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ADDITIVE DERIVATIONS ON ALGEBRAS OF MEASURABLE OPERATORS

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ABSTRACT. Given a von Neumann algebra M we introduce so called central extension $\mathrm{mix}(M)$ of M. We show that $\mathrm{mix}(M)$ is a $*$-subalgebra in the algebra $LS(M)$ of all locally measurable operators with respect to M, and this algebra coincides with $LS(M)$ if and only if M does not admit type II direct summands. We prove that if M is a properly infinite von Neumann algebra then every additive derivation on the algebra $\mathrm{mix}(M)$ is inner. In particular each derivation on the algebra $LS(M)$, where M is a type I_∞ or a type III von Neumann algebra, is inner.

KEYWORDS: *von Neumann algebras, measurable operator, locally measurable operator, algebra of mixings, derivation, inner derivation.*

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INTRODUCTION

The present paper continues the series of papers [2]-[3] devoted to the study and a description of derivations on the algebra $LS(M)$ of locally measurable operators affiliated with a von Neumann algebra M and on its various subalgebras.

Let $\mathcal{A}$ be an algebra over the field complex numbers. A linear (additive) operator $D : \mathcal{A} \to \mathcal{A}$ is called a linear (additive) *derivation* if it satisfies the identity $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathcal{A}$ (Leibniz rule). Each element $a \in \mathcal{A}$ defines a linear derivation D_a on $\mathcal{A}$ given as $D_a(x) = ax - xa$, $x \in \mathcal{A}$. Such derivations D_a are said to be *inner derivations*. If the element a implementing the derivation D_a on $\mathcal{A}$, belongs to a larger algebra $\mathcal{B}$, containing $\mathcal{A}$ (as a proper ideal as usual) then D_a is called a *spatial derivation*.

One of the main problems in the theory of derivations is to prove the automatic continuity, innerness or spatialness of derivations or to show the existence of noninner and discontinuous derivations on various topological algebras.

On this way A.F. Ber, F.A. Sukochev, V.I. Chilin [5] obtained necessary and sufficient conditions for the existence of nontrivial derivations on commutative regular algebras. In particular they have proved that the algebra $L^0(0,1)$ of all (classes of equivalence of) complex measurable functions on the interval $(0,1)$ admits nontrivial derivations. Independently A.G. Kusraev [9] by means of Boolean-valued analysis has also proved the existence of nontrivial derivations and automorphisms on $L^0(0,1)$. It is clear that these derivations are discontinuous in the measure topology, and therefore they are neither inner nor spatial. The present authors have conjectured that the existence of such pathological examples of derivations deeply depends on the commutativity of the underlying von Neumann algebra M. In this connection we have initiated the study of the above problems in the noncommutative case [2]–[3], by considering derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a von Neumann algebra M and on various subalgebras of $LS(M)$. In [1] we have proved that every derivation on so called noncommutative Arens algebras affiliated with an arbitrary von Neumann algebra and a faithful normal semi-finite trace is spatial and if the trace is finite then all derivations on this algebra are inner. In [2] and [3] we have proved the mentioned conjecture concerning derivations on $LS(M)$ for type I von Neumann algebras.

Recently this conjecture was also independently confirmed for the type I case in the paper of A.F. Ber, B. de Pagter and A.F. Sukochev [6] by means of a representation of measurable operators as operator valued functions. Another approach to similar problems in the framework of type I AW^*-algebras has been outlined in the paper of A.F. Gutman, A.G. Kusraev and S.S. Kutateladze [7].

In [3] we considered derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a type I von Neumann algebra M, and also on its subalgebras $S(M)$ — of measurable operators and $S(M,\tau)$ of τ-measurable operators, where τ is a faithful normal semi-finite trace on M. We proved that an arbitrary derivation D on each of these algebras can be uniquely decomposed into the sum $D = D_a + D_\delta$ where the derivation D_a is inner (for $LS(M)$, $S(M)$ and $S(M,\tau)$) while the derivation D_δ is an extension of derivation δ on the center of the corresponding algebra.

In the present paper we consider additive derivations on the algebra $LS(M)$, were M is a properly infinite von Neumann algebras.

In Section 1 we introduce the so called central extension $\mathrm{mix}(M)$ of a von Neumann algebra M. We show that $\mathrm{mix}(M)$ is a $*$-subalgebra in the algebra $LS(M)$ and this algebra coincides with whole $LS(M)$ if and only if M does not contain a direct summand of type II. The center $Z(M)$ of M is an abelian von Neumann algebra and hence it is $*$-isomorphic to $L^\infty(\Omega,\Sigma,\mu)$ for an appropriate measure space (Ω,Σ,μ). Therefore the algebra $LS(Z(M)) = S(Z(M))$ can be identified with the ring $L^0(\Omega,\Sigma,\mu)$ of all measurable functions on (Ω,Σ,μ). We also show that $\mathrm{mix}(M)$ is a C^*-algebra over the ring $S(Z(M)) \cong L^0(\Omega,\Sigma,\mu)$ in the sense of [4].

In Section 2 we give some necessary properties of the topology of convergence locally in measure on $LS(M)$.

Section 3 is devoted to study of derivations on the algebra mix(M). We prove that if M is a properly infinite von Neumann algebra then every additive derivation on the algebra mix(M) is inner. In particular every additive derivation on the algebra $LS(M)$, where M is of type I_∞ or III, is inner. The latter result generalizes Theorem 2.7 from [3] to additive derivations and extends it also for type III von Neumann algebras.

1. LOCALLY MEASURABLE OPERATORS AFFILIATED WITH VON NEUMANN ALGEBRAS

In this section we consider so called central extensions of von Neumann algebras and prove some auxiliary results concerning their properties. These properties can be obtained in a shorter way by referring to some general results in Boolean-valued analysis (see Chapter 8 of [8], but for readers not familiar with this theory and for the sake of completeness we give a straightforward proof of our propositions). Nevertheless at the end of the section we outline the proofs in the framework of Boolean-valued approach.

Let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. Consider a von Neumann algebra M in $B(H)$ with the operator norm $\|\cdot\|_M$. Denote by $P(M)$ the lattice of projections in M.

A linear subspace $\mathcal{D}$ in H is said to be *affiliated* with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D})\subset\mathcal{D}$ for every unitary u from the commutant

$$M' = \{y\in B(H) : xy = yx,\ \forall x\in M\}$$

of the von Neumann algebra M.

A linear operator x on H with the domain $\mathcal{D}(x)$ is said to be *affiliated* with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi\in\mathcal{D}(x)$.

