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SEMICROSSED PRODUCTS AND REFLEXIVITY

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ABSTRACT. Given a w*-closed unital algebra $\mathcal{A}$ acting on H_0 and a contractive w*-continuous endomorphism β of $\mathcal{A}$, there is a w*-closed (non-selfadjoint) unital algebra $\mathbb{Z}_+ \overline{\times}_\beta \mathcal{A}$ acting on $H_0 \otimes \ell^2(\mathbb{Z}_+)$, called the w*-semicrossed product of $\mathcal{A}$ with β. We prove that $\mathbb{Z}_+ \overline{\times}_\beta \mathcal{A}$ is a reflexive operator algebra provided $\mathcal{A}$ is reflexive and β is unitarily implemented, and that $\mathbb{Z}_+ \overline{\times}_\beta \mathcal{A}$ has the bicommutant property if and only if so does $\mathcal{A}$. Also, we show that the w*-semicrossed product generated by a commutative C^*-algebra and a continuous map is reflexive.

KEYWORDS: *C^*-envelope, reflexive subspace, semicrossed product.*

MSC (2000): Primary 47L65; Secondary 47L75.

INTRODUCTION

As is well known, to construct the C^*-crossed product of a unital C^*-algebra $\mathcal{C}$ by a $*$-isomorphism $\alpha : \mathcal{C} \to \mathcal{C}$, we begin with the Banach space $\ell^1(\mathbb{Z}, \mathcal{C}, \alpha)$ which is the closed linear span of the monomials $\delta_n \otimes x$, $n \in \mathbb{Z}$, $x \in \mathcal{C}$, under the norm $\Big| \sum\limits_{n=-k}^{k} \delta_n \otimes x_n \Big|_1 = \sum\limits_{n=-k}^{k} \|x_n\|_{\mathcal{C}}$, equipped with the (isometric) involution $(\delta_n \otimes x)^* = \delta_{-n} \otimes \alpha^{-n}(x^*)$. Now, there are two "natural" ways to define multiplication in $\ell^1(\mathbb{Z}, \mathcal{C}, \alpha)$; either the left multiplication $(\delta_n \otimes x) *_1 (\delta_m \otimes y) = \delta_{n+m} \otimes a^m(x)y$, or the right one $(\delta_n \otimes x) *_{\mathrm{r}} (\delta_m \otimes y) = \delta_{n+m} \otimes x a^n(y)$. Then the corresponding algebras are isometrically $*$-isomorphic via the map $\Psi(\delta_n \otimes x) = \delta_{-n} \otimes a^{-n}(x)$. We can see that $(\ell^1(\mathbb{Z}, \mathcal{C}, \alpha)_1)^{\mathrm{opp}} = \ell^1(\mathbb{Z}, \mathcal{C}^{\mathrm{opp}}, \alpha)_{\mathrm{r}}$, where for an algebra $\mathcal{B}$, $\mathcal{B}^{\mathrm{opp}}$ is the space $\mathcal{B}$ along with the multiplication $x \odot y := yx$; hence, in case $\mathcal{C}$ is commutative, each algebra is the opposite of the other. The left and right crossed product are the completion of the corresponding involutive Banach algebras under a universal norm induced by the $|\cdot|_1$-contractive $*$-representations (hence, they are C^*-algebras characterized by a universal property) and the map

Ψ extends to a C^*-isomorphism. Moreover, it can be proved that the crossed product is $*$-isomorphic to the reduced crossed product $C_1^*(\mathcal{C})$, i.e. the norm closure of the range of the left regular representation, and thus we end up with just one object to which we refer as *the crossed product of the dynamical system* $(\mathcal{C}, \alpha)$. The key fact is that there is a bijection between the $|\cdot|_1$-contractive $*$-representations of each of these ℓ^1-algebras and the (left or right) covariant unitary pairs (see Section 1).

If we wish to construct a non-selfadjoint analogue, we can see that there are more possibilities. For example, Peters defined the semicrossed product as the completion of the Banach algebra $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_1$ under the universal norm that arises from the left covariant isometric pairs and examined the case when α is an injective $*$-endomorphism of $\mathcal{C}$. He proved that this semicrossed product embeds isometrically in a crossed product (see [12]) and, for the commutative case, that this crossed product is the C^*-envelope of the semicrossed product (see [13]).

In Section 1 we use an alternative definition using "sufficiently many" homomorphisms of the Banach algebra $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_1$ (see also [4]). The advantage is that there is a bijection between the left covariant contractive pairs and the homomorphisms of the Banach algebra $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_1$. Moreover, there is a duality between the left covariant contractive pairs and the right covariant contractive pairs, which induce the homomorphisms of the Banach algebra $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_{\rm r}$; hence, we get similar results for the right version. Also, using a dilation theorem of [10], we can see that this definition gives the one in [12]. If we consider the maximal operator space structure, then the semicrossed products are operator algebras with a universal property that characterizes them up to completely isometric isomorphism. In Theorem 1.4 we prove that the semicrossed product is independent of the way $\mathcal{C}$ is (faithfully) represented and in Theorem 1.5 we prove that in case α is a $*$-isomorphism, its C^*-envelope is exactly the crossed product. So, in order to define a w*-analogue of the semicrossed product that arises by a w*-continuous contractive endomorphism β of a w*-closed subalgebra $\mathcal{A}$ of some $\mathcal{B}(H_0)$ (for example, a von Neumann algebra), either we take the w*-closed linear span of a non-selfadjoint left regular representation or the w*-closed linear span of the analytic polynomials of the von Neumann crossed product, depending on the properties of β.

In Section 2 we analyze the properties of the w*-semicrossed product, in case β is unitarily implemented. First of all, we study the connection between the semicrossed product and the w*-tensor product $\mathcal{A}\overline{\otimes}\mathcal{T}$, where $\mathcal{T}$ is the algebra of the analytic Toeplitz operators, and give an example when these two algebras are incomparable. A main result of this section is the reflexivity of the w*-semicrossed product, when $\mathcal{A}$ is reflexive. Recall that a subspace $\mathcal{S} \subseteq B(H)$ is *reflexive* if it coincides with its *reflexive cover*, namely $\operatorname{Ref}(\mathcal{S}) = \{T \in \mathcal{B}(H) : T\xi \in \overline{\mathcal{S}\xi}, \text{for all } \xi \in \mathcal{H}\}$ (see [8]); unlike [8], we will call $\mathcal{S}$ *hereditarily reflexive* if every w*-closed subspace of $\mathcal{S}$ is reflexive. As a consequence we have that, when a unitary implementation condition holds, the w*-closed image of $\mathrm{l}t_\pi$ (see

Example 1.1) induced by a representation (H_0, π) of $\mathcal{C}$ is reflexive. Also, we get several known results as applications. As another main result, we prove that the w*-semicrossed product is the commutant of a w*-semicrossed product and is its own bicommutant if and only if the same holds for $\mathcal{A}$.

In the last section we consider the semicrossed product of a commutative C^*-algebra $C(K)$ with a continuous map $\phi : K \to K$. As observed in Theorem 1.4, the representations induced by a character of $C(K)$, say ev_t, $t \in K$, suffice to obtain the norm of the semicrossed product and play a significant role for its study. First, we show that the w*-closure of such representations is always reflexive; in fact, it has the form $(\mathfrak{T}P_{n_0}) \oplus (\mathcal{T}P_0) \oplus \cdots \oplus (\mathcal{T}P_{p-1})$, where $\mathfrak{T}$ is the algebra of lower triangular operators in $\mathcal{B}(\ell^2(\mathbb{Z}_+))$, $\mathcal{T}$ is the algebra of analytic Toeplitz operators and $P_{n_0}, P_0, \ldots, P_{p-1}$ some projections determined by the orbit of the point $t \in K$.

In what follows we use standard notation, as in [5] for example. $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and all infinite sums are considered in the strong-convergent sense. Throughout, we use the symbol v for the unilateral shift on $\mathcal{B}(\ell^2(\mathbb{Z}_+))$, given by $v(e_n) = e_{n+1}$. A useful tool for the proofs in Sections 2 and 3 is a *Féjer-type Lemma*; consider the unitary action of $\mathbb{T}$ on $H = H_0 \overline{\otimes} \ell^2(\mathbb{Z}_+)$ induced by the operators U_s, $s \in \mathbb{R}$, given by $U_s(\xi \otimes e_n) = \mathrm{e}^{ins} \xi \otimes e_n$. For every $T \in \mathcal{B}(H)$ and every $m \in \mathbb{Z}$ we define the *"m-Fourier coefficient"*

$$G_m(T) = \int_0^{2\pi} U_s T U_s^* \mathrm{e}^{-\mathrm{i}ms} \frac{\mathrm{d}s}{2\pi},$$

the integral taken as the w*-limit of Riemann sums. If we set

$$\sigma_l(T)(t) = \frac{1}{l+1} \sum_{n=0}^{l} \sum_{m=-n}^{n} G_m(T) \exp(\mathrm{i}mt),$$

then $\sigma_l(T)(0) \xrightarrow{\mathrm{w}^*} T$. Note that $G_m(\cdot)$ is w*-continuous for every $m \in \mathbb{Z}$.

