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THE TRUNCATED TRACIAL MOMENT PROBLEM

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ABSTRACT. We present *tracial* analogs of the classical results of Curto and Fialkow on moment matrices. A sequence of real numbers indexed by words in noncommuting variables with values invariant under cyclic permutations of the indexes, is called a *tracial sequence*. We prove that such a sequence can be represented with tracial moments of matrices if its corresponding moment matrix is positive semidefinite and of finite rank. A *truncated* tracial sequence allows for such a representation if and only if one of its extensions admits a flat extension. Finally, we apply this theory via duality to investigate trace-positive polynomials in noncommuting variables.

KEYWORDS: *(Truncated) moment problem, noncommutative polynomial, sum of hermitian squares, moment matrix, free positivity.*

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1. INTRODUCTION

The moment problem is a classical question in analysis, well studied because of its importance and variety of applications. A simple example is the (univariate) Hamburger moment problem: when does a given sequence of real numbers represent the successive moments $\int x^n \, d\mu(x)$ of a positive Borel measure μ on $\mathbb{R}$? Equivalently, which linear functionals L on univariate real polynomials are integration with respect to some μ? By Haviland's theorem [13] this is the case if and only if L is nonnegative on all polynomials nonnegative on $\mathbb{R}$. Thus Haviland's theorem relates the moment problem to positive polynomials. It holds in several variables and also if we are interested in restricting the support of μ. For details we refer the reader to one of the many beautiful expositions of this classical branch of functional analysis, e.g. [1], [18], [31].

Since Schmüdgen's celebrated solution of the moment problem on compact basic closed semialgebraic sets [30], the moment problem has played a prominent role in real algebra, exploiting this duality between positive polynomials and the moment problem, cf. [19], [26], [28], [29]. The survey of Laurent [21] gives a nice presentation of up-to-date results and applications; see also [23], [27] for more on positive polynomials.

Our main motivation are trace-positive polynomials in noncommuting variables. A polynomial is called *trace-positive* if all its matrix evaluations (of *all* sizes) have nonnegative trace. Trace-positive polynomials have been employed to investigate problems on operator algebras (Connes' embedding conjecture [8], [15]) and mathematical physics (the Bessis–Moussa–Villani conjecture [4], [16]), so a good understanding of this set is desired. By duality this leads us to consider the tracial moment problem introduced below. We mention that the free noncommutative moment problem has been studied and solved by McCullough [24] and Helton [14]. Hadwin [12] considered moments involving traces on von Neumann algebras.

This paper is organized as follows. The short Section 2 fixes notation and terminology involving noncommuting variables used in the sequel. Section 3 introduces tracial moment sequences, tracial moment matrices, the tracial moment problem, and their truncated counterparts. Our main results in this section relate the truncated tracial moment problem to flat extensions of tracial moment matrices and resemble the results of Curto and Fialkow [9], [10] on the (classical) truncated moment problem. For example, we prove that a tracial sequence can be represented with tracial moments of matrices if its corresponding tracial moment matrix is positive semidefinite and of finite rank (Theorem 3.14). A truncated tracial sequence allows for such a representation if and only if one if its extensions admits a flat extension (Corollary 3.21). Finally, in Section 4 we explore the duality between the tracial moment problem and trace-positivity of polynomials. Throughout the paper several examples are given to illustrate the theory.

2. BASIC NOTIONS

Let $\mathbb{R}\langle \underline{X}\rangle$ denote the unital associative $\mathbb{R}$-algebra freely generated by $\underline{X} = (X_1, \dots, X_n)$. The elements of $\mathbb{R}\langle \underline{X}\rangle$ are polynomials in the noncommuting variables $X_1, \dots, X_n$ with coefficients in $\mathbb{R}$. An element w of the monoid $\langle \underline{X}\rangle$, freely generated by $\underline{X}$, is called a *word*. An element of the form aw, where $0 \neq a \in \mathbb{R}$ and $w \in \langle \underline{X}\rangle$ is called a *monomial* and a its *coefficient*. We endow $\mathbb{R}\langle \underline{X}\rangle$ with the *involution* $p \mapsto p^*$ fixing $\mathbb{R} \cup \{\underline{X}\}$ pointwise. Hence for each word $w \in \langle \underline{X}\rangle$, w^* is its reverse. As an example, we have $(X_1X_2^2 - X_2X_1)^* = X_2^2X_1 - X_1X_2$.

For $f \in \mathbb{R}\langle \underline{X}\rangle$ we will substitute symmetric matrices $\underline{A} = (A_1, \dots A_n)$ of the same size for the variables $\underline{X}$ and obtain a matrix $f(\underline{A})$. Since $f(\underline{A})$ is not well-defined if the A_i do not have the same size, we will assume this condition implicitly without further mention in the sequel.

Let $\operatorname{Sym}\mathbb{R}\langle \underline{X}\rangle$ denote the set of *symmetric elements* in $\mathbb{R}\langle \underline{X}\rangle$, i.e.,

$$\operatorname{Sym}\mathbb{R}\langle \underline{X}\rangle = \{f \in \mathbb{R}\langle \underline{X}\rangle : f^* = f\}.$$

Similarly, we use $\operatorname{Sym}\mathbb{R}^{t\times t}$ to denote the set of all symmetric $t \times t$ matrices.

In this paper we will mostly consider the *normalized* trace Tr, i.e.,

$$\operatorname{Tr}(A) = \frac{1}{t}\operatorname{tr}(A) \quad \text{for } A \in \mathbb{R}^{t\times t}.$$

The invariance of the trace under cyclic permutations motivates the following definition of cyclic equivalence ([15], p. 1817).

DEFINITION 2.1. Two polynomials $f, g \in \mathbb{R}\langle \underline{X}\rangle$ are *cyclically equivalent* if $f - g$ is a sum of commutators:

$$f - g = \sum_{i=1}^{k}(p_i q_i - q_i p_i) \quad \text{for some } k \in \mathbb{N} \text{ and } p_i, q_i \in \mathbb{R}\langle \underline{X}\rangle.$$

REMARK 2.2. (i) Two words $v, w \in \langle \underline{X}\rangle$ are cyclically equivalent if and only if w is a cyclic permutation of v. Equivalently: there exist $u_1, u_2 \in \langle \underline{X}\rangle$ such that $v = u_1 u_2$ and $w = u_2 u_1$.

