J. OPERATOR THEORY
67:2(2012), 335–347
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SIMPLICITY OF FINITELY ALIGNED k-GRAPH C^*-ALGEBRAS

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Communicated by Kenneth R. Davidson

ABSTRACT. It is shown that *no local periodicity* is equivalent to the *aperiodicity condition* for arbitrary finitely aligned k-graphs. This allows us to conclude that $C^*(\Lambda)$ is simple if and only if Λ is cofinal and has no local periodicity.

KEYWORDS: *k-graph, C^*-algebra, graph algebra.*

MSC (2000): 46L05.

INTRODUCTION

Kumjian and Pask introduced k-graph C^*-algebras in [3] as generalizations of the higher-rank Cuntz–Krieger algebras studied by Robertson and Steger in [8], [9], and [10]. There are two immediate difficulties that arise in the theory of k-graphs. The first difficulty is presented by sources. For directed graphs, a source is simply a vertex that receives no edge. For k-graphs, a source is a vertex that fails to receive an edge of some degree. The notion of local convexity was introduced in [4] in order to associate a C^*-algebra to certain well-behaved k-graphs with sources. The second major obstruction in studying k-graphs is presented by infinite receivers. *Finitely aligned* k-graphs were introduced in [5] in order to associate a C^*-algebra to row-infinite k-graphs graphs (possibly containing sources) that satisfy a mild condition.

In [3], Kumjian and Pask introduce an aperiodicity hypothesis for row-finite k-graphs without sources and show that if Λ satisfies this condition, then $C^*(\Lambda)$ is simple if and only if Λ is cofinal. The aperiodicity condition of Kumjian and Pask also serves as a critical hypothesis for a number of important structural results concerning k-graph C^*-algebras. A number of different aperiodicity conditions have appeared in the literature for the variety of classes of k-graphs in [4], [5], [11], [2], [6], and [7].

For row-finite k-graph without sources, Robertson and Sims introduce the notion of *no local periodicity* [6]. This formulation of aperiodicity is formally weaker than the condition introduced by Kumjian and Pask. Nonetheless, Robertson and Sims show that no local periodicity is equivalent to a number of other aperiodicity hypotheses for row-finite k-graphs without sources. The advantage of no local periodicity is that its negation is strong enough to prove that $C^*(\Lambda)$ is simple if and only if Λ is cofinal and has no local periodicity. Robertson and Sims furthermore use this condition to classify k-graph C^*-algebras in which every ideal is gauge-invariant. This work is similar to the result from directed graph algebras stating that $C^*(E)$ is simple if and only if E is cofinal and every cycle has an exit.

In [1], Farthing constructs a sourceless k-graph $\overline{\Lambda}$ from a k-graph Λ in such a way that $C^*(\overline{\Lambda})$ is Morita equivalent to $C^*(\Lambda)$ when Λ is row-finite. Robertson and Sims make use of this result in [6] to generalize their previous work to the locally convex row-finite k-graphs. Robertson and Sims' simplicity result is limited to the locally convex case because of an unexpected difficulty with projecting paths from $\overline{\Lambda}^{\infty}$ onto $\Lambda^{\leqslant\infty}$.

For the finitely aligned case, a number of aperiodicity hypotheses have appeared, often defined on different boundary path spaces. In [5], Raeburn, Sims, and Yeend use a similar condition to *Condition B* from [4] to prove their version of the Cuntz–Krieger uniqueness theorem. Farthing, Muhly, and Yeend introduce a version of Kumjian and Pasks' aperiodicity condition in [2] to prove a version of the Cuntz–Krieger uniqueness theorem using groupoid methods. The condition in [2] is much different than that in [4], partly because it operates on the closure of the boundary path space employed by Raeburn, Sims, and Yeend.

In this paper, the work of Robertson and Sims is generalized to the finitely aligned case. We show that the condition in [2] is equivalent to an appropriate formulation of no local periodicity. In Section 2, we briefly introduce the standard definitions and results from the literature. In Section 3, we introduce a condition called *strong no local periodicity* for finitely aligned k-graphs without sources and show that the condition is equivalent to no local periodicity in this situation. This allows us to exactly follow the proof of Lemma 2.2 from [6] to prove that no local periodicity implies the aperiodicity condition in [2]. We then show how to reduce the arbitrary finitely aligned case to that of no sources by introducing a sourceless $(k-a)$-graph that carries information about aperiodic paths in the original k-graph. In Section 4, we use these results to construct the usual simplicity argument as in [6] and [7].

1. PRELIMINARIES

Let $k \in \mathbb{N}$ and regard $\mathbb{N}^k$ as a monoid with identity 0. Let e_i denote the i^{th} generator of $\mathbb{N}^k$. For $m, n \in \mathbb{N}^k$ write $m \leqslant n$ to mean $m_i \leqslant n_i$ for $i = 1, 2, \ldots, k$. For $m, n \in \mathbb{N}$ let $m \vee n$ and $m \wedge n$ denote the coordinatewise maximum and minimum of m and n, respectively.

DEFINITION 1.1 ([3], Definition 1). A k-graph consists of a countable small category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$ which satisfies the *unique factorization property*: For every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there exist unique $\nu, \mu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$, and $d(\nu) = n$.

Let $\Lambda^n = d^{-1}(n)$ and let r and s denote the range and source maps of Λ respectively. Obj(Λ) is naturally identified with Λ^0 via the unique factorization property and thus $r, s : \Lambda \to \Lambda^0$. For $v \in \Lambda^0$ and $E \subseteq \Lambda$, put $vE = \{\mu \in E : r(\mu) = v\}$ and $Ev = \{\mu \in E : s(\mu) = v\}$.

For $n \in \mathbb{N}^k$, define as in Definition 3.1 from [4]

$$\Lambda^{\leqslant n} = \{\lambda \in \Lambda : d(\lambda) \leqslant n \text{ and } d(\lambda) + e_i \leqslant n \Rightarrow s(\lambda)\Lambda^{e_i} = \emptyset\}.$$

Note that $v\Lambda^{\leqslant n} \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. Furthermore, $\Lambda^{\leqslant n} = \Lambda^n$ if Λ has no sources.

