SPECTRA OF SOME COMPOSITION OPERATORS AND ASSOCIATED WEIGHTED COMPOSITION OPERATORS

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ABSTRACT. We characterize the spectrum and essential spectrum of "essentially linear fractional" composition operators acting upon the Hardy space $H^2(\mathbb{U})$ of the open unit disc $\mathbb{U}$. When the symbols of these composition operators have Denjoy–Wolff point on the unit circle, the spectrum and essential spectrum coincide. Our work permits us to describe the spectrum and essential spectrum of certain associated weighted composition operators on $H^2(\mathbb{U})$.

KEYWORDS: *Composition operator, weighted composition operator, Hardy space, spectrum, essential spectrum, compression spectrum.*

MSC (2000): Primary 47B33; Secondary 47A10.

INTRODUCTION

Let $\mathbb{U}$ be the open unit disc in the complex plane, let $H(\mathbb{U})$ be the space of analytic functions on $\mathbb{U}$, and let $H^2(\mathbb{U})$ be the classical Hardy space, consisting of those functions in $H(\mathbb{U})$ whose Maclaurin coefficients are square summable. For φ an analytic selfmap of $\mathbb{U}$, let C_φ be the composition operator with symbol φ, so that $C_\varphi f = f \circ \varphi$ for any $f \in H(\mathbb{U})$. Clearly C_φ preserves $H(\mathbb{U})$. Littlewood [26] proved C_φ also preserves $H^2(\mathbb{U})$; and thus, by the closed-graph theorem, $C_\varphi : H^2(\mathbb{U}) \to H^2(\mathbb{U})$ is a bounded linear operator.

Following p. 48 of [2], we say an analytic selfmap φ of $\mathbb{U}$ is *essentially linear fractional* provided

(a) $\varphi(\mathbb{U})$ is contained in a proper subdisc of $\mathbb{U}$ internally tangent to the unit circle at some point $\eta \in \partial \mathbb{U}$;

(b) $\varphi^{-1}(\{\eta\}) := \{\gamma \in \partial \mathbb{U} : \eta \text{ belongs to the cluster set of } \varphi \text{ at } \gamma\}$ consists of one element, say $\zeta \in \partial \mathbb{U}$; and

(c) φ''' extends continuously to $\mathbb{U} \cup \{\zeta\}$.

When a selfmap φ of $\mathbb{U}$ satisfies (a), (b), and (c) above, we write

$$\varphi \in \mathrm{LF}_e(\zeta; \eta).$$

For instance, $\varphi(z) = (2z^2 - 3z + 3)/(2z^2 - 7z + 7)$ belongs to $\mathrm{LF_e}(1;1)$ (see Example 1.5 below). Theorem 7.6 of [2] shows that for each essentially linear fractional $\varphi \in \mathrm{LF_e}(\zeta, \eta)$, there is a linear fractional selfmap ψ of $\mathbb{U}$ (satisfying $\varphi^{(j)}(\zeta) = \psi^{(j)}(\zeta)$ for $j \in \{0, 1, 2\}$) such that $C_\varphi - C_\psi$ is a compact operator. Thus, an essentially linear fractional composition operator differs from a linear fractional composition operator by a compact operator. However, this compact-difference condition obviously does not characterize essentially linear fractional composition operators: in fact, a linear fractional composition operator C_φ will be essentially linear fractional only when φ is nonautomorphic and $\varphi(\partial\mathbb{U})$ contacts $\partial\mathbb{U}$.

Much is known about spectra of composition operators on $H^2(\mathbb{U})$; see, e.g., Chapter 7 of [18]. However, for many composition operators on $H^2(\mathbb{U})$, including some essentially linear fractional composition operators, complete spectral characterizations have not been obtained. Moreover, there are interesting general questions that remain open, two of which we highlight below.

The analysis of the spectral behavior of $C_\varphi : H^2(\mathbb{U}) \to H^2(\mathbb{U})$ is typically case based, with the cases depending upon the *type* of the symbol φ: "dilation," "hyperbolic," "parabolic automorphism," and "parabolic nonautomorphism." (We review the notion of type in the "Preliminaries" section below.) Here are two of the questions addressed in this paper:

(Q1) For φ of hyperbolic type or either of the parabolic types, do the spectrum and essential spectrum of C_φ always coincide?

(Q2) Let $r(C_\varphi)$ denote the spectral radius of C_φ and $r_e(C_\varphi)$, the essential spectral radius. When φ is of dilation type and is either univalent or analytic on the closed disc, then work by Cowen and MacCluer [17] and Kamowitz [23] shows that the spectrum of C_φ consists of a disc D (possibly degenerate) centered at the origin of radius $r_e(C_\varphi)$ together with isolated eigenvalues. In this situation, must every point in D be in the essential spectrum of C_φ?

In this paper, we characterize the spectrum and essential spectrum of composition operators on $H^2(\mathbb{U})$ induced by selfmappings φ of $\mathbb{U}$ that are essentially linear fractional. For such mappings φ, our work shows that "yes" is the answer to (Q1). For question (Q2), we show that the "disc plus isolated eigenvalues" characterization of the spectrum continues to be valid for composition operators whose symbols are dilation-type essentially linear fractional maps (which need not be univalent, or analytic on the closed disc), and we show that the answer to (Q2) is also yes for such composition operators. We also show that the spectrum and essential spectrum of certain weighted composition operators coincide.

This paper is organized as follows. In the next section, we set the stage for our work, providing needed background information. In Section 2, we present some general results about spectra of composition operators; for example, we show that for any selfmapping φ of $\mathbb{U}$ of hyperbolic type or either of the parabolic types, if λ is an eigenvalue of C_φ having an outer function as a corresponding eigenvector, then λ does not belong to the compression spectrum of C_φ; i.e.,

$C_\varphi - \lambda I$ has dense range. We also show that no eigenvalue of C_φ can be isolated for certain selfmaps φ of parabolic nonautomorphism type. Using results in Section 2 as lemmas, we characterize in Section 3 the spectrum and essential spectrum of C_φ when φ is essentially linear fractional. We show, e.g., that if φ is an essentially linear fractional mapping of parabolic nonautomorphism type, then the spectrum and essential spectrum of C_φ coincide, each equaling a spiral $\{e^{-at} : t \geqslant 0\}$, where a is the second derivative of φ at its Denjoy–Wolff point. (This characterization is consistent with one obtained by Cowen ([12], Corollary 6.2) for a special family of composition operators whose symbols are univalent and of parabolic nonautomorphism type.) In Section 4, we characterize the spectrum and essential spectrum of certain weighted composition operators $f \mapsto gf \circ \varphi$, where g is bounded and analytic on $\mathbb{U}$ and φ is essentially linear fractional. For example, a consequence of Theorem 4.5 is that if $\varphi \in \mathrm{LF}_e(\omega; \omega)$ with $\varphi'(\omega) < 1$ and the radial-limit function of $g \in H^\infty(\mathbb{U})$ is continuous at ω, then the spectrum and essential spectrum of the weighted composition operator $C_{g,\varphi} : H^2(\mathbb{U}) \to H^2(\mathbb{U})$ coincide, each equaling the disc $\{z : |z| \leqslant |g(\omega)| \varphi'(\omega)^{-1/2}\}$. Here, $C_{g,\varphi} f = gf \circ \varphi$. We also obtain a sufficient condition for $C_{g,\varphi}$ to be quasinilpotent; see Theorem 4.4.

Regarding Section 4: the only prior results of which the author is aware concerning spectra of weighted composition operators on $H^2(\mathbb{U})$ pertain to the compact (or power-compact) case (see, e.g., [21]) or to the case where $C_{g,\varphi}$ is Hermitian [15], [16] or normal [4].

1. PRELIMINARIES

For detailed information about the Hardy space $H^2(\mathbb{U})$ as well as inner and outer functions, the reader may consult [19], for example. Good general references for properties of selfmaps φ of $\mathbb{U}$ and of the composition operators they induce on $H^2(\mathbb{U})$ are [18] and [27]. We concentrate here on background information crucial to our work.

1.1. REPRODUCING KERNELS FOR $H^2(\mathbb{U})$. The Hardy space $H^2(\mathbb{U})$ is a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \widehat{f}(n) \overline{\widehat{g}(n)},$$

where $(\widehat{f}(n))$ and $(\widehat{g}(n))$ are the sequences of Maclaurin coefficients for f and g respectively. The norm of $f \in H^2(\mathbb{U})$ is given by $\left(\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 \right)^{1/2}$ or, alternatively, by

$$\|f\|_{H^2(\mathbb{U})}^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^2 \, \mathrm{d}t,$$

where $f(e^{it})$ represents the radial limit of f at e^{it}, which exists for a.e. $t \in [0, 2\pi)$ (with respect to Lebesgue measure). Also, the inner product of two functions f and g in $H^2(\mathbb{U})$ may be expressed as a boundary integral:

$$\langle f, g \rangle = \frac{1}{2\pi} \int\limits_0^{2\pi} f(e^{it}) \overline{g(e^{it})} \, dt.$$

Let $h \in H^\infty(\mathbb{U})$, the Banach algebra of bounded analytic functions on $\mathbb{U}$ with $\|h\|_\infty = \sup\{|h(z)| : z \in \mathbb{U}\}$. The integral representation of the $H^2(\mathbb{U})$ norm makes it clear that $\|hf\|_{H^2(\mathbb{U})} \leqslant \|h\|_\infty \|f\|_{H^2(\mathbb{U})}$ for each $f \in H^2(\mathbb{U})$. Thus, the *multiplication operator* $M_h : H^2(\mathbb{U}) \to H^2(\mathbb{U})$, defined by $M_h f = hf$, is bounded and linear on $H^2(\mathbb{U})$.

For each $\alpha \in \mathbb{U}$, let $K_\alpha = 1/(1 - \bar{\alpha}z)$. Then $K_\alpha \in H^2(\mathbb{U})$, and it is easy to see that for each function $f \in H^2(\mathbb{U})$,

$$\langle f, K_\alpha \rangle = f(\alpha);$$

thus, K_α is the *reproducing kernel* at α for $H^2(\mathbb{U})$.