(ii) If $f \overset{\text{cyc}}{\sim} g$ then $\operatorname{Tr}(f(\underline{A})) = \operatorname{Tr}(g(\underline{A}))$ for all tuples $\underline{A}$ of symmetric matrices. Less obvious is the converse: if $\operatorname{Tr}(f(\underline{A})) = \operatorname{Tr}(g(\underline{A}))$ for all $\underline{A}$ and $f - g \in \operatorname{Sym}\mathbb{R}\langle \underline{X}\rangle$, then $f \overset{\text{cyc}}{\sim} g$; see Theorem 2.1 in [15].

(iii) Although $f \overset{\text{cyc}}{\nsim} f^*$ in general, we still have

$$\operatorname{Tr}(f(\underline{A})) = \operatorname{Tr}(f^*(\underline{A}))$$

for all $f \in \mathbb{R}\langle \underline{X}\rangle$ and all $\underline{A} \in (\operatorname{Sym}\mathbb{R}^{t\times t})^n$.

The length of the longest word in a polynomial $f \in \mathbb{R}\langle \underline{X}\rangle$ is the *degree* of f and is denoted by $\deg f$. We write $\mathbb{R}\langle \underline{X}\rangle_{\leqslant k}$ for the set of all polynomials of degree $\leqslant k$.

3. THE TRUNCATED TRACIAL MOMENT PROBLEM

In this section we define tracial (moment) sequences, tracial moment matrices, the tracial moment problem, and their truncated analogs. After a few motivating examples we proceed to show that the kernel of a tracial moment matrix has some real-radical-like properties (Proposition 3.8). We then prove that a tracial moment matrix of finite rank has a tracial moment representation, i.e., the tracial moment problem for the associated tracial sequence is solvable (Theorem 3.14). Finally, we give the solution of the truncated tracial moment problem: a truncated tracial sequence has a tracial representation if and only if one of its extensions has a tracial moment matrix that admits a flat extension (Corollary 3.21).

For an overview of the classical (commutative) moment problem in several variables we refer the reader to Akhiezer [1] (for the analytic theory) and to the survey of Laurent [22] and references therein for a more algebraic approach. The standard references on the truncated moment problems are [9], [10]. For the noncommutative moment problem with *free* (i.e., unconstrained) moments see [24], [14].

DEFINITION 3.1. A sequence of real numbers (y_w) indexed by words $w \in \langle \underline{X} \rangle$ satisfying

$$y_w = y_u \quad \text{whenever } w \overset{\text{cyc}}{\sim} u, \tag{3.1}$$

$$y_w = y_{w^*} \quad \text{for all } w, \tag{3.2}$$

and $y_\varnothing = 1$, is called a (normalized) *tracial sequence*.

EXAMPLE 3.2. Given $t \in \mathbb{N}$ and symmetric matrices $A_1, \dots, A_n \in \operatorname{Sym} \mathbb{R}^{t \times t}$, the sequence given by

$$y_w := \operatorname{Tr}(w(A_1, \dots, A_n)) = \frac{1}{t} \operatorname{tr}(w(A_1, \dots, A_n))$$

is a tracial sequence since by Remark 2.2, the traces of cyclically equivalent words coincide.

We are interested in the converse of this example (the *tracial moment problem*): *For which sequences (y_w) do there exist $N \in \mathbb{N}$, $t \in \mathbb{N}$, $\lambda_i \in \mathbb{R}_{\geqslant 0}$ with $\sum_i^N \lambda_i = 1$ and vectors $\underline{A}^{(i)} = (A_1^{(i)}, \dots, A_n^{(i)}) \in (\operatorname{Sym} \mathbb{R}^{t \times t})^n$, such that*

$$y_w = \sum_{i=1}^{N} \lambda_i \operatorname{Tr}(w(\underline{A}^{(i)}))\,? \tag{3.3}$$

We then say that (y_w) has a *tracial moment representation* and call it a *tracial moment sequence*.

The *truncated tracial moment problem* is the study of (finite) tracial sequences $(y_w)_{\leqslant k}$ where w is constrained by $\deg w \leqslant k$ for some $k \in \mathbb{N}$, and properties (3.1) and (3.2) hold for these w. For instance, which sequences $(y_w)_{\leqslant k}$ have a tracial moment representation, i.e., when does there exist a representation of the values y_w as in (3.3) for $\deg w \leqslant k$? If this is the case, then the sequence $(y_w)_{\leqslant k}$ is called a *truncated tracial moment sequence*.

REMARK 3.3. (i) To keep a perfect analogy with the classical moment problem, one would need to consider the existence of a positive Borel measure μ on $(\operatorname{Sym} \mathbb{R}^{t \times t})^n$ (for some $t \in \mathbb{N}$) satisfying

$$y_w = \int \operatorname{Tr}(w(\underline{A}))\, \mathrm{d}\mu(\underline{A}). \tag{3.4}$$

As we shall mostly focus on the *truncated* tracial moment problem in the sequel, the finitary representations (3.3) are the proper setting; cf. Theorem 3.8 in [6]. The more general representations (3.4) occur if one considers the full moment problem which, however, can be solved by solving all its truncated moment problems derived by restriction; see Theorem 3.6 in [6]. We look forward to studying representations (3.4) in more detail, see [5] for some first results.

(ii) Another natural extension of our tracial moment problem with respect to matrices would be to consider moments obtained by traces in finite *von Neumann algebras* as done by Hadwin [12]. However, our primary motivation were trace-positive polynomials defined via traces of matrices (see Definition 4.1), a theme we expand upon in Section 4. Understanding these is one of the approaches to Connes' embedding conjecture [15]. The notion dual to that of trace-positive polynomials is the tracial moment problem as defined above.

(iii) The tracial moment problem is a natural extension of the classical quadrature problem dealing with representability via atomic positive measures in the commutative case. Taking $\underline{a}^{(i)}$ consisting of 1×1 matrices $a_j^{(i)} \in \mathbb{R}$ for the $\underline{A}^{(i)}$ in (3.3), we have

$$y_w = \sum_i \lambda_i w(\underline{a}^{(i)}) = \int x^w \, \mathrm{d}\mu(x),$$

where x^w denotes the commutative collapse of $w \in \langle \underline{X} \rangle$. The measure μ is the convex combination $\sum \lambda_i \delta_{\underline{a}^{(i)}}$ of the atomic measures $\delta_{\underline{a}^{(i)}}$.

