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ON EXTENSIONS OF STABLY FINITE C^*-ALGEBRAS

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ABSTRACT. In this paper, we prove that for any C^*-algebra A with an approximate unit of projections, there is a smallest ideal I of A, in which quotient A/I is stably finite. We give a sufficient condition and a necessary condition on which I is the smallest ideal in this case for A by K-theory.

KEYWORDS: *Extension, stably finite C^*-algebra, index map.*

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1. INTRODUCTION AND MAIN RESULTS

Extension theory is important in many contexts, since it describes how more complicated C^*-algebras can be constructed out of simpler "building blocks". There are many important applications of extension theory (see [2]). A C^*-algebra A is called *finite* if it admits an approximate unit of projections and all projections in A are finite. If $A \otimes \mathcal{K}$ is finite, then A is called *stably finite*. About extensions of stably finite C^*-algebras, J.S. Spielberg gave an important result:

THEOREM 1.1 ([6]). *Let A be a C^*-algebra, let I be an ideal in A, and suppose that I and A/I are stably finite. Then A is stably finite if and only if*

$$\partial(K_1(A/I)) \cap K_0(I)_+ = 0.$$

In this short paper, we will prove that for any C^*-algebra A with an approximate unit of projections, there is a smallest ideal I in which quotient A/I is stably finite. Thus if Q is a stably finite quotient of C^*-algebra A, then there is a canonical surjective $*$-homomorphism from Q to A/I. Give a sufficient condition and a necessary condition on which I is a smallest ideal in this case for A by K-theory. When the ideal I is simple and has real rank zero, the former result is equivalent to Theorem 1.1.

THEOREM 1.2. *Let A be a C^*-algebra with an approximate unit of projections, $\{I_\lambda\}_{\lambda\in\Lambda}$ is a set of ideals of A.*

(i) *If quotient* A/I_λ *is a finite* C^**-algebra for each* $\lambda \in \Lambda$*, then* $A/\bigcap_{\lambda\in\Lambda} I_\lambda$ *is a finite* C^**-algebra;*

(ii) *If quotient* A/I_λ *is a stably finite* C^**-algebra for each* $\lambda \in \Lambda$*, then* $A/\bigcap_{\lambda\in\Lambda} I_\lambda$ *is a stably finite* C^**-algebra.*

Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be the set of all ideals I_λ of A with A/I_λ is stably finite. Throughout this paper, we denote the ideal $\bigcap_{\lambda\in\Lambda} I_\lambda$ of A by $I(A)$.

THEOREM 1.3. *Let* A *be a* C^**-algebra with an approximate unit of projections, and let* I *be an ideal of* A*, which has real rank zero. If* A/I *is stably finite and for any* $x \in K_0(I)_+$*, there is an* y *in* $\partial(K_1(A/I)) \cap K_0(I)_+$ *such that* $x \leqslant y$*, then* $I = I(A)$*.*

COROLLARY 1.4. *Let* A *be a* C^**-algebra with real rank zero. If* $K_0(A)_+ = K_0(A)$*, then* $I(A) = A$*.*

THEOREM 1.5. *Let* A *be a* C^**-algebra with an approximate unit of projections. Let* J *be the ideal of* A *generated by*

$$\{q : \text{there is a hyponormal partial isometry } v \in A \text{ such that } vv^* - v^*v = q\}.$$

Then for any $x = [p]_0$ *in* $K_0(I(A))_+$*, where* $p \in J$*, there is an* y *in* $\partial(K_1(A/I(A))) \cap K_0(I(A))_+$ *such that* $x \leqslant y$*.*

COROLLARY 1.6. *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be an extension of C^**-algebras. Suppose that* A *has an approximate unit of projections and that* I *and* B *are two stably finite* C^**-algebras. If* I *is a non-zero simple* C^**-algebra with real rank zero, then the following conditions are equivalent:*

(i) A *is not stably finite;*

(ii) $I = I(A)$*;*

(iii) *for any* $x \in K_0(I)_+$*, there is an* y *in* $\partial(K_1(A/I)) \cap K_0(I)_+$ *such that* $x \leqslant y$*.*