Now, for every $\kappa, \lambda \in \mathbb{Z}_+$, and $T \in \mathcal{B}(H)$ let the *"matrix elements"* $T_{\kappa,\lambda} \in \mathcal{B}(H_0)$ be defined by $\langle T_{\kappa,\lambda}\xi, \eta\rangle = \langle T(\xi \otimes e_\lambda), \eta \otimes e_\kappa\rangle$, $\xi, \eta \in H_0$; then we can write the Fourier coefficients explicitly by the formula

$$G_m(T) = \begin{cases} V^m \Big(\sum_{n \geqslant 0} T_{m+n,n} \otimes p_n \Big) & \text{when } m \geqslant 0, \\ \Big(\sum_{n \geqslant 0} T_{n,-m+n} \otimes p_n \Big) (V^*)^{-m} & \text{when } m < 0, \end{cases}$$

where $V = 1_{H_0} \otimes v$. For simplicity, we define the diagonal matrices

$$T_{(m)} = \begin{cases} \sum_{n \geqslant 0} T_{m+n,n} \otimes p_n & \text{when } m \geqslant 0, \\ \sum_{n \geqslant 0} T_{n,-m+n} \otimes p_n & \text{when } m < 0. \end{cases}$$

Note that the sums converge in the w*-topology as well, since the partial sums are uniformly bounded by $\|T\|$. Hence, $G_m(T)$ is the *m-diagonal* of T, when we view H as the ℓ^2-sum of copies of H_0.

1. SEMICROSSED PRODUCTS OF C^*-ALGEBRAS

Let $\mathcal{C}$ be a unital C^*-algebra and $\alpha : \mathcal{C} \to \mathcal{C}$ a $*$-morphism; define $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)$ to be the closed linear span of the monomials $\delta_n \otimes x$, $n \in \mathbb{Z}_+$, $x \in \mathcal{C}$, under the norm

$$\Big| \sum_{n=0}^{k} \delta_n \otimes x_n \Big|_1 = \sum_{n=0}^{k} \|x_n\|_{\mathcal{C}}.$$

We endow $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)$ either with the left multiplication $(\delta_n \otimes x) *_{\mathrm{l}} (\delta_m \otimes y) = \delta_{n+m} \otimes a^m(x)y$, or with the right one $(\delta_n \otimes x) *_{\mathrm{r}} (\delta_m \otimes y) = \delta_{n+m} \otimes xa^n(y)$, and denote the corresponding Banach algebras by $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_{\mathrm{l}}$ and $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_{\mathrm{r}}$, respectively. One can see that $(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_{\mathrm{l}})^{\mathrm{opp}}$ is exactly $\ell^1(\mathbb{Z}_+, \mathcal{C}^{\mathrm{opp}}, \alpha)_{\mathrm{r}}$, where, if $\mathcal{B}$ is an algebra, $\mathcal{B}^{\mathrm{opp}}$ is the space $\mathcal{B}$ with the multiplication $x \odot y := yx$. Thus, in case $\mathcal{C}$ is commutative, each algebra is the opposite of the other.

Let (H, π) be a $*$-representation of $\mathcal{C}$ and T a contraction in $\mathcal{B}(H)$. The pair (π, T) is called a *left covariant contractive (l-cov.con.)* pair, if the left covariance relation is satisfied, i.e. $\pi(x)T = T\pi(\alpha(x))$, $x \in \mathcal{C}$. If, in particular, T is an isometry, pure isometry, co-isometry or unitary, then we will call such a pair *a left covariant isometric, purely isometric, co-isometric or unitary pair*. We can see that every l-cov.con. pair induces a $|\cdot|_1$-contractive representation $(H, T \times \pi)$ of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_{\mathrm{l}}$, given by

$$(T \times \pi)\Big(\sum_{n=0}^{k} \delta_n \otimes x_n \Big) = \sum_{n=0}^{k} T^n \pi(x_n).$$

Conversely, if $\rho : \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_{\mathrm{l}} \to \mathcal{B}(H)$ is a contractive representation, then (H, ρ) restricts to a contractive representation (H, π) of the C^*-algebra $\mathcal{C}$, thus a $*$-representation. If we set $\rho(\delta_1 \otimes e) = T$, then $\|T^n\| \leqslant 1$, for every $n \in \mathbb{Z}_+$. It is easy to check that the pair (π, T) satisfies the left covariance relation.

Analogously, there is a bijection between the *right covariant contractive (r-cov.con.)* pairs (π, T), (i.e. satisfying the right covariance condition $T\pi(x) = \pi(\alpha(x))T$, $x \in \mathcal{C}$) and the $|\cdot|_1$-contractive representations $\pi \times T$ of the algebra $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_{\mathrm{r}}$. Note that if (π, T) is a l-cov.con. pair then (π, T^*) is a r-cov.con. pair. Thus TT^* commutes with $\pi(\mathcal{C})$.

EXAMPLE 1.1. Let (H_0, π) be a faithful $*$-representation of $\mathcal{C}$ and define on $H_0 \otimes \ell^2(\mathbb{Z}_+)$ the representation $\widetilde{\pi}(x) = \mathrm{diag}\{\pi(\alpha^n(x)) : n \in \mathbb{Z}_+\}$ and $V = 1_{H_0} \otimes v$, where v is the unilateral shift. Then $(\widetilde{\pi}, V)$ is a l-cov.is. pair. For simplicity we will denote the corresponding representation $V \times \widetilde{\pi}$, by lt_π. As mentioned before, the pair $(\widetilde{\pi}, V^*)$ is a r-cov.con. pair which induces the representation $\mathrm{r}t_\pi := \widetilde{\pi} \times V^*$. One can check that $\mathrm{l}t_\pi$ and $\mathrm{r}t_\pi$ are faithful.

DEFINITION 1.2. The *(left) semicrossed product* $\mathbb{Z}_+ \times_\alpha \mathcal{C}$ is the completion of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_\mathrm{l}$ under the norm

$$\|F\|_\mathrm{l} = \sup\{\|(T \times \pi)(F)\| : (\pi, T) \text{ is a l-cov.con. pair}\}.$$

The *(right) semicrossed product* $\mathcal{C} \times_\alpha \mathbb{Z}_+$ is the completion of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_\mathrm{r}$ under the norm

$$\|F\|_\mathrm{r} = \sup\{\|(\pi \times T)(F)\| : (\pi, T) \text{ is a r-cov.con. pair}\}.$$

The left semicrossed product is endowed with an operator space structure (the maximal one, see 1.2.22 of [2]) induced by the matrix norms

$$\|[F_{i,j}]\|_\mathrm{l} = \sup\{\|[(T \times \pi)(F_{i,j})]\| : (\pi, T) \text{ l-con.cov. pair}\}.$$

We note that there is a bijective correspondence between the l-cov.con. pairs (π, T) and the unital completely contractive representations of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_\mathrm{l}$. So, the left semicrossed product has the following universal property (up to completely isometric isomorphisms): for any unital operator algebra $\mathcal{B}$ and for any unital completely contractive morphism $\rho : \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_\mathrm{l} \to \mathcal{B}$, there exists a unique unital completely contractive morphism $\widetilde{\rho} : \mathbb{Z}_+ \times_\alpha \mathcal{C} \to \mathcal{B}$ that extends ρ.

In Theorem 1.4, we prove that the semicrossed product, as an operator algebra, is independent of the way $\mathcal{C}$ is (faithfully) represented. In order to do so, we use some dilations theorems of [10] and [12] and arguments similar to the ones in Theorem 6.2 of [7].

First of all, every l-cov.con. pair (π, T) on a Hilbert space H dilates to a l-cov.is. pair (η, W) on a Hilbert space $H_1 \supseteq H$, such that $\eta(x)H \subseteq H$ and $\eta(x)|_H = \pi(x)$, for every $x \in \mathcal{C}$, and $T^n = P_H W^n|_H$, for every $n \in \mathbb{Z}_+$, where W is an isometry (see [10]). Hence, by II.5 of [12] we see that the norm $\|\cdot\|_\mathrm{l}$ is the supremum over all left covariant purely isometric pairs. By Proposition I.4 of [12], for such a pair (η, W) on a Hilbert space H_1 there is a representation (H_2, π') of $\mathcal{C}$ such that $W \times \eta$ is unitarily equivalent to $\mathrm{l}t_{\pi'}$. Thus, eventually we have that, for $F \in \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_\mathrm{l}$, $\|F\|_\mathrm{l} = \sup\{\|\mathrm{l}t_\pi(F)\| : (H, \pi) \text{ a } *\text{-representation of } \mathcal{C}\}$. Moreover, $\|[F_{i,j}]\|_\mathrm{l} = \sup\{\|[\mathrm{l}t_\pi(F_{i,j})]\| : (H, \pi) \text{ a } *\text{-representation of } \mathcal{C}\}$.