Given $\lambda, \mu \in \Lambda$, a minimal common extension of λ and μ is a pair $(\alpha, \beta) \in \Lambda \times \Lambda$ such that $\lambda\alpha = \mu\beta$ and $d(\lambda\alpha) = d(\lambda) \vee d(\mu)$. The set of minimal common extensions of λ and μ is denoted by $\Lambda^{\min}(\lambda, \mu)$. Define $\mathrm{MCE}(\lambda, \mu) = \{\lambda\alpha : (\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)\}$. Recall from Definition 2.2 of [5] the following definition.

DEFINITION 1.2. A k-graph Λ is *finitely aligned* if $\Lambda^{\min}(\lambda, \mu)$ is finite for all $\lambda, \mu \in \Lambda$.

DEFINITION 1.3 ([5], Definition 2.4). Let Λ be a k-graph, $v \in \Lambda^0$, and $E \subseteq v\Lambda$. E is *exhaustive* if for every $\mu \in v\Lambda$ there is $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. Define

$$\mathrm{FE}(\Lambda) = \{F \subseteq v\Lambda \setminus \{v\} : \ v \in \Lambda^0, \ F \text{ is finite exhaustive}\}.$$

REMARK 1.4. If Λ has no sources, then Λ^n is exhaustive for all $n \in \mathbb{N}^k$. More generally, Λ is locally convex if and only if $\Lambda^{\leqslant n}$ is exhaustive for all $n \in \mathbb{N}^k$.

DEFINITION 1.5 ([2], Definition 3.10). For $\eta \in \Lambda$ and $F \subseteq r(\eta)\Lambda$,

$$\mathrm{Ext}(\eta; F) := \bigcup_{\lambda \in F} \{\alpha \in \Lambda : (\alpha, \beta) \in \Lambda^{\min}(\eta, \lambda) \text{ for some } \beta \in \Lambda\}.$$

If $F \in v\mathrm{FE}(\Lambda)$ and $\eta \in v\Lambda$, then $\mathrm{Ext}(\eta; F) \in s(\eta)\mathrm{FE}(\Lambda)$ as in Proposition 3.11 from [2].

DEFINITION 1.6 ([5], Definition 2.6). Let (Λ, d) be a finitely aligned k-graph. A *Cuntz–Krieger Λ-family* is a collection of partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra B satisfying:

(i) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;
(ii) $s_\lambda s_\mu = s_{\lambda\mu}$ when $s(\lambda) = r(\mu)$;
(iii) $s_\lambda^* s_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} s_\alpha s_\beta^*$ for all $\lambda, \mu \in \Lambda$;
(iv) $\prod_{\lambda \in E} (s_v - s_\lambda s_\lambda^*) = 0$ for all $E \in v\mathrm{FE}(\Lambda)$.

Denote by $C^*(\Lambda)$ the universal C^*-algebra containing a Cuntz–Krieger Λ family.

1.1. BOUNDARY PATHS. Given a finitely aligned k-graph, let X_Λ be the collection of graph morphisms $x : \Omega_{k,m} \to \Lambda$. For such x, define $d(x) = m$. As in Definition 2.8 from [5], let $\Lambda^{\leqslant\infty}$ be the collection of paths $x \in X_\Lambda$ for which there is $n_x \in \mathbb{N}^k$ such that $n_x \leqslant d(x)$ and

$$n \in \mathbb{N}^k, n_x \leqslant n \leqslant m \text{ and } n_i = m_i \quad \text{imply that } x(n)\Lambda^{e_i} = \emptyset.$$

Note that when Λ is locally convex, we may take $n_x = 0$.

Let $\partial\Lambda$ be the collection of paths $x \in X_\Lambda$ such that for all $n \leqslant d(x)$ and for all finite exhaustive $E \subseteq x(n)\Lambda$, there is $\lambda \in E$ such that $x(n, n + d(\lambda)) = \lambda$ ([2], Definition 5.10). We have $\Lambda^{\leqslant\infty} \subseteq \partial\Lambda$, but $\Lambda^{\leqslant\infty} \neq \partial\Lambda$ in general. If Λ is row-finite and without sources, then $\Lambda^{\leqslant\infty} = \Lambda^\infty$.

If $x \in \partial\Lambda$ and $n \leqslant d(x)$, define $\sigma^n x$ by $\sigma^n x(0, p) = x(n, n + p)$ for all $p \leqslant d(x) - n$. Then $\sigma^n x \in \partial\Lambda$ by Lemma 5.13 in [2]. If $\lambda \in \Lambda x(0)$, there is a unique path $\lambda x \in \partial\Lambda$ such that $\lambda x(0, d(\lambda)) = \lambda$ and $\lambda x(0, p) = \lambda x(0, p - d(\lambda))$ for $p \in \mathbb{N}^k$ satisfying $p + d(\lambda) \leqslant d(x)$ ([2], Lemma 5.13).

For each $\lambda \in \partial\Lambda$, define $S_\lambda \in \mathcal{B}(\ell^2(\partial\Lambda))$ by

$$S_\lambda e_x = \begin{cases} e_{\lambda x} & \text{if } r(x) = \lambda, \\ 0 & \text{else.} \end{cases}$$

PROPOSITION 1.7 ([11], Lemma 4.6). *The operators* $\{S_\lambda : \lambda \in \Lambda\}$ *form a Cuntz–Krieger* Λ*-family on* $\ell^2(\partial\Lambda)$ *such that* $S_v \neq 0$. *This is called the boundary-path representation.*

2. APERIODICITY CONDITIONS

A number of aperiodicity hypotheses have appeared in the literature for the variety of k-graphs. We collect some of them here as they have appeared for finitely aligned k-graphs.