It is easy to check that $(C_\varphi)^* K_\alpha = K_{\varphi(\alpha)}$. Also, the reproducing kernels K_α, $\alpha \in \mathbb{U}$, are eigenfunctions for adjoint multiplication operators on $H^2(\mathbb{U})$: if $h \in H^\infty(\mathbb{U})$ then $(M_h)^* K_\alpha = \overline{h(\alpha)} K_\alpha$. Finally, we have the following estimate for each $f \in H^2(\mathbb{U})$ and each $z \in \mathbb{U}$:

$$(1.1) \qquad |f(z)| = |\langle f, K_z \rangle| \leqslant \|f\|_{H^2(\mathbb{U})} \|K_z\|_{H^2(\mathbb{U})} = \frac{\|f\|_{H^2(\mathbb{U})}}{\sqrt{1 - |z|^2}}.$$

1.2. THE DENJOY–WOLFF POINT ω. Throughout this paper, φ will denote an analytic function on $\mathbb{U}$ for which $\varphi(\mathbb{U}) \subseteq \mathbb{U}$; i.e., φ will always denote an analytic *selfmap* of $\mathbb{U}$. For n a nonnegative integer, let $\varphi^{[n]}$ denote the n-th iterate of φ so that, e.g., $\varphi^{[0]}$ is the identity function on $\mathbb{U}$ and $\varphi^{[2]} = \varphi \circ \varphi$. If φ is not an elliptic automorphism of $\mathbb{U}$, then there is a (unique) point ω in the closure $\mathbb{U}^-$ of $\mathbb{U}$ such that

$$\omega = \lim_{n \to \infty} \varphi^{[n]}(z)$$

for each $z \in \mathbb{U}$. The point ω, called the *Denjoy–Wolff point* of φ, is also characterized as follows: if $|\omega| < 1$, then $\varphi(\omega) = \omega$ and $|\varphi'(\omega)| < 1$; if $\omega \in \partial\mathbb{U}$, then $\varphi(\omega) = \omega$ and $0 < \varphi'(\omega) \leqslant 1$. If $|\omega| = 1$, then $\varphi(\omega)$ represents the angular (non-tangential) limit of φ at ω and $\varphi'(\omega)$ represents the angular derivative of φ at ω. The location of the Denjoy–Wolff point and the behavior of iterate sequences $(\varphi^{[n]}(z))$ as they approach the Denjoy–Wolff point strongly influence properties of the operator C_φ. For example, no φ with Denjoy–Wolff point on $\partial\mathbb{U}$ can be the symbol of a compact composition operator on $H^2(\mathbb{U})$ (see e.g., p. 56 of [27]). The Denjoy–Wolff point plays a major role in the classification system for selfmaps of $\mathbb{U}$ presented in the next subsection.

1.3. TYPE FOR SELFMAPS OF $\mathbb{U}$. Let φ be a selfmap of $\mathbb{U}$ with Denjoy–Wolff point ω. We classify φ as follows (cf. Definition 0.3 of [5]):

(i) Dilation type: $\omega \in \mathbb{U}$.

(ii) Hyperbolic type: $\omega \in \partial\mathbb{U}$ and $\varphi'(\omega) < 1$.

(iii) Parabolic automorphism type: $\omega \in \partial\mathbb{U}$, $\varphi'(\omega) = 1$, and the iterate sequence $(\varphi^{[n]}(0))$ is separated in the hyperbolic metric on $\mathbb{U}$ (meaning that the hyperbolic distance between successive points of the orbit $(\varphi^{[n]}(0))$ stays bounded away from zero).

(iv) Parabolic nonautomorphism type: $\omega \in \partial\mathbb{U}$, $\varphi'(\omega) = 1$, and the iterate sequence $(\varphi^{[n]}(0))$ is not separated in the hyperbolic metric on $\mathbb{U}$.

Assuming $\varphi'(\omega) = 1$, distinguishing whether φ is of parabolic automorphism or nonautomorphism type can be difficult. However, there is an easy test if φ has enough smoothness near its Denjoy–Wolff point.

DEFINITION 1.1. Let n be a positive integer, let $\zeta \in \partial\mathbb{U}$, and let $0 \leqslant \varepsilon < 1$. Following p. 50 of [5], we say that the selfmap φ of $\mathbb{U}$ belongs to $C^{n+\varepsilon}(\zeta)$ provided that φ is differentiable at ζ up to order n (viewed as a function with domain $\mathbb{U} \cup \{\zeta\}$) and, for $z \in \mathbb{U}$, has the expansion

$$\varphi(z) = \sum_{k=0}^{n} \frac{\varphi^{(k)}(\zeta)}{k!}(z - \zeta)^k + \gamma(z),$$

where $\gamma(z) = o(|z - \zeta|^{n+\varepsilon})$ as $z \to \zeta$ from within $\mathbb{U}$.

It is not difficult to show that $\varphi \in C^n(\zeta)$ whenever $\varphi^{(n)}$ extends continuously to $\mathbb{U} \cup \{\zeta\}$.

The following is Theorem 4.4 of [5].

PARABOLIC-TYPE TEST. *Suppose that $\varphi \in C^2(\omega)$ and $\varphi'(\omega) = 1$. Then* $\operatorname{Re}(\omega\varphi''(\omega)) \geqslant 0$; *moreover,*

(a) *if $\varphi''(\omega) = 0$ or if $\operatorname{Re}(\omega\varphi''(\omega)) > 0$, then φ is of parabolic nonautomorphism type;*

(b) *if $\omega\varphi''(\omega)$ is pure imaginary (and nonzero) and $\varphi \in C^{3+\varepsilon}(\omega)$ for some positive ε, then φ is of parabolic automorphism type.*

1.4. HOROCYCLIC SELFMAPS OF $\mathbb{U}$. For $\zeta \in \partial\mathbb{U}$ and $\beta > 0$, let $H(\eta, \beta) = \{z : |1 - z\overline{\eta}|^2 < \beta(1 - |z|^2)\}$ be the open horodisc (of radius $\beta/(1 + \beta)$) internally tangent to the unit circle at η. Call a selfmap φ of $\mathbb{U}$ *horocyclic* at η provided $\varphi(\mathbb{U})$ lies in a horodisc $H(\eta, \beta)$, for some $\beta > 0$. Note any $\varphi \in \mathrm{LF}_e(\zeta; \eta)$ must, by definition, be horocyclic at η.

In the next subsection, we present a criterion (Proposition 1.3) for φ to be horocyclic at 1. We apply the criterion to show $\varphi(z) = 2/(\sqrt{13 - 4z} - 1)$ is horocyclic at 1 (see Example 1.4).

1.5. ESSENTIALLY LINEAR FRACTIONAL SELFMAPS OF $\mathbb{U}$. Let φ be an essentially linear-fractional selfmapping of $\mathbb{U}$ belonging to $LF_e(\zeta; \eta)$. Thus η and ζ are unimodular constants such that $\varphi \in C^3(\zeta)$, $\varphi(\zeta) = \eta$, $\varphi^{-1}(\{\eta\}) = \{\zeta\}$, and φ is horocyclic at η. Recall from the Introduction that in this context $\varphi^{-1}(\{\eta\}) = \{\zeta\}$ means that ζ is the only point in $\partial\mathbb{U}$ whose cluster set under φ includes η. It follows that if $\eta \neq \zeta$, then $\|\varphi \circ \varphi\|_\infty < 1$ (for otherwise there would be a sequence (z_n) in $\varphi(\mathbb{U})$ such that $\lim_n |\varphi(z_n)| = 1$, but since $\varphi(\mathbb{U})$ is contained in a proper subdisc of $\mathbb{U}$ internally tangent to $\partial\mathbb{U}$ at η, then both (z_n) and $(\varphi(z_n))$ approach η, making $\varphi^{-1}(\{\eta\}) \supseteq \{\zeta, \eta\}$, a contradiction). Hence if $\eta \neq \zeta$, then $(C_\varphi)^2$ is compact (see, e.g., p. 23 of [27]). When some power of C_φ is compact the essential spectrum of C_φ is just $\{0\}$ and a result of Caughran and Schwartz ([7], Theorem 3) shows that the spectrum of C_φ is $\{0\} \cup \{\varphi'(\omega)^n, n = 0, 1, 2, \ldots\}$, where ω (necessarily in $\mathbb{U}$) is the Denjoy–Wolff point of φ. *Thus, in our discussions below concerning spectra for essentially linear fractional composition operators, we will restrict our attention to the case where the essentially linear fractional symbol φ fixes a point $\zeta \in \partial\mathbb{U}$. Without loss of generality, we assume $\zeta = 1$.*

Let φ be an *arbitrary* selfmap of $\mathbb{U}$ such that $\varphi(1) = 1$ and $\varphi \in C^2(1)$. Let $p = \varphi'(1)$, $a = \varphi''(1)$, and let $T(z) = (1+z)/(1-z)$ so that T maps $\mathbb{U}$ univalently onto the right halfplane Π. Because $\varphi \in C^2(1)$, the selfmapping $\Phi := T \circ \varphi \circ T^{-1}$ of Π has the following representation:

$$(1.2) \qquad \Phi(w) = \frac{1}{p}w + \left(\frac{1}{p} - 1 + \frac{a}{p^2}\right) + \Gamma(w), \quad w \in \Pi,$$

where $\Gamma(w) = o(1)$ as $|w| \to \infty$ (cf. equation (27) of [2]). Clearly, φ is horocyclic at 1 if and only if there is a constant $c > 0$ such that

$$(1.3) \qquad \mathrm{Re}\,(\Phi(w)) \geqslant c \quad \text{for all } w \in \Pi.$$

The representation (1.2) reveals that if (1.3) holds, then $\mathrm{Re}\,(1/p - 1 + a/p^2) \geqslant c > 0$. Thus, we have established the following:

PROPOSITION 1.2. *Let φ be an arbitrary selfmap of $\mathbb{U}$ such that $\varphi(1) = 1$ and $\varphi \in C^2(1)$. If φ is horocyclic at 1, then $1/\varphi'(1) - 1 + \varphi''(1)/\varphi'(1)^2$ must be positive.*

The next proposition shows that the converse is true if φ extends to be continuous on the closed disc and $|\varphi(\zeta)| < 1$ for $\zeta \in \partial\mathbb{U} \setminus \{1\}$.

PROPOSITION 1.3. *Let φ be an analytic selfmap of $\mathbb{U}$ that extends to be continuous on $\mathbb{U}^-$. Suppose that $\varphi \in C^2(1)$, that $\varphi(1) = 1$, and that $|\varphi(\zeta)| < 1$ for $\zeta \in \partial\mathbb{U} \setminus \{1\}$. If*

$$(1.4) \qquad \mathrm{Re}\,\left(\frac{1}{\varphi'(1)} - 1 + \frac{\varphi''(1)}{\varphi'(1)^2}\right) > 0$$

then $\varphi(\mathbb{U})$ is contained in a proper subdisc of $\mathbb{U}$ internally tangent to $\partial\mathbb{U}$ at 1; i.e., φ is horocyclic at 1.