PROPOSITION 1.3. *If $F_{i,j} \in \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_\mathrm{l}$, then $\|[F_{i,j}]\|_\mathrm{l} = \|[\mathrm{l}t_{\pi_\mathrm{u}}(F_{i,j})]\|$, where $(H_\mathrm{u}, \pi_\mathrm{u})$ is the universal representation of $\mathcal{C}$. Analogously, if $F_{i,j} \in \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_\mathrm{r}$, then $\|[F_{i,j}]\|_\mathrm{r} = \|[\mathrm{r}t_{\pi_\mathrm{u}}(F_{i,j})]\|$.*

Proof. Let (H, π) be a $*$-representation of $\mathcal{C}$. By definition of the universal representation we have that $\pi_\mathrm{u}|_H = \pi$ and $\pi_\mathrm{u}(x)H \subseteq H$. Let $H_0 = H \otimes \ell^2(\mathbb{Z}_+)$. We denote by P_{H_0} the projection onto $H \otimes \ell^2(\mathbb{Z}_+) \subseteq H_\mathrm{u} \otimes \ell^2(\mathbb{Z}_+)$ and observe that $P_{H_0}(\mathbf{1}_{H_\mathrm{u}} \otimes v)^n|_{H_0} = (\mathbf{1}_{H_0} \otimes v)^n$, for every $n \in \mathbb{Z}_+$. Thus, for every $\nu \in \mathbb{Z}_+$ and for every $[F_{i,j}] \in \mathcal{M}_\nu(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha))$, we have that $[\mathrm{l}t_\pi(F_{i,j})] = (P_{H_0} \otimes I_\nu)[\mathrm{l}t_{\pi_\mathrm{u}}(F_{i,j})]|_{(H_0)^{(\nu)}}$, and so $\|[\mathrm{l}t_\pi(F_{i,j})]\| \leqslant \|[\mathrm{l}t_{\pi_\mathrm{u}}(F_{i,j})]\|$. ∎

If (H,π) is a faithful $*$-representation of $\mathcal{C}$, we denote by $C^*(\pi,V)$ the C^*-algebra generated by the representation lt_π in $\mathcal{B}(H\otimes\ell^2(\mathbb{Z}_+))$. The covariance relation shows that $C^*(\pi,V)$ is the norm-closed linear span of the monomials $V^m\widetilde{\pi}(x)(V^*)^\lambda$, $m,\lambda\in\mathbb{Z}_+$. Since, $C^*(\pi,V)$ is a direct summand of $C^*(\pi_{\mathrm{u}},V_{\mathrm{u}})$, the compression $\Phi:\mathcal{B}(H_{\mathrm{u}}\otimes\ell^2(\mathbb{Z}_+))\to\mathcal{B}(H\otimes\ell^2(\mathbb{Z}_+))$ is a $*$-epimorphism when restricted on $C^*(\pi_{\mathrm{u}},V_{\mathrm{u}})$. We will prove that it is also faithful, hence completely isometric.

To this end, for every $s\in[0,2\pi]$, we define $u_s:\ell^2(\mathbb{Z}_+)\to\ell^2(\mathbb{Z}_+)$ by $u_s(e_m)=\mathrm{e}^{2\pi\mathrm{i}s}e_m$. Let $\widetilde{U}_s=\mathbf{1}_{H_{\mathrm{u}}}\otimes u_s$ and $U_s=\mathbf{1}_H\otimes u_s$. The map $\widetilde{\gamma}_s=\mathrm{ad}_{\widetilde{U}_s}$ is a $*$-automorphism of $C^*(\pi_{\mathrm{u}},V_{\mathrm{u}})$, since $\widetilde{\gamma}_s(\widetilde{\pi}_{\mathrm{u}}(x))=\widetilde{\pi}_{\mathrm{u}}(x)$ and $\widetilde{\gamma}_s(\widetilde{V}_{\mathrm{u}}^n)=\mathrm{e}^{2\pi\mathrm{i}ns}\widetilde{V}_{\mathrm{u}}^n$. Similarly, $\gamma_s=\mathrm{ad}_{U_s}$ is a $*$-automorphism of $C^*(\pi,V)$. It is clear that $\Phi\circ\widetilde{\gamma}_s=\gamma_s\circ\Phi$, because $\Phi(\widetilde{U}_s)=U_s$. We denote by $C^*(\pi_{\mathrm{u}},V_{\mathrm{u}})^{\widetilde{\gamma}}$ the fixed point algebra of $\widetilde{\gamma}$ and define the contractive, faithful projection $\widetilde{E}:C^*(\pi_{\mathrm{u}},V_{\mathrm{u}})\to C^*(\pi_{\mathrm{u}},V_{\mathrm{u}})^{\widetilde{\gamma}}$ by

$$\widetilde{E}(X):=\int_0^{2\pi}\widetilde{\gamma}_s(X)\frac{\mathrm{d}s}{2\pi},$$

(as a Riemann integral of a norm-continuous function). Let

$$\mathcal{B}_k:=\Big\{\sum_{n=0}^{k}V_{\mathrm{u}}^n\widetilde{\pi}_{\mathrm{u}}(x_n)(V_{\mathrm{u}}^*)^n:x_n\in\mathcal{C}\Big\};$$

then we can check that $C^*(\pi_{\mathrm{u}},V_{\mathrm{u}})^{\widetilde{\gamma}}$ is the norm-closure of $\bigcup_{k\in\mathbb{Z}_+}\mathcal{B}_k$. Let X_k be an element of $\mathcal{B}_k$. Since, $V_{\mathrm{u}}^n\widetilde{\pi}_{\mathrm{u}}(x)(V_{\mathrm{u}}^*)^n=\mathrm{diag}\{\underbrace{0,\dots,0}_{n-\text{times}},\pi_{\mathrm{u}}(x),\pi_{\mathrm{u}}(\alpha(x)),\dots\}$, we see that X_k is a diagonal matrix whose (m,m)-entry is the element $(X_k)_{m,m}=\pi_{\mathrm{u}}\Big(\sum_{j=0}^{\min\{m,k\}}\alpha^{m-j}(x_{m-j})\Big)$. So, if (H,π) is a faithful $*$-representation of $\mathcal{C}$,

$$\begin{aligned}\|(X_k)_{m,m}\|&=\Big\|\pi_{\mathrm{u}}\Big(\sum_{j=0}^{\min\{m,k\}}\alpha^{m-j}(x_{m-j})\Big)\Big\|=\Big\|\sum_{j=0}^{\min\{m,k\}}\alpha^{m-j}(x_{m-j})\Big\|_{\mathcal{C}}\\&=\Big\|\pi\Big(\sum_{j=0}^{\min\{m,k\}}\alpha^{m-j}(x_{m-j})\Big)\Big\|=\|(\Phi(X_k))_{m,m}\|.\end{aligned}$$

So $\|X_k\|=\sup_m\{\|(X_k)_{m,m}\|\}=\sup_m\{\|(\Phi(X_k))_{m,m}\|\}=\|\Phi(X_k)\|$; hence $\Phi:C^*(\pi_{\mathrm{u}},V_{\mathrm{u}})\to C^*(\pi,V)$ is isometric on each $\mathcal{B}_k$. Thus, *Φ is injective when restricted to the fixed point algebra $C^*(\pi_{\mathrm{u}},V_{\mathrm{u}})^{\widetilde{\gamma}}$.*

THEOREM 1.4. *The left semicrossed product $\mathbb{Z}_+\times_\alpha\mathcal{C}$ is completely isometrically isomorphic to the norm-closed linear span of $\sum_{n=0}^{k}V^n\widetilde{\pi}(x_n)$, $x_n\in\mathcal{C}$, where (H,π) is any faithful $*$-representation of $\mathcal{C}$. Respectively, the right semicrossed product $\mathcal{C}\times_\alpha\mathbb{Z}_+$ is* *completely isometrically isomorphic to the norm-closed linear span of* $\sum_{n=0}^{k} \widetilde{\pi}(x_n)(V^*)^n$*,* $x_n \in \mathcal{C}$*, where* (H, π) *is any faithful* $*$*-representation of* $\mathcal{C}$*.*

Proof. It suffices to prove that the natural $*$-epimorphism Φ is faithful, hence a (completely) $*$-isometric isomorphism. Let $X \in \ker \Phi$, then $X^*X \in \ker \Phi$. Hence,

$$\Phi(\widetilde{E}(X^*X)) = \Phi\Big(\int_0^{2\pi} \widetilde{\gamma}_s(X^*X)\frac{\mathrm{d}s}{2\pi}\Big) = \int_0^{2\pi} \Phi(\widetilde{\gamma}_s(X^*X))\frac{\mathrm{d}s}{2\pi} = \int_0^{2\pi} \gamma_s(\Phi(X^*X))\frac{\mathrm{d}s}{2\pi} = 0.$$

Now $\widetilde{E}(X^*X)$ is in $C^*(\pi_{\mathrm{u}}, V_{\mathrm{u}})^{\widetilde{\gamma}}$ and Φ is faithful there; hence $\widetilde{E}(X^*X) = 0$ and so $X^*X = 0$. For the right semicrossed product, note that $C^*(\pi, V^*) = C^*(\pi, V)$. ∎

If, in particular, α is a $*$-isomorphism, then there is a natural way to identify the left semicrossed product as a closed subalgebra of the (reduced) crossed product, i.e. $C_1^*(\mathcal{C})$. In this case, we refer to this closed subalgebra as the *left reduced semicrossed product*. In a dual way, we can define the *right reduced semicrossed product*. The following is proved in [13], when $\mathcal{C}$ is abelian.