A linear subspace $\mathcal{D}$ in H is said to be *strongly dense* in H with respect to the von Neumann algebra M, if

(i) $\mathcal{D}\eta M$;

(ii) there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(M)$ such that $p_n\uparrow\mathbf{1}$, $p_n(H)\subset\mathcal{D}$ and $p_n^{\perp}=\mathbf{1}-p_n$ is finite in M for all $n\in\mathbb{N}$, where $\mathbf{1}$ is the identity in M.

A closed linear operator x acting in the Hilbert space H is said to be *measurable* with respect to the von Neumann algebra M, if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H. Denote by $S(M)$ the set of all measurable operators with respect to M.

A closed linear operator x in H is said to be *locally measurable* with respect to the von Neumann algebra M, if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n\uparrow\mathbf{1}$ and $z_n x\in S(M)$ for all $n\in\mathbb{N}$.

It is well-known [10] that the set $LS(M)$ of all locally measurable operators with respect to M is a unital $*$-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator, and contains $S(M)$ as a solid $*$-subalgebra.

Let τ be a faithful normal semi-finite trace on M. We recall that a closed linear operator x is said to be *τ-measurable* with respect to the von Neumann algebra M, if $x\eta M$ and $\mathcal{D}(x)$ is τ-dense in H, i.e. $\mathcal{D}(x)\eta M$ and given $\varepsilon > 0$ there exists a projection $p \in M$ such that $p(H) \subset \mathcal{D}(x)$ and $\tau(p^{\perp}) < \varepsilon$. The set $S(M,\tau)$ of all τ-measurable operators with respect to M is a solid $*$-subalgebra in $S(M)$ (see [12]).

Consider the topology t_τ of convergence in measure or *measure topology* on $S(M,\tau)$, which is defined by the following neighborhoods of zero:

$$V(\varepsilon,\delta) = \{x \in S(M,\tau) : \exists e \in P(M), \tau(e^{\perp}) \leqslant \delta, xe \in M, \|xe\|_M \leqslant \varepsilon\},$$

where ε, δ are positive numbers, and $\|\cdot\|_M$ denotes the operator norm on M.

It is well-known [12] that $S(M,\tau)$ equipped with the measure topology is a complete metrizable topological $*$-algebra.

Note that if the trace τ is a finite then $S(M,\tau) = S(M) = LS(M)$.

Given any family $\{z_i\}_{i\in I}$ of mutually orthogonal central projections in M with $\bigvee\limits_{i\in I} z_i = \mathbf{1}$ and a family of elements $\{x_i\}_{i\in I}$ in $LS(M)$ there exists a unique element $x \in LS(M)$ such that $z_i x = z_i x_i$ for all $i \in I$. This element is denoted by $x = \sum\limits_{i\in I} z_i x_i$ and it is called *the mixing* of $\{x_i\}_{i\in I}$ with respect to $\{z_i\}_{i\in I}$ (see Proposition 1.1 and further remarks in [3]).

By $\mathrm{mix}(M)$ we denote the set of all elements x from $LS(M)$ for which there exists a sequence of mutually orthogonal central projections $\{z_i\}_{i\in I}$ in M with $\bigvee\limits_{i\in I} z_i = \mathbf{1}$, such that $z_i x \in M$ for all $i \in I$, i.e.

$$\mathrm{mix}(M) = \Big\{x \in LS(M) : \exists z_i \in P(Z(M)), z_i z_j = 0, i \neq j, \bigvee_{i\in I} z_i = \mathbf{1}, z_i x \in M, i \in I\Big\},$$

where $Z(M)$ is the center of M. In other words $\mathrm{mix}(M)$ is the set of all mixings obtained by families $\{x_i\}_{i\in I}$ taken from M.

PROPOSITION 1.1. *Let M be a von Neumann algebras with the center $Z(M)$. Then*

(i) $\mathrm{mix}(M)$ *is a $*$-subalgebra in $LS(M)$ with the center $S(Z(M))$, where $S(Z(M))$ is the algebra of operators measurable with respect to $Z(M)$;*

(ii) $LS(M) = \mathrm{mix}(M)$ *if and only if M does not have direct summands of type* II.

Proof. (i) It is clear from the definition that $\mathrm{mix}(M)$ is a $*$-subalgebra in $LS(M)$ and that its center $Z(\mathrm{mix}(M))$ is contained in $S(Z(M)) = Z(LS(M))$.

Let us show the converse inclusion. Take $x \in S(Z(M))$ and let $|x| = \int\limits_0^\infty \lambda \, de_\lambda$ be the spectral resolution of $|x|$. Set

$$z_1 = e_1 \quad \text{and} \quad z_n = e_n - e_{n-1}, \quad n \geqslant 2.$$

Then it clear that $\{z_n\}_{n\in\mathbb{N}}$ is a sequence of mutually orthogonal central projections in M such that $\bigvee\limits_{n\geqslant 1} z_n = \mathbf{1}$ and $z_n x \in Z(M)$ for all $n \in \mathbb{N}$. Therefore $x \in \text{mix}(M)$. Since x commutes with each element from $LS(M) \supset \text{mix}(M)$, we have that $x \in Z(\text{mix}(M))$. Thus $Z(\text{mix}(M)) = S(Z(M))$.

(ii) If M is of type I, then by Proposition 1.6 of [3] we have $LS(M) = \text{mix}(M)$.

Let M have type III. Since any nonzero projection in M is infinite it follows that $S(M) = M$. Hence by the definitions of the algebras $LS(M)$ and $\text{mix}(M)$ we obtain that $LS(M) = \text{mix}(M)$. Thus if $M = N \oplus K$ where N is a type I and K is a type III von Neumann algebras, i.e. if M does not have type II direct summands, then $LS(M) = \text{mix}(M)$.

To prove the converse suppose that M is a type II von Neumann algebra. First assume that M is of type II$_1$ and admits a faithful normal tracial state τ on M. Let Φ be the canonical center-valued trace on M.

Since M is of type II, then there exists a projection $p_1 \in M$ such that

$$p_1 \sim \mathbf{1} - p_1.$$

Then $\Phi(p_1) = \Phi(p_1^\perp)$. From $\Phi(p_1) + \Phi(p_1^\perp) = \Phi(\mathbf{1}) = \mathbf{1}$ it follows that

$$\Phi(p_1) = \Phi(p_1^\perp) = \frac{1}{2}\mathbf{1}.$$

Suppose that there exist mutually orthogonal projections $p_1, p_2, \ldots, p_n$ in M such that

$$\Phi(p_k) = \frac{1}{2^k}\mathbf{1}, \quad k = \overline{1, n}.$$

Set $e_n = \sum\limits_{k=1}^{n} p_k$. Then $\Phi(e_n^\perp) = \frac{1}{2^n}\mathbf{1}$. Take a projection $p_{n+1} < e_n^\perp$ such that $p_{n+1} \sim e_n^\perp - p_{n+1}$. Then

$$\Phi(p_{n+1}) = \frac{1}{2^{n+1}}.$$

Hence there exists a sequence of mutually orthogonal projections $\{p_n\}_{n\in\mathbb{N}}$ in M such that

$$\Phi(p_n) = \frac{1}{2^n}\mathbf{1}, \quad n \in \mathbb{N}.$$

Note that $\tau(p_n) = \frac{1}{2^n}$. Indeed $\tau(p_n) = \tau(\Phi(p_n)) = \tau\big(\frac{1}{2^n}\mathbf{1}\big) = \frac{1}{2^n}$.