The next example shows that there are (truncated) tracial moment sequences (y_w) which cannot be written as

$$y_w = \mathrm{Tr}(w(\underline{A})).$$

EXAMPLE 3.4. Let X be a single free (noncommutative) variable. We take the index set $J = (1, X, X^2, X^3, X^4)$ and $y = (1, 1 - \sqrt{2}, 1, 1 - \sqrt{2}, 1)$. Then

$$y_w = \frac{\sqrt{2}}{2} w(-1) + \Big(1 - \frac{\sqrt{2}}{2}\Big) w(1),$$

i.e., $\lambda_1 = \sqrt{2}/2$, $\lambda_2 = 1 - \lambda_1$ and $A^{(1)} = -1$, $A^{(2)} = 1$. But there is no symmetric matrix $A \in \mathbb{R}^{t \times t}$ for any $t \in \mathbb{N}$ such that $y_w = \mathrm{Tr}(w(A))$ for all $w \in J$. The proof is given in the appendix.

The (infinite) *tracial moment matrix* (also called tracial Hankel matrix) $M(y)$ of a tracial sequence $y = (y_w)$ is defined by

$$M(y) = (y_{u^* v})_{u,v}.$$

This matrix is symmetric due to the condition (3.2) in the definition of a tracial sequence. A necessary condition for y to be a tracial moment sequence is positive semidefiniteness of $M(y)$ which in general is not sufficient.

The tracial moment matrix of *order* k is the tracial moment matrix $M_k(y)$ indexed by words u, v with $\deg u, \deg v \leqslant k$. If y is a truncated tracial moment sequence, then $M_k(y)$ is positive semidefinite. Here is an easy example showing the converse is false:

EXAMPLE 3.5. When dealing with two variables, we write (X, Y) instead of (X_1, X_2). Taking the index set

$$(1, X, Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3, X^4, X^3Y, X^2Y^2, XYXY, XY^3, Y^4)$$

the truncated moment sequence

$$y = (1,0,0,1,1,1,0,0,0,0,4,0,2,1,0,4)$$

yields the tracial moment matrix

$$M_2(y) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 4 \end{pmatrix}$$

of degree 2 with respect to the basis $(1, X, Y, X^2, XY, YX, Y^2)$. $M_2(y)$ is positive semidefinite but y has no tracial representation. Again, we postpone the proof until the appendix.

For a given polynomial $p = \sum_{w \in \langle \underline{X} \rangle} p_w w \in \mathbb{R}\langle \underline{X} \rangle$ let $\vec{x}p$ be the (column) vector of coefficients p_w in a given fixed order. One can identify $\mathbb{R}\langle \underline{X} \rangle_{\leqslant k}$ with $\mathbb{R}^\eta$ for $\eta = \eta(k) = \dim \mathbb{R}\langle \underline{X} \rangle_{\leqslant k} < \infty$ by sending each $p \in \mathbb{R}\langle \underline{X} \rangle_{\leqslant k}$ to the vector $\vec{x}p$ of its entries with $\deg w \leqslant k$. The tracial moment matrix $M(y)$ induces the linear map

$$\varphi_M : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}^{\mathbb{N}}, \quad p \mapsto M\vec{x}p.$$

The tracial moment matrices $M_k(y)$, indexed by w with $\deg w \leqslant k$, can be regarded as linear maps $\varphi_{M_k} : \mathbb{R}^\eta \to \mathbb{R}^\eta$, $\vec{x}p \mapsto M_k \vec{x}p$.

LEMMA 3.6. *Let $M = M(y)$ be a tracial moment matrix. Then the following holds:*

(i) $p(y) := \sum_w p_w y_w = \vec{x}1^* M \vec{x}p$. *In particular,* $\vec{x}1^* M \vec{x}p = \vec{x}1^* M \vec{x}q$ *if* $p \overset{\text{cyc}}{\sim} q$.

(ii) $\vec{x}p^* M \vec{x}q = \vec{x}1^* M \vec{x}p^* q$.

Proof. Let $p, q \in \mathbb{R}\langle \underline{X} \rangle$. For $k := \max\{\deg p, \deg q\}$, we have

$$\vec{x}p^* M(y) \vec{x}q = \vec{x}p^* M_k(y) \vec{x}q. \tag{3.5}$$

Both statements now follow by direct calculation. ∎

We can identify the kernel of a tracial moment matrix M with the subset of $\mathbb{R}\langle \underline{X} \rangle$ given by

$$I := \{p \in \mathbb{R}\langle \underline{X} \rangle : M\vec{x}p = 0\}. \tag{3.6}$$

PROPOSITION 3.7. *Let $M \succeq 0$ be a tracial moment matrix. Then*

$$I = \{p \in \mathbb{R}\langle \underline{X} \rangle : \langle M\vec{x}p, \vec{x}p \rangle = 0\}. \tag{3.7}$$

Further, I is a two-sided ideal of $\mathbb{R}\langle \underline{X} \rangle$ invariant under the involution.

Proof. Let $J := \{p \in \mathbb{R}\langle \underline{X} \rangle : \langle M\vec{x}p, \vec{x}p\rangle = 0\}$. The implication $I \subseteq J$ is obvious. Let $p \in J$ be given and $k = \deg p$. Since M and thus M_k for each $k \in \mathbb{N}$ is positive semidefinite, the square root $\sqrt{M_k}$ of M_k exists. Then $0 = \langle M_k\vec{x}p, \vec{x}p\rangle = \langle \sqrt{M_k}\vec{x}p, \sqrt{M_k}\vec{x}p\rangle$ implies $\sqrt{M_k}\vec{x}p = 0$. This leads to $M_k\vec{x}p = M\vec{x}p = 0$, thus $p \in I$.

To prove that I is a two-sided ideal, it suffices to show that I is a right-ideal which is closed under $*$. To do this, consider the bilinear map

$$\langle p, q\rangle_M := \langle M\vec{x}p, \vec{x}q\rangle$$

on $\mathbb{R}\langle \underline{X} \rangle$, which is a semi-scalar product. By Lemma 3.6, we get that

$$\langle pq, pq\rangle_M = ((pq)^*pq)(y) = (qq^*p^*p)(y) = \langle pqq^*, p\rangle_M.$$

Then by the Cauchy–Schwarz inequality it follows that for $p \in I$, we have

$$0 \leqslant \langle pq, pq\rangle_M^2 = \langle pqq^*, p\rangle_M^2 \leqslant \langle pqq^*, pqq^*\rangle_M \langle p, p\rangle_M = 0.$$

Hence $pq \in I$, i.e., I is a right-ideal.

Since $p^*p \overset{\text{cyc}}{\sim} pp^*$, we obtain from Lemma 3.6 that

$$\langle M\vec{x}p, \vec{x}p\rangle = \langle p, p\rangle_M = (p^*p)(y) = (pp^*)(y) = \langle p^*, p^*\rangle_M = \langle M\vec{x}p^*, \vec{x}p^*\rangle.$$

Thus if $p \in I$ then also $p^* \in I$. ∎

In the *commutative* context, the kernel of M is a real radical ideal if M is positive semidefinite as observed by Scheiderer (see p. 2974 in [21]). The next proposition gives a description of the kernel of M in the noncommutative setting, and could be helpful in defining a noncommutative real radical ideal.