THEOREM 1.5. *If* α *is a* $*$*-isomorphism, then the* C^**-envelope of the semicrossed product is the (reduced) crossed product.*

Proof. Since α is a $*$-isomorphism, we can view $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_1$ as a $|\cdot|_1$-closed subalgebra of $\ell^1(\mathbb{Z}, \mathcal{C}, \alpha)_1$. First we prove that the inclusion map $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha) \hookrightarrow \ell^1(\mathbb{Z}, \mathcal{C}, \alpha)$ is completely isometric. The key is to prove that

$$\|F\|_1 = \sup\{\|(U \times \pi)(F)\| : (\pi, U) \text{ l-cov.un. pair of } \ell^1(\mathbb{Z}, \mathcal{C}, \alpha)_1\},$$

for every $F \in \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_1$, since the right hand side is exactly the norm of the (left) crossed product. For simplicity, we denote this norm by $\|\cdot\|$. It is obvious that $\|F\| \leqslant \|F\|_1$, since every l-cov.un. pair of $\ell^1(\mathbb{Z}, \mathcal{C}, \alpha)_1$ restricts to a l-cov.un. pair of the subalgebra $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_1$. Also, if (H_0, π) is a faithful $*$-representation of $\mathcal{C}$, then $\mathrm{l}t_\pi$ is the compression of the left regular representation of $\ell^1(\mathbb{Z}, \mathcal{C}, \alpha)_1$ induced by π, denoted simply by $\mathrm{l}t$. So, $\|\mathrm{l}t_\pi(F)\| \leqslant \|\mathrm{l}t(F)\|$, thus $\|F\|_1 \leqslant \|F\|$ by Theorem 1.4. Arguing in the same way, we get that $\|[F_{i,j}]\| \leqslant \|[F_{i,j}]\|_1$ and $\|[\mathrm{l}t_\pi(F_{i,j})]\| \leqslant \|[\mathrm{l}t(F_{i,j})]\|$, for every $[F_{i,j}] \in \mathcal{M}_\nu(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_1)$. But $\mathrm{l}t$ is a $*$-morphism of the crossed product, hence completely contractive. Thus, $\|[F_{i,j}]\|_1 \leqslant \|[F_{i,j}]\|$ and equality holds. Hence, if $\widehat{\pi}(x) = \mathrm{diag}\{\pi(a^m(x)), m \in \mathbb{Z}\}$ and $U = 1_{H_0} \otimes u$, where u is the bilateral shift, then the map $\delta_n \otimes x \mapsto U^n \widehat{\pi}(x)$ extends to a complete isometry $\iota : \mathbb{Z}_+ \times_\alpha \mathcal{C} \to C_1^*(\mathcal{C})$, whose image generates $C_1^*(\mathcal{C})$ as a C^*-algebra. Let $\mathcal{B}$ be the C^*-envelope of $\mathbb{Z}_+ \times_\alpha \mathcal{C}$. Then, by the universal property of C^*-envelopes, there is a surjective C^*-homomorphism $\Psi : C_1^*(\mathcal{C}) \to \mathcal{B}$, which restricts to a completely isometry on $\iota(\mathbb{Z}_+ \times_\alpha \mathcal{C})$. Let $G \in \ker \Phi$ be of unit norm, and choose $F = \sum_{n=-k}^{k} U^n \widehat{\pi}(x_n)$ with $\|G - F\| < 1/2$.

Thus $U^kG \in \ker\Psi$, $\iota^{-1}(U^kF) \in \mathbb{Z}_+ \times_\alpha \mathcal{C}$, $\|\iota^{-1}(U^kF)\| = \|U^kF\| = \|F\| > 1/2$ and $\|U^kG - U^kF\| = \|G-F\| < 1/2$. Then $1/2 < \|\mathrm{i}^{-1}(U^kF)\| = \|\Psi(U^kF)\| = \|\Psi(U^kF - U^kG)\| \leqslant \|U^kF - U^kG\| < 1/2$, which is a contradiction. ∎

2. w*-SEMICROSSED PRODUCTS

Let $\mathcal{A} \subseteq B(H_o)$ be a unital subalgebra, closed in the w*-operator topology, and $\beta : \mathcal{A} \to \mathcal{A}$, a contractive w*-continuous endomorphism of $\mathcal{A}$. From now on we fix $H = H_o \otimes \ell^2(\mathbb{Z}_+)$ and $\pi := \widetilde{\mathrm{id}}_{\mathcal{A}}$, as in Example 1.1. Then π is a faithful representation of $\mathcal{A}$ on H, and we can write $\pi(b) = \sum\limits_{n\geqslant 0} \beta^n(b) \otimes p_n$, where $p_n \in \mathcal{B}(\ell^2(\mathbb{Z}_+))$ is the projection onto $[e_n]$. Note that the sum converges in the w*-topology as well. Hence, $\pi(b)$ belongs to the *w*-tensor product algebra* $\mathcal{A}\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{Z}_+))$. This is, by definition, the w*-closed linear span in $\mathcal{B}(H)$ of the operators $b \otimes a$, with $b \in \mathcal{A}$ and $a \in \mathcal{B}(\ell^2(\mathbb{Z}_+))$. We also represent $\mathbb{Z}_+$ on H by the isometries $V^n = \mathbf{1}_{H_o} \otimes v^n$, where v is the unilateral shift on $\ell^2(\mathbb{Z}_+)$. Thus, $V^n \in \mathcal{A}\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{Z}_+))$.

DEFINITION 2.1. The w*-semicrossed product $\mathbb{Z}_+\overline{\times}_\beta\mathcal{A}$ is the w*-closure of the linear space of the "analytic polynomials" $\sum\limits_{n=0}^{k} V^n\pi(b_n)$, $b_n \in \mathcal{A}$, $k \geqslant 0$.

It is easy to check that the left covariance relation $\pi(b)V = V\pi(\beta(b))$ holds. Hence, (π, V) is a left covariant isometric pair. Thus, the w*-semicrossed product is a unital (non-selfadjoint) subalgebra of $\mathcal{B}(H)$ and by definition, $\mathbb{Z}_+\overline{\times}_\beta\mathcal{A} \subseteq \mathcal{A}\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{Z}_+))$.

PROPOSITION 2.2. *An operator $T \in \mathcal{B}(H)$ is in the* w**-semicrossed product if and only if $T_{\kappa,\lambda} \in \mathcal{A}$ and $G_m(T) = V^m\pi(T_{m,0})$, when $m \in \mathbb{Z}_+$, while $G_m(T) = 0$ for $m < 0$. Equivalently, when $T_{\kappa,\lambda} \in \mathcal{A}$ and $\beta(T_{m+\lambda,\lambda}) = T_{m+\lambda+1,\lambda+1}$ for every $m,\lambda \in \mathbb{Z}_+$, while $T_{\kappa,\lambda} = 0$ when $\kappa < \lambda$.*

Proof. If $T = \sum\limits_{\kappa=0}^{n} V^\kappa\pi(b_\kappa)$ with $b_\kappa \in \mathcal{A}$, then $G_m(T) = V^m\pi(b_m)$ when $m \in \{0,1,\dots,n\}$ and $G_m(T) = 0$ otherwise. Let $T \in \mathbb{Z}_+\overline{\times}_\beta\mathcal{A}$ and a net $A_i = \sum\limits_{\kappa=0}^{n_i} V^\kappa\pi(b_{i,\kappa})$ of analytic polynomials converging to T in the w*-topology. Since G_m is w*-continuous, we have that $G_m(T) = \text{w*-}\lim\limits_i G_m(A_i)$ for every $m \in \mathbb{Z}$. Thus $G_m(T) = 0$ when $m < 0$. If $m \geqslant 0$, then $T_{(m)} = (V^*)^m G_m(T) = \text{w*-}\lim\limits_i (V^*)^m G_m(A_i) = \text{w*-}\lim\limits_i \pi(b_{i,m})$. Let $\phi \in \mathcal{B}(H_0)_*$ and $k \in \mathbb{Z}_+$, then $\phi\overline{\otimes}\omega_{e_\kappa,e_\kappa} \in \mathcal{B}(H)_*$; hence we get

$$\phi(T_{m+\kappa,\kappa}) = (\phi\overline{\otimes}\omega_{e_\kappa,e_\kappa})(T_{(m)}) = \lim_i(\phi\overline{\otimes}\omega_{e_\kappa,e_\kappa})(\pi(b_{i,n})) = \lim_i \phi(\beta^k(b_{i,n})).$$

Thus $T_{m+\kappa,\kappa} = \text{w*-}\lim_i \beta^\kappa(b_{i,m})$, for every $\kappa \in \mathbb{Z}_+$, so $T_{m+\kappa,\kappa} \in \mathcal{A}$. Also, since β is w*-continuous, we get that $\beta^\kappa(T_{m,o}) = \text{w*-}\lim_i \beta^\kappa(b_{i,m}) = T_{m+\kappa,\kappa}$, for every $\kappa \in \mathbb{Z}_+$. Hence, we get that $G_m(T) = V^m \pi(T_{m,o})$, for every $m \geqslant 0$. For the opposite direction, if $T \in \mathcal{B}(H)$ satisfies the conditions, we can see that $G_m(T) \in \mathbb{Z}_+ \overline{\times}_\beta \mathcal{A}$, and so by the Féjer Lemma, $T \in \mathbb{Z}_+ \overline{\times}_\beta \mathcal{A}$ as well. The last equivalence is trivial. ∎

REMARK 2.3. Note that each ad_{U_s} leaves $\mathbb{Z}_+ \overline{\times}_\beta \mathcal{A}$ invariant, and hence, being unitarily implemented, also leaves its reflexive cover invariant. Thus, so does $G_m(\cdot)$.

Suppose now that the endomorphism β is *implemented* by a unitary w acting on H_0, so that $\beta(b) = wbw^*$, for all $b \in \mathcal{A}$. Let $\rho(b) = b \otimes \mathbf{1}_{\ell^2(\mathbb{Z}_+)}$, for $b \in \mathcal{A}$ and $W = w^* \otimes v$. Then (ρ, W) is a left covariant isometric pair and we denote by $\mathbb{Z}_+ \times_w \mathcal{A}$ *the w*-closure of the linear space of the "analytic polynomials"* $\sum_{n=0}^{k} W^n \rho(b_n)$, $b_n \in \mathcal{A}, k \geqslant 0$.