Proof. Suppose that (1.4) holds. Set $\Phi := T \circ \varphi \circ T^{-1}$. We know that the proposition follows if the real part of Φ is bounded below on Π by a positive

number. Suppose, in order to obtain a contraction, that Re (Φ) is not bounded below, so that there is a sequence (w_n) in the right halfplane for which $(\mathrm{Re}\,(\Phi(w_n)))$ converges to 0.

Since Φ has the representation (1.2) and (1.4) holds, we see that (w_n) must be a bounded sequence. Thus (w_n) has a limit point b in the closure of the right halfplane. Because φ is continuous on $\mathbb{U}^-$, Φ is continuous at b and Re $(\Phi(b)) = 0$. Note that Re $(\Phi(b)) = 0$ tells us that b must be on the imaginary axis. Hence, $\varphi(T^{-1}(b)) = T^{-1}(\Phi(b))$ belongs to $\partial\mathbb{U} \setminus \{1\}$, contradicting the hypothesis that $|\varphi(\zeta)| < 1$ for $\zeta \in \partial\mathbb{U} \setminus \{1\}$. $\blacksquare$

Let $\varphi \in \mathrm{LF}_e(1;1)$ so that $\varphi(1) = 1$ and φ is horocyclic at 1. Let $p = \varphi'(1)$ and $a = \varphi''(1)$. Because $\varphi \in C^3(1) \subseteq C^2(1)$, the selfmapping $\Phi := T \circ \varphi \circ T^{-1}$ of Π has the representation (1.2). As we discussed above, because φ is horocyclic at 1, (1.2) shows that there is a positive constant c such that Re $(1/p - 1 + a/p^2) \geqslant c > 0$. Note in particular that if $p = 1$ (φ is parabolic type) then we must have Re $(a) > 0$, so that φ is of nonautomorphism type by the Parabolic Type Test. Thus, of the two types of parabolic selfmappings of $\mathbb{U}$, only those of nonautomorphism type can be essentially linear fractional. Because Re $(1/p - 1 + a/p^2) \geqslant c > 0$,

$$(1.5) \qquad \Psi(w) := \frac{w}{p} + \left(\frac{1}{p} - 1 + \frac{a}{p^2}\right)$$

is a nonautomorphic selfmap of Π. Theorem 7.6 of [2] shows that if ψ is the linear-fractional map of $\mathbb{U}$ defined by $T^{-1} \circ \Psi \circ T$, then $C_\varphi - \mathbb{C}_\psi$ is a compact operator on $H^2(\mathbb{U})$.

Here are three concrete examples of essentially linear fractional selfmappings of $\mathbb{U}$.

EXAMPLE 1.4. Let $\varphi(z) = 2/(\sqrt{13 - 4z} - 1)$, where $\sqrt{\cdot}$ represents the principal branch of the square root function — *all roots in this paper should be viewed as principal ones*. The function $z \mapsto 13 - 4z$ takes $\mathbb{U}$ to the disc centered at 13 of radius 4 and thus $z \mapsto |\sqrt{13 - 4z} - 1|$ attains its absolute minimum value 2 on $\mathbb{U}^-$ at $z = 1$ and only at $z = 1$. Thus φ is a selfmap of $\mathbb{U}$ such that $\varphi(1) = 1$, $\varphi^{-1}(\{1\}) = \{1\}$, and $\varphi \in C^3(1)$ (in fact φ is analytic on $\mathbb{U}^-$). One may apply Proposition 1.3 to see that $\varphi(\mathbb{U})$ belongs to proper subdisc of $\mathbb{U}$ internally tangent to $\mathbb{U}$ at 1. Thus $\varphi \in \mathrm{LF}_e(1;1)$. Since $\varphi(1) = 1$ and $\varphi'(1) = 1/3$, φ is of hyperbolic type.

EXAMPLE 1.5. Constructing examples of essentially linear fractional self-maps φ is quite simple using the representations like (1.2). For instance, choosing $p = 2$, $a = 6$ and $\Gamma(w) = 2/(w + 1)$, we obtain $\varphi(z) = T^{-1}(\Phi(T(z))) = (2z^2 - 3z + 3)/(2z^2 - 7z + 7)$ as an essentially linear fractional selfmap of $\mathbb{U}$ such that $\varphi(1) = 1$ and $\varphi'(1) = 2$. That φ is horocyclic at 1 is clear from the form of Φ, which shows Re $(\Phi(w)) \geqslant 1$ for every $w \in \Pi$. The form of Φ also makes it clear that $\varphi^{-1}(\{1\}) = \{1\}$. Finally, note that φ must be of dilation type since $\varphi'(1) > 1$

(in fact, $\varphi(1/2) = 1/2$.) We note that essentially linear fractional mappings need not be univalent on $\mathbb{U}$; in fact, this is the case for the the rational mapping φ we just constructed (its derivative vanishes at 0).

EXAMPLE 1.6. For $w \in \Pi$, let

$$\Phi(w) = w + 2 + \mathrm{i} - \frac{1}{4(w+1)} - \frac{1}{4(w+1)^{3/2}}.$$

Note that Φ is an analytic selfmap of Π and that, in fact, $\mathrm{Re}\,(\Phi(w)) > 3/2$ for every $w \in \Pi$ (since $|1/(4(w+1)) + 1/(4(w+1)^{3/2})| < 1/2$ for $w \in \Pi$). It follows that

$$\varphi(z) = T^{-1}(\Phi(T(z))) = \frac{z^2 - (2 - 8\mathrm{i})z - (15 + 8\mathrm{i}) + 2^{-1/2}(1 - z)^{5/2}}{z^2 + (14 + 8\mathrm{i})z - (31 + 8\mathrm{i}) + 2^{-1/2}(1 - z)^{5/2}}$$

is an analytic selfmap of $\mathbb{U}$ that is horocyclic at 1 (in fact, since $\mathrm{Re}\,(\Phi(w)) > 3/2$, the set $\varphi(\mathbb{U})$ must be contained in $T^{-1}(\{w : \mathrm{Re}\,(w) > 3/2\})$, which is the horodisc $H(1, 2/3)$ (having radius $2/5$ and center $3/5$); see the figure below). The form of Φ ensures that φ is $C^{3+\varepsilon}(1)$ for $0 \leqslant \varepsilon < 1/2$ (see pp. 50–51 of [5]) and $\varphi^{-1}(\{1\}) = \{1\}$. Since $\varphi'(1) = 1$ (and $\varphi''(1) = 2 + \mathrm{i}$ has positive real part) we see that φ is an essentially linear fractional selfmap of $\mathbb{U}$ of parabolic nonautomorphism type.
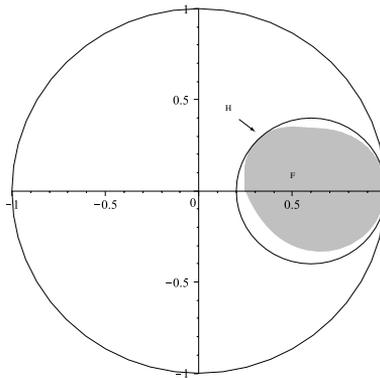

FIGURE 1. The image of the unit disk under the essentially linear fractional mapping φ of Example 1.6. Here H stands for $\partial H(1, 2/3)$ and F stands for $\varphi(\mathbb{U})$.

1.6. ADJOINTS OF COMPOSITION OPERATORS. Recall that for $h \in H^\infty(\mathbb{U})$, M_h denotes the operator of multiplication by h on $H^2(\mathbb{U})$. We let $B = (M_z)^*$ so that B is the backward shift operator on $H^2(\mathbb{U})$.

Let $\varphi(z) = (az + b)/(cz + d)$ be a nonconstant linear-fractional selfmap of $\mathbb{U}$. Cowen [13] derives the following formula for the adjoint of C_φ:

$$(1.6) \qquad\qquad C_\varphi^* = M_g C_\sigma M_h^*$$

where $g(z) = 1/(-\bar{b}z + \bar{d})$, $h(z) = cz + d$, and $\sigma(z) = (\bar{a}z - \bar{c})/(-\bar{b}z + \bar{d})$. Analogs of this adjoint formula have been developed for rational selfmappings of $\mathbb{U}$ (see, e.g., [14], [22], and [6]). We need the following variant of (1.6) (see equation (2) of [22] or equation (18) of [6]):

$$(1.7) \qquad\qquad C_\varphi^* = \Lambda_0 + M_{z\sigma'}\, C_\sigma\, B,$$

where Λ_0 is the rank-one operator defined by $(\Lambda_0 f) = f(0)/(1 - \overline{\varphi(0)}z)$.

1.7. SPECTRUM AND ESSENTIAL SPECTRUM. For an operator T on the Hilbert space H, let $\mathrm{Sp}(T)$ denote the spectrum of T and $\mathrm{Sp}_e(T)$ denote the essential spectrum. We will make use of the following well-known fact from Fredholm theory: any point λ belonging to both the spectrum of T and unbounded component of the complement of the essential spectrum must be an eigenvalue of T. The argument is simple. Because the Fredholm index is continuous on the complement of the essential spectrum and has value zero on the complement of the spectrum, the index is zero on the unbounded component of the complement of the essential spectrum. Thus, whenever λ belongs to both $\mathrm{Sp}(T)$ and the unbounded component of the complement of $\mathrm{Sp}_e(T)$, then the Fredholm index of $T - \lambda I$ is zero. Thus for such a λ, the operator $T - \lambda I$ has closed range and has kernel and co-kernel of the same dimension; it follows that the kernel must have positive dimension, for otherwise $C_\varphi - \lambda I$ would be invertible.

2. SOME GENERAL RESULTS ON COMPOSITION-OPERATOR SPECTRA

We begin with our most general result.

PROPOSITION 2.1. *Let φ be a selfmap of $\mathbb{U}$ having Denjoy–Wolff point ω in $\partial\mathbb{U}$. Suppose that λ is an eigenvalue of $C_\varphi : H^2 \to H^2$ having a corresponding eigenvector that is an outer function. Then $C_\varphi - \lambda I$ has dense range.*

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of C_φ with eigenvector g being an outer function. Note that φ is nonconstant (because its Denjoy–Wolff point belongs to $\partial\mathbb{U}$); hence $\lambda \neq 0$. We show that $\overline{\lambda}$ is not an eigenvalue of C_φ^* to complete the proof of the proposition.