PROPOSITION 3.8. *For the ideal I in* (3.6) *we have*

$$I = \{f \in \mathbb{R}\langle \underline{X} \rangle : (f^*f)^k \in I \text{ for some } k \in \mathbb{N}\}.$$

Further,

$$I = \Big\{f \in \mathbb{R}\langle \underline{X} \rangle : (f^*f)^{2k} + \sum g_i^* g_i \in I \text{ for some } k \in \mathbb{N}, g_i \in \mathbb{R}\langle \underline{X} \rangle\Big\}.$$

Proof. If $f \in I$ then also $f^*f \in I$ since I is an ideal. If $f^*f \in I$ we have $M\vec{x}f^*f = 0$ which implies by Lemma 3.6 that

$$0 = \vec{x}1^*M\vec{x}f^*f = \vec{x}f^*M\vec{x}f = \langle Mf, f\rangle.$$

Thus $f \in I$. If $(f^*f)^k \in I$ then also $(f^*f)^{k+1} \in I$. So without loss of generality let k be even. From $(f^*f)^k \in I$ we obtain

$$0 = \vec{x}1^*M\vec{x}(f^*f)^k = \vec{x}(f^*f)^{k/2^*}M\vec{x}(f^*f)^{k/2},$$

implying $(f^*f)^{k/2} \in I$. This leads to $f \in I$ by induction.

To show the second statement let $(f^*f)^{2k} + \sum g_i^* g_i \in I$. This leads to

$$\vec{x}(f^*f)^{k^*}M\vec{x}(f^*f)^k + \sum_i \vec{x}g_i{}^*M\vec{x}g_i = 0.$$

Since $M(y) \succeq 0$ we have $\vec{x}(f^*f)^{k^*} M\vec{x}(f^*f)^k \geqslant 0$ and $\vec{x}g_i{}^* M\vec{x}g_i \geqslant 0$. Thus

$$\vec{x}(f^*f)^{k^*} M\vec{x}(f^*f)^k = 0$$

(and $\vec{x}g_i{}^* M\vec{x}g_i = 0$) which implies $f \in I$ as above. ∎

In the commutative setting one uses the Riesz representation theorem for some set of continuous functions (vanishing at infinity or with compact support) to show the existence of a representing measure. We will use the Riesz representation theorem for positive linear functionals on a finite-dimensional Hilbert space.

DEFINITION 3.9. Let $\mathcal{A}$ be an $\mathbb{R}$-algebra with involution. We call a linear map $L : \mathcal{A} \to \mathbb{R}$ a *state* if $L(1) = 1$, $L(a^*a) \geqslant 0$ and $L(a^*) = L(a)$ for all $a \in \mathcal{A}$. If all the commutators have value 0, i.e., if $L(ab) = L(ba)$ for all $a, b \in \mathcal{A}$, then L is called a *tracial state*.

With the aid of the Artin–Wedderburn theorem we shall characterize tracial states on matrix $*$-algebras in Proposition 3.13. This will enable us to prove the existence of a tracial moment representation for tracial sequences with a finite rank tracial moment matrix; see Theorem 3.14.

REMARK 3.10. The only central simple algebras over $\mathbb{R}$ are full matrix algebras over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (combine e.g. the Frobenius theorem (3.12) in [20] with the Artin–Wedderburn theorem (3.5) in [20]). In order to understand ($\mathbb{R}$-linear) tracial states on these, we recall some basic Galois theory.

Let

$$\mathrm{Trd}_{\mathbb{C}/\mathbb{R}} : \mathbb{C} \to \mathbb{R}, \quad z \mapsto \frac{1}{2}(z + \bar{z})$$

denote the *field trace* and

$$\mathrm{Trd}_{\mathbb{H}/\mathbb{R}} : \mathbb{H} \to \mathbb{R}, \quad z \mapsto \frac{1}{2}(z + \bar{z})$$

the *reduced trace* ([17], p. 5). Here the Hamilton quaternions $\mathbb{H}$ are endowed with the *standard involution*

$$z = a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d \mapsto a - \mathrm{i}b - \mathrm{j}k - \mathrm{k}d = \bar{z}$$

for $a, b, c, d \in \mathbb{R}$. We extend the canonical involution on $\mathbb{C}$ and $\mathbb{H}$ to the conjugate transpose involution $*$ on matrices over $\mathbb{C}$ and $\mathbb{H}$, respectively.

Composing the field trace and reduced trace, respectively, with the normalized trace, yields an $\mathbb{R}$-linear map from $\mathbb{C}^{t\times t}$ and $\mathbb{H}^{t\times t}$, respectively, to $\mathbb{R}$. We will denote it simply by Tr. A word of caution: $\mathrm{Tr}(A)$ does not denote the (normalized) matricial trace over $\mathbb{K}$ if $A \in \mathbb{K}^{t\times t}$ and $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}\}$.

An alternative description of Tr is given by the following lemma:

LEMMA 3.11. *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. *Then the only* ($\mathbb{R}$-*linear*) *tracial state on* $\mathbb{K}^{t\times t}$ *is* Tr.

Proof. An easy calculation shows that Tr is indeed a tracial state.

Let L be a tracial state on $\mathbb{R}^{t\times t}$. By the Riesz representation theorem there exists a positive semidefinite matrix B with $\operatorname{Tr}(B) = 1$ such that, for all $A \in \mathbb{R}^{t\times t}$,

$$L(A) = \operatorname{Tr}(BA).$$

Write $B = \big(b_{ij}\big)_{i,j=1}^{t}$. Let $i \neq j$. Then $A = \lambda E_{ij}$ has zero trace for every $\lambda \in \mathbb{R}$ and is thus a sum of commutators. (Here E_{ij} denotes the $t \times t$ *matrix unit* with a one in the (i,j)-position and zeros elsewhere.) Hence

$$\lambda b_{ij} = L(A) = 0.$$

Since $\lambda \in \mathbb{R}$ was arbitrary, $b_{ij} = 0$.

Now let $A = \lambda(E_{ii} - E_{jj})$. Clearly, $\operatorname{Tr}(A) = 0$ and hence

$$\lambda(b_{ii} - b_{jj}) = L(A) = 0.$$

As before, this gives $b_{ii} = b_{jj}$. So B is scalar, and $\operatorname{Tr}(B) = 1$. Hence it is the identity matrix. In particular, $L = \operatorname{Tr}$.

