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APPROXIMATION OF CHAOTIC OPERATORS

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ABSTRACT. As it is well-known, the concept “hypercyclicity” in operator theory is the same as the concept “transitivity” in dynamical system. Now the class of hypercyclic operators is well studied. Following the idea of research in hypercyclic operators, we consider the classes of operators with other kinds of chaotic properties in this article. First, the closures and the interiors of the set of all Li–Yorke chaotic operators or all distributionally chaotic operators are discussed. Then we will show the connectedness of these sets.

KEYWORDS: *Spectrum, Fredholm index, Li–Yorke chaotic operator, distributionally chaotic operator, hypercyclic operator, closure, interior, connectedness.*

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1. INTRODUCTION

We are interested in the dynamical systems induced by continuous linear operators on Banach spaces. From Rolewicz’s article [27], hypercyclicity is widely studied. In fact, it coincides with a dynamical property "transitivity". Now there have been so many improvements in this aspect, Grosse-Erdmann’s and Shapiro’s articles [13], [29] are good surveys.

In his celebrated work [15], [16], [17], D.A. Herrero studied the chaotic properties (hypercyclicity and Devaney chaoticity) of linear operators. It is important since it shows that we can study the chaotic properties of operators in an operator theoretic way. As it is well-known, it is hard to check whether a topological system be chaotic or not for a general object. But following Herrero’s idea, we can use the technique of approximation to study the properties of chaotic operators on a Hilbert space under some compact or small perturbation. An interesting result, obtained by D.A. Herrero and Z.Y. Wang [17] or K. Chan and J. Shapiro [7], shows that the identity operator I can be perturbed by a small compact operator to be hypercyclic. This stronger result implies that a small perturbation of a simple operator can be an operator with complex dynamic properties.

These papers suggest us to consider the following question:

Question. Which kinds of operators can be approximated by chaotic operators?

From the point of approximation, we should consider the closure of the set of all operators satisfying some chaotic property. In this paper, Li–Yorke chaotic operators and distributionally chaotic operators will be studied by classical approximation tools developed in [14].

In order to explain the main results, we must introduce some definitions and properties of chaos and Hilbert space operators.

In 1975, Li and Yorke [21] observed complicated dynamical behavior for the class of interval maps with period 3. This phenomena is currently known under the name of Li–Yorke chaos. Recall that a discrete dynamical system is simply a continuous mapping $f : X \to X$ where X is a complete separable metric space. For $x \in X$, the orbit of x under f is $\mathrm{Orb}(f, x) = \{x, f(x), f^2(x), \ldots\}$ where $f^n = f \circ f \circ \cdots \circ f$ is the n^{th} iterate of f obtained by composing f with itself n times.

DEFINITION 1.1. $\{x, y\} \subset X$ is said to be a *Li–Yorke chaotic pair*, if

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0, \quad \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

Furthermore, f is called *Li–Yorke chaotic*, if there exists an uncountable subset $\Gamma \subseteq X$ such that each pair of two distinct points in Γ is a Li–Yorke chaotic pair.

In 1994, Schweizer and Smítal [28] gave the definition of distributional chaos (where it was called strong chaos), which requires more complicated statistical dependence between orbits than the existence of points which are proximal but not asymptotic.

For any pair $\{x, y\} \subset X$ and any $n \in \mathbb{N}$, define distributional function $F^n_{xy} : \mathbb{R} \to [0, 1]$:

$$F^n_{xy}(\tau) = \frac{1}{n}\#\{0 \leqslant i \leqslant n-1 : d(f^i(x), f^i(y)) < \tau\},$$

where $\#\{A\}$ is the cardinality of the set A. Furthermore, define

$$F_{xy}(\tau) = \liminf_{n\to\infty} F^n_{xy}(\tau), \quad F^*_{xy}(\tau) = \limsup_{n\to\infty} F^n_{xy}(\tau).$$

Both F_{xy} and F^*_{xy} are nondecreasing functions and may be viewed as cumulative probability distributional functions satisfying $F_{xy}(\tau) = F^*_{xy}(\tau) = 0$ for $\tau < 0$.

DEFINITION 1.2. $\{x, y\} \subset X$ is said to be a *distributionally chaotic pair*, if

$$F^*_{xy}(\tau) \equiv 1, \quad \forall\ \tau > 0 \quad \text{and} \quad F_{xy}(\varepsilon) = 0, \quad \exists\ \varepsilon > 0.$$

Furthermore, f is called *distributionally chaotic*, if there exists an uncountable subset $\Lambda \subseteq X$ such that each pair of two distinct points in Λ is a distributionally chaotic pair. Moreover, Λ is called a *distributionally ε-scrambled set*.

From the definitions, we know distributional chaos implies Li–Yorke chaos. But the converse implication is not true in general. In practice, even in the simple case of Li–Yorke chaos, it might be quite difficult to prove the chaotic behavior from the very definition. Such attempts have been made in the context of linear operators (see [10], [12]). Further results of [10] were extended in [25] to distributional chaos for the annihilation operator of a quantum harmonic oscillator. Additionally, distributional chaos for shift operators were discussed by F. Martínez-Giménez, et. al. in [24]. More about Li–Yorke chaos and distributional chaos, one can see [3], [22], [23], [30], [32]. In a recent article [18], B. Hou et. al. introduced a new dynamical property for linear operators called norm-unimodality (where it was called "special operator"), which implies distributional chaos, and obtained a sufficient condition for Cowen–Douglas operator being distributionally chaotic. We introduce the definition of norm-unimodality here.

DEFINITION 1.3. Let X be a Banach space and let $T \in B(X)$. T is called *norm-unimodal*, if we have a constant $\gamma > 1$ such that for any $m \in \mathbb{N}$, there exists $x_m \in X$ satisfying

$$\lim_{k\to\infty} \|T^k x_m\| = 0, \quad \text{and} \quad \|T^i x_m\| \geqslant \gamma^i \|x_m\|, \quad i = 1, 2, \ldots, m.$$

Furthermore, such γ is said to be a *norm-unimodal constant for the norm-unimodal operator T*.

It is not obvious that norm-unimodality is invariant under similarity, one can observe [31] for details.

Next, we introduce the notations and properties of Hilbert space operators. Let H be complex separable Hilbert space and denote by $B(H)$ the set of bounded linear operators $T : H \to H$. For $T \in B(H)$, denote the kernel of T and the range of T by $\mathrm{Ker}T$ and $\mathrm{Ran}T$ respectively. Denote by $\sigma(T)$, $\sigma_{\mathrm{e}}(T)$, $\sigma_{\mathrm{lre}}(T)$ and $\sigma_{\mathrm{w}}(T)$ the spectrum, the essential spectrum, the Wolf spectrum and the Weyl spectrum of T respectively. For $\lambda \in \rho_{s-F}(T) := \mathbb{C}\backslash\sigma_{\mathrm{lre}}(T)$, $\mathrm{ind}(\lambda - T) = \mathrm{dimKer}(\lambda - T) - \mathrm{dimKer}(\lambda - T)^*$, $\min \mathrm{ind}(\lambda - T) = \min\{\mathrm{dimKer}(\lambda - T), \mathrm{dimKer}(\lambda - T)^*\}$. Denote $\rho^{(n)}_{s-F}(T) = \{\lambda \in \rho_{s-F}(T);\ \mathrm{ind}(\lambda - T) = n\}$, where $-\infty \leqslant n \leqslant \infty$, $\rho^{(+)}_{s-F}(T) = \{\lambda \in \rho_{s-F}(T);\ \mathrm{ind}(\lambda - T) > 0\}$ and $\rho^{(-)}_{s-F}(T) = \{\lambda \in \rho_{s-F}(T);\ \mathrm{ind}(\lambda - T) < 0\}$. According to Corollary 1.14 of [14], we know that the function $\lambda \to \min \mathrm{ind}(\lambda - T)$ is constant on every component of $\rho_{s-F}(T)$ except for an at most denumerable subset $\rho^{\mathrm{s}}_{s-F}(T)$ without limit points in $\rho_{s-F}(T)$. Furthermore, if $\mu \in \rho^{\mathrm{s}}_{s-F}(T)$ and λ is a point of $\rho_{s-F}(T)$ in the same component as μ but λ is not in $\rho^{\mathrm{s}}_{s-F}(T)$, then $\min \mathrm{ind}(\mu - T) > \min \mathrm{ind}(\lambda - T)$. $\rho^{\mathrm{s}}_{s-F}(T)$ is the set of *singular points* of the semi-Fredholm domain $\rho_{s-F}(T)$ of T; $\rho^{\mathrm{r}}_{s-F}(T) = \rho_{s-F}(T)\backslash\rho^{\mathrm{s}}_{s-F}(T)$ is the set of *regular points*. Denote by $\sigma_0(T)$ the set of isolated points of $\sigma(T)\backslash\sigma_{\mathrm{e}}(T)$. Denote by $\overline{E}$ and E^0, the closure and the interior of a set E respectively. In addition, denote by $LY(H)$, $DC(H)$, $UN(H)$ the set of all Li–Yorke chaotic operators, the set of all distributionally chaotic operators and the set of all norm-unimodal operators on H respectively.

Now we are in a position to state the main results of this article. In Section 2, the closures and the interiors of the set of all distributionally chaotic operators and the set of all Li–Yorke chaotic operators are considered. Though distributionally chaotic operators require more complicated statistical dependence between orbits than Li–Yorke chaotic operators, we have:

(I). $\overline{DC(H)} = \overline{LY(H)} = \{T \in B(H);\ \partial\mathbb{D} \cap \sigma_{\mathrm{lre}}(T) \neq \emptyset\} \cup \{T \in B(H);\ \partial\mathbb{D} \subseteq \rho_{s-F}(T)$ and $\dim\mathrm{Ker}(\lambda - T) > 0,\ \forall \lambda \in \partial\mathbb{D}\}$ (Theorem 2.10).

(II). $DC(H)^0 = LY(H)^0 = \{T \in B(H),\ \exists\, \lambda \in \partial\mathbb{D}$ such that $\mathrm{ind}(\lambda - T) > 0\}$ (Theorem 2.16).

From the above two results, one can see distributionally chaotic operators and Li–Yorke chaotic operators are very similar. The closure of $DC(H)^0$ (i.e. the closure of $LY(H)^0$) is also considered.

(III). $\overline{DC(H)^0} = \overline{LY(H)^0} = \{T \in B(H);\ \partial\mathbb{D} \nsubseteq \rho^{(0)}_{s-F}(T) \cup \rho^{(-)}_{s-F}(T)\}$. Moreover, $\overline{DC(H)\backslash DC(H)^0} = \overline{LY(H)\backslash LY(H)^0} = \{T \in B(H); \partial\mathbb{D} \subseteq \rho^{(0)}_{s-F}(T) \cup \rho^{(-)}_{s-F}(T)$ and $\dim\mathrm{Ker}(\lambda - T) > 0,\ \forall \lambda \in \partial\mathbb{D}\} \cup \{T \in B(H); \partial\mathbb{D} \cap \sigma_{\mathrm{lre}}(T) \neq \emptyset$ and $\rho^{(+)}_{s-F}(T) \cap \partial\mathbb{D} = \emptyset\}$ (Theorem 2.18).