Suppose, in order to obtain a contradiction, that $\overline{\lambda}$ is an eigenvalue of C_φ^* with corresponding eigenvector h. Observe that for any nonnegative integer n and any positive integer k, we have

$$\lambda^k \langle z^n(\omega - z)g(z), h\rangle = \langle z^n(\omega - z)g(z), (C_\varphi^*)^k h\rangle = \lambda^k \langle (\varphi^{[k]})^n(\omega - \varphi^{[k]})g, h\rangle,$$

so that (since $\lambda \neq 0$)

$$(2.1) \qquad \langle z^n(\omega - z)g(z), h \rangle = \langle (\varphi^{[k]})^n(\omega - \varphi^{[k]})g, h \rangle.$$

By Theorem 3.1 of [3], the iterate sequence $(\varphi^{[k]})$ converges to ω in the norm of $H^2(\mathbb{U})$ and it follows that some subsequence $(\varphi^{[j_k]})$ converges a.e. on $\partial\mathbb{U}$ to ω. Because $|(\varphi^{[j_k]})^n(\omega - \varphi^{[j_k]})g\overline{h}| \leqslant 2|g\overline{h}| \in L^1(\partial\mathbb{U})$ and $(\varphi^{[j_k]})$ converges to ω a.e. on $\partial\mathbb{U}$, Lebesgue's Dominated Convergence Theorem shows that

$$\lim_{k \to \infty} \langle (\varphi^{[j_k]})^n(\omega - \varphi^{[j_k]})g, h \rangle = 0.$$

Thus, by (2.1), we conclude that

$$\langle z^n(\omega - z)g(z), h \rangle = 0$$

for all $n \geqslant 0$. Since $z \mapsto (\omega - z)g(z)$ is outer (being a product of outer functions), the preceding equation shows that h is orthogonal to the polynomial multiples of an outer function, a set which is a dense set in $H^2(\mathbb{U})$ by Beurling's Theorem. Thus $h \equiv 0$, a contradiction that completes the proof of the proposition. ∎

We present three applications of the preceding proposition, with first two stated as corollaries.

COROLLARY 2.2. *Suppose that φ is a selfmap of $\mathbb{U}$ having Denjoy–Wolff point ω in $\partial\mathbb{U}$; then $C_\varphi - I$ has dense range.*

Proof. The number $\lambda = 1$ is an eigenvalue for C_φ having the outer function $g(z) = 1$ ($\forall z \in \mathbb{U}$) as an eigenvector. ∎

If φ has Denjoy–Wolff point $\omega \in \mathbb{U}$, then the range of $C_\varphi - I$ is not dense because K_ω belongs to its orthogonal complement.

COROLLARY 2.3. *Suppose that φ is a selfmapping of $\mathbb{U}$ of hyperbolic type with Denjoy–Wolff point ω. Then $(C_\varphi - \lambda I)$ has dense range whenever λ belongs to the annulus $A := \{z : \varphi'(\omega)^{1/2} < |z| < \varphi'(\omega)^{-1/2}\}$.*

Proof. Let b be such that $-1/2 < \mathrm{Re}\,(b) < 1/2$. Then it is easy to see that $f_b(z) := ((1 + z)/(1 - z))^b$ belongs to $H^2(\mathbb{U})$. Moreover, f_b is outer (because, e.g., both f_b and $1/f_b$ are in $H^2(\mathbb{U})$). Cowen ([12], Theorem 4.5) shows that there is a selfmap σ of $\mathbb{U}$ such that $f_b \circ \sigma$ is an eigenvector for C_φ with corresponding eigenvalue $\varphi'(\omega)^b$. Because b is an arbitrary complex number with $\mathrm{Re}\,(b) \in (-1/2, 1/2)$, it follows that every point in the annulus A is an eigenvalue of C_φ with corresponding eigenfunction having the form $f_b \circ \sigma$ for an appropriate b. Since every composition operator preserves the collection of outer functions (see, e.g., Section 2.7 of [8]), $f_b \circ \sigma$ is outer and the corollary follows. ∎

Our third application of Proposition 2.1 concerns a family of composition operators introduced by Cowen ([12], Section 6). Let $T_\theta(z) = ((1+z)/(1-z))^{2\theta/\pi}$, so that for $0 < \theta \leqslant \pi/2$, T_θ maps $\mathbb{U}$ univalently onto the sector $\{z : |\arg(z)| < \theta\}$

in the right halfplane Π. In particular, $T_{\pi/2} = T$ maps $\mathbb{U}$ onto Π. For $a \neq 0$ such that $|\arg(a)| < \theta \leqslant \pi/2$, define the selfmap $\varphi_{a,\theta}$ of $\mathbb{U}$ by

$$\varphi_{a,\theta}(z) = (T_\theta)^{-1} \circ (T_\theta + a).$$

PROPOSITION 2.4. *The compression spectrum of $C_{\varphi_{a,\theta}}$ is empty, i.e., $(C_{\varphi_{a,\theta}} - \lambda I)$ has dense range for every λ.*

Proof. Let $\theta \in (0, \pi/2]$ and let $a \neq 0$ satisfy $|\arg(a)| < \theta$. Note that $\varphi_{a,\theta}(\mathbb{U})$ is a simply connected domain whose boundary is a Jordan curve. Thus, by Walsh's Theorem (see, e.g., p. 11 of [5]), the set of polynomials in $\varphi_{a,\theta}$ is dense $H^2(\mathbb{U})$; equivalently, $C_{\varphi_{a,\theta}}$ has dense range.

Cowen proves ([12], Corollary 6.2) that

$$\text{Sp}(C_{\varphi_{a,\theta}}) = \{e^{-\beta a} : |\arg \beta| \leqslant \pi/2 - \theta\} \cup \{0\}.$$

In fact, he shows that every nonzero spectral point is an eigenvalue of $C_{\varphi_{a,\theta}}$. We now examine the eigenfunctions that Cowen exhibits. Let $\beta \neq 0$ belong to the sector $\{z : |\arg(z)| \leqslant \pi/2 - \theta\}$ so that $e^{-\beta a}$ belongs to the spectrum of $C_{\varphi_{a,\theta}}$. Note that $\text{Re}\,(\beta T_\theta(z)) > 0$ for every $z \in \mathbb{U}$ so that $F := e^{-\beta T_\theta} \in H^\infty(\mathbb{U})$. Clearly F is an eigenfunction for $C_{\varphi_{a,\theta}}$ with corresponding eigenvalue $e^{-\beta a}$. We complete the argument by considering two cases.

Case 1. $\theta < \pi/2$. We need to show $C_{\varphi_{a,\theta}} - e^{-\beta a}I$ has dense range. This follows from Proposition 2.1, because the eigenfunction $F = e^{-\beta T_\theta}$ for $e^{-\beta a}$ is outer by the following simple computation:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|F(e^{it})|\,dt = \text{Re}\Big(\frac{-\beta}{2\pi}\int_{-\pi}^{\pi}T_\theta(e^{it})\,dt\Big) = \text{Re}(-\beta\,T_\theta(0)) = \text{Re}(-\beta) = \log|F(0)|,$$

where the second equality follows, e.g., from Theorem 3.6 of [19], because $T(z) = ((1+z)/(1-z))^{2\theta/\pi}$ is an $H^1(\mathbb{U})$ function (since $2\theta/\pi < 1$).

Case 2. $\theta = \pi/2$. In this case, $\beta > 0$ and the eigenfunction $F(z) = e^{-\beta T_\theta(z)} = e^{-\beta(1+z)/(1-z)}$ is a singular inner function; in fact, the β-power of the unit singular function $S(z) = e^{-(1+z)/(1-z)}$. Let t be any nonnegative real number. We have $(C_{\varphi_{a,\theta}} - e^{-\beta a}I)S^t = (e^{-ta} - e^{-\beta a})S^t$. It follows that the closure of the range of $(C_{\varphi_{a,\theta}} - e^{-\beta a}I)$ contains all nonnegative powers of the unit singular function. Since the linear span of the collection of all nonnegative powers of S is dense in $H^2(\mathbb{U})$ (see, e.g, Lemma 4.2 of [20]), $(C_{\varphi_{a,\theta}} - e^{-\beta a}I)$ has dense range, as desired. ∎

REMARK 2.5. The issue of when $(C_\varphi - \lambda I)$ has dense range relates to question (Q1) raised in the Introduction as follows. Suppose that φ is a selfmap of $\mathbb{U}$ of hyperbolic or parabolic automorphism type. Then by Corollary 4.4 and the remarks following Theorem 4.10 of [12], every eigenvalue of C_φ has infinite multiplicity. Thus if there is a number $\lambda \in \text{Sp}(C_\varphi) \setminus \text{Sp}_e(C_\varphi)$, then $C_\varphi - \lambda I$ must be injective with closed range and it follows that the range of $(C_\varphi - \lambda I)$ cannot be

dense. Thus for selfmaps φ of hyperbolic or parabolic automorphism type, (Q1) has an affirmative answer if one can show that for every nonzero λ, the operator $C_\varphi - \lambda I$ has dense range.

The remainder of the results in this section serve as lemmas in the next section, permitting us to characterize the spectrum and essential spectrum of essentially linear fractional composition operators (fixing a point on $\partial \mathbb{U}$).

For $\eta \in \partial \mathbb{U}$ and $\beta > 0$, recall that $H(\eta, \beta) = \{z : |1 - z\overline{\eta}|^2 < \beta(1 - |z|^2)\}$ is the open horodisc (of radius $\beta/(1+\beta)$) internally tangent to the unit circle at η.

PROPOSITION 2.6. *Suppose that φ is a selfmap of $\mathbb{U}$ of hyperbolic type that is horocyclic at its Denjoy–Wolff point $\omega \in \partial \mathbb{U}$; then $\mathrm{Sp}_e(C_\varphi) = \mathrm{Sp}(C_\varphi)$.*

Proof. Without loss of generality, we take φ's Denjoy–Wolff point to be 1. Let $\beta > 0$ be such that $\varphi(\mathbb{U}) \subseteq H(1, \beta)$. Then by Julia's Theorem ([27], p. 63), for every $z \in \mathbb{U}$ and positive integer n,

$$(2.2) \qquad |1 - \varphi^{[n+1]}(z)|^2 < \varphi'(1)^n \beta(1 - |\varphi^{[n+1]}(z)|^2) \leqslant \varphi'(1)^n \beta.$$

Thus $\|1 - \varphi^{[n+1]}\|_\infty \leqslant \varphi'(1)^n \beta$.