If L is a tracial state on $\mathbb{C}^{t\times t}$, then L induces a tracial state on $\mathbb{R}^{t\times t}$, so $L_0 := L|_{\mathbb{R}^{t\times t}} = \operatorname{Tr}$ by the above. Extend L_0 to

$$L_1 : \mathbb{C}^{t\times t} \to \mathbb{R}, \quad A + \mathrm{i}B \mapsto L_0(A) = \operatorname{Tr}(A) \quad \text{for } A, B \in \mathbb{R}^{t\times t}.$$

L_1 is a tracial state on $\mathbb{C}^{t\times t}$ as a straightforward computation shows. As $\operatorname{Tr}(A) = \operatorname{Tr}(A + \mathrm{i}B)$, all we need to show is that $L_1 = L$.

Clearly, L_1 and L agree on the vector space spanned by all commutators in $\mathbb{C}^{t\times t}$. This space is (over $\mathbb{R}$) of codimension 2. By construction, $L_1(1) = L(1) = 1$ and $L_1(\mathrm{i}) = 0$. On the other hand,

$$L(\mathrm{i}) = L(\mathrm{i}^*) = -L(\mathrm{i})$$

implying $L(\mathrm{i}) = 0$. This shows $L = L_1 = \operatorname{Tr}$.

The remaining case of tracial states over $\mathbb{H}$ is dealt with similarly and is left as an exercise for the reader. ∎

REMARK 3.12. Every complex number $z = a + \mathrm{i}b$ can be represented as a 2×2 real matrix $z' = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$. This gives rise to an $\mathbb{R}$-linear $*$-map $\mathbb{C}^{t\times t} \to \mathbb{R}^{(2t)\times(2t)}$ that commutes with Tr. A similar property holds if quaternions $a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d$ are represented by the 4×4 real matrix

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}.$$

PROPOSITION 3.13. *Let $\mathcal{A}$ be a $*$-subalgebra of $\mathbb{R}^{t\times t}$ for some $t \in \mathbb{N}$ and let $L : \mathcal{A} \to \mathbb{R}$ be a tracial state. Then there exist full matrix algebras $\mathcal{A}^{(i)}$ over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$,* *a $*$-isomorphism*

$$\mathcal{A} \to \bigoplus_{i=1}^{N} \mathcal{A}^{(i)}, \tag{3.8}$$

and $\lambda_1, \ldots, \lambda_N \in \mathbb{R}_{\geqslant 0}$ with $\sum_i \lambda_i = 1$, such that for all $A \in \mathcal{A}$,

$$L(A) = \sum_i^N \lambda_i \operatorname{Tr}(A^{(i)}).$$

Here, $\bigoplus_i A^{(i)} = \begin{pmatrix} A^{(1)} & & \\ & \ddots & \\ & & A^{(N)} \end{pmatrix}$ denotes the image of A under the isomorphism (3.8). *The size of (the real representation of) $\bigoplus_i A^{(i)}$ is at most t.*

Proof. Since L is tracial, $L(U^*AU) = L(A)$ for all orthogonal $U \in \mathbb{R}^{t\times t}$. Hence we can apply orthogonal transformations to $\mathcal{A}$ without changing the values of L. So $\mathcal{A}$ can be transformed into block diagonal form as in (3.8) according to its invariant subspaces. That is, each of the blocks $\mathcal{A}^{(i)}$ acts irreducibly on a subspace of $\mathbb{R}^t$ and is thus a central simple algebra (with involution) over $\mathbb{R}$. The involution on $\mathcal{A}^{(i)}$ is induced by the conjugate transpose involution. (Equivalently, by the transpose on the real matrix representation in the complex of quaternion case.)

Now L induces (after a possible normalization) a tracial state on the block $\mathcal{A}^{(i)}$ and hence by Lemma 3.11, we have $L_i := L|_{\mathcal{A}^{(i)}} = \lambda_i \operatorname{Tr}$ for some $\lambda_i \in \mathbb{R}_{\geqslant 0}$. Then

$$L(A) = L\Big(\bigoplus_i A^{(i)}\Big) = \sum_i L_i(A^{(i)}) = \sum_i \lambda_i \operatorname{Tr}(A^{(i)})$$

and $1 = L(1) = \sum_i \lambda_i$. ∎

The following theorem is the tracial version of the representation theorem of Curto and Fialkow for moment matrices with finite rank [9].

THEOREM 3.14. *Let $y = (y_w)$ be a tracial sequence with positive semidefinite moment matrix $M(y)$ of finite rank t. Then y is a tracial moment sequence, i.e., there exist vectors $\underline{A}^{(i)} = (A_1^{(i)}, \ldots, A_n^{(i)})$ of symmetric matrices $A_j^{(i)}$ of size at most t and $\lambda_i \in \mathbb{R}_{\geqslant 0}$ with $\sum \lambda_i = 1$ such that*

$$y_w = \sum \lambda_i \operatorname{Tr}(w(\underline{A}^{(i)})).$$

Proof. Let $M := M(y)$. We equip $\mathbb{R}\langle \underline{X} \rangle$ with the bilinear form given by

$$\langle p, q \rangle_M := \langle M\vec{x}p, \vec{x}q \rangle = \vec{x}q^* M\vec{x}p.$$

Let $I = \{p \in \mathbb{R}\langle \underline{X}\rangle : \langle p, p\rangle_M = 0\}$. Then by Proposition 3.7, I is an ideal of $\mathbb{R}\langle \underline{X}\rangle$. In particular, $I = \ker \varphi_M$ for

$$\varphi_M : \mathbb{R}\langle \underline{X}\rangle \to \operatorname{Ran} M, \quad p \mapsto M\vec{x}p.$$

Thus if we define $E := \mathbb{R}\langle \underline{X}\rangle / I$, the induced linear map

$$\overline{\varphi}_M : E \to \operatorname{Ran} M, \quad \overline{p} \mapsto M\vec{x}p$$

is an isomorphism and

$$\dim E = \dim(\operatorname{Ran} M) = \operatorname{rank} M = t < \infty.$$

Hence $(E, \langle \sqcup, \sqcup\rangle_E)$ is a finite-dimensional Hilbert space for $\langle \overline{p}, \overline{q}\rangle_E = \vec{x}q^* M\vec{x}p$.