In Section 3, we get the relation between hypercyclic operators and distributionally chaotic operators. In detail (Proposition 3.2), the set of all hypercyclic operators belongs to the closure of $DC(H)^0$. The relation between norm-unimodal operators and distributionally chaotic operators is also obtained.

(IV). $\overline{UN(H)} = \overline{DC(H)} = \overline{LY(H)}$, $DC(H)^0 = LY(H)^0 \subseteq UN(H)$, and $\overline{UN(H)\backslash DC(H)^0} = \overline{DC(H)\backslash DC(H)^0} = \overline{LY(H)\backslash LY(H)^0}$ (Theorem 3.4).

It follows from this result that, the norm-unimodal operators are very large in the class of distributionally chaotic operators. Moreover, it is useful for people to prove that an operator is distributionally chaotic as the criterion of hypercyclic operators given by Kitai [20] and refined by Grosse-Erdmann and Shapiro, et.al. [13].

In Section 4, we consider the connectedness of the sets considered above.

(V). $DC(H)^0$, $\overline{DC(H)^0}$, $\overline{DC(H)}$ and $\overline{DC(H)\backslash DC(H)^0}$ (i.e. $LY(H)^0$, $\overline{LY(H)^0}$, $\overline{LY(H)}$ and $\overline{LY(H)\backslash LY(H)^0}$) are all arcwise connected (Theorem 4.1).

2. CLOSURES AND INTERIORS OF THE SET OF ALL DISTRIBUTIONALLY CHAOTIC OPERATORS AND THE SET OF ALL LI–YORKE CHAOTIC OPERATORS

As it is well-known, the important result Theorem 2.2 of [1] obtained by C. Apostol, C. Foiaş and D. Voiculescu and improved by C. Foiaş, C.M. Pearcy and D. Voiculescu [11] is frequently used in the approximation problems. Since it will be also used throughout this article, we introduce it but only the improved version. One can observe Theorem 3.49 and Proposition 4.29 of [14] for more information. Denote by $\simeq$ the relation of unitarily equivalence between operators.

THEOREM 2.1. *Let $T \in B(H)$ and let Γ_{l} and Γ_{r} be closed subsets of $\sigma_{\mathrm{le}}(T)$ and $\sigma_{\mathrm{re}}(T)$, respectively (Γ_{l} or Γ_{r} may be absent). Then for any $\varepsilon > 0$, there exists a compact operator K such that $\|K\| < \varepsilon$ and*

$$T + K \simeq \begin{bmatrix} N_{\mathrm{l}} & * & * \\ & A & * \\ & & N_{\mathrm{r}} \end{bmatrix},$$

where N_{l} and N_{r} are diagonal normal operators of uniform infinite multiplicity such that $\sigma(N_{\mathrm{l}}) = \sigma_{\mathrm{e}}(N_{\mathrm{l}}) = \Gamma_{\mathrm{l}}$ and $\sigma(N_{\mathrm{r}}) = \sigma_{\mathrm{e}}(N_{\mathrm{r}}) = \Gamma_{\mathrm{r}}$ respectively, moreover $\sigma(T) = \sigma(A)$, $\sigma_{\mathrm{e}}(T) = \sigma_{\mathrm{e}}(A)$, $\sigma_{\mathrm{lre}}(T) = \sigma_{\mathrm{lre}}(A)$ and $\mathrm{ind}(\lambda - T) = \mathrm{ind}(\lambda - A)$, min ind $(\lambda - T) = \min \mathrm{ind}(\lambda - A)$ *for $\lambda \in \rho_{s-F}(T)$.*

REMARK 2.2. Since the properties (Li–Yorke chaoticity, distribitionally chaoticity and norm-unimodality) are all invariant under similarity and the set of all unitarily operators is arcwise connected, we can assume, without loss of generality, that

$$T + K = \begin{bmatrix} N_{\mathrm{l}} & * & * \\ & A & * \\ & & N_{\mathrm{r}} \end{bmatrix}.$$

Next, we introduce some auxiliary results for Theorem 2.10. The definition given by Cowen and Douglas [8] is well known as follows.

DEFINITION 2.3. For Ω a connected open subset of $\mathbb{C}$ and n a positive integer, let $B_n(\Omega)$ denote the operators T in $B(H)$ which satisfy:

(i) $\Omega \subseteq \sigma(T)$;

(ii) $\mathrm{ran}(T - \omega) = H$ *for* ω *in* Ω;

(iii) $\bigvee_{\omega \in \Omega} \ker(T - \omega) = H$; and

(iv) $\dim\ker(T - \omega) = n$ for ω in Ω.

One often calls the operator T in $B_n(\Omega)$ *Cowen–Douglas operator*.

Denote by $\mathbb{D}$ and $\partial\mathbb{D}$ the unit open disk and its boundary. Then we have the following theorem ([18], Theorem 3.7).

THEOREM 2.4. *Let $T \in B_n(\Omega)$. If $\Omega \cap \partial\mathbb{D} \neq \phi$, then T is norm-unimodal. Consequently, T is distributionally chaotic.*

REMARK 2.5. In fact, an extended case of this result, for $n = +\infty$, can be obtained with the same argument.

LEMMA 2.6. *Let $T \in B(H)$. Then the following statements are equivalent:*

(i) *T is not Li–Yorke chaotic.*

(ii) $\liminf_{n\to\infty} \|T^n(x)\| = 0$ *implies* $\lim_{n\to\infty} \|T^n(x)\| = 0$.

The proof is easy and left to the reader. Denote by $\sim$ the relation of similarity between operators.

LEMMA 2.7. *Let $T \in B(H)$, $\sigma(T) \cap \partial\mathbb{D} = \emptyset$. Then $\liminf_{n\to\infty} \|T^n(x)\| = 0$ implies $\lim_{n\to\infty} \|T^n(x)\| = 0$. Moreover, T is neither Li–Yorke chaotic nor distributionally chaotic.*

Proof. Since $\sigma(T) \cap \partial\mathbb{D} = \emptyset$, it readily follows from Riesz's decomposition theorem that

$$T = \begin{bmatrix} T_1 & \\ & T_2 \end{bmatrix} \begin{matrix} H_1 \\ H_2 \end{matrix},$$

where $\sigma(T_1) = \sigma(T) \cap \mathbb{D}$ and $\sigma(T_2) = \sigma(T) - \sigma(T_1)$. Thus,

$$T = \begin{bmatrix} T_1 & * \\ & \widetilde{T}_2 \end{bmatrix} \begin{matrix} H_1 \\ H_1^{\perp} \end{matrix},$$

where $\widetilde{T}_2 \sim T_2$ and then $\sigma(\widetilde{T}_2) = \sigma(T_2) = \sigma(T) - \sigma(T_1)$.

By spectral mapping theorem and spectral radius formula,

$$r_1(\widetilde{T}_2)^{-1} = r(\widetilde{T}_2^{-1}) = \lim_{n\to\infty} \|\widetilde{T}_2^{-n}\|^{1/n}, \quad (\text{where } r_1(\cdot) = \inf\{|\lambda|;\ \lambda \in \sigma(\cdot)\}).$$

Observe that $r_1(\widetilde{T}_2) \geqslant \delta > 1$, so there exists $\varepsilon > 0$ such that $r_1(\widetilde{T}_2)^{-1} + \varepsilon < 1$. Hence there exists $M \in \mathbb{N}$ such that for any $n \geqslant M$,

$$\frac{1}{\|\widetilde{T}_2^{-n}\|} \geqslant \Big(\frac{1}{r_1(\widetilde{T}_2)^{-1} + \varepsilon}\Big)^n.$$

Furthermore, for any $y \in H_1{}^{\perp}$,

$$\|\widetilde{T}_2^n(y)\| \geqslant \frac{1}{\|\widetilde{T}_2^{-n}\|}\|y\| \geqslant \Big(\frac{1}{r_1(\widetilde{T}_2)^{-1} + \varepsilon}\Big)^n \|y\| \geqslant \|y\|, \quad \text{when } n \geqslant M.$$

Let $\liminf_{n\to\infty} \|T^n(x)\| = 0$ and $x = x_1 \oplus x_2,\ x_1 \in H_1,\ x_2 \in H_1{}^{\perp}$, then one can easily obtain $x_2 = 0$ and then $T^n(x) = T_1^n(x_1)$. On the other hand $r(T_1) < 1$, so there exist $0 \leqslant \rho < 1$ and $N \in \mathbb{N}$ such that for any $n \geqslant N$, $\|T_1^n(x_1)\| \leqslant \rho^n\|x_1\|$. Therefore, $\lim_{n\to\infty} \|T^n(x)\| = \lim_{n\to\infty} \|T_1{}^n(x_1)\| = 0$. ∎

LEMMA 2.8. *Let $0 < n < \infty$ be an integer and $T \in B(\mathbb{C}^n)$. Then $\liminf_{m\to\infty} \|T^m(x)\| = 0$ implies $\lim_{m\to\infty} \|T^m(x)\| = 0$. Moreover, T is neither Li–Yorke chaotic nor distributionally chaotic.*

Proof. As it is well-known, each T in $B(\mathbb{C}^n)$ is similar to a Jordan model, i.e.

$$T \sim J = \bigoplus_{i=1}^{l} \Big\{ \bigoplus_{j=1}^{k_i} J_{n_j^i}(\mu_i) \Big\},$$

where $\{\mu_i\}_{i=1}^{l} = \sigma(T)$, $\mu_i \neq \mu_j$ for $i \neq j$, $\sum\limits_{i=1}^{l}\sum\limits_{j=1}^{k_i} n_j^i = n$ and

$$J_n(\mu) = \begin{bmatrix} \mu & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \mu \end{bmatrix}_{(n\times n)}.$$

Since the property considered is invariant under similarity, it suffices to deal with the case when $T = J_n(\mu)$.

If $|\mu| \neq 1$, then the result follows similarly from Lemma 2.7.

Assume $|\mu| = 1$. Let $\{e_i\}_{i=1}^{n}$ be the orthonormal basis corresponding to the matrix above and let $\liminf\limits_{m\to\infty} \|T^m(x)\| = 0$, then $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$.

Claim. $x_i = 0$ for $i = 1, 2, \ldots, n$ and hence $\lim\limits_{m\to\infty} \|T^m(x)\| = 0$.

If not, let x_p $(1 \leqslant p \leqslant n)$ be the last nonzero coordinate. Since for $m \geqslant n-1$,

$$\begin{aligned}
&\|T^m(x)\|^2 \\
&= \left\| \begin{bmatrix} C_m^0\mu^m & C_m^1\mu^{m-1} & \cdots & C_m^{n-1}\mu^{m-n+1} \\ & C_m^0\mu^m & \cdots & C_m^{n-2}\mu^{m-n+2} \\ & & \ddots & \vdots \\ & & & C_m^0\mu^m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \right\|^2 \\
&= |C_m^0\mu^m x_1 + C_m^1\mu^{m-1}x_2 + \cdots + C_m^{n-1}\mu^{m-n+1}x_n|^2 \\
&\qquad + |C_m^0\mu^m x_2 + C_m^1\mu^{m-1}x_3 + \cdots + C_m^{n-2}\mu^{m-n+2}x_n|^2 + \cdots + |C_m^0\mu^m x_n|^2 \\
&\geqslant |C_m^0\mu^m x_p + C_m^1\mu^{m-1}x_{p+1} + \cdots + C_m^{n-p}\mu^{m-n+p}x_n|^2 = |x_p|^2,
\end{aligned}$$

and $\liminf\limits_{m\to\infty} \|T^m(x)\| = 0$, it readily follows that $x_p = 0$, a contradiction with $x_p \neq 0$. ∎

The following result is taken from Theorem 9 of [19].