Suppose that there is a number $\lambda \in \mathrm{Sp}(C_\varphi)$ but not in $\mathrm{Sp}_e(C_\varphi)$. Since the only Fredholm composition operators are the invertible ones induced by automorphisms of $\mathbb{U}$ ([9]), we know $\lambda \neq 0$. Since φ is of hyperbolic type, every eigenvalue of C_φ has infinite multiplicity ([12], Corollary 4.4). Thus λ cannot be an eigenvalue of C_φ; i.e., $C_\varphi - \lambda I$ is injective. We arrive at a contradiction by showing that $\overline{\lambda}$ also cannot be an eigenvalue of C_φ^*, which shows that $C_\varphi - \lambda I$ has dense range and hence is surjective since its range must be closed.

Suppose, in order to obtain a contradiction, that $\overline{\lambda}$ is an eigenvalue of C_φ^* with corresponding eigenvector g. Because $\lambda \neq 0$ and $\varphi'(1) < 1$ (recall φ is of hyperbolic type), we may choose the positive integer k such that $\varphi'(1)^k < |\lambda|$. Because $z \mapsto (1-z)^k$ is an outer function and $g \in H^2(\mathbb{U})$ is nonzero function, there is a nonnegative integer m such that $\langle z^m(1-z)^k, g\rangle \neq 0$. For each nonnegative integer n, we have

$$|\lambda|^{n+1}|\langle z^m(1-z)^k, g\rangle| = |\langle z^m(1-z)^k, (C_\varphi^*)^{n+1}g\rangle| = |\langle (\varphi^{[n+1]})^m(1 - \varphi^{[n+1]})^k, g\rangle|$$
$$\leqslant \|(\varphi^{[n+1]})^m(1 - \varphi^{[n+1]})^k\|_\infty \|g\|_{H^2(\mathbb{U})} \leqslant \varphi'(1)^{nk}\beta^k\|g\|_{H^2(\mathbb{U})}.$$

Hence,

$$|\lambda|\left|\frac{\lambda}{\varphi'(1)^k}\right|^n \leqslant \frac{\beta^k\|g\|_{H^2(\mathbb{U})}}{|\langle z^m(1-z)^k, g\rangle|}$$

for every n, a contradiction since $|\lambda/\varphi'(1)^k| > 1$. ∎

The discussion of Section 1.5 shows that if $\varphi \in \mathrm{LF}_e(\omega; \omega)$ is of parabolic type, then φ satisfies the hypotheses of the following proposition.

PROPOSITION 2.7. *Suppose that* $\varphi \in C^2(\omega)$ *satisfies* $\varphi'(\omega) = 1$ *and that* $\mathrm{Re}\,(\omega\varphi''(\omega)) > 0$. *If* $\lambda \neq 1$ *is an eigenvalue of* C_φ, *then* λ *is not an isolated point of* $\mathrm{Sp}(C_\varphi)$.

Proof. Suppose that $\lambda \neq 1$ is an eigenvalue of C_φ so that $f \circ \varphi = \lambda f$ for some $f \in H^2$, which is not the zero function. Note that since φ is nonconstant, we must have $\lambda \neq 0$. Suppose that $f(z_0) = 0$ for some $z_0 \in \mathbb{U}$. Then because $f(\varphi^{[n]}(z_0)) = \lambda^n f(z_0) = 0$, we see that f vanishes at every point in the orbit $\{\varphi^{[n]}(z_0) : n \geqslant 0\}$. However, by Lemma 4.5 of [5], $(\varphi^{[n]}(z_0))$ is not a Blaschke sequence. Thus we have an $H^2(\mathbb{U})$ function f, which is not the zero function, vanishing on a non-Blaschke sequence, which is a contradiction. It follows that f has no zeros in $\mathbb{U}$. Hence there is an analytic function g on $\mathbb{U}$ such that $f = \mathrm{e}^g$ and $f^{1/x} := \mathrm{e}^{g/x}$ is an $H^2(\mathbb{U})$ function for each $x \geqslant 1$. Let $\log \lambda$ represent some fixed choice of the logarithm of λ. Because $\lambda \mathrm{e}^g = f \circ \varphi = \mathrm{e}^{g \circ \varphi}$, there is an integer k such that $g \circ \varphi = \log \lambda + g + 2\pi i k$. Then $f^{1/x} \circ \varphi = \mathrm{e}^{1/x(\log \lambda + 2k\pi i)} f^{1/x}$ so that for each $x \geqslant 1$, $\lambda^{1/x} := \mathrm{e}^{1/x(\log \lambda + 2k i \pi)}$ is an eigenvalue of C_φ. Thus λ belongs to the path $\lambda^{1/x}$, $x \geqslant 1$ of eigenvalues, so that λ is not an isolated point of $\mathrm{Sp}(C_\varphi)$.   ∎

The proof of the following lemma relies on some ideas from Section 3 of [1].

PROPOSITION 2.8. *Suppose that* $\varphi \in C^2(\omega)$ *satisfies* $\varphi'(\omega) = 1$ *and that* $\mathrm{Re}\,(\omega\varphi''(\omega)) > 0$. *Then the point spectrum of* C_φ *has no interior.*

Proof. Let k be an arbitrary positive integer and let $z_0 \in \mathbb{U}$ also be arbitrary. By Lemma 4.5 of [5], applied to $\varphi^{[k]}$ (which also satisfies $(\varphi^{[k]})'(\omega) = 1$ and $\mathrm{Re}\,(\omega(\varphi^{[k]})''(\omega) > 0)$, the sequence $(\varphi^{[kn]}(z_0))$ is not Blaschke. It follows that $(C_\varphi^*)^k = C_{\varphi^{[k]}}^*$ is a cyclic operator on H^2. In fact, K_{z_0} is a cyclic vector: if for some $f \in H^2$, we have $\langle f, (C_{\varphi^{[k]}}^*)^n K_{z_0}\rangle = 0$ for all nonnegative integers n, then since $\langle f, (C_{\varphi^{[k]}}^*)^n K_{z_0}\rangle = \langle f, K_{\varphi^{[kn]}(z_0)}\rangle = f(\varphi^{[kn]}(z_0))$, we see f vanishes on a non-Blaschke sequence so that $f \equiv 0$.

We have shown that every positive integral power of C_φ^* is cyclic and it follows that no power of C_φ can have an eigenvalue of multiplicity greater than 1. In particular this means that if λ is an eigenvalue of C_φ then $\zeta\lambda$ cannot be an eigenvalue of C_φ for ζ any root of unity different from 1. Thus, in particular, the point spectrum of C_φ has no interior.   ∎

3. SPECTRA OF ESSENTIALLY LINEAR FRACTIONAL COMPOSITION OPERATORS

As we explained in Section 1.5, when characterizing spectra for composition operators induced by essentially linear fractional selfmaps φ of $\mathbb{U}$, the situation of interest is that where φ fixes a point on the unit circle. Without loss of generality, we assume $\varphi(1) = 1$. We begin with the hyperbolic case.

THEOREM 3.1. *Suppose that φ is an essentially linear fractional selfmap of $\mathbb{U}$ of hyperbolic-type with Denjoy–Wolff point 1. Then*

$$\mathrm{Sp}(C_\varphi) = \mathrm{Sp}_e(C_\varphi) = \{z : |z| \leqslant \varphi'(1)^{-1/2}\}.$$

Proof. Let $\varphi'(1) = p$ so that $p < 1$ and let $a = \varphi''(1)$. Let Ψ be the selfmap of the right halfplane Π given by

$$\Psi(w) = \frac{1}{p}w + \left(\frac{1}{p} - 1 + \frac{a}{p^2}\right).$$

By the discussion of Section 1.5, we know that C_φ is compactly equivalent to C_ψ, where $\psi = T^{-1} \circ \Psi \circ T$ and $T(z) = (1+z)/(1-z)$. Because φ is horocyclic at 1, Proposition 1.2 ensures that the constant summand on the right in the equation defining Ψ is positive, and thus Ψ is not an automorphism. Transferring the properties of Ψ to its unit-disk incarnation ψ, we see that ψ is a linear-fractional non-automorphism of $\mathbb{U}$ with Denjoy–Wolff point 1 and $\psi'(1) = p$, so that ψ is of hyperbolic type. By, e.g., Corollary 4.8 of [12] (which characterizes the spectrum of any composition operator whose symbol is of hyperbolic type and analytic on the closed disc) the spectrum of C_ψ is the disc $D := \{z : |z| \leqslant \psi'(\omega)^{-1/2}\} = \{z : |z| \leqslant \varphi'(\omega)^{-1/2}\}$. Actually every point in the interior of $D \setminus \{0\}$ is an infinite multiplicity eigenvalue of C_ψ, so that $D = \mathrm{Sp}_e(C_\psi)$. Alternatively, $D = \mathrm{Sp}_e(C_\psi)$ because ψ is horocyclic at 1 (Proposition 2.6 above).

Because C_φ and C_ψ are equivalent modulo the compacts, we have $\mathrm{Sp}_e(C_\varphi) = D$; but since φ is horocyclic at 1, Proposition 2.6 yields $\mathrm{Sp}(C_\varphi) = \mathrm{Sp}_e(C_\varphi)$, which completes the argument. ∎

Consider $\varphi(z) = 2/(\sqrt{13 - 4z} - 1)$, the hyperbolic-type essentially linear fractional selfmap of $\mathbb{U}$ discussed in Example 1.4. By Theorem 3.1, the disc $\{z : |z| \leqslant \sqrt{3}\}$ is both the spectrum and essential spectrum of C_φ.

We now turn to the case in which the essentially linear fractional selfmap has interior fixed point (as well as boundary fixed point 1).

THEOREM 3.2. *Suppose that φ is an essentially linear fractional selfmap of $\mathbb{U}$ fixing 1. Suppose that $\varphi'(1) > 1$ so that the Denjoy–Wolff point ω of φ lies in $\mathbb{U}$. Let N be least positive integer for which $|\varphi'(\omega)|^N \leqslant \varphi'(1)^{-1/2}$. Then*

$$\mathrm{Sp}(C_\varphi) = \{z : |z| \leqslant \varphi'(1)^{-1/2}\} \cup \{\varphi'(\omega)^n : n = 0, \ldots, N-1\}$$

and

$$\mathrm{Sp}_e(C_\varphi) = \{z : |z| \leqslant \varphi'(1)^{-1/2}\}.$$

Proof. The spectrum of C_φ contains all powers of $\varphi'(\omega)$ ([12], Theorem 4.1). Moreover, work of Koenigs [25] shows that the only possible eigenvalues of C_φ are powers of $\varphi'(\omega)$.