DEFINITION 2.1. Let Λ be a finitely aligned k-graph.

(i) Λ satisfies *Condition* (A) if for all $v \in \Lambda^0$ there is $x \in v\partial\Lambda$ such that $n \vee m \leqslant d(x)$ and $\sigma^m x = \sigma^n x$ implies $m = n$ ([2], Theorem 7.1).

(ii) Λ satisfies *Condition* (B$'$) if for all $v \in \Lambda^0$ there is $x \in v\partial\Lambda$ such that $\lambda, \mu \in \Lambda v$ and $\lambda x = \mu x$ implies $\lambda = \mu$ ([2], Remarks 7.3).

(iii) Λ satisfies *Condition* (B) if for all $v \in \Lambda^0$ there is $x \in v\Lambda^{\leqslant\infty}$ such that $\lambda, \mu \in \Lambda v$ and $\lambda x = \mu x$ implies $\lambda = \mu$ ([5], Theorem 4.5).

Condition (A) has sometimes been referred to as *the aperiodicity condition* in [2], [3], [6], and [7]. We shall do so below. When Λ is row-finite without sources, $\partial\Lambda = \Lambda^\infty$. Therefore, Condition (A) presented in Theorem 7.1 from [2] for finitely aligned k-graphs reduces to the version of Condition (A) in Definition 4.3 from [3] for row-finite k-graphs without sources.

For row-finite k-graphs without sources Λ, Conditions (B) and (B$'$) reduce to precisely the same condition because $\Lambda^{\leqslant\infty} = \partial\Lambda = \Lambda^{\infty}$. When Λ is row-finite and locally convex, Condition (B) introduced in Theorem 4.5 of [4] is precisely the same as that presented in Theorem 4.3 from [5]. However, $\Lambda^{\leqslant\infty}$ is in general a proper subset of $\partial\Lambda$. Some discussion about the differences between these two conditions may be found in the Remarks 7.3 of [2].

Finally, there is the notion of *no local periodicity* (NLP), introduced by Robertson and Sims for row-finite k-graphs without sources in [6] and for row-finite locally convex k-graphs in [7]. For row-finite k-graphs without sources Λ, the two notions of no local periodicity coincide because $\Lambda^{\leqslant\infty} = \Lambda^{\infty}$.

DEFINITION 2.2. Let Λ be a row-finite k-graph.

(i) If Λ has no sources, then Λ has *no local periodicity* if for all $v \in \Lambda^0$ and for all $n \neq m \in \mathbb{N}^k$, there is $x \in v\Lambda^{\infty}$ such that $\sigma^n x \neq \sigma^m x$ ([6], Definition 1).

(ii) If Λ is locally convex, then Λ has *no local periodicity* if for all $v \in \Lambda^0$ and for all $n \neq m \in \mathbb{N}^k$, there is $x \in v\Lambda^{\leqslant\infty}$ such that either $n - n \wedge d(x) \neq m - m \wedge d(x)$ or $\sigma^{n\wedge d(x)} x \neq \sigma^{m\wedge d(x)} x$ ([7], Definition 3.2).

For finitely aligned k-graphs, we introduce the following version of no local periodicity. We do not address the extent to which our version of no local periodicity is related to the version for row-finite locally convex k-graphs. In the row-finite no sources setting, our version is equivalent to the previous versions of no local periodicity.

DEFINITION 2.3. Let Λ be a finitely aligned k-graph. Λ has *no local periodicity* (NLP) if for every $v \in \Lambda^0$ and every $m \neq n \in \mathbb{N}^k$, there exists $x \in v\partial\Lambda$ such that either $d(x) \ngeqslant m \vee n$ or $\sigma^m x \neq \sigma^n x$.

DEFINITION 2.4. Let Λ be a finitely aligned k-graph without sources. Λ has *strong no local periodicity* (SNLP) if for every $v \in \Lambda^0$ and every $m \neq n \in \mathbb{N}^k$, there exists $x \in v\partial\Lambda$ such that $d(x) \geqslant m \vee n$ and $\sigma^m x \neq \sigma^n x$.

REMARK 2.5. If no local periodicity fails at $v \in \Lambda^0$, then there are $n \neq m \in \mathbb{N}^k$ such that $\sigma^n x = \sigma^m x$ for all $x \in v\partial\Lambda$. In this case, Λ has *local periodicity n, m at $v \in \Lambda^0$*. For row-infinite finitely aligned k-graphs (with or without sources) and fixed $n \neq m \in \mathbb{N}^k$, there may exist boundary paths $x \in v\partial\Lambda$ such that $d(x) \ngeqslant n \vee m$. It is not immediately clear whether or not Λ can satisfy no local periodicity, yet satisfy $\sigma^n x = \sigma^m x$ whenever $d(x) \geqslant n \vee m$ for some $n \neq m \in \mathbb{N}^k$. The next section will establish that this is not possible for finitely aligned k-graphs without sources.

2.1. FINITELY ALIGNED, NO SOURCES.

Throughout this subsection, let Λ be a finitely aligned k-graph without sources. First we show that, in this situation, NLP is equivalent to SNLP. This will allow us to use the methods in the proof of Lemma 3.3 in [6] to show equivalence between the aperiodicity condition and no local periodicity. The main strategy is to realize that, if a boundary path has degree with some finite component, then since Λ has no sources, we can find infinite receivers along the path. Our strict assumptions in this situation will provide sufficiently many edges to construct an aperiodic boundary path.

PROPOSITION 2.6. *Λ satisfies* NLP *if and only if it satisfies* SNLP.

Proof. It is clear that SNLP implies NLP. Suppose that Λ has NLP and fails SNLP at $v \in \Lambda^0$. Then we may fix $m \neq n \in \mathbb{N}^k$ such that $\sigma^n y = \sigma^m y$ for all $y \in v\partial\Lambda$ with $d(y) \geqslant m \vee n$. We will derive a contradiction by constructing $w \in v\partial\Lambda$ satisfying $d(w) \geqslant m \vee n$ and $\sigma^m w \neq \sigma^n w$. Fix $x \in v\Lambda^\infty$. Set $n_1 = n \vee m - m$, $m_1 = n \vee m - n$, $v_1 = x(n)$, and $v_2 = x(n+n)$. Note that $n_1 \wedge m_1 = 0$.