COROLLARY 1.7. *Let* A *be a simple* C^**-algebra with real rank zero. Then* $I(A) = A$ *if and only if* $K_0(A)_+ = K_0(A)$*. Furthermore, either* $(K_0(A), K_0(A)_+)$ *is an order group or* $K_0(A)_+ = K_0(A)$*.*

2. PROOFS

LEMMA 2.1 ([3], 1.11.42). *Let* A *be a* C^**-algebra with an approximate unit of projections. Then every ideal* I *of* $M_n(A)$ *has the form* $M_n(J)$ *for some ideal* J *of* A*. So* $M_n(A)/I \cong M_n(A/J)$*.*

Proof of Theorem 1.2. (i) Let $\{p_i\}$ be an approximate unit of projections in A. π is the quotient map form A to $A/\bigcap_{\lambda\in\Lambda} I_\lambda$. Then $\pi(p_i)$ becomes an approximate unit of projections in $A/\bigcap_{\lambda\in\Lambda} I_\lambda$. For any i, we assume that $v^*v = \pi(p_i)$. There is $w \in p_iAp_i$ such that $\pi(w) = v$. Since $\pi(w^*w) = \pi(p_i)$, $w^*w \in p_i + \bigcap_{\lambda\in\Lambda} I_\lambda$. By the hypothesis of the theorem, $ww^* \in p_i + I_\lambda$ for all λ, so $ww^* \in p_i + \bigcap_{\lambda\in\Lambda} I_\lambda$, $vv^* = \pi(ww^*) = \pi(p_i)$. Therefore $A/\bigcap_{\lambda\in\Lambda} I_\lambda$ is a finite C^*-algebra.

(ii) By Lemma 2.1 and (i), (ii) is obvious. ∎

LEMMA 2.2. *Let A be a C^*-algebra with an approximate unit of projections. Then:*
(i) *if B is an ideal of A, with an approximate unit of projections, then $I(B) \subset I(A)$;*
(ii) *$I(\widetilde{A}) = I(A)$;*
(iii) *$I(M_n(A)) = M_n(I(A))$, $I(A\otimes\mathcal{K}) = I(A)\otimes\mathcal{K}$.*

Proof. Note that every ideal I of $M_n(A)$ has the form $M_n(J)$ for some ideal J of A. (iii) is trivial.

(i) Let $\{I_\lambda\}$ be the set of all ideal of A with A/I_λ is stably finite. Then $\ker \pi_\lambda \circ i = B\cap I_\lambda$. $I(B) \subset \bigcap_{\lambda\in\Lambda} I_\lambda = I(A)$.

(ii) By (i), $I(A) \subset I(\widetilde{A})$ and conversely, $I(\widetilde{A}) \subset I(A)$ is trivial. ∎

Let A be a C^*-algebra and let $M_n(A)$ denote the $n\times n$ matrices whose entries are elements of A. Let $M_\infty(A)$ denote the algebraic limit of the direct system $(M_n(A),\phi_n)$, where $\phi_n : M_n(A)\to M_{n+1}(A)$ is given by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $M_\infty(A)_+$ (respectively $M_n(A)_+$) denote the positive elements in $M_\infty(A)$ (respectively $M_n(A)$).

Given $a, b \in M_\infty(A)_+$, we say that a is Cuntz subequivalent to b, written $a \precsim b$, if there is a sequence $\{x_n\}_{n=1}^\infty$ of elements of $M_\infty(A)$ such that

$$\lim_{n\to\infty} \|x_n b x_n^* - a\| = 0.$$

We say that a and b are Cuntz equivalent (written $a \sim b$) if $a \precsim b$ and $b \precsim a$. It is easy to see that if p and q are projections, $p \precsim q$ is equivalent to the existence of a partial isometry $u \in A$ with $u^*u = p$ and $uu^* \leqslant q$.

