J. OPERATOR THEORY
67:2(2012), 349–368

HARDY–LITTLEWOOD AND UMD BANACH LATTICES VIA BESSEL CONVOLUTION OPERATORS

JORGE J. BETANCOR, JUAN C. FARIÑA, and LOURDES RODRÍGUEZ-MESA

Communicated by Nikolai K. Nikolski

ABSTRACT. In this paper we characterize the Banach lattices having the UMD and the Hardy–Littlewood properties by using Littlewood–Paley g-functions for the Poisson semigroup and maximal operators associated with convolution operators in the Bessel setting.

KEYWORDS: *Hardy–Littlewood, UMD, Bessel convolution.*

MSC (2000): Primary 42C05; Secondary 42C15.

1. INTRODUCTION

Vector valued harmonic analysis is closely connected with the geometry of Banach spaces. The fact that a certain property, for instance, the boundedness of certain classical operators, is true when Banach-valued functions are considered, is related to geometrical or topological properties of the underlying Banach space. Thus, new characterizations of old properties are obtained or new type of Banach spaces appear (see, [5], [6], [7], [8], [9], [11], [16], [20], [23], [25], amongst others).

Bourgain [6] characterized the UMD property of a Banach function space by using a version of the Hardy–Littlewood maximal function. Suppose that X is a Banach space consisting of equivalence classes, modulo equality almost everywhere, of locally integrable real functions on a complete σ-finite measure space (Ω, Σ, μ). This class of Banach spaces is named Köthe function spaces ([19] and [23]) when the following two conditions are satisfied:

(a) If $|f(w)| \leqslant |g(w)|$, a.e. $w \in \Omega$, with f measurable and $g \in X$, then $f \in X$ and $\|f\|_X \leqslant \|g\|_X$.

(b) For every $A \in \Sigma$ with $\mu(A) < \infty$ the characteristic function χ_A of A belongs to X.

Each Köthe function space is a Banach lattice with the obvious order ($f \geqslant 0 \Leftrightarrow f(w) \geqslant 0$, a.e. $w \in \Omega$). This lattice is σ-order complete. Moreover each order continuous Banach lattice with a weak unit is order isometric to a Köthe function space ([19], Theorem 1.b.14). Thus, a separable Banach lattice is order isometric to a Köthe function space if and only if it is σ-order complete. If X is a Köthe space, X' denotes the linear space of all integrals in X ([19], p. 29).

Every function $f : \mathbb{R}^n \to X$ is understood as a two variable function $f(x, w)$, $x \in \mathbb{R}^n$ and $w \in \Omega$. The operator $\mathfrak{M}$ denotes the usual Hardy–Littlewood maximal function with respect to the first variable

$$\mathfrak{M}(f)(x,w) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y,w)|\mathrm{d}y, \quad x \in \mathbb{R}^n,\ w \in \Omega.$$

Here $|B(x,r)|$ represents the Lebesgue measure of $B(x,r)$, for every $x \in \mathbb{R}^n$ and $r > 0$. Bourgain [6] proved that X has the UMD property if and only if $\mathfrak{M}$ is bounded in $L^p_X(\mathbb{R}^n)$ and also in $L^{p'}_{X^*}(\mathbb{R}^n)$, for some $1 < p < \infty$, where X^* is the dual space of X and p' is the exponent conjugated of p.

Motivated by this result of Bourgain, García-Cuerva, Macías and Torrea [11] introduced the Hardy–Littlewood property for a Banach lattice. Definitions and main properties of Banach lattices can be encountered in [19].

Let $\mathbb{Q}_+$ be the set of positive rational numbers. If X is a Banach lattice, J is a finite subset of $\mathbb{Q}_+$ and $f \in L^1_{X,\mathrm{loc}}(\mathbb{R}^n)$, the maximal function $\mathfrak{M}_J f$ is defined by

$$\mathfrak{M}_J(f)(x) = \sup_{r\in J} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

Here $|z|$ denotes, for every $z \in X$, the absolute value of z in the lattice X. Note that the supremum in the above definition exists because J is finite. If the supremum in $\mathfrak{M}_J$ is taken over an infinite set J then the supremum is not always defined. However, if X is a σ-order complete Banach lattice ([19], p. 4) we can define $\mathfrak{M}_{\mathbb{Q}_+}$ by

$$\mathfrak{M}_{\mathbb{Q}_+}(f)(x) = \sup_{r\in \mathbb{Q}_+} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|\mathrm{d}y, \quad x \in \mathbb{R}^n,$$

when $f \in L^1(\mathbb{R}^n) \otimes X$.

It is said that a Banach lattice has the Hardy–Littlewood property ([11]) when for a certain $1 < p < \infty$ there exists $C > 0$ such that

$$\|\mathfrak{M}_J f\|_p \leqslant C\|f\|_p, \quad f \in L^p_X(\mathbb{R}^n),$$

for every finite subset J of $\mathbb{Q}_+$.

If X is a Köthe function space, for every finite subset J of $\mathbb{Q}_+$, we have $\mathfrak{M}_J(f)(x)$ as a function of $w \in \Omega$, for every $x \in \mathbb{R}^n$, given by

$$[\mathfrak{M}_J(f)(x)](w) = \mathfrak{M}_J(f)(x,w) = \sup_{r\in J} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y,w)|\mathrm{d}y, \quad x \in \mathbb{R}^n,\ w \in \Omega.$$

Moreover, if X is a Köthe function space having Fatou property (see [19], p. 30) then X has the Hardy–Littlewood property if and only if the maximal operator

$$[\mathfrak{M}_{\mathbb{Q}_+}(f)(x)](w) = \mathfrak{M}_{\mathbb{Q}_+}(f)(x,w) = \sup_{r\in\mathbb{Q}_+} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y,w)| \mathrm{d}y, \quad x \in \mathbb{R}^n,$$

is bounded in $L^p_X(\mathbb{R}^n)$, for some $1 < p < \infty$ ([12], Remark 1.4).

In [11] García-Cuerva, Macías and Torrea characterized the Banach lattices having the Hardy–Littlewood property by using smooth versions of $\mathfrak{M}_J$ and $\mathfrak{M}_{\mathbb{Q}_+}$. They applied, as an important rule, the theory of vector valued singular integrals. If φ is an smooth real function defined on $[0,\infty)$ such that

$$\chi_{[0,1]} \leqslant \varphi \leqslant \chi_{[0,2]},$$

the maximal operator $\mathfrak{M}_{\varphi,J}$ and $\mathfrak{M}_\varphi$ considered in [11] are the following ones. If X is a Banach lattice and J is a finite subset of $\mathbb{Q}_+$, for every $f \in L^1_{X,\mathrm{loc}}(\mathbb{R}^n)$, $\mathfrak{M}_{\varphi,J}(f)$ is given by

$$\mathfrak{M}_{\varphi,J}(f)(x) = \sup_{t\in J} \Big| \frac{1}{t^n} \int_{\mathbb{R}^n} \varphi\Big(\frac{|x-y|}{t}\Big) f(y) \mathrm{d}y \Big|, \quad x \in \mathbb{R}^n.$$

If X is a σ-order complete Banach lattice, for every $f \in L^1(\mathbb{R}^n) \otimes X$ the maximal function $\mathfrak{M}_\varphi(f)$ is defined by

$$\mathfrak{M}_\varphi(f)(x) = \sup_{t\in\mathbb{Q}_+} \Big| \frac{1}{t^n} \int_{\mathbb{R}^n} \varphi\Big(\frac{|x-y|}{t}\Big) f(y) \mathrm{d}y \Big|, \quad x \in \mathbb{R}^n.$$

When $f \in L^p(\mathbb{R}^n) \otimes X$, $1 \leqslant p < \infty$, where X is a Köthe function space on Ω, $\mathfrak{M}_\varphi(f)$ is defined through

$$\mathfrak{M}_\varphi(f)(x,w) = \sup_{t\in\mathbb{Q}_+} \Big| \frac{1}{t^n} \int_{\mathbb{R}^n} \varphi\Big(\frac{|x-y|}{t}\Big) f(y,w) \mathrm{d}y \Big|, \quad x \in \mathbb{R}^n,\ w \in \Omega.$$

In [2] new characterizations of the UMD property and the martingale type and cotype for a Banach space were obtained in terms of harmonic analysis operators (Riesz transforms and Littlewood–Paley g-functions) associated with the Bessel operator

$$S_\lambda = -x^{-\lambda-1/2} D x^{2\lambda+1} D x^{-\lambda-1/2}, \quad \lambda > -\frac{1}{2}.$$

Moreover, in [3] the Banach spaces with the UMD property and the Hardy–Littlewood property were described by using harmonic analysis operators (Riesz transforms and maximal operators for the heat semigroup) associated with the Laguerre expansions.