Claim 1. *$\sigma^{n_1} y = \sigma^{m_1} y$ for each $y \in v_1\partial\Lambda$ or $y \in v_2\partial\Lambda$ satisfying $d(y) \geqslant n_1 \vee m_1$.*

Proof of Claim 1. Let $y \in v_1\partial\Lambda$ satisfy $d(y) \geqslant n_1 \vee m_1$. Set $w = x(0,n)y$. Then $\sigma^n w = \sigma^m w$, since $d(w) \geqslant n \vee m$. In particular,

$$\sigma^{n\vee m} w = \sigma^{n\vee m-n}\sigma^n w = \sigma^{n\vee m-n} y.$$

Also,

$$\sigma^{n\vee m} w = \sigma^{n\vee m-m}\sigma^m w = \sigma^{n\vee m-m}\sigma^n w = \sigma^{n\vee m-m} y.$$

Therefore, $\sigma^{n_1} y = \sigma^{n\vee m-m} y = \sigma^{n\vee m-n} y = \sigma^{m_1} y$, as required. A similar proof shows that the result holds for each $y \in v_2\partial\Lambda$. ∎

Claim 2. *We may assume that either $v_1\Lambda^{n_1}$ or $v_2\Lambda^{n_1}$ is finite.*

Proof of Claim 2. Suppose that both $v_1\Lambda^{n_1}$ and $v_2\Lambda^{n_1}$ are infinite sets. Then $v_1\Lambda^n$ and $v_2\Lambda^n$ are also infinite sets because $n \geqslant n_1$. Also, $x(n+m)\Lambda^{n_1}$ is an infinite set because $x(n+m) = x(n+n) = v_2$. Thus,

$$\{x(n,n+m)\alpha : \alpha \in x(n+m)\Lambda^{n_1}\}$$

is an infinite set. Notice that if $\alpha \in x(n+m)\Lambda^{n_1}$, then

$$d(x(n,n+m)\alpha) = m + n_1 = m \vee n.$$

Thus, $x(n,n+m)\alpha \in \mathrm{MCE}(x(n,n+m),\lambda)$ for some $\lambda \in v_1\Lambda^n$. This implies that

$$\bigcup_{\lambda \in v_1\Lambda^n} \mathrm{MCE}(x(n,n+m),\lambda)$$

is infinite. Because Λ is finitely aligned, $\mathrm{MCE}(x(n,n+m),\lambda)$ is finite for each $\lambda \in v_1\Lambda^n$. Hence, $\Lambda^{\min}(x(n,n+m),\lambda)$ is non-empty for infinitely many $\lambda \in v_1\Lambda^n$.

By the above work, we may choose $\lambda \in v_1\Lambda^n$ satisfying

$$\Lambda^{\min}(x(n,n+m),\lambda) \neq \emptyset \quad \text{and} \quad \lambda \neq x(m,m+n).$$

Fix $(\alpha,\beta) \in \Lambda^{\min}(x(n,n+m),\lambda)$, set $\xi = x(0,n+m)\alpha$, and choose $w \in v\Lambda^\infty$ such that $w(0,d(\xi)) = \xi$. Then we have:

$$\sigma^n w(0,n) = \lambda, \quad \sigma^m w(0,n) = x(m,m+n).$$

Therefore, $\sigma^n w \neq \sigma^m w$. This contradicts our assumption that $\sigma^n y = \sigma^m y$ for all $y \in v\partial\Lambda$ with $d(y) \geqslant m \vee n$. ∎

By Claim 1 and the fact that Λ is assumed to satisfy NLP, there is $z \in v_1\partial\Lambda$ and $i_0 \in \{1, \ldots, k\}$ such that $d(z)_{i_0} < (n_1 \vee m_1)_{i_0}$. If $v_1\Lambda^{n_1}$ is finite, then it is also exhaustive by the assumption of no sources. Hence, the definition of $\partial\Lambda$ gives $\lambda \in v_1\Lambda^{n_1}$ satisfying $z(0, d(\lambda)) = \lambda$. Thus, $d(z) \geqslant n_1$, which also implies $d(z)_{i_0} < (m_1)_{i_0}$. If $v_1\Lambda^{n_1}$ is infinite, instead take $z \in v_2\partial\Lambda$ such that $d(z)_{i_0} < (n_1 \vee m_1)_{i_0}$ for some $i_0 \in \{1, \ldots, k\}$. Since $v_2\Lambda^{n_1}$ is finite exhaustive, we may similarly conclude that $d(z) \geqslant n_1$ and $d(z)_{i_0} < (m_1)_{i_0}$. Note that in either case, $(n_1)_{i_0} = 0$ because $n_1 \wedge m_1 = 0$.

Suppose $v_1\Lambda^{n_1}$ is finite, let $z \in v_1\partial\Lambda$ be as above, and set $q = d(z)_{i_0}e_{i_0}$. We claim that $d(z) \geqslant n_1 + n_1$. To see this, assume otherwise. Fix $\overline{z} \in v_1\Lambda^\infty$ satisfying $\overline{z}(0, q + n_1) = z(0, q + n_1)$. By Claim 1, $\sigma^{n_1}\overline{z} = \sigma^{m_1}\overline{z}$. If $d(z) \not\geqslant n_1 + n_1$, then $z(n_1 + q)\Lambda^{n_1}$ is infinite (otherwise we could find $\lambda \in z(n_1 + q)\Lambda^{n_1}$ such that $z(n_1 + q, n_1 + q + d(\lambda)) = \lambda$, which would give $d(z) \geqslant n_1 + q + n_1$). Therefore, $\overline{z}(m_1 + q)\Lambda^{n_1}$ is infinite. This is a contradiction of the assumption that $v_1\Lambda^{n_1}$ is finite. To see this contradiction, recall that Lemma C.4 of [5] yields that $\operatorname{Ext}(\eta; F)$ is finite exhaustive if F is finite exhaustive. In our case, we have assumed that $v_1\Lambda^{n_1}$ is finite exhaustive and therefore $\operatorname{Ext}(\overline{z}(0, m_1 + q), v_1\Lambda^{n_1})$ is also finite exhaustive. Moreover, if $\alpha \in \overline{z}(m_1 + q)\Lambda^{n_1}$, then