PROPOSITION 2.3 ([4], [5]). *Let A be a C^*-algebra, and $a, b \in A_+$. Then*
(a) *$(a-\varepsilon)_+ \precsim a$ for every $\varepsilon > 0$.*
(b) *The following are equivalent:*
(i) *$a \precsim b$;*
(ii) *for all $\varepsilon > 0$, $(a-\varepsilon)_+ \precsim b$;*
(iii) *for all $\varepsilon > 0$, there exists $\delta > 0$ such that $(a-\varepsilon)_+ \precsim (b-\delta)_+$;*
(iv) *there are $x_n, y_n \in \widetilde{A}$ with $\lim_{n\to\infty} \|x_n b y_n - a\| = 0$, where $\widetilde{A}$ is the unitization of A.*
(c) *If $\varepsilon > 0$ and $\|a-b\| < \varepsilon$, then $(a-\varepsilon)_+ \precsim b$.*

LEMMA 2.4. *Let A be a C^*-algebra, $a, b \in A_+$, then $a+b \precsim a \oplus b$. If A has real rank zero and $a \perp b$, then $a+b \sim a \oplus b$.*

Proof. Since

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix},$$

$a+b \precsim a \oplus b$. Let A have real rank zero, and let $a \perp b$. Sine A has real rank zero, for any $\varepsilon > 0$, there is a projection $p \in \overline{aAa}$ such that $\|a - pap\| < \varepsilon$, and there is a projection $q \in \overline{bAb}$ such that $\|b - qaq\| < \varepsilon$. Note that $p \perp q$. Since

$$\begin{pmatrix} p & 0 \\ q & 0 \end{pmatrix}\begin{pmatrix} pap+qbq & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} pap & 0 \\ 0 & qbq \end{pmatrix}.$$

Hence

$$(a \oplus b - \varepsilon)_+ \precsim pap \oplus qbq \precsim pap + qbq \precsim a + b.$$

By (b) of Proposition 2.3, $a+b \sim a \oplus b$. ∎

The following lemma is a generalization of Lemma 3.3.6 in [3].

LEMMA 2.5. *If $B \subset A_+$ is a subset of a C^*-algebra A, and p is a projection in the ideal generated by B, then there are $x_1, \dots, x_k$ in A, and $a_1, \dots, a_k$ in B such that*

$$p = \sum_{i=1,k} x_i a_i x_i^*.$$

Proof. There are $y_1, \dots, y_k$ and $z_1, \dots, z_k$ in A such that

$$\Big\| \sum_{i=1}^{k} y_i a_i z_i - p \Big\| < \frac{1}{2}.$$

Let $b = p \sum\limits_{i=1}^{k} y_i a_i z_i p$. Then b is invertible in pAp. So $p = \sum\limits_{i=1}^{k} b^{-1} y_i a_i z_i p$, where the inverse is taken in pAp. To save notation, we obtain $g_1, \dots, g_k$ and $f_1, \dots, f_k$ such that $p = \sum\limits_{i=1}^{k} g_i a_i f_i$. Set

$$g = \begin{pmatrix} g_1 & g_2 & \cdots & g_k \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad f = \begin{pmatrix} f_1 & 0 & \cdots & 0 \\ f_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ f_k & 0 & \cdots & 0 \end{pmatrix},$$

$$e = \operatorname{diag}(p, 0, \dots, 0), \quad \text{and} \quad a = \operatorname{diag}(a_1, \dots, a_k).$$

Then $e = gaf$ in $M_k(A)$. So $e = egaff^*ag^*e$. We have $e \leqslant \|ff^*\| egaag^*e$. Let $c = egaag^*e$. Then c has an inverse c^{-1} in $eM_k(A)e = pAp$. Note that