THEOREM 2.9. *For any $\varepsilon > 0$, there is a compact operator $K_\varepsilon \in B(H)$ such that $\|K_\varepsilon\| < \varepsilon$ and $I + K_\varepsilon$ is distributionally chaotic.*

Now we will give a description of the closures of the set of all distributionally chaotic operators and the set of all Li–Yorke chaotic operators.

THEOREM 2.10. *Let $E_1 = \{T \in B(H);\ \partial\mathbb{D} \cap \sigma_{\mathrm{lre}}(T) \neq \emptyset\}$ and $E_2 = \{T \in B(H);\ \partial\mathbb{D} \subseteq \rho_{s-F}(T)$ and $\mathrm{dimKer}(\lambda - T) > 0,\ \forall \lambda \in \partial\mathbb{D}\}$. Then $\overline{DC(H)} = \overline{LY(H)} = E_1 \cup E_2$.*

Proof. Clearly, $\overline{DC(H)} \subseteq \overline{LY(H)}$. So it suffices to show that $E_1 \cup E_2 \subseteq \overline{DC(H)}$ and $\overline{LY(H)} \subseteq E_1 \cup E_2$.

First step, $E_1 \cup E_2 \subseteq \overline{DC(H)}$. It is enough to show that for any $T \in E_1 \cup E_2$ and $\varepsilon > 0$, there exists an operator C such that $\|C\| < \varepsilon$ and $T + C \in DC(H)$. In fact, we can obtain a compact operator K such that $\|K\| < \varepsilon$ and $T + K \in DC(H)$.

If $T \in E_1$, then choose any $\lambda_0 \in \partial\mathbb{D} \cap \sigma_{\mathrm{lre}}(T)$. By Theorem 2.1 there exists a compact operator K_1 such that $\|K_1\| < \varepsilon/2$ and

$$T + K_1 = \begin{array}{c} \begin{bmatrix} \lambda_0 I & * \\ & * \end{bmatrix} \end{array} \begin{array}{c} H_0 \\ {H_0}^{\perp}, \end{array}$$

where $\dim H_0 = \infty$.

Following Theorem 2.9, there exists a compact operator $K_\varepsilon \in B(H_0)$ such that $\|K_\varepsilon\| < \varepsilon/2$ and $\lambda_0 I + K_\varepsilon$ is distributionally chaotic. Let

$$K_2 = \begin{bmatrix} K_\varepsilon & \\ & 0 \end{bmatrix} \begin{array}{c} H_0 \\ {H_0}^{\perp}, \end{array}$$

then

$$T + K_1 + K_2 = \begin{bmatrix} \lambda_0 I + K_\varepsilon & * \\ & * \end{bmatrix} \in DC(H),$$

where $K_1 + K_2$ is a compact operator and $\|K_1 + K_2\| < \varepsilon$.

If $T \in E_2$, define

$$H_{\mathrm{r}} = \bigvee_{\lambda \in \rho^{\mathrm{r}}_{s-F}(T) \cap \Delta} \mathrm{Ker}(\lambda - T),$$

where Δ is the component of semi-Fredholm domain $\rho_{s-F}(T)$ which contains $\partial\mathbb{D}$. Then $\dim H_{\mathrm{r}} = \infty$ and

$$T = \begin{bmatrix} T_{\mathrm{r}} & * \\ & * \end{bmatrix} \begin{array}{c} H_{\mathrm{r}} \\ {H_{\mathrm{r}}}^{\perp}. \end{array}$$

Since:

(i) $\rho^{\mathrm{r}}_{s-F}(T) \cap \Delta \subseteq \sigma(T_{\mathrm{r}})$,

(ii) $\dim\mathrm{Ker}(\mu - T_{\mathrm{r}}) = \dim\mathrm{Ker}(\mu - T) = n$, $\forall \mu \in \rho^{\mathrm{r}}_{s-F}(T) \cap \Delta$, where $n \in \mathbb{N}^+ \cup \{+\infty\}$,

(iii) $\bigvee_{\mu \in \rho^{\mathrm{r}}_{s-F}(T) \cap \Delta} \mathrm{Ker}(\mu - T_{\mathrm{r}}) = \bigvee_{\mu \in \rho^{\mathrm{r}}_{s-F}(T) \cap \Delta} \mathrm{Ker}(\mu - T) = H_{\mathrm{r}}$,

(iv) $\mathrm{ran}(\mu - T_{\mathrm{r}}) = H_{\mathrm{r}}$, $\forall \mu \in \rho^{\mathrm{r}}_{s-F}(T) \cap \Delta$,

it readily follows that $T_{\mathrm{r}} \in B_n(\rho^{\mathrm{r}}_{s-F}(T) \cap \Delta)$ or $T_{\mathrm{r}} \in B_{+\infty}(\rho^{\mathrm{r}}_{s-F}(T) \cap \Delta)$. Observe that $\rho^{\mathrm{r}}_{s-F}(T) \cap \Delta \cap \partial\mathbb{D} \neq \emptyset$, so it follows from Theorem 2.4 and Remark 2.5 that T_{r} is norm-unimodal and hence distributionally chaotic, and so is T.

The first step is complete.

Second step, $\overline{LY(H)} \subseteq E_1 \cup E_2$. Observe that $\{E_1 \cup E_2\}^{\mathrm{c}} = \{T \in B(H);\ \partial\mathbb{D} \subseteq \rho_{s-F}(T)$ and $\exists \lambda \in \partial\mathbb{D}$ such that $\dim\mathrm{Ker}(\lambda - T) = 0\}$, so it follows from the stability properties of semi-Frodholm operators that $\{E_1 \cup E_2\}^{\mathrm{c}}$ is open. Since $\{\overline{LY(H)}\}^{\mathrm{c}} = \{LY(H)^{\mathrm{c}}\}^0$, it suffices to prove that $\{E_1 \cup E_2\}^{\mathrm{c}} \subseteq LY(H)^{\mathrm{c}}$.

Let $T \in \{E_1 \cup E_2\}^{\mathrm{c}}$, define

$$H_l = \bigvee_{\lambda \in \rho^{\mathrm{r}}_{s-F}(T) \cap \Phi} \mathrm{Ker}(\lambda - T)^*,$$

where Φ is the component of semi-Fredholm domain $\rho_{s-F}(T)$ which contains $\partial\mathbb{D}$. Then

$$T = \begin{bmatrix} T_0 & * \\ & T_l \end{bmatrix} \begin{matrix} H_l^{\perp} \\ H_l \end{matrix}, \quad (H_l \text{ maybe } \{0\}!).$$

Claim 1. $\rho^{\mathrm{r}}_{s-F}(T) \cap \Phi \subseteq \rho(T_0)$.

Let $\mu \in \rho^{\mathrm{r}}_{s-F}(T) \cap \Phi$. Since $\lambda \to \min \mathrm{ind}(\lambda - T)$ is constant on the semi-Fredholm domain $\rho^{r}_{s-F}(T)$ and there exists $\lambda_0 \in \partial\mathbb{D}$ such that $\dim\mathrm{Ker}(\lambda_0 - T) = 0$, we have $\mathrm{Ker}(\mu - T) = \{0\}$. Hence $\mathrm{Ker}(\mu - T_0) = \{0\}$. Observe that $\mathrm{Ker}(\mu - T)^* = \mathrm{Ker}(\mu - T_l)^*$ and $(\mu - T)^*(H_l) = H_l$, it readily follows that $\mathrm{Ker}(\mu - T_0)^* = \{0\}$. Therefore, $\mu - T_0$ is invertible.

Claim 2. $\sigma_0(T_0) \cap \Phi = \sigma(T_0) \cap \Phi = \rho^{\mathrm{s}}_{s-F}(T) \cap \Phi$.

From Claim 1, $\sigma_0(T_0) \cap \Phi \subseteq \sigma(T_0) \cap \Phi \subseteq \rho^{\mathrm{s}}_{s-F}(T) \cap \Phi$. Let $\lambda \in \rho^{\mathrm{s}}_{s-F}(T) \cap \Phi$. If $\lambda - T_0$ is invertible, then $\lambda - T_l$ is a semi-Fredholm operator and $\min \mathrm{ind}(\lambda - T_l) = \min \mathrm{ind}(\lambda - T)$. Since $(\lambda - T_l)^*(H_l) = H_l$, it is not difficult to conclude that $\dim\mathrm{Ker}(\lambda - T_l) = 0$ and $\min \mathrm{ind}(\lambda - T) = \min \mathrm{ind}(\lambda - T_l) = 0$. It contradicts with $\lambda \in \rho^{\mathrm{s}}_{s-F}(T)$. Therefore, $\lambda - T_0$ is not invertible and $\rho^{\mathrm{s}}_{s-F}(T) \cap \Phi \subseteq \sigma(T_0) \cap \Phi$. Observe that $\lambda - T$ is a left semi-Fredholm operator implies $\lambda - T_0$ is a left semi-Fredholm operator, so it is easily seen from Claim 1 that $\mathrm{ind}(\lambda - T_0) = 0$ and hence $\rho^{\mathrm{s}}_{s-F}(T) \cap \Phi \subseteq \sigma_0(T_0) \cap \Phi$.

Since the only limit points of $\rho^{\mathrm{s}}_{s-F}(T)$ belong to $\partial[\rho_{s-F}(T)]$, we can let $\sigma_0(T_0) \cap \Phi \cap \partial\mathbb{D} = \{\mu_i\}_{i=1}^m$, $m < \infty$, and it follows from Riesz's decomposition theorem and Rosenblum–Davis–Rosenthal ([14], Corollary 3.22) that,

$$T_0 = \begin{bmatrix} T_{00} & \\ & T_{01} \end{bmatrix} \begin{matrix} H_{00} \\ H_{01} \end{matrix} = \begin{bmatrix} T_{00} & * \\ & \widetilde{T}_{01} \end{bmatrix} \begin{matrix} H_{00} \\ H_l^{\perp} \ominus H_{00} \end{matrix} \sim \begin{bmatrix} T_{00} & \\ & \widetilde{T}_{01} \end{bmatrix} \begin{matrix} H_{00} \\ H_l^{\perp} \ominus H_{00} \end{matrix},$$

where $\sigma(T_{00}) = \{\mu_i\}_{i=1}^m$, $m < \infty$, $\sigma(T_{01}) \cap \partial\mathbb{D} = \emptyset$, $T_{01} \sim \widetilde{T}_{01}$ and $\dim H_{00} < \infty$. Hence

$$T \sim S := \begin{bmatrix} T_{00} & 0 & * \\ & \widetilde{T}_{01} & * \\ & & T_l \end{bmatrix} \begin{matrix} H_{00} \\ H_l^{\perp} \ominus H_{00}. \\ H_l \end{matrix}$$

Moreover

$$H_l = \bigvee_{\lambda \in \rho^{\mathrm{r}}_{s-F}(T) \cap \Phi} \mathrm{Ker}(\lambda - T)^* = \bigvee_{\lambda \in \partial\mathbb{D} \cap \rho^{\mathrm{r}}_{s-F}(T) \cap \Phi} \mathrm{Ker}(\lambda - T)^*,$$

then

$$T_l = \begin{bmatrix} \lambda_1 & & & \\ * & \lambda_2 & & \\ * & * & \lambda_3 & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3, \\ \vdots \end{matrix}$$

where $\{\lambda_i\}_{i=1}^{\infty} \subseteq \partial\mathbb{D} \cap \rho_{s-F}^{\mathrm{r}}(T) \cap \Phi$ and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of H_1.