One of our objective in this paper is to establish new characterizations of the Banach lattices with the Hardy–Littlewood property by using maximal operators defined by Hankel convolution operators in the S_λ-setting. Also we characterize the Köthe function spaces which have the UMD property in terms of the L^p-boundedness properties for Littlewood–Paley g-functions associated with Poisson semigroups for the Bessel operators S_λ.

We now recall some definitions and properties that will be useful in the sequel.

The Hankel convolution operation was studied by Hirschman [17] and Haimo [14]. We adapt the definition in [17] and [14] to the S_λ-setting. Assume that $\lambda > -\frac{1}{2}$. If f and g belong to $L^1((0,\infty), x^{\lambda+1/2}\mathrm{d}x)$, the convolution $f\#_\lambda g$ of f and g is defined by

$$(f\#_\lambda g)(x) = \int_0^\infty f(y)\,{}_\lambda\tau_x(g)(y)\mathrm{d}y, \quad x \in (0,\infty),$$

where, for every $x \in (0,\infty)$,

$$(1.1)\quad {}_\lambda\tau_x(g)(y) = \frac{(xy)^{\lambda+1/2}}{\sqrt{\pi}2^\lambda\Gamma(\lambda+1/2)}\int_0^\pi \frac{(\sin\theta)^{2\lambda}g(\sqrt{(x-y)^2+2xy(1-\cos\theta)})}{((x-y)^2+2xy(1-\cos\theta))^{(2\lambda+1)/4}}\mathrm{d}\theta,$$

$y \in (0,\infty)$, (see [13]).

In the sequel, if ϕ is a real function defined on $(0,\infty)$, we denote by $\phi_{(t)}$, $t > 0$, the function

$$\phi_{(t)}(x) = \frac{1}{t^{\lambda+3/2}}\phi\left(\frac{x}{t}\right), \quad x \in (0,\infty).$$

Suppose that ϕ is a suitable real function defined on $(0,\infty)$. If X is a Banach lattice, $f : (0,\infty) \to X$ is also good enough, and J is a finite subset of $\mathbb{Q}_+$, we define the maximal function $\mathfrak{M}^\lambda_{\phi,J}(f)$ by

$$\mathfrak{M}^\lambda_{\phi,J}(f) = \sup_{t\in J} |f\#_\lambda\phi_{(t)}|.$$

When X is a σ-order complete Banach lattice we define

$$\mathfrak{M}^\lambda_{\phi}(f) = \sup_{t\in \mathbb{Q}_+} |f\#_\lambda\phi_{(t)}|.$$

Moreover, if X is a Köthe function space on Ω we define

$$\mathfrak{M}^\lambda_{\phi}(f)(x,w) = \sup_{t\in \mathbb{Q}_+} \left|\int_0^\infty f(y,w)\,{}_\lambda\tau_x(\phi_{(t)})(y)\mathrm{d}y\right|, \quad x \in (0,\infty),\ w \in \Omega.$$

For every $y \in (0,\infty)$, the function $\phi_\lambda(\cdot;y)$ given by

$$\phi_\lambda(x;y) = \sqrt{xy}J_\lambda(xy), \quad x \in (0,\infty),$$

where J_λ denotes the Bessel function of the first kind and order λ, is an eigenfunction of the operator S_λ and

$$S_{\lambda,x}\phi_\lambda(x;y) = y^2\phi_\lambda(x;y), \quad x,y \in (0,\infty).$$

Then, the Poisson kernel associated with S_λ is defined by

$$P_t^\lambda(x,y) = \int_0^\infty e^{-zt}\phi_\lambda(x;z)\phi_\lambda(y;z)dz, \quad t,x,y \in (0,\infty).$$

The Poisson integral $P_t^\lambda(f)$ of f is given by

$$P_t^\lambda(f)(x) = \int_0^\infty P_t^\lambda(x,y)f(y)dy,$$

and according to p. 23, (4.3) of [24] we can write

$$P_t^\lambda(f) = f\#_\lambda k_{(t)}^\lambda,$$

where

$$k^\lambda(x) = \frac{2^{\lambda+1}\Gamma(\lambda+3/2)x^{\lambda+1/2}}{\sqrt{\pi}(1+x^2)^{\lambda+3/2}}, \quad x \in (0,\infty). \tag{1.2}$$

$\{P_t^\lambda\}_{t>0}$ is a non Markovian semigroup of bounded operators in $L^p(0,\infty)$, $1 \leqslant p \leqslant \infty$. The maximal operator

$$P^{\lambda,*}(f) = \sup_{t\in(0,\infty)} |f\#_\lambda k_{(t)}^\lambda|,$$

is bounded from $L^p(0,\infty)$ into itself, for every $1 < p < \infty$, and from $L^1(0,\infty)$ into $L^{1,\infty}(0,\infty)$ ([4]).

The operator S_λ can be written as $S_\lambda = D_\lambda^* D_\lambda$, where $D_\lambda = x^{\lambda+1/2}\frac{d}{dx}x^{-\lambda-1/2}$ and $D_\lambda^* = -x^{-\lambda-1/2}\frac{d}{dx}x^{\lambda+1/2}$ is the formal adjoint operator of D_λ in $L^2(0,\infty)$. According to the ideas of Muckenhoupt and Stein [22] we consider a Cauchy Riemann type equations associated with S_λ as follows

$$D_{\lambda,x}u(x,t) = \frac{\partial}{\partial t}v(x,t), \quad D_{\lambda,x}^* v(x,t) = \frac{\partial}{\partial t}u(x,t). \tag{1.3}$$

If $u, v : (0,\infty)^2 \to \mathbb{R}$, when u and v satisfy the pair of equations (1.3), we say that the function v is the S_λ-conjugate of u . If $f \in L^p(0,\infty)$, $1 \leqslant p < \infty$, the S_λ-conjugate of the Poisson integral $P_t^\lambda(f)(x)$ is the function

$$Q_t^\lambda(f)(x) = \int_0^\infty Q_t^\lambda(x,y)f(y)dy, \quad t,x \in (0,\infty),$$

where

$$Q_t^\lambda(x,y) = \frac{2\lambda+1}{\pi}(xy)^{\lambda+1/2}\int_0^\pi \frac{(x-y\cos\theta)(\sin\theta)^{2\lambda}}{(x^2+y^2+t^2-2xy\cos\theta)^{\lambda+3/2}}d\theta, \quad t,x,y \in (0,\infty).$$

The boundary value $\lim_{t\to 0^+} Q_t^\lambda(f)$ of $Q_t^\lambda(f)$ is the S_λ-Riesz transform $R_\lambda(f)$ of f defined by

$$R_\lambda(f)(x) = \lim_{\varepsilon\to 0} \int_{0,\,|x-y|>\varepsilon}^{\infty} R_\lambda(x,y)f(y)\mathrm{d}y, \quad \text{a.e. } x \in (0,\infty),$$

for every $f \in L^p(0,\infty)$, $1 \leqslant p < \infty$, where

$$R_\lambda(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty D_{\lambda,x} P_t^\lambda(x,y)\mathrm{d}t, \quad x,y \in (0,\infty).$$

R_λ defines a bounded operator from $L^p(0,\infty)$ into itself, when $1 < p < \infty$, and from $L^1(0,\infty)$ into $L^{1,\infty}(0,\infty)$ (see [1]).