Since

$$\sum_{n=1}^{\infty} n\tau(p_n) = \sum_{n=1}^{\infty} \frac{n}{2^n} < +\infty$$

it follows that the series $\sum\limits_{n=1}^{\infty} np_n$ converges in measure in $S(M, \tau)$. Therefore there exists $x = \sum\limits_{n=1}^{\infty} np_n \in S(M, \tau)$.

Let us show that $x \in LS(M) \setminus \mathrm{mix}(M)$. Suppose that $zx \in M$, where z is a nonzero central projection. Since any p_n is a faithful projection we have that $zp_n \neq 0$ for all n. Thus

$$\|zx\|_M = 1\|zx\|_M 1 = \|p_n\|_M \cdot \|zx\|_M \cdot \|p_n\|_M \geqslant \|zp_n x p_n\|_M = \|zp_n n\|_M = n,$$

i.e.

$$\|zx\|_M \geqslant n$$

for all $n \in \mathbb{N}$. From this contradiction it follows that $x \in LS(M) \setminus \mathrm{mix}(M)$.

For a general type II von Neumann algebra M take a nonzero finite projection $e \in M$ and consider the finite type II von Neumann algebra eMe which admits a separating family of normal tracial states. Now if $f \in eMe$ is the support projection of some tracial state τ on eMe then fMf is a type II_1 von Neumann algebra with a faithful normal tracial state. Hence as above one can construct an element $x \in LS(M) \setminus \mathrm{mix}(M)$. Therefore if $LS(M) = \mathrm{mix}(M)$ then M can not have a direct summand of the type II. The proof is complete. ∎

REMARK 1.2. A similar notion (i.e. the algebra $\mathrm{mix}(A)$) for arbitrary $*$-subalgebras $A \subset LS(M)$ was independently introduced recently by M.A. Muratov and V.I. Chilin [11]. They denote this algebra by $E(A)$ and called it the central extension of A. In particular if $A = M$ we have $E(M) = \mathrm{mix}(M)$. Therefore following [11] we shall say that $\mathrm{mix}(M)$ is *the central extension of* M.

An alternative proof of Proposition 1.1 follows also from Proposition 2, Theorem 1 and Theorem 3 in [11].

Let (Ω, Σ, μ) be a measure space and from now on suppose that the measure μ has the direct sum property, i.e. there is a family $\{\Omega_i\}_{i\in J} \subset \Sigma$, $0 < \mu(\Omega_i) < \infty$, $i \in J$, such that for any $A \in \Sigma$, $\mu(A) < \infty$, there exist a countable subset $J_0 \subset J$ and a set B with zero measure such that $A = \bigcup\limits_{i\in J_0} (A \cap \Omega_i) \cup B$.

We denote by $L^0(\Omega, \Sigma, \mu)$ the algebra of all (equivalence classes of) complex measurable functions on (Ω, Σ, μ) equipped with the topology of convergence in measure.

Consider the algebra $S(Z(M))$ of operators measurable with respect to the center $Z(M)$ of the von Neumann algebra M. Since $Z(M)$ is an abelian von Neumann algebra it is $*$-isomorphic to $L^\infty(\Omega, \Sigma, \mu)$ for an appropriate measure space (Ω, Σ, μ). Therefore the algebra $S(Z(M))$ can be identified with the algebra $L^0(\Omega, \Sigma, \mu)$ of all measurable functions on (Ω, Σ, μ).

PROPOSITION 1.3. *For any* $x \in \mathrm{mix}(M)$ *there exists* $f \in S(Z(M))$ *such that* $|x| \leqslant f$.

Proof. Let $x = \sum\limits_{i\in I} z_i x \in \mathrm{mix}(M), z_i x \in M$ for all $i \in I$. Put

$$f = \sum_{i\in I} z_i \|z_i x\|_M \in S(Z(M)).$$

Then we have the following and the proof is complete:

$$|x| = \Big|\sum_{i\in I} z_i x\Big| = \sum_{i\in I} z_i|z_i x| \leqslant \sum_{i\in I} z_i\|z_i x\|_M = f. \quad \blacksquare$$

Proposition 1.3 implies that for any $x \in \mathrm{mix}(M)$ there exists the vector-valued norm

$$\|x\| = \inf\{f \in S(Z(M)) : |x| \leqslant f\}. \tag{1.1}$$

By the definition we obtain that:

(i) $|x| \leqslant \|x\|$ for all $x \in \mathrm{mix}(M)$;

(ii) if $x \in \mathrm{mix}(M)$ then $\|x\| = \inf\{f \in S(Z(M)) : f \geqslant 0, f^{-1}x \in M, \|f^{-1}x\|_M \leqslant 1\}$;

(iii) if $z \in M$ is a central projection then $\|zx\| = z\|x\|$;

(iv) if $x \in M$ then $\|x\|_M = \|\,\|x\|\,\|_M$.

PROPOSITION 1.4. *Let $x \in M$. Then $\|x\| = \mathbf{1}$ if and only if $\|zx\|_M = 1$ for each nonzero central projection $z \in M$.*

Proof. Let $x \in M$, $\|x\| = \mathbf{1}$. Then $\|zx\| = z\|x\| = z$ for each nonzero central projection $z \in M$. Thus

$$\|zx\|_M = \|\,\|zx\|\,\|_M = \|z\|_M = 1.$$

Now let $\|zx\|_M = 1$ for each nonzero central projection $z \in M$, in particular, $\|x\|_M = 1$. Thus $\|x\| \leqslant \mathbf{1}$. Suppose that $\|x\| \neq \mathbf{1}$. Then there exist a nonzero central projection $z \in M$ and a number $0 < \varepsilon < 1$ such that $z\|x\| \leqslant \varepsilon z$. Thus

$$\|zx\|_M \leqslant \varepsilon\|z\|_M = \varepsilon < 1,$$

and this contradicts to the equality $\|zx\|_M = 1$. Hence $\|x\| = \mathbf{1}$. The proof is complete. $\blacksquare$

A complex linear space E is said to be normed by $L^0(\Omega, \Sigma, \mu)$ if there is a map $\|\cdot\| : E \to L^0(\Omega, \Sigma, \mu)$ such that for any $x, y \in E, \lambda \in \mathbb{C}$, the following conditions are fulfilled:

(i) $\|x\| \geqslant 0$; $\|x\| = 0 \Leftrightarrow x = 0$;

(ii) $\|\lambda x\| = |\lambda|\|x\|$;

(iii) $\|x + y\| \leqslant \|x\| + \|y\|$.