Now we come to end the proof. By Lemma 2.6, it suffices to show that $\liminf_{n\to\infty} \|S^n(x)\| = 0$ implies $\lim_{n\to\infty} \|S^n(x)\| = 0$. Let $\liminf_{n\to\infty} \|S^n(x)\| = 0$, then there exists a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ such that $\lim_{n_k\to\infty} \|S^{n_k}(x)\| = 0$. Observe that $x = x_0 \oplus \widetilde{x}_0 \oplus x_1$, $x_0 \in H_{00}$, $\widetilde{x}_0 \in H_1{}^{\perp} \ominus H_{00}$, $x_1 \in H_1$, we have $\lim_{n_k\to\infty} \|T_1^{n_k}(x_1)\| = 0$. Following the matrix representation of T_1, $x_1 = 0$. Hence,

$$S^{n_k}(x) = \begin{bmatrix} T_{00} & \\ & \widetilde{T}_{01} \end{bmatrix}^{n_k} \begin{bmatrix} x_0 \\ \widetilde{x}_0 \end{bmatrix}.$$

It readily follows from Lemma 2.8 and Lemma 2.7 that $\lim_{n\to\infty} \|T_{00}^n(x_0)\| = 0$ and $\lim_{n\to\infty} \|\widetilde{T}_{01}^n(\widetilde{x}_0)\| = 0$, whence we conclude that $\lim_{n\to\infty} \|S^n(x)\| = 0$.

The second step is complete. ∎

Theorem 2.10 also includes the information of the interior of $DC(H)^{\mathrm{c}}$ (i.e. the interior of $LY(H)^{\mathrm{c}}$). Obviously, the operator T satisfying, $\sigma(T) \cap \partial\mathbb{D} = \emptyset$, is in $\{DC(H)^{\mathrm{c}}\}^0$. But there exists an operator T in $\{DC(H)^{\mathrm{c}}\}^0$, whose spectrum $\sigma(T)$ intersects the unit circle, i.e. $\sigma(T) \cap \partial\mathbb{D} \neq \emptyset$.

EXAMPLE 2.11. Let $A \in B(H)$ satisfy

$$\begin{cases} Ae_i = \frac{1}{2}e_{i+1} & i \leqslant -2, \\ Ae_i = 2e_{i+1} & i > -2, \end{cases}$$

where $\{e_i\}_{i=-\infty}^{\infty}$ is an orthonormal basis of H. Then A is in $\{DC(H)^{\mathrm{c}}\}^0$.

Proof. First, let us consider the dynamical property of A. For any x in H, $x = \sum_{i=-\infty}^{\infty} x_i e_i$ and

$$A^{2n+1}(x) = \Big(\dots, \frac{1}{2^{2n+1}}x_{-(2n+2)}, \widehat{\frac{1}{2^{2n-1}}x_{-(2n+1)}}, \frac{1}{2^{2n-3}}x_{-2n}, \dots, \frac{1}{2}x_{-(n+2)}, 2x_{-(n+1)}, 2^3x_{-n}, \dots \Big),$$

$$A^{2n}(x) = \Big(\dots, \frac{1}{2^{2n}}x_{-2n-1}, \widehat{\frac{1}{2^{2n-2}}x_{-2n}}, \frac{1}{2^{2n-4}}x_{-2n+1}, \dots, x_{-(n+1)}, 2^2x_{-n}, 2^4x_{-n+1}, \dots \Big),$$

where the position under $\wedge$ is the zeroth position corresponding to the orthonormal basis $\{e_i\}_{i=-\infty}^{\infty}$. A straightforward computation shows that if $x \neq 0$, then $\|A^n(x)\| \to \infty$. Hence A is not distributionally chaotic. But the dynamical property of the small perturbations of A is not obvious in this direct manner.

Through easy computations, one can obtain $\sigma(A) = \{z \in \mathbb{C};\ (1/2) \leqslant |z| \leqslant 2\}$ and $\mathrm{ind}(\lambda - A) = -1$, $\dim\mathrm{Ker}(\lambda - A) = 0$ for $\lambda \in \{z \in \mathbb{C};\ (1/2) < |z| < 2\}$. It readily follows from Theorem 2.10 that $A \in \{\overline{DC(H)}\}^{\mathrm{c}} = \{DC(H)^{\mathrm{c}}\}^{0}$, i.e. there exists $\varepsilon > 0$ such that $A + B$ is not distributionally chaotic for any B such that $\|B\| < \varepsilon$. ∎

Next, we consider the interiors of the set of all Li–Yorke chaotic operators and the set of all distributionally chaotic operators. Before proving Theorem 2.16, it is convenient to cite in full length a result of Apostol and Morrel ([2] or [14]). Let $\Gamma = \partial\Omega$, where Ω is an analytic Cauchy domain, and let $L^2(\Gamma)$ be the Hilbert space of (equivalent classes of) complex functions on Γ which are square integrable with respect to $(1/2\pi)$-times the arc-length measure on Γ; $M(\Gamma)$ will stand for the operator defined as multiplication by λ on $L^2(\Gamma)$. The subspace $H^2(\Gamma)$ spanned by the rational functions with poles outside $\overline{\Omega}$ is invariant under $M(\Gamma)$. By $M_+(\Gamma)$ and $M_-(\Gamma)$ we shall denote the restriction of $M(\Gamma)$ to $H^2(\Gamma)$ and its compression to $L^2(\Gamma) \ominus H^2(\Gamma)$, respectively, i.e.

$$M(\Gamma) = \begin{array}{c} \begin{bmatrix} M_+(\Gamma) & Z \\ & M_-(\Gamma) \end{bmatrix} \end{array} \begin{array}{c} H^2(\Gamma) \\ H^2(\Gamma)^{\perp} \end{array}.$$

DEFINITION 2.12. $S \in B(H)$ is a *simple model*, if

$$S \simeq \begin{bmatrix} S_+ & * & * \\ & A & * \\ & & S_- \end{bmatrix},$$

where

(i) $\sigma(S_+)$, $\sigma(S_-)$, $\sigma(A)$ are pairwise disjoint;

(ii) A is similar to a normal operator with finite spectrum;

(iii) S_+ is (either absent or) unitarity equivalent to $\bigoplus\limits_{i=1}^{m} M_+(\partial\Omega_i)^{(k_i)}$, $1 \leqslant k_i \leqslant \infty$, where $\{\Omega_i\}_{i=1}^{m}$ is a finite family of analytic Cauchy domains with pairwise diajoint closures;

(iv) S_- is (either absent or) unitarity equivalent to $\bigoplus\limits_{j=1}^{n} M_-(\partial\Phi_j)^{(h_j)}$, $1 \leqslant h_j \leqslant \infty$, where $\{\Phi_j\}_{j=1}^{n}$ is a finite family of analytic Cauchy domains with pairwise diajoint closures.

THEOREM 2.13. *The simple models are dense in $B(H)$. More precisely: Given $T \in B(H)$ and $\varepsilon > 0$ there exists a simple model S such that:*

(i) $\sigma(S_+) \subseteq \rho_{s-F}^{(-)}(T) \subseteq \sigma(S_+)_\varepsilon$, $\sigma(S_-) \subseteq \rho_{s-F}^{(+)}(T) \subseteq \sigma(S_-)_\varepsilon$, *and* $\sigma(A) \subseteq \sigma(T)_\varepsilon$, *where* $(\cdot)_\varepsilon = \{z \in \mathbb{C}; \mathrm{dist}[z, \cdot] \leqslant \varepsilon\}$.

(ii) $\mathrm{ind}(\lambda - S) = \mathrm{ind}(\lambda - T)$, *for each* $\lambda \in \rho_{s-F}^{(-)}(S_+) \cup \rho_{s-F}^{(+)}(S_-)$.

(iii) $\|T - S\| < \varepsilon$.

Additionally, we need a lemma which appeared in another article [19]. But for convenience to read this article, we also give the details of the proof.

LEMMA 2.14. *Let $N \in B(H)$ be a normal operator. Then* $\liminf_{n\to\infty} \|N^n(x)\| = 0$ *implies* $\lim_{n\to\infty} \|N^n(x)\| = 0$. *Moreover, N is neither Li–Yorke chaotic nor distributionally chaotic.*

Proof. Since N is a normal operator, then there exist a locally compact space X, a finite positive regular Borel measure μ and a Borel function $\eta \in L^\infty(X,\mu)$ such that N and M_η are unitarily equivalent. M_η is the operator defined as multiplication by η on $L^2(X,\mu)$. Let $\liminf_{n\to\infty} \|M_\eta^n(f)\| = 0$ and

$$\Delta_1 = \{z \in X; |\eta(z)| \geqslant 1 \text{ a.e. } [\mu]\}, \qquad \Delta_2 = \{z \in X; |\eta(z)| < 1 \text{ a.e. } [\mu]\},$$
$$\Delta_3 = \{z \in X; f(z) = 0 \text{ a.e. } [\mu]\}, \qquad \Delta_4 = \{z \in X; f(z) \neq 0 \text{ a.e. } [\mu]\}.$$

Obviously, Δ_i $(i = 1,2,3,4)$ are measurable subsets. Then there exists a sequence of positive integers $\{n_k\}_{k=1}^\infty$ such that $\lim_{n_k\to\infty} \|M_\eta^{n_k}(f)\| = 0$ and

$$\|M_\eta^{n_k}(f)\|^2 = \int_X |\eta^{n_k} f|^2 \mathrm{d}\mu = \int_{\Delta_1\cap\Delta_4} |\eta^{n_k} f|^2 \mathrm{d}\mu + \int_{\Delta_2\cap\Delta_4} |\eta^{n_k} f|^2 \mathrm{d}\mu \geqslant \int_{\Delta_1\cap\Delta_4} |f|^2 \mathrm{d}\mu + \int_{\Delta_2\cap\Delta_4} |\eta^{n_k} f|^2 \mathrm{d}\mu.$$

Consequently $\mu(\Delta_1 \cap \Delta_4) = 0$. For any $n \in \mathbb{N}$, there exists a positive integer k such that $n_k \leqslant n < n_{k+1}$. Therefore,

$$\|M_\eta^n(f)\|^2 = \int_{\Delta_2\cap\Delta_4} |\eta^n f|^2 \mathrm{d}\mu = \int_{\Delta_2\cap\Delta_4} |\eta^{n_k} f|^2 |\eta^{n-n_k}|^2 \mathrm{d}\mu \leqslant \int_{\Delta_2\cap\Delta_4} |\eta^{n_k} f|^2 \mathrm{d}\mu = \|M_\eta^{n_k}(f)\|^2,$$

and hence $\lim_{n\to\infty} \|M_\eta^n(f)\| = 0$. We are done. ∎

COROLLARY 2.15. *Let $T \in B(H)$ be a subnormal operator. Then* $\liminf_{n\to\infty} \|T^n(x)\| = 0$ *implies* $\lim_{n\to\infty} \|T^n(x)\| = 0$. *Moreover, T is neither Li–Yorke chaotic nor distributionally chaotic.*

THEOREM 2.16. *Let $F = \{T \in B(H),\ \exists\, \lambda \in \partial\mathbb{D}$ such that $\operatorname{ind}(\lambda - T) > 0\}$. Then $DC(H)^0 = LY(H)^0 = F$.*

Proof. Obviously, $DC(H)^0 \subseteq LY(H)^0$. So it suffices to show that $F \subseteq DC(H)^0$ and $LY(H)^0 \subseteq F$.