$$d(\overline{z}(0, m_1 + q)\alpha) = m_1 + q + n_1 = (m_1 + q) \vee n_1.$$

Therefore, $\alpha \in \operatorname{Ext}(\overline{z}(0, m_1 + q); v_1\Lambda^{n_1})$ so that

$$\overline{z}(m_1 + q)\Lambda^{n_1} \subseteq \operatorname{Ext}(\overline{z}(0, m_1 + q); v_1\Lambda^{n_1}).$$

Thus, we can conclude $d(z) \geqslant n_1 + n_1$.

Similarly, if $v_2\Lambda^{n_1}$ is finite, we may take $z \in v_2\partial\Lambda$ and conclude that $d(z) \geqslant n_1 + n_1$. Without loss of generality, assume $v_1\Lambda^{n_1}$ is finite and fix $z, \overline{z}$ as above.

We have $\sigma^{n_1}\overline{z} = \sigma^{m_1}\overline{z}$ by Claim 1, so $\overline{z}(n_1 + q)\Lambda^{e_{i_0}} = \overline{z}(m_1 + q)\Lambda^{e_{i_0}}$ is an infinite set. Also, the above work shows that $\overline{z}(m_1 + q + n_1)\Lambda^{e_{i_0}}$ is infinite. Arguing similarly to the proof of Claim 2

$$\bigcup_{\lambda \in \overline{z}(m_1+q)\Lambda^{e_{i_0}}} \operatorname{MCE}(\overline{z}(m_1 + q, m_1 + q + n_1), \lambda)$$

is an infinite set.

This implies that $\Lambda^{\min}(\overline{z}(m_1 + q, m_1 + q + n_1), \lambda)$ is non-empty for infinitely many $\lambda \in \overline{z}(m_1 + q)\Lambda^{e_{i_0}}$. Therefore, we may choose $\lambda \in \overline{z}(m_1 + q)\Lambda^{e_{i_0}}$ such that

$$\lambda \neq \overline{z}(n_1 + q, n_1 + q + e_{i_0}) \quad \text{and} \quad \Lambda^{\min}(\overline{z}(m_1 + q, m_1 + q + n_1), \lambda) \neq \emptyset.$$

Let $(\alpha, \beta) \in \Lambda^{\min}(\overline{z}(m_1 + q, m_1 + q + n_1), \lambda)$ and set

$$\xi = \overline{z}(0, m_1 + q + n_1)\alpha.$$

Choose $w \in v_1\Lambda^\infty$ such that $w(0, d(\xi)) = \xi$. Then we have

$$\begin{aligned}\sigma^{m_1} w(q, q + e_{i_0}) &= \xi(m_1 + q, m_1 + q + e_{i_0}) = \lambda,\\ \sigma^{n_1} w(q, q + e_{i_0}) &= w(n_1 + q, n_1 + q + e_{i_0}) = \overline{z}(n_1 + q, n_1 + q + e_{i_0}).\end{aligned}$$

However, λ is chosen such that $\lambda \neq \overline{z}(n_1 + q, n_1 + q + e_{i_0})$. Therefore, $\sigma^{m_1} w \neq \sigma^{n_1} w$, as required. ∎

PROPOSITION 2.7. *Let Λ be a finitely aligned k-graph without sources. The following are equivalent:*

(i) *Λ has no local periodicity;*

(ii) *Λ satisfies the aperiodicity condition.*

Proof. It is clear that the aperiodicity condition implies no local periodicity. Assume that Λ satisfies NLP and fails the aperiodicity condition. The above work shows we may assume that for every $v \in \Lambda^0$ and $n \neq m \in \mathbb{N}^k$, there is $x \in v\Lambda^\infty$ such that $d(x) \geqslant n \vee m$ and $\sigma^n x \neq \sigma^m x$. A proof identical to that for Lemma 3.3 of [6] now shows that Λ satisfies the aperiodicity condition. ∎

2.2. FINITELY ALIGNED, WITH SOURCES. This subsection is dedicated to proving the following proposition.

PROPOSITION 2.8. *Let Λ be a finitely aligned k-graph. Then Λ satisfies the aperiodicity condition if and only if Λ has no local periodicity.*

Suppose that Λ has no local periodicity but fails the aperiodicity condition at some $v_1 \in \Lambda^0$.

Assume there exists $x_1 \in v_1\Lambda^{\leqslant\infty}$ such that $d(x_1)_{i_1} < \infty$ for some $i_1 \in \{1, \ldots, k\}$. If no such $x_1 \in v_1\Lambda^{\leqslant\infty}$ exists, then $v_1\Lambda^{\leqslant\infty} = v_1\Lambda^\infty$. Fix $t_1 \in \mathbb{N}^k$ such that $x_1(t_1)\Lambda^{e_{i_1}} = \emptyset$. Set $v_2 = x_1(t_1)$ and note that $d(y)_{i_1} = 0$ for every $y \in v_2\Lambda^{\leqslant\infty}$.