Let $D = \{z : |z| \leqslant \varphi'(1)^{-1/2}\}$. We prove that the disc D equals $\mathrm{Sp}_e(C_\varphi)$, and the proposition follows from the discussion of the preceding paragraph and

that of Section 1.7 because any spectral point λ outside of D is in the unbounded component of the essential resolvent, making λ an eigenvalue.

Let ψ be the linear-fractional selfmapping of $\mathbb{U}$ given by $T^{-1} \circ \Psi \circ T$, where

$$\Psi(w) = \frac{w}{p} + \left(\frac{1}{p} - 1 + \frac{a}{p^2}\right),$$

$p = \varphi'(1) > 1$, and $a = \varphi''(1)$. From the discussion of Section 1.5, we know there is a constant $c > 0$ such that

(3.1)
$$\mathrm{Re}\left(\frac{1}{p} - 1 + \frac{a}{p^2}\right) \geqslant c,$$

and since $p > 1$, it follows that $\mathrm{Re}\,(a) > 0$ (just as in parabolic non-automorphism case). We prove that the essential spectrum of C_ψ is the disc D, which implies the same is true of C_φ since C_φ and C_ψ differ by a compact operator.

Applying either results of Kamowitz (see, e.g. p. 296 of [18]) or Theorem 7.31 of [18], we see that the essential spectral radius of C_ψ is $\psi'(1)^{-1/2} = \varphi'(1)^{-1/2}$. Thus we need only show that every point in D is in the essential spectrum of C_ψ.

Applying the modified version (1.7) of Cowen's adjoint formula for linear fractional composition operators, we see

$$C_\psi^* = \Lambda_0 + M_{z\sigma'(z)} C_\sigma B,$$

where B is the backward shift and

$$\sigma(z) = \frac{(-2p^2 + \bar{a})z - \bar{a}}{(2p - 2p^2 + \bar{a})z - 2p - \bar{a}} \quad \text{while} \quad z\sigma'(z) = \frac{4p^3 z}{((2p - 2p^2 + \bar{a})z - 2p - \bar{a})^2}.$$

Thus, since Λ_0 is a rank-one operator, we see that C_ψ^* is equivalent to the operator

$$M_{z\sigma'(z)} C_\sigma B$$

modulo the compacts.

By Lemma 3.3 of [2] (or Corollary 2.2 of [24]), the weighted composition operator $M_{\sigma'(z)} C_\sigma$ is equivalent to $\sigma'(1) C_\sigma = (1/p) C_\sigma$ modulo the compacts, making C_ψ^* equivalent to $M_{z/p} C_\sigma B$ modulo the compacts. We show that every point in the interior of the punctured disc $D \setminus \{0\}$ is an infinite multiplicity eigenvalue of $M_{z/p} C_\sigma B$. Thus each point of D is in the essential spectrum of $M_{z/p} C_\sigma B$, hence of C_ψ^*, hence of C_ψ, as desired.

The selfmapping σ of $\mathbb{U}$ is of hyperbolic type with Denjoy–Wolff point 1 and Denjoy–Wolff derivative equal to $\sigma'(1) = 1/p$. Translated to the right-half plane Π via T, the function σ takes the form $\Sigma(w) = pw + 1 - p + \bar{a}/p$ (i.e. $\Sigma = T \circ \sigma \circ T^{-1}$). Note $1 - p + \bar{a}/p = p(1/p - 1 + \bar{a}/p^2)$ has positive real part so that Σ is indeed a self-mapping of Π. It is easy to see that

$$\Gamma(w) = w + \frac{p - p^2 + \bar{a}}{p(p-1)}$$

is a self-mapping of Π for which

(3.2) $$\Gamma \circ \Sigma = p\Gamma.$$

Note well that $(p - p^2 + \bar{a})/(p(p-1))$ has strictly positive real part (the denominator is positive and $(p - p^2 + \bar{a}) = p^2(1/p - 1 + \bar{a}/p^2)$ has real part exceeding $p^2 c$ by (3.1)), which means that $\gamma(z) := (T^{-1} \circ \Gamma \circ T)(z)$ will take the unit disc $\mathbb{U}$ univalently onto a proper subdisc of $\mathbb{U}$ internally tangent to the unit disc at 1. In particular $\gamma(\mathbb{U})$ will be bounded away from -1. The intertwining relationship (3.2) translates to the disc as follows

$$\gamma \circ \sigma = \nu \circ \gamma$$

where $\nu = T^{-1} \circ (pT)$ is a hyperbolic automorphism of $\mathbb{U}$ with $\omega = 1$ and $\nu'(1) = 1/p$. Let ln be the principal branch of the logarithm function and recall that $f_b(z) = e^{b \ln((1+z)/(1-z))}$ belongs to $H^2(\mathbb{U})$ whenever $-1/2 < \mathrm{Re}\,(b) < 1/2$. Since $\gamma(\mathbb{U})$ is bounded away from -1, we conclude that $f_b \circ \gamma$ belongs to $H^2(\mathbb{U})$ whenever $-\infty < \mathrm{Re}\,(b) < 1/2$. Let $g(z) = zf_b(\gamma(z))$ so that $Bg = f_b \circ \gamma$. For every b satisfying $-\infty < \mathrm{Re}\,(b) < 1/2$, we have

$$M_{z/p}(C_\sigma Bg)(z) = M_{z/p}(f_b \circ \gamma \circ \sigma)(z) = \frac{z}{p}(f_b \circ \nu)(\gamma(z)) = \frac{z}{p}(T \circ \nu)^b(\gamma(z))$$

$$= \frac{z}{p}p^b f_b(\gamma(z)) = p^{b-1}g(z).$$

Fix b with $-\infty < \mathrm{Re}\,(b) < 1/2$ and let k be an integer. Replacing b with $b + 2\pi ki/\ln(p)$ in the preceding computation, we see that

$$z \mapsto ze^{(b + 2\pi ki/\ln(p))\ln((1+\gamma(z))/(1-\gamma(z)))}$$

is an eigenvector for $M_{z/p}C_\sigma B$ with corresponding eigenvalue p^{b-1}. It follows that every point in the punctured disc $\{z : |z| < p^{-1/2}\} \setminus \{0\}$ is an infinite-multiplicity eigenvalue of the operator $M_{z/p}C_\sigma B$. Hence the essential spectrum of $M_{z/p}C_\sigma B$ contains the closure of this punctured disc, which is D, as desired. ∎

By the the preceding theorem, $\mathrm{Sp}(C_\varphi) = \{z : |z| \leqslant 1/\sqrt{2}\} \cup \{1\}$ and $\mathrm{Sp}_e(C_\varphi) = \{z : |z| \leqslant 1/\sqrt{2}\}$ when $\varphi(z) = (2z^2 - 3z + 3)/(2z^2 - 7z + 7)$ is the essentially linear fractional selfmap of Example 1.5 ($\varphi(1/2) = 1/2$, $\varphi'(1/2) = 3/8$, and $\varphi'(1) = 2$).

As we have discussed, if φ is essentially linear fractional and of parabolic type with Denjoy–Wolff point 1, then $\mathrm{Re}\,(\varphi''(1)) > 0$. We now complete our characterization of spectra and essential spectra for essentially linear fractional composition operators.

THEOREM 3.3. *Suppose that φ is an essentially linear fractional selfmap of $\mathbb{U}$ fixing 1. Suppose that $\varphi'(1) = 1$ so that the Denjoy–Wolff point ω of φ is 1. Let $a = \varphi''(1)$ so that $\mathrm{Re}\,(a) > 0$. Then*

$$\mathrm{Sp}(C_\varphi) = \mathrm{Sp}_e(C_\varphi) = \{e^{-at} : t \geqslant 0\} \cup \{0\}.$$

Proof. Let ψ be the linear fractional selfmap of $\mathbb{U}$ given by $\psi = T^{-1} \circ \Psi \circ T$, where Ψ is the selfmap of Π given by

$$\Psi(w) = w + a.$$

By Corollary 6.2 of [12], the spectrum of C_ψ is precisely the spiral $S := \{e^{-at} : t \geqslant 0\} \cup \{0\}$. Because S has no interior and no isolated points, S is the essential spectrum of C_ψ as well. Because C_φ and C_ψ are equivalent modulo the compact operators, $\mathrm{Sp}_e(C_\varphi) = S$, as desired.

That $\mathrm{Sp}(C_\varphi) = \mathrm{Sp}_e(C_\varphi)$ follows quickly from Propositions 2.7 and 2.8. Let $\lambda \in \mathrm{Sp}(C_\varphi) \setminus \mathrm{Sp}_e(C_\varphi)$ be arbitrary. Note λ is neither 0 nor 1. Because the complement of S is connected, $C_\varphi - \lambda I$ must be Fredholm of index 0. Thus λ must belong to the point spectrum of C_φ and from Proposition 2.7, we see that λ is not an isolated point of the spectrum of C_φ. Since λ is in the essential resolvent and not isolated, it cannot be a boundary point of $\mathrm{Sp}(C_\varphi)$ (see, e.g, Theorem 6.8, p. 366 of [11]). This contradicts Proposition 2.8: since $\mathrm{Sp}(C_\varphi) \setminus \mathrm{Sp}_e(C_\varphi)$ consists of eigenvalues, every point in $\mathrm{Sp}(C_\varphi) \setminus \mathrm{Sp}_e(C_\varphi)$ must be a boundary point of $\mathrm{Sp}(C_\varphi)$. We conclude $\mathrm{Sp}(C_\varphi) \setminus \mathrm{Sp}_e(C_\varphi)$ is empty, as desired. ∎

Applying the preceding theorem, we see that the spiral $\{e^{-2t-it} : t \geqslant 0\} \cup \{0\}$ is the spectrum and essential spectrum of the parabolic-type selfmap φ of Example 1.6.