We now consider the operator $\mathbb{S}_\lambda = D_\lambda D_\lambda^*$. Note that $\mathbb{S}_\lambda = S_{\lambda+1}$. We denote

$$\mathbb{P}_t^\lambda(x,y) = P_t^{\lambda+1}(x,y), \quad t,x,y \in (0,\infty), \quad \text{and} \quad \mathbb{P}_t^\lambda(f) = P_t^{\lambda+1}(f).$$

As Cauchy–Riemman type equations for the operator $S_{\lambda+1}$ we consider the pair of equations

$$D_{\lambda,x}^* u(x,t) = \frac{\partial}{\partial t} v(x,t), \quad D_{\lambda,x} v(x,t) = \frac{\partial}{\partial t} u(x,t), \tag{1.4}$$

and if $u, v : (0,\infty)^2 \to \mathbb{R}$ we say that v is $\mathbb{S}_\lambda$-conjugate of u when u and v satisfy the equations in (1.4). Then, v is $\mathbb{S}_\lambda$-conjugate of u if and only if u is S_λ-conjugate of v. If $f \in L^p(0,\infty)$, $1 \leqslant p < \infty$, the $\mathbb{S}_\lambda$ conjugate of the Poisson integral $\mathbb{P}_t^\lambda(f)(x)$ is the function

$$\mathbb{Q}_t^\lambda(f)(x) = \int_0^\infty \mathbb{Q}_t^\lambda(x,y)f(y)\mathrm{d}y, \quad t,x \in (0,\infty),$$

where

$$\mathbb{Q}_t^\lambda(x,y) = Q_t^\lambda(y,x), \quad t,x,y \in (0,\infty).$$

By proceeding as in [1] we can see that the boundary value $\lim_{t\to 0^+} \mathbb{Q}_t^\lambda(f)$ of $\mathbb{Q}_t^\lambda(f)$ is the $\mathbb{S}_\lambda$-Riesz transform $\mathbb{R}_\lambda(f)$ of f defined by

$$\mathbb{R}_\lambda(f)(x) = \lim_{\varepsilon\to 0} \int_{0,\,|x-y|>\varepsilon}^{\infty} \mathbb{R}_\lambda(x,y)f(y)\mathrm{d}y, \quad \text{a.e. } x \in (0,\infty),$$

for every $f \in L^p((0,\infty),\mathrm{d}x)$, $1 \leqslant p < \infty$, where

$$\mathbb{R}_\lambda(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty D_{\lambda,x}^* \mathbb{P}_t^\lambda(x,y)\mathrm{d}t = R_\lambda(y,x), \quad x,y \in (0,\infty).$$

$\mathbb{R}_\lambda$ defines a bounded operator from $L^p(0,\infty)$ into itself, when $1 < p < \infty$, and from $L^1(0,\infty)$ into $L^{1,\infty}(0,\infty)$.

The Littlewood–Paley g-functions associated with Poisson semigroups $\{P_t^\lambda\}_{t>0}$ and $\{\mathbb{P}_t^\lambda\}_{t>0}$ are defined as follows. If X is a Köthe function space on Ω and $f : (0,\infty) \to X$ we consider

$$g_\lambda(f)(x,w) = \Big(\int_0^\infty |t\nabla_\lambda P_t^\lambda(f)(x,w)|^2 \frac{dt}{t}\Big)^{1/2}, \quad x \in (0,\infty),\ w \in \Omega,$$

where $\nabla_\lambda h(x,t) = (|D_{\lambda,x}h(x,t)|^2 + |\frac{\partial}{\partial t}h(x,t)|^2)^{1/2}$, and

$$\mathfrak{g}_\lambda(f)(x,w) = \Big(\int_0^\infty |t\widetilde{\nabla}_\lambda \mathbb{P}_t^\lambda(f)(x,w)|^2 \frac{dt}{t}\Big)^{1/2}, \quad x \in (0,\infty),\ w \in \Omega,$$

where

$$\widetilde{\nabla}_\lambda h(x,t) = \Big(|D^*_{\lambda,x}h(x,t)|^2 + \Big|\frac{\partial}{\partial t}h(x,t)\Big|^2\Big)^{1/2}.$$

Here $P_t^\lambda(f)(x,w)$ and $\mathbb{P}_t^\lambda(f)(x,w)$ are defined in a natural way.

The main results of this paper are the following ones.

THEOREM 1.1. *Let X be a Banach lattice and $\lambda > -\frac{1}{2}$. Assume that ϕ is a nonnegative and not identically zero real function defined on $(0,\infty)$ such that $x^{-\lambda-1/2}\phi$ can be extended to $\mathbb{R}$ as an even and smooth function having bounded support. Then, the following assertions are equivalent:*

(i) *X has the Hardy–Littlewood property.*

(ii) *There exist $1 < p < \infty$ and a constant $C > 0$ such that*

$$\|\mathfrak{M}^\lambda_{\phi,J}(f)\|_{L^p_X(0,\infty)} \leqslant C\|f\|_{L^p_X(0,\infty)}, \quad f \in L^p_X(0,\infty),$$

for every finite subset J of Q_+.

(iii) *There exists $C > 0$ such that*

$$|\{x \in (0,\infty) : \|\mathfrak{M}^\lambda_{\phi,J}(f)\|_X > \gamma\}| \leqslant \frac{C\|f\|_{L^1_X(0,\infty)}}{\gamma}, \quad f \in L^1_X(0,\infty),$$

for every $\gamma > 0$ and every finite subset J of Q_+.

(iv) *For every $1 < p < \infty$ there exists a constant $C > 0$ such that*

$$\|\mathfrak{M}^\lambda_{\phi,J}(f)\|_{L^p_X(0,\infty)} \leqslant C\|f\|_{L^p_X(0,\infty)}, \quad f \in L^p_X(0,\infty),$$

for every finite subset J of Q_+.

THEOREM 1.2. *Let X be a Köthe function space such that X' is a norming subspace of X^* and $\lambda > -\frac{1}{2}$. Assume that ϕ is a nonnegative and not identically zero real function defined on $(0,\infty)$ such that $x^{-\lambda-1/2}\phi$ can be extended to $\mathbb{R}$ as an even and smooth function belonging to Schwartz class. Then, the following assertions are equivalent:*

(i) *X has the Hardy–Littlewood property.*

(ii) *There exist $1 < p < \infty$ and a constant $C > 0$ such that*

$$\|\mathfrak{M}^\lambda_{\phi}(f)\|_{L^p_X(0,\infty)} \leqslant C\|f\|_{L^p_X(0,\infty)}, \quad f \in L^p_X(0,\infty).$$

(iii) *There exists* $C > 0$ *such that*

$$|\{x \in (0,\infty) : \|\mathfrak{M}^\lambda_\phi(f)\|_X > \gamma\}| \leqslant \frac{C\|f\|_{L^1_X(0,\infty)}}{\gamma}, \quad f \in L^1_X(0,\infty),$$

for every $\gamma > 0$.

(iv) *For every* $1 < p < \infty$ *there exists a constant* $C > 0$ *such that*

$$\|\mathfrak{M}^\lambda_\phi(f)\|_{L^p_X(0,\infty)} \leqslant C\|f\|_{L^p_X(0,\infty)}, \quad f \in L^p_X(0,\infty).$$

(v) *For every* $f \in L^p_X(0,\infty)$, $1 \leqslant p < \infty$, $\mathfrak{M}^\lambda_\phi(f)(x) \in X$, *for almost all* $x \in (0,\infty)$.

Theorems 1.1 and 1.2 also hold when the function ϕ is replaced by the Poisson kernel k^λ defined in (1.2).

THEOREM 1.3. *Let X be a Köthe function space. Then the following assertions are equivalent:*

(i) *X is a* UMD *space.*

(ii) *There exists* $1 < p < \infty$ *and a constant* $C > 0$ *such that*

$$C^{-1}\|f\|_{L^p_X(0,\infty)} \leqslant \|g_\lambda(f)\|_{L^p_X(0,\infty)} \leqslant C\|f\|_{L^p_X(0,\infty)}, \tag{1.5}$$

and

$$C^{-1}\|f\|_{L^p_X(0,\infty)} \leqslant \|\mathfrak{g}_\lambda(f)\|_{L^p_X(0,\infty)} \leqslant C\|f\|_{L^p_X(0,\infty)}, \tag{1.6}$$

for every $f \in L^p_X(0,\infty)$.

(iii) *For every* $1 < p < \infty$ *there exists* $C > 0$ *such that*

$$C^{-1}\|f\|_{L^p_X(0,\infty)} \leqslant \|g_\lambda(f)\|_{L^p_X(0,\infty)} \leqslant C\|f\|_{L^p_X(0,\infty)},$$

and

$$C^{-1}\|f\|_{L^p_X(0,\infty)} \leqslant \|\mathfrak{g}_\lambda(f)\|_{L^p_X(0,\infty)} \leqslant C\|f\|_{L^p_X(0,\infty)},$$

for every $f \in L^p_X(0,\infty)$.

In Section 2 we prove Theorems 1.1 and 1.2. Section 3 is devoted to the proof of Theorem 1.3.

Throughout this paper C always denotes a suitable constant that can change in each occurrence.