Suppose there is $x_2 \in v_2\Lambda^{\leqslant\infty}$ such that $0 < d(x_2)_{i_2} < \infty$ for some $i_2 \in \{1, \ldots, k\}$. Then $i_1 \neq i_2$ and we may find $t_2 \in \mathbb{N}^k$ such that $x_2(t_2)\Lambda^{e_{i_2}} = \emptyset$. Set $v_3 = x_2(t_2)$. We may continue in this fashion to find $v_a \in \Lambda^0$ and an arrangement

$$\{i_1, \ldots, i_a, i_{a+1}, \ldots, i_k\}$$

of $\{1, \ldots, k\}$ such that, for every $x \in v_a\Lambda^{\leqslant\infty}$,

$$d(x)_i = \begin{cases} 0 & \text{if } i = i_j,\ j \leqslant a, \\ \infty & \text{if } i = i_j,\ a + 1 \leqslant j \leqslant k. \end{cases}$$

Let $\{f_j\}$ be the standard generators in $\mathbb{N}^{k-a}$. Define $\pi : \mathbb{N}^k \to \mathbb{N}^{k-1}$ and $\iota : \mathbb{N}^{k-a} \to \mathbb{N}^k$ by:

$$\pi\Big(\sum_{j=1}^{k} b_{i_j} e_{i_j}\Big) = \sum_{j=1}^{k-a} b_{i_{j+a}} f_j, \quad \iota\Big(\sum_{j=1} b_j f_j\Big) = \sum_{j=1}^{k-a} b_j e_{j+a}.$$

Define a category Γ by setting:

$$\mathrm{Obj}(\Gamma) = \{w \in \Lambda^0 : v_a\Lambda w \neq \emptyset\}, \quad \mathrm{Hom}(\Gamma) = \{\lambda \in \Lambda : v_a\Lambda r(\lambda) \neq \emptyset\}.$$

Define a degree functor $d' : \Gamma \to \mathbb{N}^{k-a}$ by

$$d'(\lambda) = \pi(d(\lambda)).$$

Claim 1. *(Γ, d') is a finitely aligned $(k-a)$-graph without sources.*

Proof of Claim 1. It is clear that Γ is a category, with range and source maps coming from Λ. It must be checked that d' is a well-defined functor satisfying unique factorization.

That d' is a well-defined functor follows immediately from its definition. To see that d' satisfies unique factorization, let $\lambda \in \Gamma$ and suppose $d'(\lambda) = m' + n'$, where $m', n' \in \mathbb{N}^{k-a}$. Set $m = \iota(m')$ and $n = \iota(n')$. Note that $d(\lambda) = m + n$, since otherwise $d(\lambda)_{i_j} > 0$ for some $j \in \{1, \ldots, a\}$, a contradiction of the fact that $v_a\Lambda^{e_{i_j}} = \emptyset$ for $j \in \{1, \ldots, a\}$. Thus, there are $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$, and $d(\nu) = n$. It is clear that $d'(\mu) = m'$ and $d'(\nu) = n'$, so d' satisfies unique factorization.

Γ is finitely aligned because $|\Gamma^{\min}(\lambda, \mu)| = \infty$ readily implies that $\Lambda^{\min}(\lambda, \mu)$ is infinite.

Finally, suppose that $w \in \Gamma^0$ is such that $w\Gamma^{f_j} = \emptyset$ for some $j \in \{1, \ldots, k-a\}$. Fix $\lambda \in v_a\Lambda w \neq \emptyset$ and choose $x \in \Lambda^{\leqslant\infty}$ such that $x(0, d(\lambda)) = \lambda$. Then $d(x)_{j+a} = \infty$, a contradiction. Therefore, Γ has no sources ∎

Claim 2. *Γ has* NLP.

Proof of Claim 2. Fix $w \in \Gamma^0$ and $m' \neq n' \in \mathbb{N}^{k-a}$. Let $m = \iota(m'), n = \iota(n')$, where $\iota : \mathbb{N}^{k-a} \to \mathbb{N}^k$ is standard injection. Because Λ is assumed to satisfy NLP, there is $x \in w\partial\Lambda$ such that $d(x) \ngeq m \vee n$ or $\sigma^m x \neq \sigma^n x$.

Suppose that $d(x) \ngeq m \vee n$ for some $x \in w\partial\Lambda$. Then $d(x)_i < (m \vee n)_i$ for some $i \in \{1, \ldots, k\}$. Since $(m \vee n)_i = 0$ for $i \in \{i_1, \ldots, i_a\}$, this implies that $d(x)_{i_j} < (m \vee n)_{i_j}$ for some $j \in \{a+1, \ldots, k\}$. Define $y \in w\partial\Gamma$ by $y(0, l) = x(0, \iota(l))$. Then $d'(y)_{i_j} < (m' \vee n')_{i_j}$, so that $d'(y) \ngeq m' \vee n'$.

Suppose that $\sigma^n x \neq \sigma^m x$ for some $x \in w\partial\Lambda$. Define $y \in w\partial\Gamma$ by $y(0, l) = x(0, \iota(l))$. Note that $d(x)_i = 0$ for $i \in \{i_1, \ldots, i_a\}$. It follows immediately that $\sigma^{n'} y \neq \sigma^{m'} y$. ∎

Claim 3. *Γ fails the aperiodicity condition.*

Proof of Claim 3. It is assumed that Λ fails the aperiodicity condition at $v_1 \in \Lambda^0$. Let $y \in v_a\partial\Gamma$. We will find $n' \neq m' \in \mathbb{N}^{k-a}$ such that $\sigma^{n'} y = \sigma^{m'} y$.

For $t \in \mathbb{N}^k$ define $x \in v_a\partial\Lambda$ by $x(0, t) = y(0, \pi(t))$ and fix $\mu \in v_1\Lambda v_a$ (using the fact that $v_1\Lambda v_a \neq \emptyset$ by construction of v_a). Since Λ fails the aperiodicity condition at $v_1 \in \Lambda^0$, there are $n \neq m \in \mathbb{N}^k$ such that $\sigma^n(\mu x) = \sigma^m(\mu x)$. Notice that $d(x)_i = 0$ when $i \in \{i_1, \ldots, i_a\}$ and that $d(x)_i = \infty$ whenever $m_i \neq n_i$. Thus, $n_i \neq m_i$ for some $i \in \{i_{a+1}, \ldots, i_k\}$.