Let $\widehat{X}_i$ be the right multiplication with X_i on E, i.e., $\widehat{X}_i\overline{p} := \overline{pX_i}$. Since I is a right ideal of $\mathbb{R}\langle \underline{X}\rangle$, the operator $\widehat{X}_i$ is well defined. Further, $\widehat{X}_i$ is symmetric since

$$\langle \widehat{X}_i\overline{p}, \overline{q}\rangle_E = \langle M\vec{x}pX_i, \vec{x}q\rangle = (X_i p^* q)(y) = (p^* q X_i)(y) = \langle M\vec{x}p, \vec{x}qX_i\rangle = \langle \overline{p}, \widehat{X}_i\overline{q}\rangle_E.$$

Thus each $\widehat{X}_i$, acting on a t-dimensional vector space, has a representation matrix $A_i \in \operatorname{Sym} \mathbb{R}^{t\times t}$.

Let $\mathcal{B} = B(\widehat{X}_1, \dots, \widehat{X}_n) = B(A_1, \dots, A_n)$ be the algebra of operators generated by $\widehat{X}_1, \dots, \widehat{X}_n$. These operators can be written as

$$\widehat{p} = \sum_{w\in\langle \underline{X}\rangle} p_w \widehat{w}$$

for some $p_w \in \mathbb{R}$, where $\widehat{w} = \widehat{X}_{w_1} \cdots \widehat{X}_{w_s}$ for $w = X_{w_1} \cdots X_{w_s}$. Observe that $\widehat{w} = w(A_1, \dots, A_n)$. We define the linear functional

$$L : \mathcal{B} \to \mathbb{R}, \quad \widehat{p} \mapsto \vec{x}1^* M\vec{x}p = p(y),$$

which is a state on $\mathcal{B}$. Since $y_w = y_u$ for $w \overset{\text{cyc}}{\sim} u$, it follows that L is tracial. Thus by Proposition 3.13 (and Remark 3.12), there exist $\lambda_1, \dots \lambda_N \in \mathbb{R}_{\geqslant 0}$ with $\sum_i \lambda_i = 1$ and real symmetric matrices $A_j^{(i)}$ $(i = 1, \dots, N)$ for each $A_j \in \operatorname{Sym} \mathbb{R}^{t\times t}$, such that for all $w \in \langle \underline{X}\rangle$, we have, as desired:

$$y_w = w(y) = L(\widehat{w}) = \sum_i \lambda_i \operatorname{Tr}(w(\underline{A}^{(i)})). \quad \blacksquare$$

The sufficient conditions on $M(y)$ in Theorem 3.14 are also necessary for y to be a tracial moment sequence. Thus we get our first characterization of tracial moment sequences:

COROLLARY 3.15. *Let $y = (y_w)$ be a tracial sequence. Then y is a tracial moment sequence if and only if $M(y)$ is positive semidefinite and of finite rank.*

Proof. If $y_w = \operatorname{Tr}(w(\underline{A}))$ for some $\underline{A} = (A_1, \dots, A_n) \in (\operatorname{Sym} \mathbb{R}^{t\times t})^n$, then

$$L(p) = \sum_w p_w y_w = \sum_w p_w \operatorname{Tr}(w(\underline{A})) = \operatorname{Tr}(p(\underline{A})).$$

Hence, for all $p \in \mathbb{R}\langle \underline{X} \rangle$,

$$\vec{x}p^* M(y)\vec{x}p = L(p^*p) = \mathrm{Tr}(p^*(\underline{A})p(\underline{A})) \geqslant 0.$$

Further, the tracial moment matrix $M(y)$ has rank at most t^2. This can be seen as follows: M induces a bilinear map

$$\Phi : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}\langle \underline{X} \rangle^*, \quad p \mapsto (q \mapsto \mathrm{Tr}((q^*p)(\underline{A}))),$$

where $\mathbb{R}\langle \underline{X} \rangle^*$ is the dual space of $\mathbb{R}\langle \underline{X} \rangle$. This implies

$$\operatorname{rank} M = \dim(\operatorname{Ran} \Phi) = \dim(\mathbb{R}\langle \underline{X} \rangle / \ker \Phi).$$

The kernel of the evaluation map $\varepsilon_{\underline{A}} : \mathbb{R}\langle \underline{X} \rangle \to \mathbb{R}^{t \times t}$, $p \mapsto p(\underline{A})$ is a subset of $\ker \Phi$. In particular,

$$\dim(\mathbb{R}\langle \underline{X} \rangle / \ker \Phi) \leqslant \dim(\mathbb{R}\langle \underline{X} \rangle / \ker \varepsilon_{\underline{A}}) = \dim(\operatorname{Ran} \varepsilon_{\underline{A}}) \leqslant t^2.$$

The same holds true for each convex combination $y_w = \sum\limits_i \lambda_i \mathrm{Tr}(w(\underline{A}^{(i)}))$.

The converse is Theorem 3.14. ∎

DEFINITION 3.16. Let $A \in \operatorname{Sym} \mathbb{R}^{t \times t}$ be given. A (symmetric) extension of A is a matrix $\widetilde{A} \in \operatorname{Sym} \mathbb{R}^{(t+s)\times(t+s)}$ of the form

$$\widetilde{A} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

for some $B \in \mathbb{R}^{t \times s}$ and $C \in \mathbb{R}^{s \times s}$. Such an extension is *flat* if $\operatorname{rank} A = \operatorname{rank} \widetilde{A}$, or, equivalently, if $B = AW$ and $C = W^*AW$ for some matrix W.

The kernel of a flat extension M_k of a tracial moment matrix M_{k-1} has some (truncated) *ideal-like properties* as shown in the following lemma.

LEMMA 3.17. *Let $f \in \mathbb{R}\langle \underline{X} \rangle$ with $\deg f \leqslant k-1$ and let M_k be a flat extension of M_{k-1}. If $f \in \ker M_k$ then $fX_i, X_i f \in \ker M_k$.*

Proof. Let $f = \sum\limits_w f_w w$. Then for $v \in \langle \underline{X} \rangle_{k-1}$, we have

$$(M_k \vec{x} f X_i)_v = \sum_w f_w y_{v^* w X_i} = \sum_w f_w y_{(vX_i)^* w} = (M_k \vec{x} f)_{vX_i} = 0. \tag{3.9}$$

The matrix M_k is of the form $M_k = \begin{pmatrix} M_{k-1} & B \\ B^* & C \end{pmatrix}$. Since M_k is a flat extension, $\ker M_k = \ker \begin{pmatrix} M_{k-1} & B \end{pmatrix}$. Thus by (3.9), $fX_i \in \ker \begin{pmatrix} M_{k-1} & B \end{pmatrix} = \ker M_k$. For $X_i f$ we obtain analogously that

$$(M_k \vec{x} X_i f)_v = \sum_w f_w y_{v^* X_i w} = \sum_w f_w y_{(X_i v)^* w} = (M_k \vec{x} f)_{X_i v} = 0$$

for $v \in \langle \underline{X} \rangle_{k-1}$, which implies $X_i f \in \ker M_k$. ∎

We are now ready to prove the tracial version of the flat extension theorem of Curto and Fialkow [10].