The pair $(E, \|\cdot\|)$ is called a lattice-normed space over $L^0(\Omega, \Sigma, \mu)$. A lattice-normed space E is called d-decomposable, if for any $x \in E$ with $\|x\| = \lambda_1 + \lambda_2$, $\lambda_1, \lambda_2 \in L^0(\Omega, \Sigma, \mu)$, $\lambda_1\lambda_2 = 0, \lambda_1, \lambda_2 \geqslant 0$, there exist $x_1, x_2 \in E$ such that $x = x_1 + x_2$ and $\|x_i\| = \lambda_i$, $i = 1, 2$.

A net (x_α) in E is said to be (bo)-convergent to $x \in E$, if the net $\{\|x_\alpha - x\|\}$ (o)-converges (i.e. almost everywhere converges) to zero in $L^0(\Omega, \Sigma, \mu)$.

A lattice-normed space E which is d-decomposable and complete with respect to the (bo)-convergence is called a *Banach–Kantorovich space*.

It is known that every Banach–Kantorovich space E over $L^0(\Omega, \Sigma, \mu)$ is a module over $L^0(\Omega, \Sigma, \mu)$ and $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in L^0(\Omega, \Sigma, \mu)$, $x \in E$ (see [8]).

Let $\mathcal{A}$ be an arbitrary Banach–Kantorovich space over $L^0(\Omega, \Sigma, \mu)$ and let $\mathcal{A}$ be an $*$-algebra such that $(\lambda x)^* = \overline{\lambda} x^*$, $(\lambda x)y = \lambda(xy) = x(\lambda y)$ for all $\lambda \in L^0(\Omega, \Sigma, \mu)$, $x, y \in \mathcal{A}$. $\mathcal{A}$ is called C^*-algebra over $L^0(\Omega, \Sigma, \mu)$ if $\|xy\| \leqslant \|x\|\|y\|$, $\|xx^*\| = \|x\|^2$ for all $x, y \in \mathcal{A}$ (see [4]).

The main result of this section is the following.

PROPOSITION 1.5. *Let M be a von Neumann algebra with the center $Z(M) \cong L^\infty(\Omega, \Sigma, \mu)$ and let $\|\cdot\|$ be the $S(Z(M))$-valued norm on* $\mathrm{mix}(M)$ *defined by* (1.1). *Then* $(\mathrm{mix}(M), \|\cdot\|)$ *is a C^*-algebra over $S(Z(M)) \cong L^0(\Omega, \Sigma, \mu)$.*

Proof. Let $x \in \mathrm{mix}(M), x \neq 0$ and let $|x| = \int\limits_0^\infty \lambda \, \mathrm{d}e_\lambda$ be the spectral resolution of $|x|$. Then there exists $\lambda_0 > 0$ such that $e_{\lambda_0} \neq 0$. Take an element $f \in S(Z(M))$ such that $|x| \leqslant f$. Then

$$\lambda_0 e_{\lambda_0} \leqslant |x| e_{\lambda_0} \leqslant f e_{\lambda_0}, \quad \text{i.e. } \lambda_0 e_{\lambda_0} \leqslant f e_{\lambda_0}.$$

Thus $\lambda_0 z(e_{\lambda_0}) \leqslant f z(e_{\lambda_0})$, where $z(e_{\lambda_0})$ is the central support of the projection e_{λ_0}. Thus $\lambda_0 z(e_{\lambda_0}) \leqslant \|x\| z(e_{\lambda_0})$. This means that $\|x\| \neq 0$.

Let $g \in S(Z(M))$, $x \in \mathrm{mix}(M)$. We have

$$\begin{aligned}\|gx\| = \inf\{f \in S(Z(M)) : |gx| \leqslant f\} &= \inf\{|g| f \in S(Z(M)) : |x| \leqslant f\} \\ &= |g| \inf\{f \in S(Z(M)) : |x| \leqslant f\} = |g|\|x\|,\end{aligned}$$

i.e.

$$\|gx\| = |g|\|x\|.$$

Now let $x, y \in \mathrm{mix}(M)$. By Theorem 2.4.5 of [10] there exist partial isometries $u, v \in M$ such that $|x + y| \leqslant u|x|u^* + v|y|v^*$. Thus

$$|x+y| \leqslant u|x|u^* + v|y|v^* \leqslant u\|x\|u^* + v\|y\|v^* = \|x\|uu^* + \|y\|vv^* \leqslant \|x\| + \|y\|,$$

and therefore $\|x + y\| \leqslant \|x\| + \|y\|$.

Take $x, y \in \mathrm{mix}(M)$. We may assume that $\|x\| = \|y\| = \mathbf{1}$. Then $x, y \in M$, $\|x\|_M = \|y\|_M = 1$, and therefore $\|xy\|_M \leqslant 1$. Hence $|xy| \leqslant \mathbf{1}$. Thus $\|xy\| \leqslant \mathbf{1}$, i.e. $\|xy\| \leqslant \|x\|\|y\|$.

Let $x \in M$, $\|x\| = \mathbf{1}$. By Proposition 1.4 we obtain $\|zx\|_M = 1$ for every nonzero central projection $z \in M$. Thus

$$\|zxx^*\|_M = \|(zx)(zx)^*\|_M = \|zx\|_M^2 = 1.$$

Therefore by Proposition 1.4 we obtain that $\|xx^*\| = \mathbf{1}$, i.e. $\|xx^*\| = \|x\|^2$.

Finally we shall prove the completeness of the space $\mathrm{mix}(M)$. First we consider the case where the center $S(Z(M)) \cong L^0(\Omega, \Sigma, \mu)$ satisfies the condition $\mu(\Omega) < \infty$.

Let $\{x_n\}$ be a (bo)-fundamental sequence in $\mathrm{mix}(M)$, i.e. $\|x_n - x_m\| \to 0$ as $n, m \to \infty$. By the inequality

$$|\|x_n\| - \|x_m\|| \leqslant \|x_n - x_m\|$$

we obtain that the sequence $\{\|x_n\|\}$ is (o)-fundamental in $S(Z(M))$, in particular, $\{\|x_n\|\}$ is order bounded in $S(Z(M))$, i.e. there exists $c \in S(Z(M))$ such that $\|x_n\| \leqslant c$ for all $n \in \mathbb{N}$.