It is easy to check that $\mathbb{Z}_+ \times_w \mathcal{A}$ is unitarily equivalent to $\mathbb{Z}_+ \overline{\times}_\beta \mathcal{A}$, via $Q = \sum_{n \geqslant 0} w^{-n} \otimes p_n$. Thus we refer to $\mathbb{Z}_+ \overline{\times}_w \mathcal{A}$ as the w*-semicrossed product, as well. Using the unitary operator Q and Proposition 2.2 we get the following characterization.

PROPOSITION 2.4. *An operator $T \in \mathcal{B}(H)$ is in $\mathbb{Z}_+ \overline{\times}_w \mathcal{A}$ if and only if $G_m(T) = W^m \rho(b_m)$, for some $b_m \in \mathcal{A}$, when $m \in \mathbb{Z}_+$ and $G_m(T) = 0$ for $m < 0$. Equivalently, when $T_{m+\lambda,\lambda} = (w^*)^m b_m$, for every $m, \lambda \in \mathbb{Z}_+$ and $T_{\kappa,\lambda} = 0$, when $\kappa < \lambda$.*

The relation between the w*-tensor product $\mathcal{A} \overline{\otimes} \mathcal{T}$ and $\mathbb{Z}_+ \overline{\times}_w \mathcal{A}$ depends on some properties of w. Specifically,

- $\mathcal{A} \overline{\otimes} \mathcal{T} = \mathbb{Z}_+ \overline{\times}_w \mathcal{A}$ if and only if $w, w^* \in \mathcal{A}$.
- $\mathcal{A} \overline{\otimes} \mathcal{T} \subsetneq \mathbb{Z}_+ \overline{\times}_w \mathcal{A}$ if and only if $w^* \notin \mathcal{A}, w \in \mathcal{A}$.
- $\mathbb{Z}_+ \overline{\times}_w \mathcal{A} \subsetneq \mathcal{A} \overline{\otimes} \mathcal{T}$ if and only if $w \notin \mathcal{A}, w^* \in \mathcal{A}$.
- $(\mathbb{Z}_+ \overline{\times}_w \mathcal{A}) \cap (\mathcal{A} \overline{\otimes} \mathcal{T}) = \rho(\mathcal{A})$, if and only if $(w^n \mathcal{A}) \cap \mathcal{A} = \{0\}, \forall\, n \in \mathbb{Z}_+$.

It is easy to verify that, when $(w^n \mathcal{A}) \cap \mathcal{A} = \{0\}$ for every $n \in \mathbb{Z}_+$, then $w, w^* \notin \mathcal{A}$, but the converse is not always true.

EXAMPLE 2.5. Take $\mathcal{A} = L^\infty(\mathbb{T})$ acting on $L^2(\mathbb{T})$ and $\beta(f)(z) = f(\lambda z)$, where λ is a q-th root of unity. Then β is unitarily implemented by $w \in \mathcal{B}(L^2(\mathbb{T}))$, with $(w(g))(z) = g(\lambda z)$. Then $w^{mq} = I_{H_0}$, for every $m \in \mathbb{Z}_+$, hence $w^{mq} \mathcal{A} \cap \mathcal{A} = \mathcal{A}$. In this case, $(\mathbb{Z}_+ \overline{\times}_w \mathcal{A}) \cap (\mathcal{A} \overline{\otimes} \mathcal{T})$ contains the w*-closed algebra generated by $\sum_{n=0}^{k} W^{nq} \rho(b_n)$, $b_n \in \mathcal{A}$, which properly contains $\rho(\mathcal{A})$.

The following lemma will be superseded below (Theorem 2.9).

LEMMA 2.6. *The* w*-*semicrossed product* $\mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0)$ *is reflexive, for every unitary* $w\in\mathcal{B}(H_0)$.

Proof. Let $T\in\operatorname{Ref}(\mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0))$. Then, by Remark 2.3, each $G_m(T)$ belongs to the reflexive cover of the w*-semicrossed product. Thus, for $\kappa<\lambda$ and $\xi,\eta\in H_0$, there is a sequence $A_n\in\mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0)$ such that $\langle T(\xi\otimes e_\lambda),\eta\otimes e_\kappa\rangle=\lim_n\langle A_n(\xi\otimes e_\lambda),\eta\otimes e_\kappa\rangle$. Hence, $\langle T_{\kappa,\lambda}\xi,\eta\rangle=\lim_n\langle (A_n)_{\kappa,\lambda}\xi,\eta\rangle=0$, since each $(A_n)_{\kappa,\lambda}=0$, for $\kappa<\lambda$. So $G_m(T)=0$ for every $m<0$. Now, fix $m\in\mathbb{Z}_+$ and consider $\xi\in H_0$, $g_r=\sum_n r^n e_n$, $0\leqslant r<1$. We can check that the subspace $\mathcal{F}=\overline{[(b\xi)\otimes g_r: b\in\mathcal{B}(H_0)]}$ is $(\mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0))^*$-invariant, and as a consequence, $G_m(T)^*$-invariant. Since $\xi\otimes g_r\in\mathcal{F}$, there is a sequence (b_n) in $\mathcal{B}(H_0)$ such that $G_m(T)^*(\xi\otimes g_r)=\lim_n(b_n\xi)\otimes g_r$. Thus, $\sum_\kappa r^{m+\kappa}T^*_{m+\kappa,\kappa}\xi\otimes e_\kappa=\lim_n(b_n\xi)\otimes g_r$. Taking scalar product with $\eta\otimes e_\kappa$, where $\eta\in H_0$ and $\kappa\geqslant 0$, we have that $r^{m+\kappa}\langle T^*_{m+\kappa,\kappa}\xi,\eta\rangle=\lim_n r^\kappa\langle b_n\xi,\eta\rangle$. Hence, $r^m\langle T^*_{m+\kappa,\kappa}\xi,\eta\rangle=\lim_n\langle b_n\xi,\eta\rangle=r^m\langle T^*_{m,0}\xi,\eta\rangle$, for every η. Thus, $T^*_{m+\kappa,\kappa}\xi=T^*_{m,0}\xi$, for arbitrary $\xi\in H_0$, so $T_{m+\kappa,\kappa}=T_{m,0}$ for every $\kappa\in\mathbb{Z}_+$. Hence, $G_m(T)\in\mathcal{B}(H_0)\overline{\otimes}\mathcal{T}$, which coincides with $\mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0)$ since $w\in\mathcal{B}(H_0)$. ∎

Let $\mathcal{S}$ be a w*-closed subspace of $\mathcal{B}(H)$. We say that $\mathcal{S}$ is *G-invariant* if $G_m(\mathcal{S})\subseteq\mathcal{S}$ for every $m\in\mathbb{Z}$. If, in particular, $\mathcal{S}$ is a w*-closed subspace of $\mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0)$, then $G_m(\mathcal{S})=0$, for every $m<0$. In the next proposition we prove that we can associate a sequence $(\mathcal{S}_m)_{m\geqslant 0}$ of w*-closed subspaces of $\mathcal{B}(H_0)$ to such an $\mathcal{S}$, and vice versa.

PROPOSITION 2.7. *A* w*-*closed subspace* $\mathcal{S}$ *of* $\mathcal{B}(H)$ *is a G-invariant subspace of* $\mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0)$ *if and only if it is the* w*-*closure of the linear space of the analytic polynomials* $\sum_{n=0}^{k}W^n\rho(x_n)$, $x_n\in\mathcal{S}_n, k\in\mathbb{Z}_+$, *where* $\mathcal{S}_n$ *are* w*-*closed subspaces of* $\mathcal{B}(H_0)$.

Proof. Let $\mathcal{S}$ be a G-invariant w*-closed subspace of $\mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0)$ and let $\mathcal{S}_m=\{w^mT_{m,0}: T\in\mathcal{S}\}$, for every $m\geqslant 0$. Then $\mathcal{S}_m$ is a w*-closed subspace of $\mathcal{B}(H_0)$. Indeed, let $x=\text{w*-}\lim_i w^m(T_i)_{m,0}$, for $T_i\in S$. Then $\rho((w^*)^m x)=\text{w*-}\lim_i\rho((T_i)_{m,0})$, so $W^m\rho(x)=\text{w*-}\lim_i V^m\rho((T_i)_{m,0})=\text{w*-}\lim_i G_m(T_i)$, since $T_i\in\mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0)$. But $\mathcal{S}$ is G-invariant, hence $W^m\rho(x)\in\mathcal{S}$, which gives that $x=w^m(V^m\rho(x))_{m,0}\in\mathcal{S}_m$. A use of the Féjer Lemma and Proposition 2.4, completes the forward implication. For the converse, let $\mathcal{S}$ be a w*-closed subspace as in the statement and $A\in\mathcal{S}$; so $A=\text{w*-}\lim_i A_i$, where $A_i=\sum_{\kappa=0}^{n_i}W^\kappa\rho(x_{i,\kappa})$, with $x_{i,\kappa}\in\mathcal{S}_\kappa$. Then $A\in\mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0)$ and $G_m(A)=\text{w*-}\lim_i G_m(A_i)=\text{w*-}\lim_i W^m\rho(x_{i,\kappa})$. So, $w^mA_{m,0}=\text{w*-}\lim_i x_{i,m}\in\mathcal{S}_m$. Hence, we have that $G_m(A)=W^m\rho(w^mA_{m,0})\in\mathcal{S}$. ∎