4. SPECTRA OF SOME WEIGHTED COMPOSITION OPERATORS

In this section, we consider the spectrum of weighted composition operators on $H^2(\mathbb{U})$ of the form $C_{g,\varphi}$, where φ is an analytic selfmap of $\mathbb{U}$ and $g \in H^\infty(\mathbb{U})$. Here, we have $C_{g,\varphi}f = gf \circ \varphi$ for $f \in H^2(\mathbb{U})$. Clearly, $C_{g,\varphi}$ is a bounded linear operator on $H^2(\mathbb{U})$ and $\|C_{g,\varphi} : H^2(\mathbb{U}) \to H^2(\mathbb{U})\| \leqslant \|g\|_\infty \|C_\varphi : H^2(\mathbb{U}) \to H^2(\mathbb{U})\|$.

We begin with a couple results that do not require φ to be essentially linear fractional.

LEMMA 4.1. *Suppose that φ is an arbitrary analytic selfmap of $\mathbb{U}$ with Denjoy–Wolff point ω. Suppose that $g \in H^\infty(\mathbb{U})$ extends to be continuous on $\mathbb{U} \cup \{\omega\}$ (if $\omega \in \partial\mathbb{U}$). If λ is an eigenvalue of $C_{g,\varphi}$, then $|\lambda| \leqslant |g(\omega)| r(C_\varphi)$, where $r(C_\varphi)$ is the spectral radius of φ. If $g(\omega) = 0$ and φ and g are nonconstant, then $C_{g,\varphi}$ has no eigenvalues.*

Proof. Suppose that λ is a *nonzero* eigenvalue for $C_{g,\varphi}$ with corresponding eigenvector f. For each positive integer n and $z \in \mathbb{U}$, we have

$$(4.1) \qquad \lambda^n f(z) = \prod_{j=0}^{n-1} g(\varphi^{[j]}(z)) f(\varphi^{[n]}(z)).$$

Observe that for any fixed $z \in \mathbb{U}$ and positive integer n, we have from (1.1):

(4.2) $$|f(\varphi^{[n]}(z))| \leqslant \frac{\|f \circ \varphi^{[n]}\|_{H^2(\mathbb{U})}}{\sqrt{1 - |z|^2}} \leqslant \|C_\varphi^n\| \frac{\|f\|_{H^2(\mathbb{U})}}{\sqrt{1 - |z|^2}}.$$

Choose $z \in \mathbb{U}$ such that $f(z) \neq 0$ and notice from (4.1) that $g(\varphi^{[j]}(z))$ must be nonzero for every non-negative integer j. Since $(\varphi^{[j]}(z))$ approaches ω as $j \to \infty$, we know $g(\varphi^{[j]}(z))$ approaches $g(\omega)$ as $j \to \infty$. Upon (i) taking n-th roots of the absolute value each side of (4.1), (ii) using the estimate (4.2), and (iii) letting $n \to \infty$, we obtain

(4.3) $$|\lambda| \leqslant |g(\omega)| \, r(C_\varphi),$$

as desired.

Now assume $g(\omega) = 0$ while g and φ are nonconstant. Then by (4.3), $\lambda = 0$ is the only possible eigenvalue for $C_{g,\varphi}$. However, g, being nonconstant, is not the zero function; since φ is also nonconstant, 0 cannot be an eigenvalue of $C_{g,\varphi}$. ∎

We can weaken the hypothesis on g by making further assumptions on φ.

LEMMA 4.2. *Suppose that*
  (i) *φ is an arbitrary selfmapping of $\mathbb{U}$ of hyperbolic type or*
 (ii) *$\varphi \in C^2(\omega)$ satisfies $\varphi'(\omega) = 1$ and $\mathrm{Re}\,(\omega \varphi''(\omega)) > 0$.*
  If λ is an eigenvalue of $C_{g,\varphi}$ and $g \in H^\infty(\mathbb{U})$ has finite radial (equivalently non-tangential) limit $g(\omega)$ at ω, then

$$|\lambda| \leqslant |g(\omega))| \, r(C_\varphi);$$

moreover, $C_{g,\varphi}$ will have no eigenvalues if $g(\omega) = 0$, and φ and g are nonconstant functions.

Proof. In cases (i) and (ii) for each $z \in \mathbb{U}$, the sequence $(\varphi^{[j]}(z))$ converges nontangentially to ω (for case (i) see Lemma 2.66 of [18]; for case (ii), see Lemma 4.5 of [5]). Thus $(g(\varphi^{[j]}(z))$ will converge to $g(\omega)$ as $j \to \infty$ and the argument of Lemma 4.1 may be applied to complete the proof. ∎

We now turn our attention to weighted composition operators $C_{g,\varphi}$, where φ is essentially linear fractional. Suppose that $\varphi \in \mathrm{LF}_e(\zeta; \eta)$ so that η and ζ are unimodular constants such that $\varphi \in C^3(\zeta)$, $\varphi(\zeta) = \eta$, $\varphi^{-1}(\{\eta\}) = \{\zeta\}$, and $\varphi(\mathbb{U})$ lies in a proper subdisc of $\mathbb{U}$ internally tangent to the unit circle at η. Just as in Section 1.5, we see that if $\eta \neq \zeta$, then $\|\varphi \circ \varphi\|_\infty < 1$ and $(C_\varphi)^2$ is thus compact. It follows that $(C_{g,\varphi})^2 = g \, g \circ \varphi \, (C_\varphi)^2$ is also compact and its spectrum is $\{g(\omega)\varphi'(\omega)^n : n = 0, 1, 2, \ldots\} \cup \{0\}$ by, e.g, Corollary 1 of [21] or p. 187 of [10]. Thus, just as in the preceding section, we focus on the situation where φ fixes a point on $\partial\mathbb{U}$ and we assume, without loss of generality, that this point is 1.

Let $\zeta \in \partial\mathbb{U}$. We say that $g \in H^\infty(\mathbb{U})$ is continuous at ζ provided that g's radial limit function is continuous at ζ (which implies g is continuous on $\mathbb{U} \cup \{\zeta\}$).

We require the following lemma, which is a quick consequence of Corollary 2.2 of [24].

LEMMA 4.3. *Suppose that $\varphi \in \mathrm{LF}_e(1;1)$ and that $g \in H^\infty(\mathbb{U})$ is continuous at 1. Then*

$$C_{g,\varphi} \equiv g(1)C_\varphi \quad \textit{modulo the compact operators.}$$

Proof. The essentially linear fractional map φ satisfies $\varphi(1) = 1$. Let $p = \varphi'(1)$, $a = \varphi''(1)$, and ψ be the selfmap of $\mathbb{U}$ defined by $\psi = T^{-1} \circ \Psi \circ T$, where Ψ is given by (1.5). We know that $C_\varphi - C_\psi$ is compact. Hence, for $g \in H^\infty(\mathbb{U})$, $M_g C_\varphi - M_g C_\psi$ is compact; that is

$$C_{g,\varphi} \equiv C_{g,\psi} \quad \textit{modulo the compact operators.}$$

Our linear-fractional selfmap ψ is not an automorphism and satisfies $\psi(1) = 1$. By Corollary 2.2 of [24], we have $C_{g-g(1),\psi}$ is compact and the lemma follows.   ∎

The next theorem generalizes Theorem 3 of [21].

THEOREM 4.4. *Suppose that φ and g are as in Lemma 4.3 and that $g(1) = 0$. Then $\mathrm{Sp}(C_{g,\varphi}) = \{0\}$.*

Proof. By Lemma 4.3, $C_{g,\varphi}$ is compact and thus $\mathrm{Sp}_e(C_{g,\varphi}) = \{0\}$. Hence any nonzero spectral point must be an eigenvalue of $C_{g,\varphi}$. Apply Lemma 4.1 to see that $C_{g,\varphi}$ has no nonzero eigenvalues and the theorem follows.   ∎

We now characterize the spectrum and essential spectrum of $C_{g,\varphi}$ when φ and g satisfy the hypotheses of Lemma 4.3 and either $\varphi'(1) < 1$ or $\varphi'(1) > 0$. Our final result will be a spectral characterization for $C_{g,\varphi}$ in the parabolic case $\varphi'(1) = 1$, but this result requires significant additional assumptions on φ.

THEOREM 4.5. *Suppose that φ and g are as in Lemma 4.3 and $\varphi'(1) < 1$. Then*

$$\mathrm{Sp}(C_{g,\varphi}) = \mathrm{Sp}_e(C_{g,\varphi}) = \{z : |z| \leqslant |g(1)|\varphi'(1)^{-1/2}\}.$$

Proof. By Lemma 4.3, $C_{g,\varphi}$ is equivalent to $g(1)C_\varphi$ modulo the compact operators. By Theorem 3.1, the essential spectrum of $g(1)C_\varphi$ is $D := \{z : |z| \leqslant |g(1)|\varphi'(1)^{-1/2}\}$, and thus $\mathrm{Sp}_e(C_{g,\varphi}) = D$ as well.

Suppose that $C_{g,\varphi}$ has a spectral point λ outside D, then λ must be an eigenvalue of $C_{g,\varphi}$. However, Lemma 4.1 (or Lemma 4.2) ensures that all eigenvalues of $C_{g,\varphi}$ must belong to D (because $r(C_\varphi) = \varphi'(1)^{-1/2}$ by Theorem 2.1 of [12]), and we conclude the spectrum of $C_{g,\varphi} = D$, as desired.   ∎

Let $\varphi(z) = 2/(\sqrt{13-4z}-1)$ and $g(z) = 1/\sqrt{3}+(1-z)^{3/2}$. Then, by Theorem 4.5, the closed unit disc is both the spectrum and essential spectrum of $C_{g,\varphi}$.

THEOREM 4.6. *Suppose that φ and g are as in Lemma 4.3 and $\varphi'(1) > 1$ so that the Denjoy–Wolff point ω of φ belongs to $\mathbb{U}$. Let N be the least nonnegative integer such*

that $|g(\omega)||\varphi'(\omega)|^N \leqslant |g(1)||\varphi'(1)|^{-1/2}$. Then

$$\mathrm{Sp}(C_{g,\varphi}) = \{z : |z| \leqslant |g(1)||\varphi'(1)|^{-1/2}\} \cup \{g(w)\varphi'(\omega)^n : n = 0, 1, \dots, N-1\}$$

and

$$\mathrm{Sp_e}(C_{g,\varphi}) = \{z : |z| \leqslant |g(1)||\varphi'(1)|^{-1/2}\}.$$

Proof. The proof is similar to that of Theorem 4.5. Just as before, $C_{g,\varphi}$ is equivalent to $g(1)C_\varphi$ modulo the compact operators. By Theorem 3.2, the essential spectrum of $g(1)C_\varphi$ is $D := \{z : |z| \leqslant |g(1)||\varphi'(1)|^{-1/2}\}$ and thus, $\mathrm{Sp_e}(C_{g,\varphi}) = D$ as well.