Now replacing x_n with $(\mathbf{1}+c)^{-1}x_n$ we may assume that $x_n \in M$, $\|x_n\| \leqslant \mathbf{1}$ and $\{x_n\}$ is (bo)-fundamental.

Since $\mu(\Omega) < \infty$ by Egorov's theorem for any $k \in \mathbb{N}$ there exists $A_k \in \Sigma$ with $\mu(\Omega \setminus A_k) \leqslant \frac{1}{k}$ such that $\|\chi_{A_k}(x_n - x_m)\|_M \to 0$ as $n, m \to \infty$. Since M is complete, one has that $\chi_{A_k}x_n \to a_k$ as $n \to \infty$ for an appropriate $a_k \in M$.

Put

$$z_1 = \chi_{A_1},\ z_k = \chi_{A_k} \bigwedge \Big(\bigvee_{i=1}^{k-1} z_i\Big)^{\perp}, \quad k \geqslant 2.$$

Then $z_i \wedge z_j = 0,\ i \neq j,\ \bigvee\limits_{k \geqslant 1} z_k = \mathbf{1}$. Set

$$a = \sum_{k=1}^{\infty} z_k a_k.$$

Then $x_n \to a$. This means that the space $\operatorname{mix}(M)$ is (bo)-complete.

Now we consider the general case for the center $S(Z(M)) \cong L^0(\Omega, \Sigma, \mu)$. There exists a mutually orthogonal system $\{\Omega_i : i \in I\}$ in Σ such that $\mu(\Omega_i) < \infty$. As above we have that for every $i \in I$ there exists $a_i \in \operatorname{mix}(M)$ such that $\chi_{\Omega_i}x_n \to a_i$. Set

$$a = \sum_{i \in I} \chi_{\Omega_i} a_i.$$

Then $x_n \to a$. This means that the space $\operatorname{mix}(M)$ is (bo)-complete. The proof is complete. ∎

From Propositions 1.1 and 1.5 we obtain the following result.

COROLLARY 1.6. *Let M be a von Neumann algebra without direct summands of type* II. *Then $(LS(M), \|\cdot\|)$ is a C^*-algebra over $S(Z(M)) \cong L^0(\Omega, \Sigma, \mu)$.*

REMARK 1.7. The following Boolean-valued approach to central extensions was kindly suggested by the referee, to whom the authors are deeply indebted.

Let M be a von Neumann algebra and $\mathbb{B}$ denotes the Boolean algebra of central projections in M. In Boolean-valued universe $V^{(\mathbb{B})}$ there exists a von Neumann factor $\mathcal{M}$ such that the restricted descent $\mathcal{M} \Downarrow$ of $\mathcal{M}$ can be identified with M (see Theorem 8.4.4 (2) of [8]). Unrestricted descent $\mathcal{M} \downarrow$ of $\mathcal{M}$ can be identified with the central extension $\operatorname{mix}(M)$ (see Theorem 7.5.5 and 8.3.2 of [8]). The Boolean-valued representation preserves classification into types, thus M and $\mathcal{M}$ are of the same type ([8], Theorem 8.4.6). Moreover it can be easily seen that $LS(M) \downarrow$ can be interpreted as $LS(M) \simeq LS(M \Downarrow)$. Now Proposition 1.1(i), 1.4 and 1.5 follow immediately from Theorems 8.3.1 and 8.4.2 of [8].

As to Proposition 1.1(ii), we should interpret in Boolean-valued model $V^{(\mathbb{B})}$ Theorem 2.5.4 from [10] according to which $LS(M) = \mathcal{M}$ if and only if $\mathcal{M}$ is a factor of type I or III. Since $LS(M) \downarrow\simeq LS(M)$ and $\mathcal{M} \downarrow\simeq \mathrm{mix}(M)$, it follows that $LS(M) = \mathrm{mix}(M)$ if and only if M does not have direct summands of type II.

2. THE TOPOLOGY OF CONVERGENCE LOCALLY IN MEASURE

Let M be an arbitrary commutative von Neumann algebra. Then as we have mentioned above M is $*$-isomorphic to the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$, while the algebra $LS(M) = S(M)$ is $*$-isomorphic to the $*$-algebra $L^0(\Omega, \Sigma, \mu)$.

The basis of neighborhoods of zero in the topology of convergence locally in measure on $L^0(\Omega, \Sigma, \mu)$ consists of the following sets, where $\varepsilon, \delta > 0$, $A \in \Sigma$, $\mu(A) < +\infty$:

$$W(A, \varepsilon, \delta) = \{f \in L^0(\Omega, \Sigma, \mu) : \exists B \in \Sigma,\ B \subseteq A,\ \mu(A \setminus B) \leqslant \delta,$$
$$f \cdot \chi_B \in L^\infty(\Omega, \Sigma, \mu),\ \|f \cdot \chi_B\|_{L^\infty(\Omega,\Sigma,\mu)} \leqslant \varepsilon\}.$$

Recall the definition of the dimension functions on the lattice $P(M)$ of projection from M (see [10]).

By L_+ we denote the set of all measurable functions $f : (\Omega, \Sigma, \mu) \to [0, \infty]$ (modulo functions equal to zero almost everywhere).

Let M be an arbitrary von Neumann algebra with the center $Z(M) = L^\infty(\Omega, \Sigma, \mu)$. Then there exists a map $d : P(M) \to L_+$ with the following properties:

(i) $d(e)$ is a finite function if only if the projection e is finite;
(ii) $d(e + q) = d(e) + d(q)$ for $p, q \in P(M)$, $eq = 0$;
(iii) $d(uu^*) = d(u^*u)$ for every partial isometry $u \in M$;
(iv) $d(ze) = zd(e)$ for all $z \in P(Z(M))$, $e \in P(M)$;
(v) if $\{e_\alpha\}_{\alpha \in J}$, $e \in P(M)$ and $e_\alpha \uparrow e$, then

$$d(e) = \sup_{\alpha \in J} d(e_\alpha).$$

This map $d : P(M) \to L_+$, is called the dimension functions on $P(M)$.

The basis of neighborhoods of zero in the topology of convergence locally in measure on $LS(M)$ consists (in the above notations) of the following sets, where $\varepsilon, \delta > 0$, $A \in \Sigma$, $\mu(A) < +\infty$:

$$V(A, \varepsilon, \delta) = \{x \in LS(M) : \exists p \in P(M),\ \exists z \in P(Z(M)),\ xp \in M,$$
$$\|xp\|_M \leqslant \varepsilon,\ z^\perp \in W(A, \varepsilon, \delta),\ d(zp^\perp) \leqslant \varepsilon z\}.$$

We need following assertion from pp. 242, 261, 265 of [10].