First step, $F \subseteq DC(H)^0$. It readily follows from the stability properties of semi-Frodholm operators that F is open. So it suffices to prove that $F \subseteq DC(H)$.

Let $T \in F$, define

$$H_{\mathrm{r}} = \bigvee_{\mu\in\rho_{s-F}^{\mathrm{r}}(T)\cap\Delta} \operatorname{Ker}(\mu - T),$$

where Δ is the component of $\rho_{s-F}^{(+)}(T)$ which contains a point in $\partial\mathbb{D}$. Then $\dim H_{\mathrm{r}} = \infty$ and

$$T = \begin{bmatrix} T_{\mathrm{r}} & * \\ & * \end{bmatrix} \begin{array}{c} H_{\mathrm{r}} \\ H_{\mathrm{r}}{}^{\perp} \end{array}.$$

Applying the argument of the proof of Theorem 2.10, we have $T_{\mathrm{r}} \in B_n(\rho_{s-F}^{\mathrm{r}}(T) \cap \Delta)$ or $T_{\mathrm{r}} \in B_{+\infty}(\rho_{s-F}^{\mathrm{r}}(T) \cap \Delta)$, and hence T is norm-unimodal and distributionally chaotic.

The first step is complete.

Second step, $LY(H)^0 \subseteq F$. Since $\{LY(H)^0\}^{\mathrm{c}} = \overline{LY(H)^{\mathrm{c}}}$, it is enough to show that for any $T \in F^{\mathrm{c}}$ and $\varepsilon > 0$, there exists $C \in B(H)$ such that $\|C\| < \varepsilon$ and $T + C$ is not Li–Yorke chaotic.

Let $T \in F^{\mathrm{c}}$ and $\varepsilon > 0$. It readily follows from Theorem 2.13 that there exists a simple model

$$S \simeq \begin{bmatrix} S_+ & * & * \\ & A & * \\ & & S_- \end{bmatrix},$$

such that $\sigma(S_-) \subseteq \rho_{s-F}^{(+)}(T) \subseteq \sigma(S_-)_\varepsilon$ (it together with $\rho_{s-F}^{(+)}(T) \cap \partial\mathbb{D} = \emptyset$ indicates $\sigma(S_-) \cap \partial\mathbb{D} = \emptyset$) and $\|T - S\| < \varepsilon$. So it suffices to prove S cannot be Li–Yorke chaotic.

Observe that $\sigma(S_+)$, $\sigma(S_-)$, $\sigma(A)$ are pairwise disjoint and A is similar to a normal operator N with finite spectrum, it readily follows that $S \sim S_+ \oplus N \oplus S_-$. So we directly let $S = S_+ \oplus N \oplus S_-$.

Let $\liminf\limits_{n\to\infty} \|S^n(x)\| = 0$. Since $x = x_+ \oplus x_0 \oplus x_-$ corresponding to the space decomposition, it readily follows from Lemma 2.7 and Corollary 2.15 that

$$\lim_{n\to\infty} \|S_-{}^n(x_-)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \left\| \begin{bmatrix} S_+ & \\ & N \end{bmatrix}^n \begin{bmatrix} x_+ \\ x_0 \end{bmatrix} \right\| = 0,$$

whence we conclude that $\lim\limits_{n\to\infty} \|S^n(x)\| = 0$.

The second step is complete. ∎

The following example shows that $DC(H)^0 \subsetneqq DC(H)$ and hence $LY(H)^0 \subsetneqq LY(H)$.

EXAMPLE 2.17. Let $A \in B(H)$ satisfy

$$\begin{cases} Ae_i = 2e_{i-1} & i \neq 0, \\ Ae_0 = 0, \end{cases}$$

where $\{e_i\}_{i=-\infty}^{\infty}$ is an orthonormal basis of H. Then A is distributionally chaotic but not in $DC(H)^0$.

Proof. Since $H_0 = \bigvee\limits_{i=0}^{\infty} \{e_i\}$ is an invariant space of A and $A|_{H_0} = 2B$ is distributionally chaotic, where B is backward unilateral shift, it readily follows that A

is distributionally chaotic. One can easily obtain $\mathrm{ind}(\lambda - A) = 0$ for $|\lambda| < 2$, so A is not in $DC(H)^0$.

Let us consider it directly. For any $\varepsilon > 0$, let $K \in B(H)$ satisfying $Ke_0 = \varepsilon e_{-1}$, $Ke_i = 0$, $i \neq 0$, then K is compact and $\|K\| = \varepsilon$. Since $\sigma(A+K) = \{z \in \mathbb{C};\ |z| = 2\}$, then $A+K$ is not distributionally chaotic. Hence A is not in $DC(H)^0$. ∎

Though $DC(H)^0 \subsetneqq DC(H)$ ($LY(H)^0 \subsetneqq LY(H)$), one may ask $\overline{DC(H)^0} = \overline{DC(H)}$? ($\overline{LY(H)^0} = \overline{LY(H)}$?). Unfortunately, it is not true. It means that there exists a class of distributionally chaotic operators (Li–Yorke chaotic operators) which are more complicated. We give the descriptions.

THEOREM 2.18. *Let* $G_0 = \{T \in B(H);\ \partial\mathbb{D} \not\subseteq \rho^{(0)}_{s-F}(T) \cup \rho^{(-)}_{s-F}(T)\}$, $G_1 = \{T \in B(H); \partial\mathbb{D} \subseteq \rho^{(0)}_{s-F}(T) \cup \rho^{(-)}_{s-F}(T)$ *and* $\mathrm{dimKer}(\lambda - T) > 0,\ \forall \lambda \in \partial\mathbb{D}\}$ *and* $G_2 = \{T \in B(H); \partial\mathbb{D} \cap \sigma_{\mathrm{lre}}(T) \neq \emptyset$ *and* $\rho^{(+)}_{s-F}(T) \cap \partial\mathbb{D} = \emptyset\}$. *Then* $\overline{DC(H)^0} = \overline{LY(H)^0} = G_0$ *and* $\overline{DC(H)\backslash DC(H)^0} = \overline{LY(H)\backslash LY(H)^0} = G_1 \cup G_2$.

Proof. First, we will show that $\overline{DC(H)^0} = \overline{LY(H)^0} = G_0$. Clearly, $DC(H)^0 = LY(H)^0 = F \subseteq G_0$, where F is denoted in Theorem 2.16. Since it follows from the stability properties of semi-Frodholm operators that G_0 is closed, we have $\overline{DC(H)^0} = \overline{LY(H)^0} \subseteq G_0$. So it suffices to show that for any $T \in G_0$ and $\varepsilon > 0$, there exists $C \in B(H)$ such that $\|C\| < \varepsilon$ and $T + C \in DC(H)^0$ (i.e. $LY(H)^0$).

Let $T \in G_0$, then

(i) there exists $\lambda \in \partial\mathbb{D}$ such that $\mathrm{ind}(\lambda - T) > 0$; or

(ii) $\rho^{(+)}_{s-F}(T) \cap \partial\mathbb{D} = \emptyset$, but there exists $\lambda \in \partial\mathbb{D}$ such that $\mathrm{ind}(\lambda - T) = 0$; or

(iii) $[\rho^{(+)}_{s-F}(T) \cup \rho^{(0)}_{s-F}(T)] \cap \partial\mathbb{D} = \emptyset$, but $\sigma_{\mathrm{lre}}(T) \cap \partial\mathbb{D} \neq \emptyset$.

Case (i) is obvious.

Case (ii). It readily follows from the hypothesis that $\sigma_{\mathrm{lre}}(T) \cap \partial\mathbb{D} \neq \emptyset$. Choose a $\lambda_0 \in \sigma_{\mathrm{lre}}(T) \cap \partial[\rho^{(0)}_{s-F}(T) \cap \partial\mathbb{D}]$, then by Theorem 2.1 there exists a compact operator K_1 such that $\|K_1\| < \varepsilon/2$ and

$$T + K_1 = \begin{array}{c} \begin{bmatrix} \lambda_0 I & * \\ & A \end{bmatrix} \end{array} \begin{array}{l} H_0 \\ H_0{}^{\perp}, \end{array}$$

where $\mathrm{dim}H_0 = \infty$, $\sigma_{\mathrm{lre}}(A) = \sigma_{\mathrm{lre}}(T)$ and $\mathrm{ind}(\lambda - A) = \mathrm{ind}(\lambda - T)$ for $\lambda \in \rho_{s-F}(T)$. Let

$$B_\varepsilon = \begin{bmatrix} 0 & \varepsilon/2 & & \\ & 0 & \varepsilon/2 & \\ & & \ddots & \ddots \end{bmatrix} \begin{array}{l} e_1 \\ e_2, \\ \vdots \end{array}$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of H_0, then B_ε is a Cowen–Douglas operator with $\|B_\varepsilon\| = \varepsilon/2$. Let

$$B_2 = \begin{bmatrix} B_\varepsilon & \\ & 0 \end{bmatrix} \begin{array}{l} H_0 \\ H_0{}^{\perp}, \end{array}$$

then it follows from easy computations that there exists $\lambda \in \partial\mathbb{D}$ such that

$$\operatorname{ind}(T + K_1 + B_2 - \lambda) = \operatorname{ind}\left(\begin{bmatrix}\lambda_0 I + B_\varepsilon & * \\ & A\end{bmatrix} - \lambda\right) = 1,$$

and hence $T + K_1 + B_2 \in DC(H)^0$, where $\|K_1 + B_2\| < \varepsilon$.

Case (iii). It readily follows from Theorem 2.13 that there exists C_1 such that $\|C_1\| < \varepsilon/2$ and

$$T + C_1 \simeq \begin{bmatrix} S_+ & * & * \\ & A & * \\ & & S_- \end{bmatrix},$$

where S_+ is either absent or unitarity equivalent to a subnormal operator and $\partial\mathbb{D}\backslash\sigma(S_+)$ contains a small arc in $\partial\mathbb{D}$, A is similar to a normal operator with finite spectrum and $\sigma_{\mathrm{lre}}(A) \cap \partial\mathbb{D} \neq \emptyset$, S_- is either absent or unitarity equivalent to the adjoint of a subnormal operator and $\sigma(S_-) \cap \partial\mathbb{D} = \emptyset$; $\sigma(S_+)$, $\sigma(A)$, $\sigma(S_-)$ are pairwise disjoint. Observe that

$$\rho(T{+}C_1) = \rho(S_+) \cap \rho(A) \cap \rho(S_-) \quad \text{and} \quad \sigma_{\mathrm{lre}}(T{+}C_1) = \sigma_{\mathrm{lre}}(S_+) \cup \sigma_{\mathrm{lre}}(A) \cup \sigma_{\mathrm{lre}}(S_-),$$

so that $\sigma_{\mathrm{lre}}(T + C_1) \cap \partial\mathbb{D} \neq \emptyset$ and $\rho(T + C_1) \cap \partial\mathbb{D} \neq \emptyset$. Then we can obtain C_2 through the technique of Case (ii) such that $\|C_2\| < \varepsilon/2$ and $\operatorname{ind}(T + C_1 + C_2 - \lambda_1) > 0$ for some $\lambda_1 \in \partial\mathbb{D}$. Hence $T + C_1 + C_2 \in DC(H)^0$, where $\|C_1 + C_2\| < \varepsilon$.