THEOREM 2.8. *Let $(\mathcal{S}_m)_{m\geqslant 0}$ be the sequence associated to a G-invariant* w*-*closed subspace $\mathcal{S}$ of $\mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0)$. If every $\mathcal{S}_m$ is reflexive then $\mathcal{S}$ is reflexive.*

Proof. By Lemma 2.6, $\operatorname{Ref}(\mathcal{S}) \subseteq \operatorname{Ref}(\mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0)) = \mathbb{Z}_+\overline{\times}_w\mathcal{B}(H_0)$. So, for every T in the reflexive cover of $\mathcal{S}$ and every $m,\lambda \in \mathbb{Z}_+$, we have that $T_{m+\lambda,\lambda} = (w^*)^m b_m$, where $b_m \in \mathcal{B}(H_0)$. Thus, it suffices to prove that $b_m \in \mathcal{S}_m$. Since $T \in \operatorname{Ref}(\mathcal{S})$, then, for every $\xi,\eta \in H_0$, there is a sequence (A_n) in $\mathcal{S}$ such that $\langle T(\xi\otimes e_\lambda),(w^*)^m\eta\otimes e_{m+\lambda}\rangle = \lim_n\langle A_n(\xi\otimes e_\lambda),(w^*)^m\eta\otimes e_{m+\lambda}\rangle$. So, $\langle b_m\xi,\eta\rangle = \langle T_{m+\lambda,\lambda}\xi,(w^*)^m\eta\rangle = \lim_n\langle (A_n)_{m+\lambda,\lambda}\xi,(w^*)^m\eta\rangle$. Since each $A_n \in \mathcal{S}$, we get that $(A_n)_{m+\lambda,\lambda} = (w^*)^m b_{n,m}$ for some $b_{n,m} \in \mathcal{S}_m$. Thus $\langle b_m\xi,\eta\rangle = \lim_n\langle b_{n,m}\xi,\eta\rangle$, which means that $b_m \in \operatorname{Ref}(\mathcal{S}_m) = \mathcal{S}_m$. ∎

THEOREM 2.9. *If $\mathcal{A}$ is a reflexive algebra, then $\mathbb{Z}_+\overline{\times}_w\mathcal{A}$ is reflexive. In addition, if $\mathcal{A}$ is hereditarily reflexive, then every G-invariant* w*-*closed subspace of $\mathbb{Z}_+\overline{\times}_w\mathcal{A}$ is reflexive.*

Proof. The algebra $\mathbb{Z}_+\overline{\times}_w\mathcal{A}$ is associated to the sequence $(\mathcal{A})_{m\geqslant 0}$; hence it is reflexive by the previous theorem. ∎

APPLICATIONS 2.10. **A.** (*Sarason's result*, Theorem 3 in [15]). Consider the case of a reflexive subalgebra $\mathcal{A}$ of $M_n(\mathbb{C})$ and a unitary $w \in M_n(\mathbb{C})$ such that $w\mathcal{A}w^* \subseteq \mathcal{A}$. Then $\mathbb{Z}_+\overline{\times}_w\mathcal{A}$ is reflexive. Note that $\mathbb{Z}_+\overline{\times}_w\mathcal{A} = \mathcal{T}$ when $n = 1$ and $w = I_{H_0}$.

B. (*Ptak's result*, Theorem 2 in [14]). More generally, $\mathcal{A}\overline{\otimes}\mathcal{T}$ coincides with $\mathbb{Z}_+\overline{\times}_{I_{H_0}}\mathcal{A}$. So $\mathcal{A}\overline{\otimes}\mathcal{T}$ is reflexive, when $\mathcal{A}$ is reflexive.

C. If $\mathcal{M}$ is a maximal abelian selfadjoint algebra and β is a $*$-automorphism, then $\mathbb{Z}_+\overline{\times}_\beta\mathcal{M}$ is reflexive, since every $*$-automorphism of a m.a.s.a. is unitarily implemented. For example let $\mathcal{M} = L^\infty(\mathbb{T})$ acting on $L^2(\mathbb{T})$ and β the rotation by $\theta \in \mathbb{R}$. Also $\mathbb{Z}_+\overline{\times}_\beta\mathcal{A}$ is reflexive whenever $\mathcal{A}$ is a β-invariant w*-closed subalgebra of $\mathcal{M}$, since $\mathcal{M}$ is hereditarily reflexive (see [8]).

D. Consider $\mathcal{T}$ acting on $H^2(\mathbb{T})$ and β as in the previous example. Then, $\mathcal{T}$ is reflexive and so $\mathbb{Z}_+\overline{\times}_\beta\mathcal{T}$ is a reflexive subalgebra of $\mathcal{B}(H^2(\mathbb{T}))\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{Z}_+))$.

E. If $\mathcal{A}$ is a nest algebra and β is an isometric automorphism, then it is unitarily implemented (see [3]). Thus, $\mathbb{Z}_+\overline{\times}_\beta\mathcal{A}$ is reflexive.

F. Consider a C^*-algebra $\mathcal{C}$ and a $*$-morphism $\alpha : \mathcal{C} \to \mathcal{C}$. Let (H_0,σ) be a faithful $*$-representation of $\mathcal{C}$ such that the induced $*$-morphism

$$\beta : \sigma(\mathcal{C}) \to \sigma(\mathcal{C}) : \sigma(x) \mapsto \beta(\sigma(x)) = \sigma(\alpha(x))$$

is implemented by a unitary $w \in \mathcal{B}(H_0)$. Then the induced representation $\mathrm{l}t_\sigma$ is faithful on $\mathbb{Z}_+ \times_\alpha \mathcal{C}$. Thus, $\overline{\mathrm{l}t_\sigma(\mathbb{Z}_+ \times_\alpha \mathcal{C})}^{\mathrm{w}^*}$ is the w*-closed linear span of the analytic polynomials $\sum_{n=0}^{k} V^n \pi(\sigma(x))$, and it is unitarily equivalent to the algebra $\mathfrak{C} := \overline{\mathrm{span}\{\rho(\sigma(x)), W^n : x \in \mathcal{C}, n \in \mathbb{Z}_+\}}^{\mathrm{w}^*}$, via $Q = \sum_{n \geqslant 0} w^{-n} \otimes p_n$. But $\mathfrak{C}$ is exactly the w*-semicrossed product $\mathbb{Z}_+ \overline{\times}_w \overline{\sigma(\mathcal{C})}^{\mathrm{w}^*}$. Thus, $\overline{\mathrm{lt}_\sigma(\mathbb{Z}_+ \times_\alpha \mathcal{C})}^{\mathrm{w}^*}$ is reflexive. In particular, let K be a compact, Hausdorff space, μ a positive, regular Borel measure on K and $\sigma : C(K) \to B(L^2(K,\mu)) : f \mapsto M_f$. Consider a homeomorphism ϕ of K, such that ϕ and ϕ^{-1} preserve the μ-null sets and let $\alpha(f) = f \circ \phi$. Then the map $M_f \to M_{f\circ\phi}$ extends to a $*$-automorphism of $L^\infty(K,\mu)$, hence it is unitarily implemented. Thus, $\overline{\mathrm{lt}_\sigma(\mathbb{Z}_+ \times_\alpha C(K))}^{\mathrm{w}^*}$ is reflexive.

G. Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with a faithful, normal, tracial state τ and let $L^2(\mathcal{M},\tau)$ be the Hilbert space associated to $(\mathcal{M},\tau)$. Let $\beta : \mathcal{M} \to \mathcal{M}$ be a trace-preserving $*$-automorphism and consider $\mathcal{M}$ acting on $L^2(\mathcal{M},\tau)$ by left multiplication. Then β is unitarily implemented and it can be verified that the w*-semicrossed product $\mathbb{Z}_+ \overline{\times}_w \mathcal{M}$ coincides with the adjoint of the analytic semicrossed product defined in [9] and [11]. Hence, we obtain Proposition 4.5 of [11] for $p = 2$.

REMARK 2.11. An analogous result to Theorem 2.8 is proved in [1]. They also obtain Ptak's result (see 2.10 **B**).

We conclude the analysis of the w*-semicrossed product $\mathbb{Z}_+ \overline{\times}_w \mathcal{A}$ by finding its commutant. We know that $U_s(\mathbb{Z}_+ \overline{\times}_w \mathcal{A})U_s^* = \mathbb{Z}_+ \overline{\times}_w \mathcal{A}$, for all $s \in [0, 2\pi]$, hence, $U_s(\mathbb{Z}_+ \overline{\times}_w \mathcal{A})' U_s^* = (\mathbb{Z}_+ \overline{\times}_w \mathcal{A})'$. Thus, $T \in (\mathbb{Z}_+ \overline{\times}_w \mathcal{A})'$ if and only if $G_m(T) \in (\mathbb{Z}_+ \overline{\times}_w \mathcal{A})'$, for every $m \in \mathbb{Z}$. Now, recall that $w\mathcal{A}w^* \subseteq \mathcal{A}$, hence $w^*\mathcal{A}'w \subseteq \mathcal{A}'$. So, we can define the w*-semicrossed product $\mathbb{Z}_+ \overline{\times}_\gamma \mathcal{A}'$, where $\gamma \equiv \mathrm{ad}_{w^*} : \mathcal{A}' \to \mathcal{A}'$.

THEOREM 2.12. *If* $\gamma \equiv \mathrm{ad}_{w^*}$, *then* $(\mathbb{Z}_+ \overline{\times}_w \mathcal{A})' = \mathbb{Z}_+ \overline{\times}_\gamma \mathcal{A}'$.