PROPOSITION 2.1. *Let $\varepsilon, \delta > 0$, $A \in \Sigma$, $\mu(A) < +\infty$. Then:*
(i) $\lambda V(A, \varepsilon, \delta) = V(A, |\lambda|\varepsilon, \delta)$ *for all* $\lambda \in \mathbb{C}$, $\lambda \neq 0$;
(ii) $x \in V(A, \varepsilon, \delta) \Leftrightarrow |x| \in V(A, \varepsilon, \delta)$;
(iii) $x \in V(A, \varepsilon, \delta) \Rightarrow x^* \in V(A, 2\varepsilon, \delta)$;

(iv) $x \in V(A, \varepsilon, \delta)$, $y \in M \Rightarrow yx \in \|y\|_M V(A, \varepsilon, \delta)$;
(v) *for each* $x \in LS(M)$ *there exist* $\varepsilon_1, \delta_1 > 0$, $B \in \Sigma$, $\mu(B) < +\infty$, *such that*

$$x \cdot V(B, \varepsilon_1, \delta_1) \subseteq V(A, \varepsilon, \delta).$$

In the next section we shall also use the following properties of the topology of convergence locally in measure.

LEMMA 2.2. *Let* $\varepsilon, \delta > 0$, $A \in \Sigma$, $\mu(A) < +\infty, \lambda \in \mathbb{C}$. *If* $|\lambda| \leqslant \varepsilon$, *then* $\lambda \mathbf{1} \in V(A, \varepsilon, \delta)$.

Proof. Put $p = \mathbf{1}$, $z = \mathbf{1}$. Then $\|\lambda \mathbf{1} p\|_M = |\lambda| \leqslant \varepsilon$, $z^\perp = 0 \in W(A, \varepsilon, \delta)$, $d(zp^\perp) = d(0) = 0 \leqslant \varepsilon z$, and therefore $\lambda \mathbf{1} \in V(A, \varepsilon, \delta)$. The proof is complete. ∎

LEMMA 2.3. *Let* $x \in LS(M)$, $\varepsilon, \delta > 0$, $A \in \Sigma$, $\mu(A) < +\infty$. *Then there exists* $\lambda_0 > 0$ *such that* $x \in \lambda_0 V(A, \varepsilon, \delta)$.

Proof. By Proposition 2.1(v) there exist $\varepsilon_1, \delta_1 > 0, B \in \Sigma, \mu(B) < +\infty$, such that

$$x \cdot V(B, \varepsilon_1, \delta_1) \subseteq V(A, \varepsilon, \delta).$$

From Lemma 2.2 it follows that $\varepsilon_1 \mathbf{1} \in V(B, \varepsilon_1, \delta_1)$. Therefore $x\varepsilon_1 \mathbf{1} \in V(A, \varepsilon, \delta)$, i.e. $x \in \lambda_0 V(A, \varepsilon, \delta)$, where $\lambda_0 = \varepsilon_1^{-1}$. The proof is complete. ∎

LEMMA 2.4. *If* $x \in V(A, \varepsilon, \delta)$ *and* $u, v \in M$ *are partial isometries, then* $uxv \in V(A, 4\varepsilon, \delta)$.

Proof. The case when $u = 0$ or $v = 0$ is trivial. Assume that $u, v \neq 0$. Then $\|u\|_M = \|v\|_M = 1$. By Proposition 2.1(iv) we obtain that $vx \in V(A, \varepsilon, \delta)$. From Proposition 2.1(iii) it follows that $x^* v^* = (vx)^* \in V(A, 2\varepsilon, \delta)$. Applying Proposition 2.1(iv) once more we have that $v^* x^* u^* \in V(A, 2\varepsilon, \delta)$ and $uxv = (v^* x^* u^*)^* \in V(A, 4\varepsilon, \delta)$. The proof is complete. ∎

LEMMA 2.5. *If* $f_i \in S(Z(M))$, $i = 1, 2$, $|f_1| \leqslant |f_2|$ *and* $f_2 \in V(A, \varepsilon, \delta)$, *then* $f_1 \in V(A, \varepsilon, \delta)$.

Proof. Let $f_2 \in V(A, \varepsilon, \delta)$. Then $|f_2| \in V(A, \varepsilon, \delta)$. Therefore there exist $p_0 \in P(M)$, $z_0 \in P(Z(M))$ such that

$$|f_2| p_0 \in M, \quad \||f_2| p_0\|_M \leqslant \varepsilon, \quad z_0^\perp \in W(A, \varepsilon, \delta),\, d(z_0 p_0^\perp) \leqslant \varepsilon z_0.$$

From $|f_1| \leqslant |f_2|$ we get $p_0 |f_1| p_0 \leqslant p_0 |f_2| p_0$ and $|f_1| p_0 \leqslant |f_2| p_0$. Hence $|f_1| p_0 \in M$ and $\||f_1| p_0\|_M \leqslant \||f_2| p_0\|_M \leqslant \varepsilon$, i.e. $\||f_1| p_0\|_M \leqslant \varepsilon$. Since $z_0^\perp \in W(A, \varepsilon, \delta)$, $d(z_0 p_0^\perp) \leqslant \varepsilon z_0$ we see that $|f_1| \in V(A, \varepsilon, \delta)$ or $f_1 \in V(A, \varepsilon, \delta)$. The proof is complete. ∎

Recall [13] that a von Neumann algebra M is said to be *properly infinite*, if any nonzero central projection z in M is infinite.

LEMMA 2.6. *Let M be a properly infinite von Neumann algebra and let $\varepsilon, \delta > 0$, $A \in \Sigma$, $\delta < \mu(A) < +\infty$, $0 < \varepsilon < 1$. Then from $\lambda \mathbf{1} \in V(A, \varepsilon, \delta)$, where $\lambda \in \mathbb{C}$, it follows that $|\lambda| \leqslant \varepsilon$.*

Proof. Let $\lambda \mathbf{1} \in V(A, \varepsilon, \delta)$. Then there exist $p \in P(M)$, $z \in P(Z(M))$ such that $zp^{\perp}$ is a finite and $\|\lambda p\|_M \leqslant \varepsilon$, $z^{\perp} \in W(A, \varepsilon, \delta)$. Set $z = \chi_E$, where $E \in \Sigma$. Since $z^{\perp} \in W(A, \varepsilon, \delta)$ there exists $B \in \Sigma$, $B \subset A$ such that

$$\mu(A \setminus B) \leqslant \delta, \ \|z^{\perp}\chi_B\|_M \leqslant \varepsilon.$$

Since $0 < \varepsilon < 1$, from the inequality $\|z^{\perp}\chi_B\|_M \leqslant \varepsilon$ we have that $(\mathbf{1} - \chi_E)\chi_B = 0$. From $\mu(A) > \delta$ and $\mu(A \setminus B) \leqslant \delta$, we get $\chi_B \neq 0$, and therefore from $(\mathbf{1} - \chi_E)\chi_B = 0$ we obtain that $\chi_E \neq 0$, i.e. $z \neq 0$. Since $zp^{\perp}$ is finite and M is properly infinite, then projection zp is an infinite. Therefore $p \neq 0$. Thus $|\lambda| = |\lambda|\|p\|_M = \|\lambda p\|_M \leqslant \varepsilon$, i.e. $|\lambda| \leqslant \varepsilon$. The proof is complete. ∎

3. DERIVATIONS ON THE CENTRAL EXTENSIONS OF PROPERLY INFINITE VON NEUMANN ALGEBRAS

The following theorem is the main result of this paper.