2. PROOFS OF THEOREMS 1.1 AND 1.2

In this section we present proofs for Theorems 1.1 and 1.2. The strategy of the proofs is as follows. We split the region $(0,\infty) \times (0,\infty)$ in three parts: two parts far away from the diagonal (global parts) and one part close to the diagonal (local part). Then, we decompose the convolution operators in three parts that correspond to each of the above regions. The key of the proofs is the comparison of Hankel and classical convolution operators in the local part. In the global parts the Hankel convolution operators are bounded by Hardy type operators whose L^p-boundedness properties are well-known (see, for instance, [21]).

Proof of Theorem 1.1. Let us consider $\psi(z) = z^{-\lambda-1/2}\phi(z)$, $z \in \mathbb{R}$. According to [10] there exists Φ in the Schwartz class such that $\psi(z) = \Phi(z^2)$, $z \in \mathbb{R}$. We define

$$\Psi(x) = \frac{1}{\sqrt{\pi}2^{\lambda+1}\Gamma(\lambda+1/2)}\int_0^\infty u^{\lambda-1/2}\Phi(x^2+u)\mathrm{d}u, \quad x \in \mathbb{R}.$$

Firstly, we are going to prove that, if $1 \leqslant p < \infty$ and $J \subset \mathbb{Q}_+$, the following assertions are equivalent:

(a) The operator $\mathfrak{M}^\lambda_{\phi,J}$ is of strong type (respectively, weak type) (p,p), with respect to $((0,\infty),\mathrm{d}x)$.

(b) The operator $\mathfrak{M}_{\Psi,J,+}$ defined by

$$\mathfrak{M}_{\Psi,J,+}(f)(x) = \sup_{t\in J}\Big|\frac{1}{t}\int_0^\infty \Psi\Big(\frac{|x-y|}{t}\Big)f(y)\mathrm{d}y\Big|, \quad x \in (0,\infty),$$

is of strong type (respectively, weak type) (p,p), with respect to $((0,\infty),\mathrm{d}x)$.

Note that the operators $\mathfrak{M}_{\Psi,J,+}$ and $\mathfrak{M}^\lambda_{\phi,J}$ are acting on functions $f:(0,\infty)\to X$. Then the strong and weak type (p,p), $1\leqslant p<\infty$, are understood in a vector valued setting.

According to (1.1) we can write

$$\begin{aligned}{}_\lambda\tau_x(\phi_{(t)})(y) &= \frac{(xy)^{\lambda+1/2}}{\sqrt{\pi}2^\lambda\Gamma(\lambda+1/2)}t^{-2\lambda-2}\int_0^\pi(\sin\theta)^{2\lambda}\psi\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t}\Big)\mathrm{d}\theta\\ &= H_1(t,x,y)+H_2(t,x,y), \quad t,x,y\in(0,\infty),\end{aligned}$$

where, for every $t,x,y\in(0,\infty)$,

$$H_1(t,x,y) = \frac{(xy)^{\lambda+1/2}}{\sqrt{\pi}2^\lambda\Gamma(\lambda+1/2)}t^{-2\lambda-2}\int_0^{\pi/2}(\sin\theta)^{2\lambda}\psi\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t}\Big)\mathrm{d}\theta$$

and

$$H_2(t,x,y) ={}_\lambda\,\tau_x(\phi_{(t)})(y) - H_1(t,x,y).$$

Since ψ is in the Schwartz class it follows that

$$\begin{aligned}|H_2(t,x,y)| &\leqslant C(xy)^{\lambda+1/2}t^{-2\lambda-2}\int_{\pi/2}^\pi(\sin\theta)^{2\lambda}\Big|\psi\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t}\Big)\Big|\mathrm{d}\theta\\ &\leqslant C(xy)^{\lambda+1/2}t^{-2\lambda-2}\Big(\frac{\sqrt{x^2+y^2}}{t}\Big)^{-2\lambda-2} \leqslant \frac{(xy)^{\lambda+1/2}}{(x+y)^{2\lambda+2}} \leqslant C\begin{cases}\frac{1}{y} & 0<x<y,\\ \frac{1}{x} & 0<y<x.\end{cases}\end{aligned}$$

We now analyze $H_1(t,x,y)$. Firstly, we note that by using again that ψ is in the Schwartz class, it has that

$$\begin{aligned}|H_1(t,x,y)| &\leqslant C(xy)^{\lambda+1/2}t^{-2\lambda-2}\int_0^{\pi/2}(\sin\theta)^{2\lambda}\Big|\psi\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t}\Big)\Big|d\theta\\ &\leqslant C(xy)^{\lambda+1/2}t^{-2\lambda-2}\int_0^{\pi/2}(\sin\theta)^{2\lambda}\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t}\Big)^{-2\lambda-2}d\theta\\ &\leqslant \frac{(xy)^{\lambda+1/2}}{|x-y|^{2\lambda+2}}\leqslant C\begin{cases}\frac{1}{y} & 0<2x<y,\\ \frac{1}{x} & 0<y<\frac{x}{2}.\end{cases}\end{aligned}$$

Assume that $x/2<y<2x$ and $t>0$. We define

$$H_{1,1}(t,x,y)=\frac{(xy)^{\lambda+1/2}}{t^{2\lambda+2}\sqrt{\pi}2^\lambda\Gamma(\lambda+1/2)}\int_0^{\pi/2}\theta^{2\lambda}\psi\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t}\Big)d\theta.$$

Mean value theorem leads to

$$\begin{aligned}&\Big|H_1(t,x,y)-H_{1,1}(t,x,y)\Big|\\ &\leqslant C\frac{(xy)^{\lambda+1/2}}{t^{2\lambda+2}}\int_0^{\pi/2}|\theta^{2\lambda}-(\sin\,\theta)^{2\lambda}|\,\Big|\psi\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t}\Big)\Big|d\theta\\ &\leqslant C\frac{(xy)^{\lambda+1/2}}{t^{2\lambda+2}}\int_0^{\pi/2}\theta^{2\lambda+2}\Big(\frac{\sqrt{(x-y)^2+2xy(1-\cos\theta)}}{t}\Big)^{-2\lambda-2}d\theta\\ &\leqslant C\frac{(xy)^{\lambda+1/2}}{t^{2\lambda+2}}\int_0^{\pi/2}\theta^{2\lambda+2}\Big(\frac{xy\theta^2}{t^2}\Big)^{-\lambda-1}d\theta\leqslant C\frac{1}{x}.\end{aligned}$$

We can write

$$H_{1,1}(t,x,y)=\frac{(xy)^{\lambda+1/2}}{t^{2\lambda+2}\sqrt{\pi}2^\lambda\Gamma(\lambda+1/2)}\int_0^{\pi/2}\theta^{2\lambda}\Phi\Big(\frac{(x-y)^2+2xy(1-\cos\theta)}{t^2}\Big)d\theta.$$

We consider

$$H_{1,2}(t,x,y)=\frac{(xy)^{\lambda+1/2}}{t^{2\lambda+2}\sqrt{\pi}2^\lambda\Gamma(\lambda+1/2)}\int_0^{\pi/2}\theta^{2\lambda}\Phi\Big(\frac{(x-y)^2+xy\theta^2)}{t^2}\Big)d\theta.$$

By using again mean value theorem, since Φ belongs to the Schwartz class, we get

$$\begin{aligned}
&|H_{1,1}(t,x,y) - H_{1,2}(t,x,y)| \\
&\leqslant C\frac{(xy)^{\lambda+1/2}}{t^{2\lambda+2}}\int_0^{\pi/2}\theta^{2\lambda}\Big|\Phi\Big(\frac{(x-y)^2+2xy(1-\cos\theta)}{t^2}\Big)-\Phi\Big(\frac{(x-y)^2+xy\theta^2}{t^2}\Big)\Big|\mathrm{d}\theta \\
&\leqslant C\frac{(xy)^{\lambda+1/2}}{t^{2\lambda+2}}\int_0^{\pi/2}\theta^{2\lambda}\Big|\frac{xy(1-\cos\theta-\theta^2/2)}{t^2}\Big|\Big(\frac{t^2}{(x-y)^2+xy\theta^2}\Big)^{\lambda+2}\mathrm{d}\theta \\
&\leqslant C(xy)^{\lambda+3/2}\int_0^{\pi/2}\theta^{2\lambda+4}(xy\theta^2)^{-\lambda-2}\mathrm{d}\theta \leqslant C\frac{1}{x}.
\end{aligned}$$

