathbf{1}$, then S extends continuously to a bounded semigroup on $L_\infty(\Omega \sqcup \widetilde{\Gamma})$.*

Proof. (i) Let $C = \{(u,\varphi) \in H : u \geqslant 0 \text{ and } \varphi \geqslant 0\}$. Then C is closed and convex in H. Define $P\colon H \to C$ by $P(u,\varphi) = ((\operatorname{Re} u)^+, (\operatorname{Re}\varphi)^+)$. Then P is the orthogonal projection onto C.

"$\Rightarrow$" Let $u \in H^1_{b,\sigma}(\Omega)$. Then $(\operatorname{Re} u)^+ \in H^1_{b,\sigma}(\Omega)$ and

$$j((\operatorname{Re} u)^+) = ((\operatorname{Re} u)^+, B(\operatorname{Tr}((\operatorname{Re} u)^+))) = ((\operatorname{Re} u)^+, (\operatorname{Re} B(\operatorname{Tr} u))^+) = Pj(u)$$

since B is a lattice homomorphism. Moreover,

$$\operatorname{Re} a((\operatorname{Re} u)^+, u - (\operatorname{Re} u)^+) = a((\operatorname{Re} u)^+, -(\operatorname{Re} u)^-) = 0.$$

So C is invariant under S by Proposition 2.9.

"$\Leftarrow$" If S is positive, then C is invariant under S. Let $u \in H^1_{b,\sigma}(\Omega)$. It follows from Proposition 2.9 that there exists a $w \in H^1_{b,\sigma}(\Omega)$ such that $Pj(u) = j(w)$. Then $((\operatorname{Re} u)^+, (\operatorname{Re} B(\operatorname{Tr} u))^+) = Pj(u) = j(w) = (w, B(\operatorname{Tr} w))$. Therefore $w = (\operatorname{Re} u)^+$ and

$$(\operatorname{Re} B(\operatorname{Tr} u))^+ = B(\operatorname{Tr} w) = B(\operatorname{Tr}((\operatorname{Re} u)^+)) = B((\operatorname{Re}\operatorname{Tr} u)^+).$$

This is for all $u \in H^1_{b,\sigma}(\Omega)$. Since $\operatorname{Tr} H^1_{b,\sigma}(\Omega)$ is dense in $L_2(\widetilde{\Gamma})$ one deduces that $(B\varphi)^+ = B(\varphi^+)$ for all $\varphi \in L_2(\widetilde{\Gamma},\mathbb{R})$. So B is a lattice homomorphism.

(ii) Let $C = \{(u,\varphi) \in H : u \leqslant \mathbf{1} \text{ and } \varphi \leqslant B\mathbf{1}\}$. Then C is closed and convex. Define $P\colon H \to C$ by $P(u,\varphi) = ((\operatorname{Re} u) \wedge \mathbf{1}, (\operatorname{Re}\varphi) \wedge B\mathbf{1})$. Then P is the orthogonal projection of H onto C. Let $u \in H^1_{b,\sigma}(\Omega)$. Define $w = (\operatorname{Re} u) \wedge \mathbf{1}$. Then $w \in H^1_{b,\sigma}(\Omega)$ and

$$Pj(u) = ((\operatorname{Re} u) \wedge \mathbf{1}, (\operatorname{Re} B(\operatorname{Tr} u)) \wedge B\mathbf{1}) = ((\operatorname{Re} u) \wedge \mathbf{1}, B(\operatorname{Tr}((\operatorname{Re} u) \wedge \mathbf{1}))) = j(w).$$

Moreover,

$$\begin{aligned}\operatorname{Re} a(w, u-w) &= \operatorname{Re} a((\operatorname{Re} u) \wedge \mathbf{1}, \mathrm{i}\operatorname{Im} u + (\operatorname{Re} u - \mathbf{1})^+) = a((\operatorname{Re} u) \wedge \mathbf{1}, (\operatorname{Re} u - \mathbf{1})^+) \\ &= \int_{\widetilde{\Gamma}} \alpha \operatorname{Tr}((\operatorname{Re} u) \wedge \mathbf{1}) \operatorname{Tr}((\operatorname{Re} u - \mathbf{1})^+) = \int_{\widetilde{\Gamma}} \alpha \operatorname{Tr}((\operatorname{Re} u - \mathbf{1})^+) \geqslant 0\end{aligned}$$

So by Proposition 2.9 the set C is invariant under S.

Finally, let $(u,\varphi) \in H$ and suppose that $u \leqslant \mathbf{1}$ and $\varphi \leqslant \mathbf{1}$. Then $\frac{1}{c}\varphi \leqslant B\mathbf{1}$ and $\frac{1}{c}(u,\varphi) \in C$. Let $t > 0$ and write $(v,\psi) = S_t(u,\varphi)$. Then $\frac{1}{c}(v,\psi) \in C$. Hence $v \leqslant c\mathbf{1}$ and $\psi \leqslant cB\mathbf{1} \leqslant c^2\mathbf{1}$. So S extends to a continuous semigroup on L_∞ and $\|S_t\|_{\infty\to\infty} \leqslant c^2$ for all $t > 0$. ∎

Using the operator A one can define another semigroup generator which looks different. If $u \in D(\overline{b_N})$, then we say that $\mathcal{A}u \in H^1_{b,\sigma}(\Omega)$ *weakly on* Ω if there exists an $f \in H^1_{b,\sigma}(\Omega)$ such that $\mathcal{A}u = f$ weakly on Ω.