The proof of $\overline{DC(H)^0} = \overline{LY(H)^0} = G_0$ is complete.

Second, we will show that $\overline{DC(H)\backslash DC(H)^0} = \overline{LY(H)\backslash LY(H)^0} = G_1 \cup G_2$. Clearly, it follows from Theorem 2.10 and Theorem 2.16 that

$$\begin{aligned} G_1 &= \overline{DC(H)}\backslash\overline{DC(H)^0} \subseteq \overline{DC(H)\backslash DC(H)^0} \subseteq \overline{LY(H)\backslash LY(H)^0} \\ &\subseteq \overline{LY(H)}\backslash LY(H)^0 = G_1 \cup G_2. \end{aligned}$$

In order to obtain the result, it suffices to show that $G_2 \subseteq \overline{DC(H)\backslash DC(H)^0}$, i.e. for any $T \in G_2$ and $\varepsilon > 0$, there exists C such that $\|C\| < \varepsilon$ and $T + C \in DC(H)\backslash DC(H)^0$. In fact, we can obtain a compact operator K such that $\|K\| < \varepsilon$ and $T + K \in DC(H)\backslash DC(H)^0$.

Let $T \in G_2$ and $\varepsilon > 0$, it readily follows from Theorem 2.1 that there exists a compact operator K_1 such that $\|K_1\| < \varepsilon/2$ and

$$T + K_1 = \begin{bmatrix} \lambda_0 I & * \\ & * \end{bmatrix} \begin{matrix} H_0 \\ H_0{}^\perp \end{matrix},$$

where $\lambda_0 \in \partial\mathbb{D} \cap \sigma_{\mathrm{lre}}(T)$ and $\dim H_0 = \infty$. By Theorem 2.9, there exists a compact operator $K_\varepsilon \in B(H_0)$ such that $\|K_\varepsilon\| < \varepsilon/2$ and $\lambda_0 I + K_\varepsilon$ is distributionally chaotic. Let

$$K_2 = \begin{bmatrix} K_\varepsilon & \\ & 0 \end{bmatrix} \begin{matrix} H_0 \\ H_0{}^\perp \end{matrix},$$

then

$$T + K_1 + K_2 = \begin{bmatrix} \lambda_0 I + K_\varepsilon & * \\ & * \end{bmatrix} \in DC(H),$$

where $K_1 + K_2$ is a compact operator and $\|K_1 + K_2\| < \varepsilon$. Observe that $\rho_{s-F}^{(+)}(T + K_1 + K_2) \cap \partial\mathbb{D} = \rho_{s-F}^{(+)}(T) \cap \partial\mathbb{D} = \emptyset$, it readily follows that $T + K_1 + K_2 \in DC(H)\backslash DC(H)^0$.

The proof of $\overline{DC(H)\backslash DC(H)^0} = \overline{LY(H)\backslash LY(H)^0} = G_1 \cup G_2$ is complete. ∎

3. SOME OTHER RESULTS

In this section, we consider the relation between hypercyclic operators and distributionally chaotic operators, and the closure of the set of all norm-unimodal operators.

Recall the definition of chaos given by Devaney [9] as follows.

DEFINITION 3.1. Suppose that $f : X \to X$ is a continuous function on a complete separable metric space X, then f is *Devaney chaotic* if:

(i) the periodic points for f are dense in X,

(ii) f is transitive,

(iii) f has sensitive dependence on initial conditions.

It was shown by Banks et. al. [4] that if f satisfies (i) and (ii), then f must have sensitive dependence on initial conditions. Hence only the first two conditions of the definition need to be verified.

Denote by $HC(H)$ and $DE(H)$ the set of all hypercyclic operators and the set of all Devaney chaotic operators on H respectively. Obviously, $DE(H) \subseteq HC(H)$.

PROPOSITION 3.2. $\overline{DE(H)} = \overline{HC(H)} \subseteq \overline{DC(H)^0} = \overline{LY(H)^0}$.

Proof. It follows from Proposition 4 of [16] and [15] that $\overline{DE(H)} = \overline{HC(H)} = \{T \in B(H);\ \sigma_{\mathrm{w}}(T) \cup \partial\mathbb{D}$ is connected, $\sigma_0(T) = \emptyset$ and $\mathrm{ind}(\lambda - T) \geqslant 0,\ \forall \lambda \in \rho_{s-F}(T)\}$. Observe Theorem 2.18, we obtain the result. ∎

REMARK 3.3. $DE(H)$ and $HC(H)$ are dense in $B(H)$ in the strong operator topology [5], and hence Proposition 3.2 tells us that $DC(H)^0$ and $LY(H)^0$ are both dense in $B(H)$ in the strong operator topology. One can observe [5] for more information. In addition, Prajitura [26] showed that the closure of $HC(H)$ contains all the weakly hypercyclic operators, which are the bounded linear operators having a point x in H such that $\{x, Tx, T^2x, \dots\}$ is dense in the weak topology of H. He gave a complete spectral characterization of the closure of the set of weakly hypercyclic operators, and also showed that the set has an empty interior. One can observe [26] for more information.

Next, we can see the set of all norm-unimodal operators is large in the set of all distributionally chaotic operators.

THEOREM 3.4. *$\overline{UN(H)} = \overline{DC(H)} = \overline{LY(H)}$, $DC(H)^0 = LY(H)^0 \subseteq UN(H)$ and $\overline{UN(H)\backslash DC(H)^0} = \overline{DC(H)\backslash DC(H)^0} = \overline{LY(H)\backslash LY(H)^0}$.*

Proof. First it will be shown that $\overline{UN(H)} = \overline{DC(H)} = \overline{LY(H)}$. Observe that $\overline{UN(H)} \subseteq \overline{DC(H)} = \overline{LY(H)}$ (where E_1, E_2 are denoted in Theorem 2.10), it is enough to show that for any $T \in E_1 \cup E_2$ and $\varepsilon > 0$, there exists C such that $\|C\| < \varepsilon$ and $T + C \in UN(H)$. But different to the first step of Theorem 2.10, there does not generally exist a compact operator satisfying the property.

If $T \in E_1$, then choose any $\lambda_0 \in \partial\mathbb{D} \cap \sigma_{\rm lre}(T)$. By Theorem 2.1, there exists a compact operator K_1 such that $\|K_1\| < \varepsilon/2$ and

$$T + K_1 = \begin{array}{cc} \begin{bmatrix} \lambda_0 I & * \\ & * \end{bmatrix} & \begin{array}{c} H_0 \\ H_0{}^{\perp}, \end{array} \end{array}$$

where $\dim H_0 = \infty$. Let

$$C_\varepsilon = \begin{array}{cc} \begin{bmatrix} 0 & \varepsilon/2 & & \\ & 0 & \varepsilon/2 & \\ & & \ddots & \ddots \end{bmatrix} & \begin{array}{c} e_0 \\ e_1, \\ \vdots \end{array} \end{array}$$

where $\{e_i\}_{i=0}^{\infty}$ is an orthonormal basis of H_0, then C_ε is a Cowen–Douglas operator with $\|C_\varepsilon\| = \varepsilon/2$. Let

$$C_2 = \begin{array}{cc} \begin{bmatrix} C_\varepsilon & \\ & 0 \end{bmatrix} & \begin{array}{c} H_0 \\ H_0{}^{\perp}, \end{array} \end{array}$$

then it follows from Theorem 2.4 that

$$T + K_1 + C_2 = \begin{bmatrix} \lambda_0 I + C_\varepsilon & * \\ & * \end{bmatrix} \in UN(H),$$

where $\|K_1 + C_2\| < \varepsilon$.

Observe that $K_1 + C_2$ is not compact. In fact, the operator T in E_1 satisfying $\sigma_{\rm lre}(T) \subseteq \mathbb{D}^-$ can not be perturbed into $UN(H)$ by a compact operator (it is easy and left to the reader).

If $T \in E_2$, by proceeding as the first step of the proof of Theorem 2.10, we know T is norm-unimodal.

The proof of $\overline{UN(H)} = \overline{DC(H)} = \overline{LY(H)}$ is complete.

It is immediately obtained from the first step of the proof of Theorem 2.16 that $DC(H)^0 = LY(H)^0 \subseteq UN(H)$.

Next we show that $\overline{UN(H)\backslash DC(H)^0} = \overline{DC(H)\backslash DC(H)^0} = \overline{LY(H)\backslash LY(H)^0}$.

Clearly, $\overline{UN(H)\backslash DC(H)^0} \subseteq \overline{DC(H)\backslash DC(H)^0} = \overline{LY(H)\backslash LY(H)^0}$. So it suffices to show that $G_1 \cup G_2 \subseteq \overline{UN(H)\backslash DC(H)^0}$, i.e. for any $T \in G_1 \cup G_2$ and $\varepsilon > 0$, there exists C such that $\|C\| < \varepsilon$ and $T + C \in UN(H)\backslash DC(H)^0$, where G_1, G_2 are denoted in Theorem 2.18. But different to the second step of Theorem 2.18, there does not generally exist a compact operator satisfying the property.

If $T \in G_1$, then it follows from the first step of the proof of Theorem 2.10 that $T \in UN(H)\backslash DC(H)^0$.

If $T \in G_2$ and $\varepsilon > 0$, by Theorem 2.1, there exists a compact operator K_1 such that $\|K_1\| < \varepsilon/3$ and

$$T + K_1 = \begin{bmatrix} \lambda_0 I & * & * \\ & A & * \\ & & \lambda_0 I \end{bmatrix} \begin{array}{l} H_0 \\ H_1, \\ H_2 \end{array}$$

where $\dim H_0 = \dim H_2 = \infty$, $\lambda_0 \in \partial\mathbb{D} \cap \sigma_{\mathrm{lre}}(T)$ and $\rho_{s-F}^{(+)}(A) \cap \partial\mathbb{D} = \rho_{s-F}^{(+)}(T) \cap \partial\mathbb{D} = \emptyset$. Let $N \in B(H_2)$ be an uniform infinite multiplicity normal operator such that $\sigma(N) = \{z \in \mathbb{C};\ |z| \leqslant \varepsilon/3\}$ and let

$$B_\varepsilon = \begin{bmatrix} 0 & \varepsilon/3 & & \\ & 0 & \varepsilon/3 & \\ & & \ddots & \ddots \end{bmatrix} \begin{array}{l} e_0 \\ e_1, \\ \vdots \end{array}$$

where $\{e_i\}_{i=0}^{\infty}$ is an orthonormal basis of H_0, then $\|N\| = \varepsilon/3$, $\|B_\varepsilon\| = \varepsilon/3$, $B_\varepsilon \in B_1(\mathbb{D}_{\varepsilon/3})$ and $\sigma(B_\varepsilon) = \mathbb{D}_{\varepsilon/3}{}^{-}$, where $\mathbb{D}_{\varepsilon/3} = \{z \in \mathbb{C};\ |z| < \varepsilon/3\}$ and $\mathbb{D}_{\varepsilon/3}{}^{-}$ is the closure of $\mathbb{D}_{\varepsilon/3}$.