Proof. Obviously $T \in (\mathbb{Z}_+ \overline{\times}_w \mathcal{A})'$ if and only if $T \in \{b \otimes \mathbf{1}, w^* \otimes v : b \in \mathcal{A}\}'$; note also that $V \in (\mathbb{Z}_+ \overline{\times}_w \mathcal{A})'$. Let $T \in (\mathbb{Z}_+ \overline{\times}_w \mathcal{A})'$, then for $m \geqslant 0$ and $b \in \mathcal{A}$, $n \geqslant 0$,

$$\begin{aligned} G_m(T)(b \otimes \mathbf{1}) = (b \otimes \mathbf{1})G_m(T) \quad &\text{and} \quad G_m(T)(w^* \otimes v) = (w^* \otimes v)G_m(T), \\ \text{hence, } T_{m+n,n} b = b T_{m+n,n} \quad &\text{and} \quad T_{m+n+1,n+1}(w^*)^n = (w^*)^n T_{m+n,n}, \\ \text{so, } T_{m+n,n} \in \mathcal{A}' \quad &\text{and} \quad T_{m+n,n} = \gamma^n(T_{m,0}). \end{aligned}$$

Thus, if we set $\pi'(T_{m,0}) = \sum_{n\geqslant 0} \gamma^n(T_{m,0}) \otimes p_n$, we get that $G_m(T) = V^m \pi'(T_{m,0})$, for $m \geqslant 0$. Now, let $m < 0$, hence $G_m(T) = T_{(m)}(V^*)^{-m}$. Since, $V \in (\mathbb{Z}_+ \overline{\times}_w \mathcal{A})'$, we have that $T_{(m)} = G_m(T)V^{-m} \in (\mathbb{Z}_+ \overline{\times}_w \mathcal{A})'$. Thus, $G_0(T_{(m)}) = T_{(m)} \in (\mathbb{Z}_+ \overline{\times}_w \mathcal{A})'$ and so, by what we have proved, $T_{n,-m+n} = \gamma^n(T_{0,-m})$. Since $G_m(T)(\xi \otimes e_0) = 0$, then

$$G_m(T)(w^* \otimes v)^{-m}(\xi \otimes e_0) = (w^* \otimes v)^{-m} G_m(T)(\xi \otimes e_0) = 0,$$

so $(T_{0,-m}(w^*)^{-m}\xi)\otimes e_0 = 0$; hence $T_{0,-m} = 0$. Therefore $T_{n,-m+n} = \gamma^n(T_{0,-m}) = 0$, for every $n \geqslant 0$; hence $G_m(T) = 0$, for every $m < 0$. Hence, by Proposition 2.2, we get that $T \in \mathbb{Z}_+\overline{\times}_\gamma\mathcal{A}'$. For the converse, let $T \in \mathbb{Z}_+\overline{\times}_\gamma\mathcal{A}'$, then $G_m(T) \in \mathbb{Z}_+\overline{\times}_\gamma\mathcal{A}'$ for every $m \in \mathbb{Z}$, and we can see that $G_m(T) \in \{b\otimes\mathbf{1}, w^*\otimes v : b \in \mathcal{A}\}'$. Hence, $G_m(T) \in (\mathbb{Z}_+\overline{\times}_w\mathcal{A})'$ for every $m \in \mathbb{Z}$, so $T \in (\mathbb{Z}_+\overline{\times}_w\mathcal{A})'$. ∎

THEOREM 2.13. *The double commutant of* $\mathbb{Z}_+\overline{\times}_w\mathcal{A}$ *is* $\mathbb{Z}_+\overline{\times}_w\mathcal{A}''$. *Thus, the* w*-*semicrossed product is its own bicommutant if and only if* $\mathcal{A} = \mathcal{A}''$.

Proof. We recall that $Q(\mathbb{Z}_+\overline{\times}_\beta\mathcal{A})Q^* = \mathbb{Z}_+\overline{\times}_w\mathcal{A}$, where $Q = \sum\limits_n w^{-n}\otimes p_n$; hence $Q^*(\mathbb{Z}_+\overline{\times}_\gamma\mathcal{A}')Q = \mathbb{Z}_+\overline{\times}_{w^*}\mathcal{A}'$. Thus, $(\mathbb{Z}_+\overline{\times}_w\mathcal{A})'' = (\mathbb{Z}_+\overline{\times}_\gamma\mathcal{A}')' = (Q(\mathbb{Z}_+\overline{\times}_{w^*}\mathcal{A}')Q^*)' = Q(\mathbb{Z}_+\overline{\times}_{w*}\mathcal{A}')'Q^* = Q(\mathbb{Z}_+\overline{\times}_\beta\mathcal{A}'')Q^* = \mathbb{Z}_+\overline{\times}_w\mathcal{A}''$. ∎

We end this section with a note on the *reduced w*-semicrossed products* (see the definition below). Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H_0, β a $*$-automorphism of $\mathcal{M}$ and consider $\mathbb{Z}\overline{\rtimes}_\beta\mathcal{M}$ to be the usual w*-crossed product, a von Neumann subalgebra of $\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{Z}))$. This is by definition the von Neuman algebra $\{\widehat{\pi}(b), U : b \in \mathcal{M}\}''$, where $\widehat{\pi}(b) = \sum\limits_{n\in\mathbb{Z}} \beta^n(b)\otimes p_n$ and $U = 1_{H_0}\otimes u$, the ampliation of the bilateral shift $u \in \mathcal{B}(\ell^2(\mathbb{Z}))$.

DEFINITION 2.14. The reduced w*-semicrossed product $\mathbb{Z}_+\overline{\rtimes}_\beta\mathcal{M}$ is the w*-closure of the linear space of "analytic polynomials" $\sum\limits_{n=0}^{k} U^n\widehat{\pi}(b_n)$, $b_n \in \mathcal{M}$, $k \geqslant 0$.

Since $(\widehat{\pi}, U)$ is a l-cov.un. pair, the reduced w*-semicrossed product is a (w*-closed) subalgebra of the w*-crossed product. In fact, note that $\mathbb{Z}_+\overline{\rtimes}_\beta\mathcal{M}$ is the intersection of $\mathbb{Z}\overline{\rtimes}_\beta\mathcal{M}$ with the "lower triangular" matrices. Hence, we have the following proposition.

PROPOSITION 2.15. *The reduced* w*-*semicrossed product of a von Neumann algebra is reflexive.*

Now, take $\mathcal{A}$ to be a w*-closed subalgebra of $\mathcal{M}$ which is invariant under β. We define $\mathbb{Z}_+\overline{\rtimes}_\beta\mathcal{A}$ to be *the w*-closure of the linear space of "analytic polynomials"* $\sum\limits_{n=0}^{k} U^n\widehat{\pi}(b_n)$, $b_n \in \mathcal{A}$, $k \geqslant 0$. Using the technique of Theorem 2.9 one can show the following.

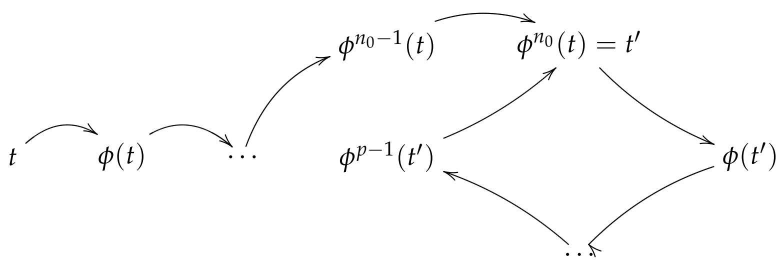
COROLLARY 2.16. *If* $\mathcal{A}$ *is reflexive subalgebra of* $\mathcal{M}$ *which is invariant under* β, *then* $\mathbb{Z}_+\overline{\rtimes}_\beta\mathcal{A}$ *is reflexive.*

3. THE COMMUTATIVE CASE

Now, we examine the case where $\mathcal{C}$ is a commutative, unital C^*-algebra, $\mathcal{C} = C(K)$, and the $*$-endomorphism α is induced by a continuous map $\phi : K \to$

K. Let ev_t be the evaluation at $t \in K$, i.e. $\mathrm{ev}_t(f) = f(t)$; then $(\ell^2(K), \bigoplus_t \mathrm{ev}_t)$ is a faithful $*$-representation of $C(K)$. If some $t \in K$ has dense orbit, we obtain a faithful representation of $C(K)$ on $\ell^2(\mathbb{Z}_+)$. As observed in Theorem 1.4, such representations play a fundamental role for the semicrossed product $\mathbb{Z}_+ \times_\alpha C(K)$, since they are "enough" to obtain the norm. Let $\pi_t := \widetilde{\mathrm{ev}}_t$, as in Example 1.1. So, $\pi_t : C(K) \to \mathcal{B}(\ell^2(\mathbb{Z}_+))$ is given by $\pi_t(f) := \sum_{n \geqslant 0} f(\phi^n(t)) p_n$, where p_n is the one-dimensional projection on $[e_n]$. Then (π_t, v) is a left covariant isometric pair. We define the *one point w*-semicrossed product* to be $\mathcal{C}_t = \overline{\mathrm{lt}_{\pi_t}(\mathbb{Z}_+ \times_\alpha C(K))}^{\mathrm{w}^*}$, i.e. the w*-closed linear span in $\mathcal{B}(\ell^2(\mathbb{Z}_+))$ of the "analytic polynomials" $\sum_{n=0}^{k} v^n \pi_t(f_n)$, $f_n \in C(K)$.