Define $p \in \mathbb{N}^k$ by

$$p_i = \begin{cases} d(\sigma^n(\mu x))_i & \text{if } i = i_j,\ j \leqslant a, \\ d(\mu)_i & \text{if } i = i_j,\ a+1 \leqslant j \leqslant k. \end{cases}$$

Then $p \leqslant d(\sigma^n(\mu x))$, $p + n \geqslant d(\mu)$, and

$$\sigma^{n \vee m - n} \sigma^{n+p}(\mu x) = \sigma^{n \vee m - m} \sigma^{n+p}(\mu x).$$

Let $q = n + p - d(\mu)$. Then

$$(n \vee m - n + q) \vee (n \vee m - m + q) \leqslant d(x)$$

because $((n \vee m - n) \vee (n \vee m - m))_i > 0$ implies $d(x)_i = \infty$ and $((n \vee m - n + q) \vee (n \vee m - m + q))_i = q_i \leqslant d(x)_i$ otherwise. Moreover,

$$\sigma^{n \vee m - n + q} x = \sigma^{n \vee m + p}(\mu x) = \sigma^{n \vee m - n} \sigma^{n+p}(\mu x) = \sigma^{n \vee m - m} \sigma^{n+p}(\mu x) = \sigma^{n \vee m - m + q} x.$$

Set $n' = \pi(n \vee m - m + q)$ and $m' = \pi(n \vee m - n + q)$. Notice that $n' \neq m'$, since otherwise $(n \vee m - n)_{i_j} = (n \vee m - m)_{i_j}$ for each $j \in \{a+1, \ldots, k\}$. Finally, the above work shows that $\sigma^{n'} y = \sigma^{m'} y$. Therefore, Γ fails the aperiodicity condition at $v_a \in \Gamma^0$. ∎

Proof of Proposition 2.8. It is clear that the aperiodicity condition implies no local periodicity.

If Λ has no local periodicity but fails the aperiodicity condition, then Claims 2 and 3 establish the existence of a lower rank graph Γ without sources, which has both no local periodicity and fails the aperiodicity condition. This is a contradiction of Proposition 2.6. ∎

2.3. EQUIVALENT CONDITIONS. The following lemma (and its proof) is more or less identical to Lemma 3.4 in [6].

LEMMA 2.9. *Suppose Λ is a finitely aligned k-graph which has local periodicity n, m at $v \in \Lambda^0$. Then $d(x) \geqslant n \vee m$ and $\sigma^n x = \sigma^m x$ for every $x \in v\partial\Lambda$. Fix $x \in v\partial\Lambda$ and set $\mu = x(0, m), \alpha = x(m, m \vee n)$, and $\nu = \mu\alpha(0, n)$. Then $\mu\alpha y = \nu\alpha y$ for every $y \in s(\alpha)\partial\Lambda$.*

Proof. Let $y \in s(\alpha)\partial\Lambda$ and set $w = \mu\alpha y$. Then we have $d(w) \geqslant n \vee m$ and $\sigma^n w = \sigma^m w$ by assumption. Moreover, $w(0, n) = \nu$, so $w = \nu\sigma^n w$. Since $\sigma^m w = \sigma^n w$, it follows that $\sigma^n w = \alpha y$, so $\mu\alpha y = w = \nu\alpha y$. ∎

DEFINITION 2.10 ([2]). Let Λ be a finitely aligned k-graph. Λ satisfies Condition B if for each $v \in \Lambda^0$, there is $x \in v\partial\Lambda$ such that $\lambda \neq \mu \in \Lambda v$ implies $\lambda x \neq \mu x$.

PROPOSITION 2.11. *Let Λ be a finitely aligned k-graph. The following are equivalent:*

(i) *Λ has no local periodicity;*
(ii) *Λ satisfies the aperiodicity condition;*
(iii) *Λ satisfies Condition* (B′).

Proof. Proposition 2.8 establishes that (i) is equivalent to (ii).

(iii) $\Rightarrow$ (i). Suppose Λ has local periodicity n, m at $v \in \Lambda^0$. Choose μ, ν, α as in Lemma 2.9 and note $d(\mu\alpha) = m \vee n$, $d(\nu\alpha) = n + m \vee n - m$, and that $n + m \vee n - m \neq m \vee n$ if $m \neq n$. Thus, $\mu\alpha \neq \nu\alpha$ and $\mu\alpha y = \nu\alpha y$ for each $y \in s(\alpha)\partial\Lambda$. Therefore, Λ fails Condition B at $s(\alpha)$.

(ii) $\Rightarrow$ (iii). Suppose that Λ fails Condition B at $v \in \Lambda^0$. Then for each $x \in v\partial\Lambda$, there are $\lambda_x \neq \mu_x \in \Lambda v$ such that $\lambda_x x = \mu_x x$. Notice $d(\lambda_x) \neq d(\mu_x)$, since then $\lambda_x = (\lambda_x x)(0, d(\lambda_x)) = (\mu_x x)(0, d(\mu_x)) = \mu_x$.