THEOREM 3.18. *Let $y = (y_w)_{\leqslant 2k}$ be a truncated tracial sequence of order $2k$. If* $\operatorname{rank} M_k(y) = \operatorname{rank} M_{k-1}(y)$, *then there exists a unique tracial extension $\widetilde{y} = (\widetilde{y}_w)_{\leqslant 2k+2}$ of y such that $M_{k+1}(\widetilde{y})$ is a flat extension of $M_k(y)$.*

Proof. Let $M_k := M_k(y)$. We will construct a flat extension $M_{k+1} := \left(\begin{smallmatrix} M_k & B \\ B^* & C \end{smallmatrix}\right)$ such that M_{k+1} is a tracial moment matrix. Since M_k is a flat extension of $M_{k-1}(y)$ we can find a basis b of $\operatorname{Ran}(M_k)$ consisting of columns of M_k labeled by w with $\deg w \leqslant k-1$. Thus the range of M_k is completely determined by the range of $M_k|_{\operatorname{span} b}$, i.e., for each $p \in \mathbb{R}\langle \underline{X} \rangle$ with $\deg p \leqslant k$ there exists a *unique* $r \in \operatorname{span} b$ such that $M_k \vec{x} p = M_k \vec{x} r$; equivalently, $p - r \in \ker M_k$.

Let $v \in \langle \underline{X} \rangle$, $\deg v = k+1$, $v = v' X_i$ for some $i \in \{1, \dots, n\}$ and $v' \in \langle \underline{X} \rangle$ with $\deg v' = k$. For v' there exists an $r \in \operatorname{span} b$ such that $v' - r \in \ker M_k$.

If there exists a flat extension M_{k+1}, then by Lemma 3.17, from $v' - r \in \ker M_k \subseteq \ker M_{k+1}$ it follows that $(v' - r)X_i \in \ker M_{k+1}$. Hence the desired flat extension has to satisfy

$$M_{k+1} \vec{x} v = M_{k+1} \vec{x} r X_i = M_k \vec{x} r X_i. \tag{3.10}$$

Therefore we define

$$B \vec{x} v := M_k \vec{x} r X_i. \tag{3.11}$$

More precisely, let $(w_1, \dots, w_\ell)$ be the words in the basis of M_k of degree k. Let r_{w_i} be the unique element in $\operatorname{span} b$ with $w_i - r_{w_i} \in \ker M_k$. Then $B = M_k W$ with $W = (\vec{x} r_{w_1 X_1}, \vec{x} r_{w_1 X_2}, \dots, \vec{x} r_{w_\ell X_n})$ and we define

$$C := W^* M_k W. \tag{3.12}$$

Since the r_{w_i} are uniquely determined,

$$M_{k+1} = \begin{pmatrix} M_k & B \\ B^* & C \end{pmatrix} \tag{3.13}$$

is well-defined. The constructed M_{k+1} is a flat extension of M_k, and $M_{k+1} \succeq 0$ if and only if $M_k \succeq 0$; see Proposition 2.1 in [10]. Moreover, once B is chosen, there is only one C making M_{k+1} as in (3.13) a flat extension of M_k. This follows from general linear algebra; see e.g. p. 11 in [10]. Hence M_{k+1} is the *only* candidate for a flat extension.

Therefore we are done if M_{k+1} is a tracial moment matrix, i.e.,

$$(M_{k+1})_{v,w} = (M_{k+1})_{v_1,w_1} \quad \text{whenever } v^* w \overset{\text{cyc}}{\sim} v_1^* w_1. \tag{3.14}$$

To show this we prove first that $(M_{k+1})_{u, X_i v} = (M_{k+1})_{u X_i, v}$ for $u, v \in \langle \underline{X} \rangle_k$, which implies the tracial property for block B and B^* (by symmetry) in M_{k+1}.

If $\deg u, \deg v X_i \leqslant k$ there is nothing to show since M_k is a tracial moment matrix. If $\deg u \leqslant k$ and $\deg v X_i = k+1$ there exists an $r \in \operatorname{span} b$ such that

$r - v \in \ker M_{k-1}$, and by Lemma 3.17, also $vX_i - rX_i \in \ker M_k$. Then we get

$$\begin{aligned}(M_{k+1})_{u,vX_i} &= \vec{x}u^* M_{k+1} \vec{x} v X_i = \vec{x}u^* M_{k+1}\vec{x} r X_i = \vec{x}u^* M_k \vec{x} r X_i \\ &= y_{u^* r X_i} = y_{X_i u^* r} = y_{(uX_i)^* r} \overset{(*)}{=} \vec{x} u X_i{}^* M_{k+1} \vec{x} v = (M_{k+1})_{uX_i, v},\end{aligned}$$

where equality $(*)$ holds by (3.10) and by construction.

We are left to show that $(M_{k+1})_{X_j u, vX_i} = (M_{k+1})_{uX_i, X_j v}$. Then (3.14) follows recursively for all entries in block C and M_{k+1} is a tracial moment matrix. Let $\deg u = \deg v = k$, then there exist $s, r \in \operatorname{span} b$ with $u - s \in \ker M_{k-1}$ and $r - v \in \ker M_{k-1}$. Then

$$\begin{aligned}(M_{k+1})_{X_j u, vX_i} &= \vec{x} X_j u^* M_{k+1} \vec{x} v X_i = \vec{x} X_j s^* M_k \vec{x} r X_i = y_{s^* X_j r X_i} = y_{(sX_i)^*(X_j r)} \\ &\overset{(*)}{=} \vec{x} u X_i{}^* M_{k+1} \vec{x} X_j v = (M_{k+1})_{(uX_i)^* X_j v} = (M_{k+1})_{uX_i, X_j v}.\end{aligned}$$

Finally, the construction of $\widetilde{y}$ from M_{k+1} is clear. ∎

COROLLARY 3.19. *Let $y = (y_w)_{\leqslant 2k}$ be a truncated tracial sequence. If $M_k(y)$ is positive semidefinite and $M_k(y)$ is a flat extension of $M_{k-1}(y)$, then y is a truncated tracial moment sequence.*

Proof. By Theorem 3.18 we can extend $M_k(y)$ inductively to a positive semidefinite moment matrix $M(\widetilde{y})$ with $\operatorname{rank} M(\widetilde{y}) = \operatorname{rank} M_k(y) < \infty$. Thus $M(\widetilde{y})$ has finite rank and by Theorem 3.14, there exists a tracial moment representation of $\widetilde{y}$. Therefore y is a truncated tracial moment sequence. ∎

The following two corollaries give characterizations of tracial moment matrices coming from tracial moment sequences.