Let $t' = \phi^{n_0}(t)$ be the first periodic element of the orbit of t with period p (as in the diagram that follows). Then $\mathrm{orb}(t) = \{t, \dots, \phi^{n_0-1}(t), t', \dots, \phi^{p-1}(t')\}$ induces a family of projections $\{P_{n_0}, P_0, \dots, P_{p-1}\}$ such that $I = P_{n_0} \oplus P_0 \oplus \cdots \oplus P_{p-1}$. Indeed, let P_{n_0} be the projection on $[e_0, \dots, e_{n_0-1}]$ and P_i be the projection on $[e_{n_0+i+pj} : j \in \mathbb{Z}_+]$ for $i = 0, \dots, p-1$. Note that if $f \in C(K)$, then $\pi_t(f)(e_{n_0+i+pj}) = f(\phi^{n_0+i+pj}(t)) e_{n_0+i+pj} = f(\phi^i(t')) e_{n_0+i+pj}$, for $j \in \mathbb{Z}_+$. Hence, $\pi_t(f) P_i = f(\phi^i(t')) P_i$, for every $i = 0, \dots, p-1$.



PROPOSITION 3.1. *The algebra $\mathcal{C}_t$ is the linear sum $(\mathfrak{T} P_{n_0}) \oplus (\mathcal{T} P_0) \oplus \cdots \oplus (\mathcal{T} P_{p-1})$, where $\mathfrak{T}$ is the algebra of lower triangular operators in $\mathcal{B}(\ell^2(\mathbb{Z}_+))$, $\mathcal{T}$ is the algebra of analytic Toeplitz operators and $P_{n_0}, P_0, \dots, P_{p-1}$ are the projections induced by the orbit of t.*

Proof. For any $n \in \mathbb{Z}_+$ and $f \in C(K)$, we have

$$v^n \pi_t(f) = v^n \pi_t(f) P_{n_0} \oplus f(t') v^n P_0 \oplus f(\phi^{p-1}(t')) v^n P_{p-1}.$$

Thus, $\mathcal{C}_t \subseteq (\mathfrak{T} P_{n_0}) \oplus (\mathcal{T} P_0) \oplus \cdots \oplus (\mathcal{T} P_{p-1})$. For the converse, first let $TP_{n_0} \in \mathfrak{T} P_{n_0}$ and note that $(TP_{n_0})_{\kappa,\lambda} = 0$ when $\kappa < \lambda$ or $n_0 - 1 < \lambda$. Then we get that $G_m(TP_{n_0}) = 0$, when $m < 0$, and $G_m(TP_{n_0}) = v^m \Big(\sum_{n=0}^{n_0-1} (TP_{n_0})_{m+n,n} p_n \Big)$, when $m \geqslant 0$. Note that $(TP_{n_0})_{\kappa,\lambda} \in \mathbb{C}$ for every $\kappa, \lambda \in \mathbb{Z}_+$. Fix $m \geqslant 0$ and let $n \in \{0, \dots, n_0 - 1\}$. Then by Urysohn's Lemma there is a sequence $(f_{n,j})_j$ of continuous functions on K, such that $\lim_j f_{n,j}(\phi^n(t)) = (TP_{n_0})_{m+n,n}$ and $f_{n,j}(s) =$

0 for $s \in \operatorname{orb}(t) \setminus \{\phi^n(t)\}$. Hence, $(TP_{n_0})_{m+n,n}p_n = \text{w*-}\lim_j \pi_t(f_{n,j}) \in \mathcal{C}_t$ and so $v^m(TP_{n_0})_{m+n,n}p_n \in \mathcal{C}_t$. Thus $G_m(TP_{n_0}) \in \mathcal{C}_t$, and, by the Féjer Lemma, $TP_{n_0} \in \mathcal{C}_t$. So, $\mathfrak{T}P_{n_0} \subseteq \mathcal{C}_t$. Also, for fixed $i \in \{0, \ldots, p-1\}$ and $m \in \mathbb{Z}_+$, consider $v^m P_i \in \mathcal{T}P_i$. Again by Urysohn's Lemma, there is a sequence $(f_{i,j})_j$ of continuous functions on K, such that $\lim_j f_{i,j}(\phi^i(t')) = 1$ and $f_{i,j}(s) = 0$ for $s \in \operatorname{orb}(t) \setminus \{\phi^i(t')\}$. Then $\text{w*-}\lim_j \pi_t(f_{i,j}) = P_i$, so $v^m P_i \in \mathcal{C}_t$. Hence, $\mathcal{T}P_i \subseteq \mathcal{C}_t$, for every $i \in \{0, \ldots, p-1\}$. Thus, $(\mathfrak{T}P_{n_0}) \oplus (\mathcal{T}P_0) \oplus \cdots \oplus (\mathcal{T}P_{p-1}) \subseteq \mathcal{C}_t$. ∎

Note that if $\operatorname{orb}(t)$ has no periodic points, then $\mathcal{C}_t = \mathfrak{T}$, since $P_{n_0} = \mathbf{1}_{\ell^2(\mathbb{Z}_+)}$. Also, if $\operatorname{orb}(t)$ has exactly one periodic point t', then $\phi^n(t) = t'$ for every $n \geqslant n_0$ (i.e. t' is a fixed point); thus $\mathcal{C}_t = \mathfrak{T}P_{n_0} \oplus \mathcal{T}P_{n_0}^{\perp}$. If t is itself a fixed point, then $\mathcal{C}_t = \mathcal{T}$.

REMARK 3.2. Let $\mathcal{D}$ be the algebra of diagonal operators in $\mathcal{B}(\ell^2(\mathbb{Z}_+))$ and $\mathcal{D}_\phi = \{T \in \mathcal{D} : T_{\kappa,\kappa} = T_{n,n} \text{ when } \phi^\kappa(t) = \phi^n(t)\}$ which is a w*-closed subalgebra of $\mathcal{D}$. Hence, $T \in \mathcal{D}_\phi$ if and only if T is of the form

$$T = \operatorname{diag}\{y_0, \ldots, y_{n_0-1}, y_{n_0}, \ldots, y_{p-1}, y_{n_0}, \ldots, y_{p-1}, \ldots\}.$$

It is immediate from the previous proposition that $\mathcal{C}_t$ is generated by the unilateral shift in $\mathcal{B}(\ell^2(\mathbb{Z}_+))$ and the diagonal matrices id $\mathcal{D}_\phi$. Thus, an operator $T \in \mathcal{B}(\ell^2(\mathbb{Z}_+))$ is in $\mathcal{C}_t$ if and only if for every $m < 0$, $G_m(T) = 0$, and for every $m \geqslant 0$, $G_m(T) = v^m \sum_n T_{m+n,n}p_n$ where $T_{m+\kappa,\kappa} = T_{m+n,n}$, whenever $\phi^\kappa(t) = \phi^n(t)$.

THEOREM 3.3. *The algebra $\mathcal{C}_t$ is reflexive.*

Proof. If $T \in \operatorname{Ref}(\mathcal{C}_t)$, then $G_m(T) \in \operatorname{Ref}(\mathcal{C}_t)$; thus $G_m(T) = 0$, for $m < 0$. Let $g_r = \sum_{n \geqslant 0} r^n e_n$, with $0 \leqslant r < 1$, and $\mathcal{F} = \overline{[\pi_t(f)g_r : f \in C(K)]}$. Then $\mathcal{F}$ is $(\mathcal{C}_t)^*$-invariant; thus $G_m(T)^*$-invariant, for $m \in \mathbb{Z}_+$. So, there is a sequence of $f_j \in C(K)$ such that $G_m(T)^* g_r = \lim_j \pi_t(f_j) g_r$. Hence $r^m \overline{T}_{m+n,n} = \lim_j (f_j(\phi^n(t)))$, for every $n \in \mathbb{Z}_+$. Thus, $T_{m+n,n} = T_{m+\kappa,\kappa}$, if $\phi^\kappa(t) = \phi^n(t)$. So, by Remark 3.2, $T \in \mathcal{C}_t$. ∎

REMARK 3.4. In order to construct $\mathcal{C}_t$, *it is sufficient to take coefficients from any uniform algebra $\mathfrak{A}$ on K.* Indeed, let $\mathfrak{A}$ be a norm closed subalgebra of $C(K)$ containing the constant functions which separates the points of K and form the polynomials $\sum_{n=0}^{k} v^n \pi_t(f_n)$, $f_n \in \mathfrak{A}$. By Remark 3.2, it suffices to prove that $\pi_t(\operatorname{ball}(\mathfrak{A}))$ is w*-dense in $\operatorname{ball}(\mathcal{D}_\phi)$. Fix $z \in \mathbb{T}$ and $n_0 \in \mathbb{Z}_+$, and take $T \in \mathcal{D}_\phi$, such that $T_{n_0,n_0} = z$ and $T_{n,n} = 1$, if $\phi^n(t) \neq \phi^{n_0}(t)$. Using the argument of the claim of Theorem 2.9 of [6] we can find a sequence of $(f_j)_j$ in $\operatorname{ball}(\mathfrak{A})$ such that $\text{w*-}\lim_j \pi_t(f_j) = T$. To complete the proof, observe that products of elements of this form approximate the unitaries in $\mathcal{D}_\phi$ in the w*-topology and that the strong closure of $\pi_t(\mathrm{ball}(\mathfrak{A}))$ is closed under multiplication.

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