If $d(\lambda_x)_i \neq d(\mu_x)_i$ for some $i \in \{1, \dots, k\}$, then $d(x)_i + d(\lambda_x)_i = d(x)_i + d(\mu_x)_i$ implies $d(x)_i = \infty$. Hence,

$$(d(\lambda_x) \vee d(\mu_x) - d(\mu_x)) \vee (d(\lambda_x) \vee d(\mu_x) - d(\lambda_x)) \leqslant d(x).$$

Therefore,

$$\sigma^{d(\lambda_x) \vee d(\mu_x) - d(\mu_x)} x = \sigma^{d(\lambda_x) \vee d(\mu_x) - d(\mu_x)} \sigma^{d(\mu_x)}(\mu x) = \sigma^{d(\lambda_x) \vee d(\mu_x)} \mu x.$$

Similarly,

$$\sigma^{d(\lambda_x) \vee d(\mu_x) - d(\lambda_x)} x = \sigma^{d(\lambda_x) \vee d(\mu_x)} \lambda x.$$

Since we have $\lambda_x x = \mu_x x$, this yields

$$\sigma^{d(\lambda_x) \vee d(\mu_x) - d(\mu_x)} x = \sigma^{d(\lambda_x) \vee d(\mu_x) - d(\lambda_x)} x.$$

Hence, Λ fails the aperiodicity condition at $v \in \Lambda^0$. ∎

3. MAIN RESULT

DEFINITION 3.1 ([12], Definition 5.1). Let Λ be a finitely aligned k-graph and let $H \subseteq \Lambda^0$. H is *hereditary* if, for all $\lambda \in \Lambda$, $r(\lambda) \in H$ implies $s(\lambda) \in H$. H is *saturated* if for all $v \in \Lambda^0$, $F \in v\mathrm{FE}(\Lambda)$ and $s(F) \subseteq H$ imply $v \in H$.

Given a saturated and hereditary set H, I_H denotes the ideal generated by $\{s_v : v \in H\}$.

DEFINITION 3.2 ([12], Definition 8.4). Let Λ be a k-graph. Λ is *cofinal* if, for every $v \in \Lambda^0$ and $x \in \partial\Lambda$, there is $n \leqslant d(x)$ such that $v\Lambda x(n) \neq \emptyset$.

PROPOSITION 3.3. *Let Λ be a finitely aligned k-graph. The following are equivalent:*

(i) *Λ is cofinal;*
(ii) *If I is an ideal of $C^*(\Lambda)$ and $s_v \in I$ for some $v \in \Lambda^0$, then $I = C^*(\Lambda)$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that Λ is cofinal and let $H \subseteq \Lambda^0$ be a non-empty, saturated, and hereditary set. Suppose that $H \neq \Lambda^0$. By Claim 8.6 of [12], there is a path $x \in \partial\Lambda$ such that $x(n) \notin H$ for all $n \leqslant d(x)$. This, however, is a contradiction: Let $v \in H$. By the assumption that Λ is cofinal, there is $n \leqslant d(x)$ for which $v\Lambda x(n) \neq \emptyset$. Let $\lambda \in v\Lambda x(n)$. Then $r(\lambda) \in H$ and hence $x(n) = s(\lambda) \in H$ by the assumption that H is hereditary.

Suppose that I is an ideal of $C^*(\Lambda)$ and that $s_v \in I$ for some $v \in \Lambda^0$. Let $H_I = \{v \in \Lambda^0 : s_v \in I\}$. Then H_I is non-empty and Lemma 3.3 from [12] shows that H_I is a saturated and hereditary subset of Λ^0, whence $H_I = \Lambda^0$. This implies $s_v \in I$ for all $v \in \Lambda^0$, which yields $I = C^*(\Lambda)$. Hence, the only non-empty saturated hereditary subset of Λ^0 is Λ^0 itself.

(ii) $\Rightarrow$ (i). Assume that Λ is not cofinal. Then there is a vertex $v \in \Lambda$ and a path $x \in \partial\Lambda$ such that $v\Lambda x(n) = \emptyset$ for all $n \in \mathbb{N}^k$ with $n \leqslant d(x)$. Let

$$H_x = \{w \in \Lambda^0 : w\Lambda x(n) = \emptyset \; \forall \; n \in \mathbb{N}^k \text{ such that } n \leqslant d(x)\}.$$

The proof of Proposition 8.5 in [12] shows that H_x is a non-trivial saturated and hereditary set in Λ^0. Hence, I_{H_x} is a non-trivial ideal of $C^*(\Lambda)$ containing a vertex projection. ∎

PROPOSITION 3.4. *Let Λ be a finitely aligned k-graph. The following are equivalent:*

(i) *Λ has no local periodicity;*
(ii) *every non-zero ideal of $C^*(\Lambda)$ contains a vertex projection;*
(iii) *the boundary-path representation π_S is faithful.*

Proof. (i) $\Rightarrow$ (ii). Suppose that Λ has no local periodicity. Then Λ satisfies the aperiodicity condition. Therefore, the Cuntz–Krieger uniqueness theorem given in Theorem 7.1 of [2] yields that every ideal of $C^*(\Lambda)$ contains a vertex projection.

(ii) $\Rightarrow$ (i). If $\ker(\pi_S) \neq \{0\}$, then $s_v \in \ker(\pi_S)$ for some $v \in \Lambda^0$. This is a contradiction because π_S is non-zero on vertex projections given that $v\partial\Lambda$ is non-empty for each $v \in \Lambda^0$.

(iii) $\Rightarrow$ (i). Suppose that Λ has local periodicity n, m at $v \in \Lambda^0$. Let μ, ν, α be as in Lemma 2.9 and put $a := s_{\mu\alpha}s^*_{\mu\alpha} - s_{\nu\alpha}s^*_{\nu\alpha}$. A proof identical to that for Proposition 3.5 in [6] now shows that $a \in \ker(\pi_S)\backslash\{0\}$ after replacing Λ^∞ by $\partial\Lambda$. ∎

THEOREM 3.5. *Let Λ be a finitely aligned k-graph. Then $C^*(\Lambda)$ is simple if and only if Λ is cofinal and has no local periodicity.*

Proof. Proposition 3.4 shows that every non-zero ideal of $C^*(\Lambda)$ contains a vertex projection. Proposition 3.3 shows that every such ideal is equal to all of $C^*(\Lambda)$. Therefore, $C^*(\Lambda)$ has no non-trivial ideals. If $C^*(\Lambda)$ is simple, then Proposition 3.3 shows that Λ is cofinal and Proposition 3.4 gives that Λ has no local periodicity. ∎

Acknowledgements. I would like to thank my advisor, Jack Spielberg, for his help with these results.

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Received May 25, 2009.