Moreover, let

$$\widetilde{B_\varepsilon} = \begin{bmatrix} B_\varepsilon & & \\ & 0 & \\ & & 0 \end{bmatrix} \begin{array}{l} H_0 \\ H_1 \\ H_2 \end{array} \quad \text{and} \quad \widetilde{N} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & N \end{bmatrix} \begin{array}{l} H_0 \\ H_1, \\ H_2 \end{array}$$

then it follows from Theorem 2.4 that

$$T + K_1 + \widetilde{B_\varepsilon} + \widetilde{N} = \begin{bmatrix} \lambda_0 I + B_\varepsilon & * & * \\ & A & * \\ & & \lambda_0 I + N \end{bmatrix} \in UN(H),$$

where $\|K_1 + \widetilde{B_\varepsilon} + \widetilde{N}\| < \varepsilon$.

Observe that

$$\begin{aligned} \rho_{s-F}^{(+)}(T + K_1 + \widetilde{B_\varepsilon} + \widetilde{N}) \cap \partial\mathbb{D} &= \rho_{s-F}^{(+)}\left(\begin{bmatrix} \lambda_0 I + B_\varepsilon & * & * \\ & A & * \\ & & \lambda_0 I + N \end{bmatrix}\right) \cap \partial\mathbb{D} \\ &= [\rho_{s-F}^{(+)}(A) \backslash \{z \in \mathbb{C};\ |z - \lambda_0| \leqslant \varepsilon/3\}] \cap \partial\mathbb{D} = \emptyset, \end{aligned}$$

so that $T + K_1 + \widetilde{B_\varepsilon} + \widetilde{N} \in UN(H) \backslash DC(H)^0$.

Notice that $K_1 + \widetilde{B_\varepsilon} + \widetilde{N}$ is not compact. In fact, the operator T in G_2 satisfying $\sigma_{\mathrm{lre}}(T) \subseteq \mathbb{D}^-$ can not be perturbed into $UN(H) \backslash DC(H)^0$ by a compact operator.

The proof of $\overline{UN(H) \backslash DC(H)^0} = \overline{DC(H) \backslash DC(H)^0} = \overline{LY(H) \backslash LY(H)^0}$ is complete. ∎

EXAMPLE 3.5. Let $A \in B(H)$ satisfy

$$\begin{cases} Ae_i = 2e_{i-1} & i \geqslant 1, \\ Ae_0 = e_{-1}, & \\ Ae_i = \frac{|i|}{|i|+1} e_{i-1}, & i \leqslant -1. \end{cases}$$

where $\{e_i\}_{i=-\infty}^{\infty}$ is an orthonormal basis of H. Then A is norm-unimodal, but not in $DC(H)^0$.

Proof. For any $m \in \mathbb{N}$, $\|A^i(e_m)\| \geqslant 2^i \|e_m\|$, $1 \leqslant i \leqslant m$ and $\lim\limits_{n\to\infty} \|A^n(e_m)\| = 0$. So A is norm-unimodal. Since $\sigma(A) = \{z \in \mathbb{C};\ 1 \leqslant |z| \leqslant 2\}$ implies $\partial\mathbb{D} \subseteq \sigma_{\text{lre}}(A)$, then A is not in $DC(H)^0$. ∎

4. CONNECTEDNESS

The main purpose in this section is to discuss the connectedness for the sets considered in Section 2.

THEOREM 4.1. *$DC(H)^0$, $\overline{DC(H)^0}$, $\overline{DC(H)}$ and $\overline{DC(H)\backslash DC(H)^0}$ (i.e. $LY(H)^0$, $\overline{LY(H)^0}$, $\overline{LY(H)}$ and $\overline{LY(H)\backslash LY(H)^0}$) are all arcwise connected.*

Proof. It will be only shown that $DC(H)^0$ is arcwise connected, since others follow by the same arguments. This will be done in two steps.

First step, any $T \in DC(H)^0$ can be connected to $\widetilde{T} \in DC(H)^0$, where $\sigma_{\text{lre}}(\widetilde{T}) \cap \partial\mathbb{D} \neq \emptyset$.

If $\sigma_{\text{lre}}(T) \cap \partial\mathbb{D} \neq \emptyset$, then the result is obvious.

If $\sigma_{\text{lre}}(T) \cap \partial\mathbb{D} = \emptyset$, then $\partial\mathbb{D} \subseteq \rho_{s-F}^{(+)}(T)$. Choose $\lambda_0 \in \sigma_{\text{lre}}(T)$, by Theorem 2.1 there exists a compact operator K such that

$$T + K = \begin{array}{c} \begin{bmatrix} \lambda_0 I & * & * \\ & A & * \\ & & \lambda_0 I \end{bmatrix} \end{array} \begin{array}{l} H_0 \\ H_1, \\ H_2 \end{array}$$

where $\dim H_0 = \dim H_2 = \infty$, $\sigma_{\text{lre}}(T) = \sigma_{\text{lre}}(A)$ and $\text{ind}(\lambda - T) = \text{ind}(\lambda - A)$ for all $\lambda \in \rho_{s-F}(A)$. Choose $\mu_0 \in \partial\mathbb{D}$, let

$$\delta(t) = \begin{bmatrix} \alpha(t) I & * & * \\ & A & * \\ & & \alpha(t) I \end{bmatrix}, \quad 1 < t \leqslant 2,$$

where $\alpha(t) = (t-1)(\mu_0 - \lambda_0) + \lambda_0, 1 < t \leqslant 2$ and

$$\beta(t) = \begin{cases} T + tK & 0 \leqslant t \leqslant 1, \\ \delta(t) & 1 < t \leqslant 2. \end{cases}$$

Obviously, $\beta(0) = T$,

$$\beta(2) = \widetilde{T} := \begin{bmatrix} \mu_0 I & * & * \\ & A & * \\ & & \mu_0 I \end{bmatrix},$$

$\sigma_{\mathrm{lre}}(\widetilde{T}) \cap \partial\mathbb{D} = \{\mu_0\} \neq \emptyset$ and $\beta(t)$ is continuous on [0,2].

Since $\rho_{s-F}^{(+)}(\delta(t)) \cap \partial\mathbb{D} = [\rho_{s-F}^{(+)}(A) \backslash \{\alpha(t)\}] \cap \partial\mathbb{D} = [\rho_{s-F}^{(+)}(T) \backslash \{\alpha(t)\}] \cap \partial\mathbb{D} \neq \emptyset$ for $1 < t \leqslant 2$ and $\rho_{s-F}^{(+)}(T + tK) = \rho_{s-F}^{(+)}(T) \supseteq \partial\mathbb{D}$ for $0 \leqslant t \leqslant 1$, it readily follows that $\{\beta(t);\ 0 \leqslant t \leqslant 2\} \subseteq DC(H)^0$.

The first step is complete.

Second step, any two operators T and S in $DC(H)^0$ which satisfy $\sigma_{\mathrm{lre}}(T) \cap \partial\mathbb{D} \neq \emptyset$ and $\sigma_{\mathrm{lre}}(S) \cap \partial\mathbb{D} \neq \emptyset$ respectively can be connected.

Let $\lambda_0 \in \partial\rho_{s-F}^{(+)}(T) \cap \partial\mathbb{D}$ be the point such that there exists $\theta_0 > 0$ such that $\{\lambda_0 \mathrm{e}^{\mathrm{i}\theta};\ 0 < \theta < \theta_0\} \subseteq \rho_{s-F}^{(+)}(T)$. Similarly, let $\lambda_1 \in \partial\rho_{s-F}^{(+)}(S) \cap \partial\mathbb{D}$ and $\theta_1 > 0$ such that $\{\lambda_1 \mathrm{e}^{\mathrm{i}\theta};\ 0 < \theta < \theta_1\} \subseteq \rho_{s-F}^{(+)}(S)$. Observe that there exists θ' such that $\lambda_0 = \mathrm{e}^{\mathrm{i}\theta'}\lambda_1$, define $\widetilde{S} = \mathrm{e}^{\mathrm{i}\theta'} S$. Then $\lambda_0 \in \partial\rho_{s-F}^{(+)}(\widetilde{S}) \cap \partial\mathbb{D}$ and $\{\lambda_0 \mathrm{e}^{\mathrm{i}\theta};\ 0 < \theta < \theta_1\} \subseteq \rho_{s-F}^{(+)}(\widetilde{S})$.

By Theorem 2.1, there exist compact operators $K_1,\ K_{21}$ such that

$$T + K_1 = \begin{bmatrix} \lambda_0 I & C_1 \\ & A_1 \end{bmatrix} \begin{matrix} H_1 \\ H_1{}^{\perp} \end{matrix}, \quad \widetilde{S} + K_{21} = \begin{bmatrix} \widetilde{A}_2 & C_2 \\ & \lambda_0 I \end{bmatrix} \begin{matrix} H_2{}^{\perp} \\ H_2 \end{matrix},$$

where $\dim H_1 = \infty$, $\sigma_{\mathrm{lre}}(A_1) = \sigma_{\mathrm{lre}}(T)$ and $\mathrm{ind}(\lambda - A_1) = \mathrm{ind}(\lambda - T)$ for all $\lambda \in \rho_{s-F}(T)$; $\dim H_2 = \infty$, $\sigma_{\mathrm{lre}}(\widetilde{A}_2) = \sigma_{\mathrm{lre}}(\widetilde{S})$, $\mathrm{ind}(\lambda - \widetilde{A}_2) = \mathrm{ind}(\lambda - \widetilde{S})$ for all $\lambda \in \rho_{s-F}(\widetilde{S})$. Moreover, one can obtain from Theorem 3.48 in [14] that there exists a compact operator $\widetilde{K}_{22} \in B(H_2{}^{\perp})$ such that $\min \mathrm{ind}(\widetilde{A}_2 + \widetilde{K}_{22} - \lambda) = 0$ for all $\lambda \in \rho_{s-F}(\widetilde{A}_2)$. Let $K_2 = K_{21} + K_{22},\ A_2 = \widetilde{A}_2 + \widetilde{K}_{22}$, where

$$K_{22} = \begin{bmatrix} \widetilde{K}_{22} & \\ & 0 \end{bmatrix} \begin{matrix} H_2{}^{\perp} \\ H_2 \end{matrix},$$

then

$$\widetilde{S} + K_2 = \begin{bmatrix} A_2 & C_2 \\ & \lambda_0 I \end{bmatrix} \begin{matrix} H_2{}^{\perp} \\ H_2 \end{matrix}.$$

Without loss of generality, we can assume that $H_1 = H_2{}^{\perp}$. Define

$$\gamma(t) = \begin{cases} \mathrm{e}^{\mathrm{i}t} S & 0 \leqslant t < \theta', \\ \widetilde{S} + (t - \theta')K_2 & \theta' \leqslant t \leqslant \theta' + 1, \\ (t - (\theta' + 1))(T + K_1) + (\theta' + 2 - t)(\widetilde{S} + K_2) & \theta' + 1 < t < \theta' + 2, \\ T + [\theta' + 3 - t]K_1 & \theta' + 2 \leqslant t \leqslant \theta' + 3. \end{cases}$$

Obviously, $\gamma(0) = S$, $\gamma(\theta' + 3) = T$ and $\gamma(t)$ is continuous on $[0,\ \theta' + 3]$. It suffices to show that $\{\gamma(t);\ 0 \leqslant t \leqslant \theta' + 3\} \subseteq DC(H)^0$.