By making the change of variables $u = xy\theta^2/t^2$ it has that

$$\begin{aligned}
H_{1,2}(t,x,y) &= \frac{1}{t\sqrt{\pi}2^{\lambda+1}\Gamma(\lambda+1/2)}\int_0^{xy\pi^2/(4t^2)} u^{\lambda-1/2}\Phi\Big(\Big(\frac{x-y}{t}\Big)^2+u\Big)\mathrm{d}u \\
&= \frac{1}{t\sqrt{\pi}2^{\lambda+1}\Gamma(\lambda+1/2)}\Big(\int_0^{\infty}-\int_{xy\pi^2/(4t^2)}^{\infty}\Big)u^{\lambda-1/2}\Phi\Big(\Big(\frac{x-y}{t}\Big)^2+u\Big)\mathrm{d}u \\
&= H_{1,3}(t,x,y) - H_{1,4}(t,x,y).
\end{aligned}$$

We can write

$$\begin{aligned}
|H_{1,4}(t,x,y)| &\leqslant C\frac{1}{t}\int_{xy\pi^2/(4t^2)}^{\infty} u^{\lambda-1/2}\Big|\Phi\Big(\Big(\frac{x-y}{t}\Big)^2+u\Big)\Big|\mathrm{d}u \\
&\leqslant C\frac{1}{t}\int_{xy\pi^2/(4t^2)}^{\infty} u^{\lambda-1/2}\Big(\Big(\frac{x-y}{t}\Big)^2+u\Big)^{-\lambda-1}\mathrm{d}u \leqslant C\frac{1}{t}\int_{xy\pi^2/(4t^2)}^{\infty} u^{-3/2}\mathrm{d}u \leqslant C\frac{1}{x}.
\end{aligned}$$

Note that $H_{1,3}(t,x,y) = \Psi_t(x-y)$, $x \in \mathbb{R}$ and $t > 0$. Since Φ is in the Schwartz class, Ψ also is in the Schwartz class. Moreover, Ψ is even. Then

$$|\Psi(x-y)| \leqslant C\begin{cases} \frac{1}{y} & 0 < 2x < y, \\ \frac{1}{x} & 0 < y < \frac{x}{2}. \end{cases}$$

By the above estimates, defining

$$\Psi_t(x) = \frac{1}{t}\Psi\Big(\frac{x}{t}\Big), \quad x \in \mathbb{R},\ t > 0, \quad \text{and} \quad \Psi_t *_+ f(x) = \int_0^{\infty} f(y)\Psi_t(x-y)\mathrm{d}y, \quad x,t > 0,$$

we get, for every $J \subset \mathbb{Q}_+$,

$$\begin{aligned}
&\Big\| \sup_{t\in J} |\Psi_t *_+ f(x)| - \sup_{t\in J} |\phi_{(t)} \#_\lambda f(x)| \Big\|_X \\
&\leqslant C\Big(I_0(\|f\|_X)(x) + I_\infty(\|f\|_X)(x) \\
&\qquad + \Big\| \sup_{t\in J} \Big| \int_{x/2}^{2x} \Psi_t(x-y) f(y) \mathrm{d}y \Big| - \sup_{t\in J} \Big| \int_{x/2}^{2x} f(y)_\lambda \tau_x(\phi_{(t)})(y) \mathrm{d}y \Big| \Big\|_X \Big) \\
&\leqslant C(I_0(\|f\|_X)(x) + I_\infty(\|f\|_X)(x) + N(\|f\|_X)(x)), \quad x \in (0,\infty),
\end{aligned}$$

where, for every $x \in (0,\infty)$

$$I_0(g)(x) = \frac{1}{x}\int_0^x g(y)\mathrm{d}y, \quad I_\infty(g)(y) = \int_x^\infty \frac{g(y)}{y}\mathrm{d}y, \quad N(g)(y) = \int_{x/2}^{2x} \frac{1}{y} g(y)\mathrm{d}y.$$

Jensen inequality allows us to see that N is a bounded operator from $L^p(0,\infty)$ into itself, for every $1 \leqslant p < \infty$. Moreover, the well-known Hardy inequalities ([21]) say that the operators I_0 and I_∞ are bounded from $L^p(0,\infty)$ into itself, for every $1 < p < \infty$, and that they are bounded from $L^1(0,\infty)$ into $L^{1,\infty}(0,\infty)$.

Thus the equivalence (a) $\Leftrightarrow$ (b) is established.

Also, if $1 \leqslant p < \infty$ and $J \subset \mathbb{Q}_+$, the following assertions are equivalent:

(c) The operator $\mathfrak{M}_{\Psi,J,+}$ is of strong type (respectively, weak type) (p,p), $1 < p < \infty$, (respectively $(1,1)$) with respect to $((0,\infty), \mathrm{d}x)$.

(d) The operator $\mathfrak{M}_{\Psi,J}$ is of strong type (respectively, weak type) (p,p), $1 < p < \infty$, (respectively, $(1,1)$) with respect to $(\mathbb{R}, \mathrm{d}x)$.

To see that, we write

$$\begin{aligned}
&\int_{-\infty}^\infty \Big\| \sup_{t\in J} \Big| \frac{1}{t}\int_{-\infty}^\infty \Psi\Big(\frac{|x-y|}{t}\Big) f(y)\mathrm{d}y \Big| \Big\|_X^p \mathrm{d}x \\
&\leqslant \int_0^\infty \Big\| \sup_{t\in J} \Big| \frac{1}{t}\int_0^\infty \Psi\Big(\frac{|x-y|}{t}\Big) f(y)\mathrm{d}y \Big| \Big\|_X^p \mathrm{d}x + \int_0^\infty \Big\| \sup_{t\in J} \Big| \frac{1}{t}\int_0^\infty \Psi\Big(\frac{|x+y|}{t}\Big) f(y)\mathrm{d}y \Big| \Big\|_X^p \mathrm{d}x \\
&\quad + \int_0^\infty \Big\| \sup_{t\in J} \Big| \frac{1}{t}\int_0^\infty \Psi\Big(\frac{|x-y|}{t}\Big) f(-y)\mathrm{d}y \Big| \Big\|_X^p \mathrm{d}x + \int_0^\infty \Big\| \sup_{t\in J} \Big| \frac{1}{t}\int_0^\infty \Psi\Big(\frac{|x+y|}{t}\Big) f(-y)\mathrm{d}y \Big| \Big\|_X^p \mathrm{d}x.
\end{aligned}$$

Moreover, since Ψ is in the Schwartz class, it has

$$\Big| \Psi\Big(\frac{x+y}{t}\Big) \Big| \leqslant C \begin{cases} \frac{t}{y} & y > x, \\ \frac{t}{x} & 0 < y < x. \end{cases}$$

Then, the operator

$$f \to \sup_{t>0} \Big| \frac{1}{t} \int_0^\infty \Psi\Big(\frac{x+y}{t}\Big) f(y)\mathrm{d}y \Big|, \quad x \in (0,\infty)$$

is bounded from $L^p_X(0,\infty)$ into itself, when $1 < p < \infty$.

Hence, if $\mathfrak{M}_{\Psi,J,+}$ is bounded from $L^p_X(0,\infty)$ into itself, then $\mathfrak{M}_{\Psi,J}$ is bounded from $L^p_X(\mathbb{R})$ into itself, when $1 < p < \infty$. The converse property is clear.

The corresponding property for $p = 1$ can be seen in a similar way.