THEOREM 3.1. *Let M be a properly infinite von Neumann algebra. Then every additive derivation on the algebra* $\mathrm{mix}(M)$ *is inner.*

For the proof of the Theorem 3.1 we need several preliminary assertions.

Let $\mathcal{A}$ be an algebra and denote by $Z(\mathcal{A})$ its center. If D is an additive derivation on $\mathcal{A}$ and $\Delta = D|_{Z(\mathcal{A})}$ is its restriction on to the center of $\mathcal{A}$, then Δ maps $Z(\mathcal{A})$ into itself ([3], Remark 1; see also Lemma 4.2 of [6]).

Let M be a commutative von Neumann algebra and let $\mathcal{A}$ be an arbitrary subalgebra of $LS(M) = S(M)$ containing M. Further we shall identify the algebra $LS(M) = S(M)$ with an appropriate $L^0(\Omega, \Sigma, \mu)$.

Consider a derivation $D : \mathcal{A} \to S(M)$ and let us show that D can be extended to a derivation $\widetilde{D}$ on the whole $S(M)$.

Since M is a commutative, for an arbitrary element $x \in S(M)$ there exists a sequence $\{z_n\}$ of mutually orthogonal projections with $\bigvee\limits_{n \in \mathbb{N}} z_n = \mathbf{1}$ and $z_n x \in M$ for all $n \in \mathbb{N}$. Set

$$\widetilde{D}(x) = \sum_{n \geqslant 1} z_n D(z_n x). \tag{3.1}$$

Since every derivation $D : \mathcal{A} \to S(M)$ is identically zero on projections of M, the equality (3.1) gives a well-defined derivation $\widetilde{D} : S(M) \to S(M)$ which coincides with D on $\mathcal{A}$.

Given an arbitrary additive derivation Δ on $S(M)$=$L^0(\Omega, \Sigma, \mu)$ the element

$$z_{\Delta} = \inf\{\pi \in \nabla : \pi\Delta = \Delta\}$$

is called the support of the derivation Δ, where ∇ is the complete Boolean algebra of all idempotents from $L^0(\Omega, \Sigma, \mu)$ (i.e. characteristic functions of sets from Σ).

For any nontrivial additive derivation $\Delta : L^0(\Omega, \Sigma, \mu) \to L^0(\Omega, \Sigma, \mu)$ there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in $L^{\infty}(\Omega, \Sigma, \mu)$ with $|\lambda_n| \leqslant \mathbf{1}$, $n \in \mathbb{N}$, such that

$$|\Delta(\lambda_n)| \geqslant n z_{\Delta}$$

for all $n \in \mathbb{N}$ (see Lemma 2.6 of [3]). In [3] this assertion was proved for linear derivations, but the proof is the same for additive derivations.

LEMMA 3.2. *Let M be a properly infinite von Neumann algebra, and let $\mathcal{A} \subseteq LS(M)$ be a $*$-subalgebra such that $M \subseteq \mathcal{A}$ and suppose that $D : \mathcal{A} \to \mathcal{A}$ is an additive derivation. Then $D|_{Z(\mathcal{A})} \equiv 0$, in particular, D is $Z(\mathcal{A})$-linear.*

Proof. Let D be an additive derivation on $\mathcal{A}$, and let Δ be its restriction onto $Z(\mathcal{A})$. Since $M \subset \mathcal{A} \subset LS(M)$ it follows that $Z(M) \subset Z(\mathcal{A}) \subset S(Z(M)) = L^0(\Omega, \Sigma, \mu)$. Let us extend the derivation Δ onto whole $S(Z(M))$ as in (3.1) above, and denote the extension also by Δ.

Since M is properly infinite there exists a sequence of mutually orthogonal projections $\{p_n\}_{n=1}^{\infty}$ in M such that $p_n \sim \mathbf{1}$ for all $n \in \mathbb{N}$, and $\bigvee_{n=1}^{\infty} p_n = \mathbf{1}$.

For any bounded sequence $\Lambda = \{\lambda_n\}_{n \in \mathbb{N}}$ in $Z(M)$ define an operator x_Λ by

$$x_\Lambda = \sum_{n=1}^{\infty} \lambda_n p_n.$$

Then, for all $n \in \mathbb{N}$,

$$x_\Lambda p_n = p_n x_\Lambda = \lambda_n p_n. \tag{3.2}$$

Take $\lambda \in Z(\mathcal{A})$ and $n \in \mathbb{N}$. From the identity $D(\lambda p_n) = D(\lambda) p_n + \lambda D(p_n)$ multiplying it by p_n from the both sides we obtain

$$p_n D(\lambda p_n) p_n = p_n D(\lambda) p_n + \lambda p_n D(p_n) p_n.$$

Since p_n is a projection, one has that $p_n D(p_n) p_n = 0$, and since $D(\lambda) = \Delta(\lambda) \in Z(\mathcal{A})$, we have

$$p_n D(\lambda p_n) p_n = \Delta(\lambda) p_n. \tag{3.3}$$

Now from the identity $D(x_\Lambda p_n) = D(x_\Lambda) p_n + x_\Lambda D(p_n)$, in view of (3.2) one has similarly

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n + \lambda_n p_n D(p_n) p_n,$$

i.e.