(a) $\rho_{s-F}^{(+)}(\mathrm{e}^{\mathrm{i}t}S) = \mathrm{e}^{\mathrm{i}t}\rho_{s-F}^{(+)}(S)$, $0 \leqslant t < \theta'$ and $\rho_{s-F}^{(+)}(S) \cap \partial\mathbb{D} \neq \emptyset$ imply $\{\mathrm{e}^{\mathrm{i}t}S;\ 0 \leqslant t < \theta'\} \subseteq DC(H)^0$.

(b) $\rho_{s-F}^{(+)}(\widetilde{S} + (t-\theta')K_2) = \rho_{s-F}^{(+)}(\widetilde{S}) = \mathrm{e}^{\mathrm{i}\theta'}\rho_{s-F}^{(+)}(S)$, $\theta' \leqslant t \leqslant \theta'+1$ and $\rho_{s-F}^{(+)}(S) \cap \partial\mathbb{D} \neq \emptyset$ imply $\{\widetilde{S} + (t-\theta')K_2;\ \theta' \leqslant t \leqslant \theta'+1\} \subseteq DC(H)^0$.

(c) For any given $\theta'+1 < t < \theta'+2$, since

$$\begin{aligned}\rho_{s-F}^{(+)}([t-(\theta'+1)]\lambda_0 + [\theta'+2-t]A_2) &= [t-(\theta'+1)]\lambda_0 + [\theta'+2-t]\rho_{s-F}^{(+)}(A_2)\\ &= \lambda_0 + (\theta'+2-t)(\rho_{s-F}^{(+)}(A_2) - \lambda_0)\\ &= \lambda_0 + (\theta'+2-t)(\rho_{s-F}^{(+)}(\widetilde{S}) - \lambda_0),\end{aligned}$$

and

$$\begin{aligned}\rho_{s-F}^{(+)}([t-(\theta'+1)]A_1 + [\theta'+2-t]\lambda_0) &= [t-(\theta'+1)]\rho_{s-F}^{(+)}(A_1) + [\theta'+2-t]\lambda_0\\ &= \lambda_0 + [t-(\theta'+1)](\rho_{s-F}^{(+)}(A_1) - \lambda_0)\\ &= \lambda_0 + [t-(\theta'+1)](\rho_{s-F}^{(+)}(T) - \lambda_0),\end{aligned}$$

we know there exists $\theta_t > 0$ such that

$$\begin{aligned}&\{\lambda_0\mathrm{e}^{\mathrm{i}\theta};\ 0<\theta<\theta_t\}\\ &\quad\subseteq \rho_{s-F}^{(+)}([t-(\theta'+1)]\lambda_0 + [\theta'+2-t]A_2) \cap \rho_{s-F}^{(+)}([t-(\theta'+1)]A_1 + [\theta'+2-t]\lambda_0).\end{aligned}$$

For each $\lambda \in \{\lambda_0\mathrm{e}^{\mathrm{i}\theta};\ 0<\theta<\theta_t\}$, since $\min\operatorname{ind}(\mu - A_2) = 0$ for $\mu \in \rho_{s-F}(\widetilde{A}_2) = \rho_{s-F}(A_2)$ implies $\min\operatorname{ind}([t-(\theta'+1)]\lambda_0 + [\theta'+2-t]A_2 - \lambda) = 0$, and $\operatorname{ind}([t-(\theta'+1)]\lambda_0 + [\theta'+2-t]A_2 - \lambda) > 0$, it readily follows that $[t-(\theta'+1)]\lambda_0 + [\theta'+2-t]A_2 - \lambda$ is surjective.

Consequently

$$\begin{aligned}&\operatorname{ind}([t-(\theta'+1)](T+K_1) + [\theta'+2-t](\widetilde{S}+K_2) - \lambda)\\ &\qquad= \operatorname{ind}([t-(\theta'+1)]\lambda_0 + [\theta'+2-t]A_2 - \lambda)\\ &\qquad\qquad+ \operatorname{ind}([t-(\theta'+1)]A_1 + [\theta'+2-t]\lambda_0 - \lambda) > 0.\end{aligned}$$

Hence $\{(t-(\theta'+1))(T+K_1) + (\theta'+2-t)(\widetilde{S}+K_2);\ \theta'+1 < t < \theta'+2\} \subseteq DC(H)^0$.

(d) $\rho_{s-F}^{(+)}(T + [\theta'+3-t]K_1) = \rho_{s-F}^{(+)}(T)$, $\theta'+2 \leqslant t \leqslant \theta'+3$ and $\rho_{s-F}^{(+)}(T) \cap \partial\mathbb{D} \neq \emptyset$ imply $\{T + [\theta'+3-t]K_1;\ \theta'+2 \leqslant t \leqslant \theta'+3\} \subseteq DC(H)^0$.

Therefore, $\{\gamma(t);\ 0 \leqslant t \leqslant \theta'+3\} \subseteq DC(H)^0$.

The second step is complete.

Thus, $DC(H)^0$ is arcwise connected. ∎

REMARK 4.2. From Chan and Sanders's paper [6], $HC(H)$ is a connected subset of $B(H)$ in the strong operator topology. Though we do not known the arcwise connectedness of $DC(H)$ and $LY(H)$ in the norm topology, it follows from the strong operator topology is weaker than the norm topology and $\overline{HC(H)} \subseteq \overline{DC(H)} = \overline{LY(H)}$ that, $DC(H)$ and $LY(H)$ are both connected in the strong operator topology.

EXAMPLE 4.3. Let B be the backward unilateral shift. Then there exists an arc $\alpha(t)$ in $DC(H)^0$ which connects $5B$ and $5B^2$.

Proof. Clearly, $\sigma(5B) = \sigma(5B^2) = 5\mathbb{D}^-$ and $\operatorname{ind}(\lambda - 5B) = 1$, $\operatorname{ind}(\lambda - 5B^2) = 2$ for $|\lambda| < 5$.

First there exist compact operators $K_1,\ K_2$ such that

$$5B + K_1 = \begin{bmatrix} -5I & C_1 \\ & A_1 \end{bmatrix} \begin{matrix} H_1 \\ H_1{}^{\perp} \end{matrix} \quad \text{and} \quad 5B^2 + K_2 = \begin{bmatrix} A_2 & C_2 \\ & 5I \end{bmatrix} \begin{matrix} H_2{}^{\perp} \\ H_2 \end{matrix},$$

where $\dim H_1 = \infty$, $\sigma_{\mathrm{lre}}(A_1) = \sigma_{\mathrm{lre}}(5B)$ and $\operatorname{ind}(\lambda - A_1) = \operatorname{ind}(\lambda - 5B)$ for all $\lambda \in \rho_{s-F}(5B)$; $\dim H_2 = \infty$, $\sigma_{\mathrm{lre}}(A_2) = \sigma_{\mathrm{lre}}(5B^2)$ and $\operatorname{ind}(\lambda - A_2) = \operatorname{ind}(\lambda - 5B^2)$ for all $\lambda \in \rho_{s-F}(5B^2)$.

Without loss of generality, we assume $H_1 = H_2{}^{\perp}$. Let

$$\delta(t) = \begin{bmatrix} -5(1-t) + tA_2 & (1-t)C_1 + tC_2 \\ & (1-t)A_1 + 5t \end{bmatrix}, \quad 0 < t < 1,$$

and

$$\alpha(t) = \begin{cases} 5B + (1+t)K_1 & -1 \leqslant t \leqslant 0, \\ \delta(t) & 0 < t < 1, \\ 5B^2 + (2-t)K_2 & 1 \leqslant t \leqslant 2. \end{cases}$$

Obviously, $\alpha(-1) = 5B$, $\alpha(2) = 5B^2$ and $\alpha(t)$ is continuous on $[-1, 2]$. It suffices to show $\alpha(t) \in DC(H)^0$ for any $-1 \leqslant t \leqslant 2$.

(1) $\rho_{s-F}^{(+)}(5B + (1+t)K_1) = \rho_{s-F}^{(+)}(5B)$, $-1 \leqslant t \leqslant 0$ and $\partial\mathbb{D} \subseteq \rho_{s-F}^{(+)}(5B)$ imply $\{5B + (1+t)K_1;\ -1 \leqslant t \leqslant 0\} \subseteq DC(H)^0$.

(2) Since for $0 < t < 1$,

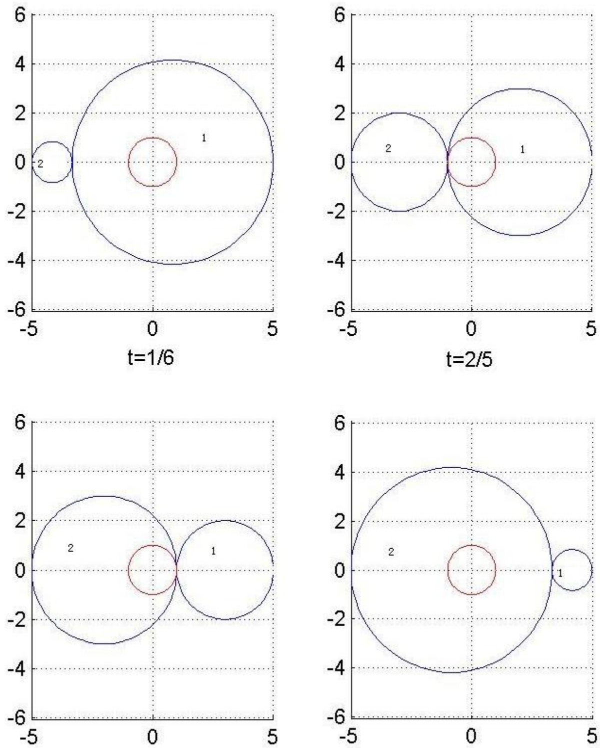
$$\begin{aligned} \rho_{s-F}^{(+)}(\delta(t)) &= [\rho_{s-F}^{(+)}(-5(1-t) + tA_2)] \cup [\rho_{s-F}^{(+)}((1-t)A_1 + 5t)] \\ &= \{-5(1-t) + 5t\mathbb{D}\} \cup \{5(1-t)\mathbb{D} + 5t\}, \end{aligned}$$

we know $\{\delta(t);\ 0 < t < 1\} \subseteq DC(H)^0$.

One can read the spectrum information from the next page picture as follows. The number 1 or 2 in the picture means the Fredholm index of $\lambda - \delta(t)$ for any λ belonging to the open disk which the number lies on respectively. We choose five moments.

(3) $\rho_{s-F}^{(+)}(5B^2 + (2-t)K_2) = \rho_{s-F}^{(+)}(5B^2)$, $1 \leqslant t \leqslant 2$ and $\partial\mathbb{D} \subseteq \rho_{s-F}^{(+)}(5B^2)$ imply $\{5B^2 + (2-t)K_2;\ 1 \leqslant t \leqslant 2\} \subseteq DC(H)^0$.

Hence $\{\alpha(t),\ -1 \leqslant t \leqslant 2\} \subseteq DC(H)^0$. ∎



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