Note that, since ψ has compact support, Ψ also has compact support. Moreover, $\Psi(x) > 0$, $|x| \leqslant b$, for a certain $b \in \mathbb{Q}^+$, because ϕ is nonnegative and not identically zero. Then, there exists $C > 0$ such that

$$\chi_{(x-bt,x+bt)}(y) \leqslant C\Psi\Big(\frac{|x-y|}{t}\Big), \quad x, y \in \mathbb{R},\ t > 0.$$

Assume that J is a finite subset of $\mathbb{Q}_+$. Then, if $f : \mathbb{R} \to X$ is a nonnegative function

$$\mathfrak{M}_{J_b}(f)(x) \leqslant C\mathfrak{M}_{\Psi,J}(f)(x), \quad x \in \mathbb{R}, \tag{2.1}$$

where $J_b = \{bq : q \in J\}$. Moreover, if $f : \mathbb{R} \to X$ is a nonnegative function, we can write, for a certain $a \in \mathbb{Q}_+$,

$$\begin{aligned}\frac{1}{t}\int_{\mathbb{R}} \Psi\Big(\frac{x-y}{t}\Big) f(y)\mathrm{d}y &= \frac{1}{t}\int_{|x-y|<at} \Psi\Big(\frac{x-y}{t}\Big) f(y)\mathrm{d}y \\ &\leqslant \frac{C}{t}\int_{|x-y|<at} f(y)\mathrm{d}y \leqslant C\mathfrak{M}_{J_a}(f)(x), \quad x \in \mathbb{R}.\end{aligned}$$

Then

$$\mathfrak{M}_{\Psi,J}(f)(x) \leqslant C\mathfrak{M}_{J_a}(f)(x), \quad x \in \mathbb{R}. \tag{2.2}$$

Since $\| |b| \|_X = \|b\|_X$, for every $b \in X$, we conclude from (2.1) and (2.2) that $\mathfrak{M}_{\Psi,J}$ is a bounded operator from $L^p_X(\mathbb{R})$, $1 < p < \infty$, (respectively, $L^1_X(\mathbb{R})$) into itself (respectively, $L^{1,\infty}_X(\mathbb{R})$), for every finite subset J of $\mathbb{Q}_+$, if and only if $\mathfrak{M}_J$ is a bounded operator from $L^p_X(\mathbb{R})$, $1 < p < \infty$, (respectively, $L^1_X(\mathbb{R})$) into itself (respectively, $L^{1,\infty}_X(\mathbb{R})$), for every finite subset J of $\mathbb{Q}_+$.

Thus the proof of Theorem 1.1 is complete by using Theorem 1.7 of [11]. ∎

Proof of Theorem 1.2. Suppose that X is a Köthe function space such that X' is a norming subspace of X^* and $f : \mathbb{R} \to X$ is a nonnegative function belonging to $L^p(\mathbb{R}) \otimes X$, $1 \leqslant p < \infty$. As in (2.1), for a certain $b > 0$, we have

$$\mathfrak{M}_{J_b}(f)(x,w) \leqslant C\mathfrak{M}_{\Psi,J}(f)(x,w), \quad x \in \mathbb{R},\ w \in \Omega, \tag{2.3}$$

for every finite subset J of $\mathbb{Q}_+$. Then

$$\mathfrak{M}(f)(x,w) \leqslant C\mathfrak{M}_\Psi(f)(x,w), \quad x \in \mathbb{R}.$$

Also, we can write

$$
\begin{aligned}
&\Big|\frac{1}{t}\int_{\mathbb{R}}\Psi\Big(\frac{x-y}{t}\Big)f(y,w)\mathrm{d}y\Big| \\
&\leqslant\sum_{k=0}^{\infty}\Big|\frac{1}{t}\int_{2^k t<|x-y|\leqslant 2^{k+1}t}\Psi\Big(\frac{x-y}{t}\Big)f(y,w)\mathrm{d}y\Big|+\Big|\frac{1}{t}\int_{|x-y|\leqslant t}\Psi\Big(\frac{x-y}{t}\Big)f(y,w)\mathrm{d}y\Big| \\
&\leqslant C\Big(\sum_{k=0}^{\infty}\frac{1}{t}\int_{2^k t<|x-y|\leqslant 2^{k+1}t}\Big(1+\frac{|x-y|}{t}\Big)^{-2}f(y,w)\mathrm{d}y+\frac{1}{t}\int_{|x-y|\leqslant t}\Big(1+\frac{|x-y|}{t}\Big)^{-2}f(y,w)\mathrm{d}y\Big) \\
&\leqslant C\Big(\sum_{k=0}^{\infty}\frac{1}{2^{2k}t}\int_{|x-y|\leqslant 2^{k+1}t}f(y,w)\mathrm{d}y+\frac{1}{t}\int_{|x-y|\leqslant t}f(y,w)\mathrm{d}y\Big) \\
&\leqslant C\sup_{r\in\mathbb{Q}_+}\frac{1}{r}\int_{|x-y|<r}f(y,w)\mathrm{d}y, \quad t\in\mathbb{Q}_+,\ x\in\mathbb{R},\ w\in\Omega.
\end{aligned}
$$

Then, we obtain that

$$
\mathfrak{M}_{\Psi}(f)(x,w)\leqslant C\mathfrak{M}(f)(x,w), \quad x\in\mathbb{R}. \tag{2.4}
$$

Now, by taking into account that for our Köthe function space to have the Hardy–Littlewood property is equivalent to the maximal operator $\mathfrak{M}$ is bounded from $L^p_X(\mathbb{R})$ into itself, for some $1<p<\infty$, or from $L^1_X(\mathbb{R})$ into $L^{1,\infty}_X(\mathbb{R})$, the arguments developed in the proof of Theorem 1.1 allow us to establish the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

To see that (i) $\Leftrightarrow$ (v) it is sufficient to proceed as in the proof of Proposition 4.12 in [16] by using (2.3) and (2.4).

The proof of Theorem 1.2 is thus finished. ∎

3. PROOF OF THEOREM 1.3

We now prove Theorem 1.3. Suppose firstly that X has the UMD property. We consider the partial g-Littlewood–Paley functions associated with $\{P_t^\lambda\}_{t>0}$ defined by

$$
g_{\lambda,1}(f)(x,w)=\Big(\int_0^\infty\Big|t\frac{\partial}{\partial t}P_t^\lambda(f)(x,w)\Big|^2\frac{\mathrm{d}t}{t}\Big)^{1/2},
$$

and

$$
g_{\lambda,2}(f)(x,w)=\Big(\int_0^\infty|tD_{\lambda,x}P_t^\lambda(f)(x,w)|^2\frac{\mathrm{d}t}{t}\Big)^{1/2}.
$$

It is clear that

$$
\|g_\lambda(f)\|_{L^p_X(0,\infty)}\leqslant C\|f\|_{L^p_X(0,\infty)},
$$

if and only if the same inequality holds when g_λ is replaced by $g_{\lambda,1}$ and $g_{\lambda,2}$.

In Lemmas 5.1 and 5.2 of [2] it was proved that

$$\Big\|t\frac{\partial}{\partial t}\Big(P_t^\lambda(x,y)-P_t(x,y)\Big)\Big\|_{L^2((0,\infty),\,dt/t)} \leqslant C\begin{cases}\frac{1}{y} & y>2x,\\ \frac{1}{y}\Big(1+\Big(\log\Big(1+\frac{xy}{|x-y|^2}\Big)\Big)^{1/2}\Big) & \frac{x}{2}<y<2x,\\ \frac{1}{x} & 0<y<\frac{x}{2},\end{cases} \tag{3.1}$$

where $P_t(x,y)=\dfrac{1}{\pi}\dfrac{t}{(x-y)^2+t^2}$, $t,x,y\in(0,\infty)$.

We can write, by using the Minkowski inequality

$$\begin{aligned} g_{\lambda,1}(f)(x,w) &= \Big(\int_0^\infty\Big|t\frac{\partial}{\partial t}\int_0^\infty P_t^\lambda(x,y)f(y,w)\,dy\Big|^2\frac{dt}{t}\Big)^{1/2}\\ &\leqslant \Big(\int_0^\infty\Big|t\int_0^\infty\frac{\partial}{\partial t}(P_t^\lambda(x,y)-P_t(x,y))f(y,w)\,dy\Big|^2\frac{dt}{t}\Big)^{1/2}\\ &\quad+\Big(\int_0^\infty\Big|t\int_0^\infty\frac{\partial}{\partial t}P_t(x,y)f(y,w)\,dy\Big|^2\frac{dt}{t}\Big)^{1/2}\\ &\leqslant \int_0^\infty|f(y,w)|\Big\|t\frac{\partial}{\partial t}(P_t^\lambda(x,y)-P_t(x,y))\Big\|_{L^2((0,\infty),dt/t)}dy+\mathbf{g}_1(f)(x,w), \end{aligned}$$

being

$$\mathbf{g}_1(f)(x,w)=\Big(\int_0^\infty\Big|t\frac{\partial}{\partial t}\int_0^\infty P_t(x,y)f(y,w)\,dy\Big|^2\frac{dt}{t}\Big)^{1/2}.$$

From (3.1) it follows that, for every $1\leqslant p<\infty$, there exists $C_p>0$ for which

$$\Big\|\int_0^\infty|f(y,w)|\Big\|t\frac{\partial}{\partial t}(P_t^\lambda(x,y)-P_t(x,y))\Big\|_{L^2((0,\infty),dt/t)}dy\Big\|_{L^p_X(0,\infty)}\leqslant C_p\|f\|_{L^p_X(0,\infty)},$$

for each $f\in L^p_X(0,\infty)$.