$$gaag^* = \sum_{i=1}^{k} (g_i a^{1/2}) a (g_i a^{1/2})^*.$$

Therefore

$$p = \sum_{i=1}^{k} c^{-1/2} e(g_i a^{1/2}) a (g_i a^{1/2}) e c^{-1/2}.$$

Set $x_i = c^{-1/2} e(g_i a^{1/2})$. The lemma then follows. ∎

Proof of Theorem 1.3. For any projection p in I, there is a projection q in $M_n(I)$ for some positive integer n, such that $[p]_0 \leqslant [q]_0$ and $[q]_0$ belongs to $\partial(K_1(A/I))$. If $q = 0$, then $p \in I(A)$. We may assume that $q \neq 0$. Let u be a unitary in $M_m(\widetilde{A}/I)$ with $\partial([u]_1) = [q]_0$. By K-theory, there is a unitary w in $M_{2m}(\widetilde{A})$ such that $\pi(w) = u \oplus u^*$, where π is the quotient map from A to A/I. Then

$$\partial([u]_1) = [w(1_m \oplus 0_m)w^*]_0 - [1_m \oplus 0_m]_0.$$

Therefore

$$[w(1_m \oplus 0_m)w^* \oplus 0_n]_0 = [1_m \oplus 0_m \oplus q]_0.$$

So there are integer r, s and a unitary v in $M_{2m+n+r+s}(\widetilde{I})$ such that

$$v(w(1_m \oplus 0_m)w^* \oplus 0_n \oplus 1_r \oplus 0_s)v^* = 1_m \oplus 0_m \oplus q \oplus 1_r \oplus 0_s.$$

Let $R = v(w(1_m \oplus 0_m) \oplus 0_n \oplus 1_r \oplus 0_s)$ in $M_{2m+n+r+s}(\widetilde{A})$, then R is cohyponormal partial isometry and $RR^* - R^*R = 0_{2m} \oplus q \oplus 0_{r+s}$ belongs to $I(M_{2m+n+r+s}(\widetilde{A}))$. By Lemma 2.2, q belongs to $I(M_n(A))$. Since $[p]_0 \leqslant [q]_0$, there is a projection p' in $M_l(I)$ such that $[p]_0 + [p']_0 = [q]_0$, without loss of generality, we may assume that $p \oplus p'$ and q belong to $M_k(I)$. There are integers i, j and a unitary x in $M_{k+i+j}(\widetilde{I})$, such that

$$p \oplus p' \oplus 1_i \oplus 0_j = x(q \oplus 1_i \oplus 0_j)x^*.$$

Let π' be the quotient map from $\widetilde{I}$ to $\widetilde{I}/I(A)$. Then

$$\pi'(q \oplus 1_i \oplus 0_j) = 0_k \oplus 1_i \oplus 0_j,$$

and

$$\pi'(x(q \oplus 1_i \oplus 0_j)x^*) = \pi'(p \oplus p' \oplus 1_i \oplus 0_j) = \pi'(p \oplus p') \oplus 1_i \oplus 0_j.$$

Since $\widetilde{I}/I(A)$ is stably finite, $\pi'(p \oplus p') = 0_k$. So $p \in I(M_k(A))$. By Lemma 2.2, $p \in I(A)$. ∎

Proof of Theorem 1.5. By Lemma 2.2(iii), without loss of generality, we may assume that I, A and A/I are stable. Note that J is the ideal of A generated by

$$C = \{q : \text{there is a hyponormal partial isometry } v \in A \text{ such that } vv^* - v^*v = q\}.$$

$J \subset I(A)$. For any $p \in J$, by Lemma 2.5, there are projections $q_1, \ldots, q_k$ in C and there are $x_1, \ldots, x_k$ in A such that

$$p = \sum_{i=1}^{k} x_i q_i x_i^*.$$

By Lemma 2.4,

$$p \precsim \bigoplus_{i=1}^{k} x_i q_i x_i^* \precsim \bigoplus_{i=1}^{k} q_i.$$

So $[p]_0 \leqslant \sum_{i=1}^{k} [q_i]_0$. Note from the construction of C, that $\sum_{i=1}^{k} [q_i]_0$ belongs to $\partial(K_1(A/I)) \cap K_0(I)_+$. ∎

Finally, we end with the following question.

QUESTION 2.6. *Let A be a C^*-algebra which has real rank zero. For any $x \in K_0(I(A))_+$, is there an y in $\partial(K_1(A/I(A))) \cap K_0(I(A))_+$ such that $x \leqslant y$?*

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