The Laplacian with Wentzell boundary conditions can be realized in the Sobolev space H^1. This has been carried out in Theorem 2.1 of [15]. We generalize this approach for the elliptic operator A.

PROPOSITION 4.19. *Define the operator A_1 on the Hilbert space $H^1_{b,\sigma}(\Omega)$ by taking as domain $D(A_1)$ the set of all $u \in H^1_{b,\sigma}(\Omega)$ such that $\mathcal{A}u \in H^1_{b,\sigma}(\Omega)$ weakly on Ω and $(B^*B(\mathrm{Tr}\,\mathcal{A}u) - \alpha \mathrm{Tr}\,u)$ is the (a_{ij})-normal derivative of u and letting $A_1 u = \mathcal{A}u$ for all $u \in D(A_1)$. Then $-A_1$ generates a holomorphic semigroup on $H^1_{b,\sigma}(\Omega)$.*

Proof. Let a_c be the classical form associated with (a, j) (see Theorem 2.5). Then A is associated with the closed sectorial form a_c. Define the operator A_0 in H by $D(A_0) = \{w \in D(A) : Aw \in D(a_c)\}$ and $A_0 w = Aw$ for all $w \in D(A_0)$. Then $-A_0$ generates a holomorphic semigroup in the Hilbert space $(D(a_c), \|\cdot\|_{a_c})$. The map $j \colon H^1_{b,\sigma}(\Omega) \to D(a_c)$ is a isomorphism of normed spaces. Hence the operator $-j^{-1}A_0 j$ generates a holomorphic semigroup on $H^1_{b,\sigma}(\Omega)$. Therefore it suffices to show that $A_1 = j^{-1}A_0 j$.

Let $u \in D(j^{-1}A_0 j)$. Then $j(u) \in D(A)$, $Aj(u) \in j(H^1_{b,\sigma}(\Omega))$ and $A_0 j(u) = Aj(u)$. It follows from Proposition 4.17 that $\mathcal{A}u \in L_2(\Omega)$ weakly on Ω and there exists a $\psi \in L_2(\widetilde{\Gamma})$ such that $(B^*\psi - \alpha \mathrm{Tr}\,u)$ is the (a_{ij})-normal derivative of u and $Aj(u) = (\mathcal{A}u, \psi)$. Since $Aj(u) \in j(H^1_{b,\sigma}(\Omega))$ one deduces that $\mathcal{A}u \in H^1_{b,\sigma}(\Omega)$ and $j(\mathcal{A}u) = Aj(u) = (\mathcal{A}u, \psi)$. In particular, $\psi = B(\mathrm{Tr}\,\mathcal{A}u)$. Therefore $(B^*B(\mathrm{Tr}\,\mathcal{A}u) - \alpha \mathrm{Tr}\,u)$ is the (a_{ij})-normal derivative of u and $u \in D(A_1)$. Then $A_1 u = \mathcal{A}u = j^{-1}A_0 j(u)$. Conversely, suppose that $u \in D(A_1)$. Then $j(u) \in D(a_c)$ and it follows from Proposition 4.17 that $j(u) \in D(A)$ with $Aj(u) = (\mathcal{A}u, B\mathrm{Tr}\,\mathcal{A}u) = j(\mathcal{A}u)$. So $j(u) \in D(A_0)$ and $u \in D(j^{-1}A_0 j)$. ∎

In case of the Laplacian, i.e. if $a_{ij} = \delta_{ij}$, the set Ω is bounded and Lipschitz, and if B is the multiplication operator with a bounded measurable function β, then $D(A_1)$ is the set of all $u \in H^1(\Omega)$ such that $\Delta u \in H^1(\Omega)$ weakly on Ω and the normal derivative satisfies

$$\frac{\partial u}{\partial \nu} + |\beta|^2 \mathrm{Tr}\,(\Delta u) + \alpha \mathrm{Tr}\,u = 0.$$

Moreover, $A_1 u = -\Delta u$, cf. Remark 2.9 of [6].

Acknowledgements. The authors thank Hendrik Vogt for critical comments on earlier versions of Corollary 2.2 and Lemma 4.1. In addition the authors thank the referee for his careful reading of the manuscript and his many improving comments. The first named author is most grateful for the hospitality during a research stay at the University of Auckland. Both authors highly appreciated the stimulating atmosphere and hospitality at the University of Torun in the framework of the Marie Curie "Transfer of Knowledge" programme TODEQ.

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Received February 17, 2009.