Also, it has that

$$\begin{aligned}
g_{\lambda,2}(f)(x,w) &= \Big(\int_0^\infty \Big| t\int_0^\infty D_{\lambda,x}P_t^\lambda(x,y)f(y,w)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t}\Big)^{1/2}\\
&\leqslant \Big(\int_0^\infty \Big|\int_0^\infty \Big(tD_{\lambda,x}P_t^\lambda(x,y) - t\frac{\partial}{\partial x}P_t(x,y)\Big)f(y,w)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t}\Big)^{1/2}\\
&\quad + \Big(\int_0^\infty \Big| t\int_0^\infty \frac{\partial}{\partial x}P_t(x,y)f(y,w)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t}\Big)^{1/2}\\
&\leqslant \int_0^\infty |f(y,w)|\Big\|t\Big(D_{\lambda,x}P_t^\lambda(x,y)-\frac{\partial}{\partial x}P_t(x,y)\Big)\Big\|_{L^2((0,\infty),\mathrm{d}t/t)}\mathrm{d}y+\mathbf{g}_2(f)(x,w),
\end{aligned}$$

where

$$\mathbf{g}_2(f)(x,w) = \Big(\int_0^\infty \Big|t\frac{\partial}{\partial x}\int_0^\infty P_t(x,y)f(y,w)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t}\Big)^{1/2}.$$

From Lemmas 5.4 and 5.5 of [2] we deduce that, for every $1 \leqslant p < \infty$, there exists $C_p > 0$ such that

$$\Big\|\int_0^\infty |f(y,w)|\Big\|t\Big(D_{\lambda,x}P_t^\lambda(x,y)-\frac{\partial}{\partial x}P_t(x,y)\Big)\Big\|_{L^2((0,\infty),\mathrm{d}t/t)}\mathrm{d}y\Big\|_{L^p_X(0,\infty)}\leqslant C_p\|f\|_{L^p_X(0,\infty)},$$

for each $f \in L^p_X(0,\infty)$.

Then, g_λ is bounded from $L^p_X(0,\infty)$ into itself, with $1 < p < \infty$, provided that $\mathbf{g}_j$, $j = 1,2$, are bounded from $L^p_X(0,\infty)$ into itself.

To analyze $\mathfrak{g}_\lambda$ we proceed in a similar way. We define

$$\mathfrak{g}_{\lambda,1}(f)(x,w) = \Big(\int_0^\infty \Big|t\frac{\partial}{\partial t}\mathbb{P}_t^\lambda(f)(x,w)\Big|^2\frac{\mathrm{d}t}{t}\Big)^{1/2},$$

and

$$\mathfrak{g}_{\lambda,2}(f)(x,w) = \Big(\int_0^\infty |tD^*_{\lambda,x}\mathbb{P}_t^\lambda(f)(x,w)|^2\frac{\mathrm{d}t}{t}\Big)^{1/2}.$$

By using (5.3.5) of [18] we obtain

$$D^*_{\lambda,x}\mathbb{P}_t^\lambda(x,y) = D_{\lambda,y}P_t^\lambda(y,x), \quad t,x,y \in (0,\infty).$$

Hence, since $\mathfrak{g}_{\lambda,1} = g_{\lambda+1,1}$ and by using again Lemmas 5.4 and 5.5 of [2] and symmetries, from the above arguments we deduce that $\mathfrak{g}_{\lambda,1}$ (respectively, $\mathfrak{g}_{\lambda,2}$) is bounded from $L^p_X(0,\infty)$ into itself, with $1 < p < \infty$, provided that $\mathbf{g}_1$ (respectively, $\mathbf{g}_2$) is bounded from $L^p_X(0,\infty)$ into itself.

Also it is not hard to see that $\mathbf{g}_1$ (respectively, $\mathbf{g}_2$) is bounded from $L^p_X(0,\infty)$ into itself, with $1 < p < \infty$, if and only if G_1 (respectively, G_2) is bounded from $L^p_X(\mathbb{R})$ into itself, where

$$G_1(f)(x,w) = \Big(\int_0^\infty \Big| t\frac{\partial}{\partial t} \int_{-\infty}^\infty P_t(x,y) f(y,w) \Big|^2 \frac{dt}{t} \Big)^{1/2}$$

and

$$G_2(f)(x,w) = \Big(\int_0^\infty \Big| t\frac{\partial}{\partial x} \int_{-\infty}^\infty P_t(x,y) f(y,w) \Big|^2 \frac{dt}{t} \Big)^{1/2}.$$

It is known that the (sublinear) operators

$$\mathfrak{G}_1(f)(x) = \Big(\int_0^\infty \Big| t\frac{\partial}{\partial t} \int_{-\infty}^\infty P_t(x,y) f(y) \Big|^2 \frac{dt}{t} \Big)^{1/2}, \quad x \in \mathbb{R},$$

and

$$\mathfrak{G}_2(f)(x) = \Big(\int_0^\infty \Big| t\frac{\partial}{\partial x} \int_{-\infty}^\infty P_t(x,y) f(y) \Big|^2 \frac{dt}{t} \Big)^{1/2}, \quad x \in \mathbb{R},$$

are bounded from $L^p(\mathbb{R}, u(x)dx)$ into itself, for every $1 < p < \infty$ and $u \in A_p$, where as usual A_p denotes the Muckenhoupt class of weights. Then, according to Theorem 5 of [23] G_j, $j = 1,2$, are bounded from $L^p_X(\mathbb{R})$ into itself, for every $1 < p < \infty$.

We have proved that for every $1 < p < \infty$ there exists $C_p > 0$ such that

$$\|g_\lambda(f)\|_{L^p_X(0,\infty)} \leqslant C_p \|f\|_{L^p_X(0,\infty)}, \quad f \in L^p_X(0,\infty),$$

and

$$\|\mathfrak{g}_\lambda(f)\|_{L^p_X(0,\infty)} \leqslant C_p \|f\|_{L^p_X(0,\infty)}, \quad f \in L^p_X(0,\infty).$$

By Proposition 6.1 of [2] we have that, for every $f, g \in L^2(0,\infty)$,

$$\frac{1}{4} \int_0^\infty f(x) g(x) dx = \int_0^\infty \int_0^\infty t\frac{\partial}{\partial t}(P_t^\lambda(f)(x)) t\frac{\partial}{\partial t}(P_t^\lambda(g)(x)) \frac{dt}{t} dx. \tag{3.2}$$

Let $1 < p < \infty$. Since both of the sides define bounded bilinear operators from $L^p(0,\infty) \times L^{p'}(0,\infty)$ into $\mathbb{C}$, where p' denotes the exponent conjugate of p, we conclude that (3.2) holds for every $f \in L^p(0,\infty)$ and $g \in L^{p'}(0,\infty)$. Hence (3.2) holds for every $f \in L^p(0,\infty) \otimes X$ and $g \in L^{p'}(0,\infty) \otimes X^*$, that is, if $n \in \mathbb{N}$,

$f_i \in L^p(0,\infty)$, $g_i \in L^{p'}(0,\infty)$, $\alpha_i \in X$ and $\beta_i \in X^*$, $i = 1, 2, \dots, n$, then

$$\frac{1}{4}\int_0^\infty \Big\langle \sum_{i=1}^n f_i(x)\alpha_i, \sum_{m=1}^n g_m(x)\beta_m \Big\rangle \mathrm{d}x$$
$$= \int_0^\infty \int_0^\infty \Big\langle \sum_{i=1}^n t\frac{\partial}{\partial t}P_t^\lambda(f_i)(x)\alpha_i, \sum_{m=1}^n t\frac{\partial}{\partial t}P_t^\lambda(g_m)(x)\beta_m \Big\rangle \frac{\mathrm{d}t}{t}\mathrm{d}x.$$

By taking into account that X' is an UMD space (here $X' = X^*$) and $L^{p'}(0,\infty) \otimes X'$ is dense in $L_{X'}^{p'}(0,\infty)$ we conclude that,

$$\|f\|_{L_X^p(0,\infty)} \leqslant C\|g_{\lambda,1}(f)\|_{L_X^p(0,\infty)}, \quad f \in L^p(0,\infty) \otimes X.$$

Again density implies that

$$\|f\|_{L_X^p(0,\infty)} \leqslant C\|g_{\lambda,1}(f)\|_{L_X^p(0,\infty)}, \quad f \in L_X^p(0,\infty).$$

Thus the proof of (i) $\Rightarrow$ (iii) is finished.