Any spectral point of $C_{g,\varphi}$ outside D must be an eigenvalue of $C_{g,\varphi}$. However, it's not difficult to see that every eigenvalue of $C_{g,\varphi}$ must have the form $g(\omega)\varphi'(\omega)^n$ for some nonnegative integer n (see, e.g, proof of Lemma 1 of [21]) and that all points of this form are in the spectrum of $C_{g,\varphi}$ ([21], Lemma 3). The theorem follows. ∎

For a concrete example illustrating the preceding theorem, set $g(z) = (3 - 2z)^3$ and $\varphi(z) = (2z^2 - 3z + 3)/(2z^2 - 7z + 7)$. Then φ is the essentially linear fractional map of Example 1.5, which satisfies $\varphi(1) = 1$, $\varphi'(1) = 2$, $\varphi(1/2) = 1/2$, and $\varphi'(1/2) = 3/8$. Here $g(1) = 1$ while $g(1/2) = 8$. Thus $|g(1/2)||\varphi'(1/2)|^n > |g(1)||\varphi'(1)|^{-1/2}$ for precisely $n = 0, 1$, and 2. Hence,

$$\mathrm{Sp}(C_{g,\varphi}) = \{z : |z| < 1/\sqrt{2}\} \cup \{8, 3, 9/8\}$$

and $\mathrm{Sp_e}(C_{g,\varphi}) = \{z : |z| < 1/\sqrt{2}\}$.

Only with significantly stronger hypotheses on φ can we obtain a spectral characterization for $C_{g,\varphi}$ when φ is of parabolic type. We suppose that φ satisfies the hypotheses of Lemma 4.3 and in addition that φ is continuous on the closed disc $\mathbb{U}^-$, one-to-one on $\mathbb{U}^-$, and is $C^{3+\varepsilon}(1)$ for some $\varepsilon > 0$. Moreover, we assume that

$$\mathrm{Re}\left(\overline{\varphi''(1)}(\mathcal{S}\varphi)(1)\right) \geqslant 0,$$

where

$$\mathcal{S}\varphi(1) = \left(\frac{\varphi''}{\varphi'}\right)'(1) - \frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2(1)$$

is the Schwarzian derivative of φ at 1. With all these hypotheses on φ, the "Parabolic Models" Theorem 4.12 of [5] may be applied to φ, or, more precisely to the right halfplane incarnation Φ of φ (so that $\Phi = T \circ \varphi \circ T^{-1}$). The Parabolic Models Theorem from [5] tells us that there is an analytic mapping ν defined on the right halfplane Π such that $\nu \circ \Phi = \nu + \varphi''(1)$. Moreover by part (c) of Theorem 4.12 of [5] (see also the discussion of Schwarzian derivatives on pages 51 and 52 of [5]), we can assume that ν is a selfmap of the right halfplane Π. Thus $\nu \circ T \circ \varphi = \nu \circ T + \varphi''(1)$ and since ν is a selfmap of Π, for each $t \geqslant 0$, we see that $h(z) = \mathrm{e}^{-t\nu \circ T}$ belongs to $H^\infty(\mathbb{U})$ and

(4.4) $$C_\varphi h = \mathrm{e}^{-t\varphi''(1)}h.$$

THEOREM 4.7. *Suppose that φ and g are as in Lemma 4.3, that φ is of parabolic type, and that φ is continuous on the closed disc $\mathbb{U}^-$, one-to-one on $\mathbb{U}^-$, and is $C^{3+\varepsilon}(1)$ for some $\varepsilon > 0$. Moreover, assume that*

$$\operatorname{Re}\left(\overline{\varphi''(1)}(\mathcal{S}\varphi)(1)\right) \geqslant 0,$$

where $(S\varphi)(1)$ is the Schwarzian derivative of φ at 1. Then

$$\operatorname{Sp}(C_{g,\varphi}) = \operatorname{Sp}_e(C_{g,\varphi}) = \{g(1)e^{-bt} : t \geqslant 0\} \cup \{0\},$$

where $b = \varphi''(1)$.

Proof. If $g(1) = 0$, the theorem follows from Theorem 4.4 above. Thus, we assume $g(1) \neq 0$.

Just as in the proofs of Theorems 4.5 and 4.6 above, $C_{g,\varphi}$ is equivalent to $g(1)C_\varphi$ modulo the compact operators. By Theorem 3.3, the essential spectrum of $g(1)C_\varphi$ is $S := \{g(1)e^{-bt} : t \geqslant 0\} \cup \{0\}$; thus, $\operatorname{Sp}_e(C_{g,\varphi}) = S$ as well.

The argument that there are no points in the spectrum of $C_{g,\varphi}$ outside of S is similar to that of Theorem 3.3.

As we have discussed (see Section 1.5), since φ is essentially linear fractional and of parabolic type, $\operatorname{Re}(\varphi''(1)) > 0$. Hence, by Lemma 4.5 of [5], applied to $\varphi^{[k]}$ the iterate sequence $(\varphi^{[kn]}(z))$ fails to be Blaschke for every $z \in \mathbb{U}$. We claim that every power of $C_{g,\varphi}^*$ is cyclic on $H^2(\mathbb{U})$. Since g is continuous on $\mathbb{U} \cup \{1\}$ with $g(1) \neq 0$ and since $(\varphi^{[j]}(0))$ approaches 1, there is a positive integer N such that for all $j \geqslant N$, $g(\varphi^{[j]}(0))$ is nonzero. Set $z_0 = \varphi^{[N]}(0)$. Then K_{z_0} will be cyclic for the k-th power of $C_{g,\varphi}^*$: $((C_{g,\varphi}^*)^{kn}K_{z_0} = (C_\varphi^*(M_g)^*)^{kn} = \prod\limits_{j=0}^{kn-1} \overline{g(\varphi^{[j]}(z_0))}K_{\varphi^{[kn]}(z_0)}$ so that if $f \in H^2(\mathbb{U})$ is orthogonal to all powers of $(C_{g,\varphi})^*$, it must vanish on the non-Blaschke sequence $(\varphi^{[kn]}(z_0))$ and hence $f \equiv 0$. Thus, we have established our claim: every power of $C_{g,\varphi}$ is cyclic.

If follows, just as in the proof of Proposition 2.8, that every spectral point of $C_{g,\varphi}$ outside of the essential spectrum must be boundary point of the spectrum and thus must be an isolated point of the spectrum. We show no spectral point of $C_{g,\varphi}$ outside of S can be isolated, completing the proof of the theorem.

Suppose that $\lambda \in \operatorname{Sp}(C_{g,\varphi}) \setminus S$ so that λ is an eigenvalue of $C_{g,\sigma}$. Note $\lambda \neq 0$. Let $f \in H^2(\mathbb{U})$ be an eigenfunction associated with λ. Since $\operatorname{Re}\left(\overline{\varphi''(1)}(\mathcal{S}\varphi)(1)\right) \geqslant 0$, there is an analytic self map ν of the right halfplane such that $\nu \circ T \circ \varphi = \nu \circ T + \varphi''(1)$. Let $t \geqslant 0$, let $h(z) = e^{-t\nu \circ T}$. Since $h \in H^\infty(\mathbb{U})$, $hf \in H^2(\mathbb{U})$; moreover, applying (4.4) and $gf \circ \varphi = \lambda f$, we see for each $t \geqslant 0$,

$$C_{g,\varphi}hf = (h \circ \varphi)(gf \circ \varphi) = e^{-t\varphi''(1)}\lambda hf.$$

Thus for each $t \geqslant 0$, we see $e^{-t\varphi''(1)}\lambda$ is an eigenvalue of $C_{g,\varphi}$ so λ is not an isolated spectral point, which completes the proof of the theorem. ∎

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For a concrete example illustrating the preceding theorem consider the essentially linear fractional mapping φ of Example 1.6. Recall $\varphi = T^{-1} \circ \Phi \circ T$ where $\Phi(w) = w + 2 + i - 1/(4(w+1)) - 1/(4(w+1)^{3/2})$. It is easy to see that Φ' has positive real part on a neighborhood of the closed right halfplane and thus φ is one-to-one on the closed disc. Also $\varphi \in C^{3+\varepsilon}(1)$ for, say, $\varepsilon = 1/4$. Finally, using $\varphi''(1) = 2 + i$ and $\varphi'''(1) = 39/8 + 6i$, one calculates the Schwarzian derivative of φ at 1 to be $3/8$. Thus $\mathrm{Re}\,\overline{(\varphi''(1)}(\mathcal{S}\varphi)(1)) = 3/4 \geqslant 0$. Letting $g(z) = i\sqrt{5 - z}$, e.g, we see $\mathrm{Sp}(C_{g,\varphi}) = \mathrm{Sp}_e(C_{g,\varphi}) = \{2ie^{-2t-it} : t \geqslant 0\} \cup \{0\}$.

5. OPEN QUESTIONS

Aside from questions (Q1) and (Q2) raised in the Introduction, the following questions seem interesting:

(i) What is the compression spectrum of $C_\varphi : H^2(\mathbb{U}) \to H^2(\mathbb{U})$?

(ii) If φ is of hyperbolic type or parabolic automorphism type, is the compression spectrum of C_φ at most $\{0\}$?

(iii) To what extent can the spectral characterizations of Theorems 3.1, 3.2, and 3.3 for essentially linear fractional composition operators be generalized? For example, can analogs be shown to hold under only the assumption that φ is horocyclic at a boundary fixed point?

(iv) For a weighted composition operator $C_{g,\varphi}$, where g and φ satisfy the hypotheses of either Theorem 4.5 or Theorem 4.7, the spectrum and essential spectrum of $C_{g,\varphi}$ are the same. Can $\mathrm{Sp}(C_{g,\varphi}) = \mathrm{Sp}_e(C_{g,\varphi})$ be established for a wider variety of combinations of g and φ?

As we mentioned in Section 2, if the answer to the second question above is "yes", then (Q1) from the Introduction has an affirmative answer when φ is of hyperbolic or parabolic-automorphism type. We remark that for maps φ of parabolic automorphism type (but not automorphisms), Cowen ([12], Section 7) conjectures that $\mathrm{Sp}(C_\varphi) = \mathrm{Sp}_e(C_\varphi) = \mathbb{U}^-$.

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