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n. \tag{3.4}$$

Now (3.3) and (3.4) imply

$$p_n D(x_\Lambda) p_n = \Delta(\lambda_n) p_n. \tag{3.5}$$

If we suppose that $\Delta \neq 0$ then $z_\Delta \neq 0$. By Lemma 2.6 of [3] there exists a bounded sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ in $Z(M)$ such that, for all $n \in \mathbb{N}$,

$$|\Delta(\lambda_n)| \geqslant n z_\Delta.$$

Replacing the algebra M by the algebra $z_\Delta M$, and the derivation D by $z_\Delta D$, we may assume that $z_\Delta = \mathbf{1}$, i.e., for all $n \in \mathbb{N}$,

$$|\Delta(\lambda_n)| \geqslant n\mathbf{1}. \tag{3.6}$$

Now take $\varepsilon, \delta > 0$, $A \in \Sigma$, $\delta < \mu(A) < +\infty$. By Lemma 2.3 there exists a number $\lambda_0 > 0$ such that $D(x_\Lambda) \in \lambda_0 V(A, \varepsilon, \delta)$. From Lemma 2.4 it follows that $p_n D(x_\Lambda) p_n \in \lambda_0 V(A, 4\varepsilon, \delta)$ for all $n \in \mathbb{N}$. If we combine this with (3.5) we obtain

$$\Delta(\lambda_n) p_n \in \lambda_0 V(A, 4\varepsilon, \delta) \tag{3.7}$$

for all $n \in \mathbb{N}$. Since $p_n \sim \mathbf{1}$ for each $n \in \mathbb{N}$, there exists a sequence of partial isometries $\{u_n\}_{n\in\mathbb{N}}$ in M such that $u_n u_n^* = p_n$ and $u_n^* u_n = \mathbf{1}$ for all $n \in \mathbb{N}$. Using (3.7) and Lemma 2.4 we have $u_n^* \Delta(\lambda_n) p_n u_n \in \lambda_0 V(A, 16\varepsilon, \delta)$ for all $n \in \mathbb{N}$. Thus from the equality $u_n^* p_n u_n = u_n^* u_n u_n^* u_n = \mathbf{1}$, we obtain that $\Delta(\lambda_n) \in \lambda_0 V(A, 16\varepsilon, \delta)$ for all $n \in \mathbb{N}$. Thus by Lemma 2.5 and from the inequality (3.6) we have

$$n\mathbf{1} \in \lambda_0 V(A, 16\varepsilon, \delta) \tag{3.8}$$

for all $n \in \mathbb{N}$. Take the number $n_0 \in \mathbb{N}$ such that $n_0 > 16\lambda_0\varepsilon$. From Proposition 2.1(i) and (3.8) we obtain that

$$\mathbf{1} \in V(A, 16\lambda_0\varepsilon n_0^{-1}, \delta).$$

Since $\delta < \mu(A)$ and $16\lambda_0\varepsilon n_0^{-1} < 1$, from Lemma 2.6 we get $1 \leqslant 16\lambda_0\varepsilon n_0^{-1}$, which contradicts the inequality $n_0 > 16\lambda_0\varepsilon$. This contradiction implies that $\Delta \equiv 0$, i.e. D is identically zero on the center of $\mathcal{A}$, and therefore it is $Z(\mathcal{A})$-linear. The proof is complete. ∎

REMARK 3.3. A result similar to Lemma 3.2 for the case of linear derivations has been announced without proof in Proposition 6.22 of [6].

In the case of linear derivations on the algebras $\mathcal{A} = S(M)$ or $S(M, \tau)$ a shorter proof of Lemma 3.2 can be obtained also from the following result.

PROPOSITION 3.4. *Let M be a properly infinite von Neumann algebra with the center $Z(M)$. Then the centers of the algebras $S(M)$ and $S(M, \tau)$ coincide with $Z(M)$.*

Proof. Suppose that $z \in S(M)$, $z \geqslant 0$, is a central element and let $z = \int_0^\infty \lambda \, \mathrm{d}e_\lambda$ be its spectral resolution. Then $e_\lambda \in Z(M)$ for all $\lambda > 0$. Assume that $e_n^\perp \neq 0$ for all $n \in \mathbb{N}$. Since M is properly infinite, $Z(M)$ does not contain non-zero finite projections. Thus $e_n^\perp$ is infinite for all $n \in \mathbb{N}$, which contradicts the condition $z \in S(M)$. Therefore there exists $n_0 \in \mathbb{N}$ such that $e_n^\perp = 0$ for all $n \geqslant n_0$, i.e. $z \leqslant n_0\mathbf{1}$. This means that $z \in Z(M)$, i.e. $Z(S(M)) = Z(M)$. Similarly $Z(S(M, \tau)) = Z(M)$. The proof is complete. ∎

Let M be a properly infinite von Neumann algebra with the center $Z(M)$ and let D be a linear derivation on the algebra $\mathcal{A} = S(M)$ or $S(M,\tau)$. By Proposition 3.4 it follows that $Z(\mathcal{A}) = Z(M)$, and therefore $\Delta = D|_{Z(\mathcal{A})}$ is a linear derivation on the algebra $Z(M)$. By Lemma 4.1.2 of [13] we obtain that $\Delta = 0$ as it was asserted in Lemma 3.2.

Proof of Theorem 3.1. Let $D : \mathrm{mix}(M) \to \mathrm{mix}(M)$ be an additive derivation. By Lemma 3.2 it follows that D is $S(Z(M))$-linear. From Proposition 1.5 we have that $\mathrm{mix}(M)$ is a C^*-algebra over $S(Z((M)) \cong L^0(\Omega,\Sigma,\mu)$. Since D is $S(Z(M))$-linear, by Theorem 5 of [4] we obtain that D is a $S(Z(M))$-bounded, i.e. there exists $c \in S(Z(M))$ such that $\|D(x)\| \leqslant c\|x\|$ for all $x \in \mathrm{mix}(M)$. Take a sequence of pairwise orthogonal central projections $\{z_n\}_{n\in\mathbb{N}}$ in M with $\bigvee\limits_{n\geqslant 1} z_n = \mathbf{1}$ such that $z_n c \in Z(M)$ for all n. Then for any $x \in M$ we have

$$\|D(z_n x)\| = z_n\|D(x)\| \leqslant z_n c\|x\|,$$

i.e. $\|D(z_n x)\| \in Z(M)$. Thus

$$z_n D(x) \in z_n M.$$

Therefore the operator $z_n D$ maps each subalgebra $z_n M$ into itself for all $n \in \mathbb{N}$. By Sakai's theorem ([13], Theorem 4.1.6) there exists $a_n \in z_n M$ such that

$$z_n D(x) = a_n x - x a_n,\ x \in z_n M.$$

Set $a = \sum\limits_{n\geqslant 1} z_n a_n$. Then $a \in \mathrm{mix}(M)$ and $D(x) = ax - xa$ for all $x \in \mathrm{mix}(M)$. This means that D is inner. The proof is complete. ∎

From Theorem 3.1 and Proposition 1.1 we obtain the following which generalizes and extends Theorem 2.7 from [3].

COROLLARY 3.5. *Let M be a direct sum of von Neumann algebras of type* I_∞ *and* III. *Then every additive derivation on the algebra $LS(M)$ is inner.*

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