Suppose now that, for some $1 < p < \infty$, the inequalities (1.5) and (1.6) hold. According to Theorem 2.1 of [2] to see that X has the UMD property it is sufficient to prove that the Riesz transform R_λ^*, the adjoint of R_λ, associated to the Bessel operator S_λ can be extended to $L_X^p(0,\infty)$ as a bounded operator from $L_X^p(0,\infty)$ into itself.

Let $f \in L^2(0,\infty)$. Then

$$P_t^\lambda(R_\lambda^* f) = \mathbb{Q}_t^\lambda(f).$$

Hence according to Cauchy–Riemann equations (1.3) and (1.4) we obtain

$$\frac{\partial}{\partial t}P_t^\lambda(R_\lambda^* f) = \frac{\partial}{\partial t}\mathbb{Q}_t^\lambda(f) = D_{\lambda,x}^*\mathbb{P}_t^\lambda(f), \tag{3.3}$$

and

$$D_{\lambda,x}P_t^\lambda(R_\lambda^* f) = D_{\lambda,x}\mathbb{Q}_t^\lambda(f) = \frac{\partial}{\partial t}\mathbb{P}_t^\lambda(f). \tag{3.4}$$

If we extend R_λ^* to $L^2(0,\infty) \otimes X$ in the obvious way, then $R_\lambda^*(f) \in L^2(0,\infty) \otimes X$, for every $f \in L^2(0,\infty) \otimes X$, and the formulas (3.3) and (3.4) hold for every $f \in L^2(0,\infty) \otimes X$. Thus we conclude that

$$g_\lambda(R_\lambda^* f) = \mathfrak{g}_\lambda(f), \quad f \in L^2(0,\infty) \otimes X.$$

By combining inequalities (1.5)–(1.6) we get, for every $f \in (L^2(0,\infty) \cap L^p(0,\infty)) \otimes X$,

$$\frac{1}{C}\|R_\lambda^* f\|_{L_X^p(0,\infty)} \leqslant \|g_\lambda(R_\lambda^* f)\|_{L_X^p(0,\infty)} = \|\mathfrak{g}_\lambda(f)\|_{L_X^p(0,\infty)} \leqslant C\|f\|_{L_X^p(0,\infty)}.$$

Hence, since $(L^2(0,\infty) \cap L^p(0,\infty)) \otimes X$ is dense in $L_X^p(0,\infty)$, R_λ^* can be extended to $L_X^p(0,\infty)$ as a bounded operator from $L_X^p(0,\infty)$ into itself.

Thus it is established that (ii) $\Rightarrow$ (i) and the proof of Theorem 1.3 finishes.

Acknowledgements. The authors would like to thank to José Luis Torrea for introducing us to vector valued harmonic analysis.

REFERENCES

[1] J.J. BETANCOR, D. BURACEWSKI, J.C. FARIÑA, M.T. MARTÍNEZ, J.L. TORREA, Riesz transforms related to Bessel operators, *Proc. Roy. Soc. Edinburgh* **137**(2007), 701–725.

[2] J.J. BETANCOR, J.C. FARIÑA, M.T. MARTÍNEZ, J.L. TORREA, Riesz transform and *g*-function associated with Bessel operators and their appropriate Banach spaces, *Israel J. Math.* **157**(2007), 259–282.

[3] J.J. BETANCOR, J.C. FARIÑA, L. RODRÍGUEZ-MESA, A. SANABRIA, J.L. TORREA, Transference between Laguerre and Hermite settings, *J. Funct. Anal.* **254**(2008), 826–850.

[4] J.J. BETANCOR, K. STEMPAK, On Hankel conjugate functions, *Studia Sci. Math. Hungar.* **41**(2004), 59–91.

[5] J. BOURGAIN, Some remarks on Banach spaces in which martingale difference sequences are unconditional, *Ark. Mat.* **21**(1983), 163–168.

[6] J. BOURGAIN, Extension of a result of Benedek, Calderón and Panzone, *Ark. Mat.* **22**(1984), 91–95.

[7] J. BOURGAIN, Vector valued singular integrals and the H^1-BMO duality, in *Probability Theory and Harmonic Analysis*, Decker, New York 1986, pp. 1–19.

[8] D.L. BURKHOLDER, A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional, *Ann. Probab.* **9**(1981), 997–1011.

[9] D.L. BURKHOLDER, A geometric condition that implies the existence of certain singular integrals of Banach space-valued functions, *Conference on Harmonic Analysis in Honor of A. Zygmund, Vol. I, II (Chicago, Ill., 1981)*, Wadsworth Math. Ser., Wadsworth, Inc., Belmont, CA 1983, pp. 270–286.

[10] S.J.L. VAN EIJNDHOVEN, J. DE GRAAF, Some results on Hankel invariant distribution spaces, *Nederl. Akad. Wetensch. Indag. Math.* **45**(1983), 77–87.

[11] J. GARCÍA-CUERVA, R. MACÍAS, J.L. TORREA, The Hardy–Littlewood property of Banach lattices, *Israel J. Math.* **83**(1993), 177–201.

[12] J. GARCÍA-CUERVA, R. MACÍAS, J.L. TORREA, Maximal operators and BMO for Banach lattices, *Proc. Edinburgh Math. Soc.* **41**(1998), 585–690.

[13] J. GOSSELIN, K. STEMPAK, A weak-type estimate for Fourier–Bessel multipliers, *Proc. Amer. Math. Soc.* **106**(1989), 655–662.

[14] D.T. HAIMO, Integral equations associated with Hankel convolutions, *Trans. Amer. Math. Soc.* **116**(1965), 330–375.

[15] E. HARBOURE, R. MACÍAS, C. SEGOVIA, J.L. TORREA, Some estimates for maximal functions on Köthe function spaces, *Israel J. Math.* **90**(1995), 349–371.

[16] E. HARBOURE, J.L. TORREA, B. VIVIANI, Vector-valued extensions of operators related to the Ornstein–Uhlenbeck semigroup, *J. Anal. Math.* **91**(2003), 1–29.

[17] I.I. HIRSCHMAN, JR., Variation diminishing Hankel transforms, *J. Anal. Math.* **8**(1960/1961), 307–336.

[18] N.N. LEBEDEV, *Special Functions and their Applications*, Revised edition, Dover Publ. Inc., New York 1972.

[19] J. LINDENSTRAUSS, L. TZAFRINI, *Classical Banach Spaces*. II. *Function Spaces*, Springer-Verlag, Berlin 1979.

[20] T. MARTÍNEZ, J.L. TORREA, Q. XU, Vector-valued Littlewood–Paley–Stein theory for semigroups, *Adv. Math.* **203**(2006), 430–475.

[21] B. MUCKENHOUPT, Hardy's inequality with weights, *Studia Math.* **44**(1972), 31–38.

[22] B. MUCKENHOUPT, E. STEIN, Classical expansions and their relation to conjugate harmonic functions, *Trans. Amer. Math. Soc.* **118**(1965), 17–92.

[23] J.L. RUBIO DE FRANCIA, Martingale and integral transforms of Banach space valued functions, *Lecture Notes in Math.* **1221**(1986), 195–222.

[24] K. STEMPAK, The Littlewood–Paley theory for the Fourier–Bessel transform, preprint, Mathematical Institute of Wroclaw, n. 45, Wroclaw, Poland 1985.

[25] Q. XU, Littlewood–Paley theory for functions with values in uniformly convex spaces, *J. Reine Angew. Math.* **504**(1998), 195–226.

JORGE J. BETANCOR, DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE LA LAGUNA, CAMPUS DE ANCHIETA, AVDA. ASTROFÍSICO FCO. SÁNCHEZ, S/N, 38271 LA LAGUNA (STA. CRUZ DE TENERIFE), SPAIN
E-mail address: jbetanco@ull.es

JUAN C. FARIÑA, DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE LA LAGUNA, CAMPUS DE ANCHIETA, AVDA. ASTROFÍSICO FCO. SÁNCHEZ, S/N, 38271 LA LAGUNA (STA. CRUZ DE TENERIFE), SPAIN
E-mail address: jcfarina@ull.es

LOURDES RODRíGUEZ-MESA, DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE LA LAGUNA, CAMPUS DE ANCHIETA, AVDA. ASTROFÍSICO FCO. SÁNCHEZ, S/N, 38271 LA LAGUNA (STA. CRUZ DE TENERIFE), SPAIN
E-mail address: lrguez@ull.es

Received June 6, 2009.