COROLLARY 3.20. *Let $y = (y_w)$ be a tracial sequence. Then y is a tracial moment sequence if and only if $M(y)$ is positive semidefinite and there exists some $N \in \mathbb{N}$ such that $M_{k+1}(y)$ is a flat extension of $M_k(y)$ for all $k \geqslant N$.*

Proof. If y is a tracial moment sequence then by Corollary 3.15, $M(y)$ is positive semidefinite and has finite rank t. Thus there exists an $N \in \mathbb{N}$ such that $t = \operatorname{rank} M_N(y)$. In particular, $\operatorname{rank} M_k(y) = \operatorname{rank} M_{k+1}(y) = t$ for all $k \geqslant N$, i.e., $M_{k+1}(y)$ is a flat extension of $M_k(y)$ for all $k \geqslant N$.

For the converse, let N be given such that $M_{k+1}(y)$ is a flat extension of $M_k(y)$ for all $k \geqslant N$. By Theorem 3.18, the (iterated) unique extension $\widetilde{y}$ of $(y_w)_{\leqslant 2k}$ for $k \geqslant N$ is equal to y. Otherwise there exists a flat extension $\widetilde{y}$ of $(y_w)_{\leqslant 2\ell}$ for some $\ell \geqslant N$ such that $M_{\ell+1}(\widetilde{y}) \succeq 0$ is a flat extension of $M_\ell(y)$ and $M_{\ell+1}(\widetilde{y}) \neq M_{\ell+1}(y)$ contradicting the uniqueness of the extension in Theorem 3.18.

Thus $M(y) \succeq 0$ and $\operatorname{rank} M(y) = \operatorname{rank} M_N(y) < \infty$. Hence by Theorem 3.14, y is a tracial moment sequence. ∎

COROLLARY 3.21. *Let $y = (y_w)$ be a tracial sequence. Then y has a tracial moment representation with matrices of size at most $t :=$ rank $M(y)$ if $M_N(y)$ is positive semidefinite and $M_{N+1}(y)$ is a flat extension of $M_N(y)$ for some $N \in \mathbb{N}$ with* rank $M_N(y) = t$.

Proof. Since $\operatorname{rank} M(y) = \operatorname{rank} M_N(y) = t$, each $M_{k+1}(y)$ with $k \geqslant N$ is a flat extension of $M_k(y)$. As $M_N(y) \succeq 0$, all $M_k(y)$ are positive semidefinite. Thus $M(y)$ is also positive semidefinite. Indeed, let $p \in \mathbb{R}\langle \underline{X} \rangle$ and $\ell = \max\{\deg p, N\}$. Then $\vec{x}p^* M(y) \vec{x}p = \vec{x}p^* M_\ell(y) \vec{x}p \geqslant 0$.

Thus by Corollary 3.20, y is a tracial moment sequence. The representing matrices can be chosen to be of size at most $\operatorname{rank} M(y) = t$. ∎

4. POSITIVE DEFINITE MOMENT MATRICES AND TRACE-POSITIVE POLYNOMIALS

In this section we explain how the representability of *positive definite* tracial moment matrices relates to sum of hermitian squares representations of trace-positive polynomials. We start by introducing some terminology.

An element of the form g^*g for some $g \in \mathbb{R}\langle \underline{X} \rangle$ is called a *hermitian square* and we denote the set of all sums of hermitian squares by

$$\Sigma^2 = \Big\{ f \in \mathbb{R}\langle \underline{X} \rangle : f = \sum g_i^* g_i \text{ for some } g_i \in \mathbb{R}\langle \underline{X} \rangle \Big\}.$$

A polynomial $f \in \mathbb{R}\langle \underline{X} \rangle$ is *matrix-positive* if $f(\underline{A})$ is positive semidefinite for all tuples $\underline{A}$ of symmetric matrices $A_i \in \operatorname{Sym} \mathbb{R}^{t \times t}$, $t \in \mathbb{N}$. Helton [14] proved that $f \in \mathbb{R}\langle \underline{X} \rangle$ is matrix-positive if and only if $f \in \Sigma^2$ by solving a noncommutative moment problem; see also [24].

We are interested in a different type of positivity induced by the trace.

DEFINITION 4.1. A polynomial $f \in \mathbb{R}\langle \underline{X} \rangle$ is called *trace-positive* if

$$\operatorname{Tr}(f(\underline{A})) \geqslant 0 \text{ for all } \underline{A} \in (\operatorname{Sym} \mathbb{R}^{t \times t})^n,\ t \in \mathbb{N}.$$

Trace-positive polynomials are intimately connected to deep open problems from e.g. operator algebras (Connes' embedding conjecture [15]) and mathematical physics (the Bessis–Moussa–Villani conjecture [16]), so a good understanding of this set is needed. A distinguished subset is formed by sums of hermitian squares and commutators.

DEFINITION 4.2. Let Θ^2 be the set of all polynomials which are cyclically equivalent to a sum of hermitian squares, i.e.,

$$\Theta^2 = \Big\{ f \in \mathbb{R}\langle \underline{X} \rangle : f \overset{\text{cyc}}{\sim} \sum g_i^* g_i \text{ for some } g_i \in \mathbb{R}\langle \underline{X} \rangle \Big\}. \tag{4.1}$$

Obviously, all $f \in \Theta^2$ are trace-positive. However, in contrast to Helton's sum of squares theorem mentioned above, the following noncommutative version of the well-known Motzkin polynomial ([23], p. 5) shows that a trace-positive polynomial need not be a member of Θ^2 [15].

EXAMPLE 4.3. Let

$$M_{\mathrm{nc}} = XY^4X + YX^4Y - 3XY^2X + 1 \in \mathbb{R}\langle X, Y\rangle.$$

Then $M_{\mathrm{nc}} \notin \Theta^2$ since the commutative Motzkin polynomial is not a (commutative) sum of squares ([23], p. 5). The fact that $M_{\mathrm{nc}}(A, B)$ has nonnegative trace for all symmetric matrices A, B has been shown by Schweighofer and the second author in Example 4.4 in [15] using Putinar's Positivstellensatz [28].

Let $\Sigma_k^2 := \Sigma^2 \cap \mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k}$ and $\Theta_k^2 := \Theta^2 \cap \mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k}$. These are convex cones in $\mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k}$. By duality there exists a connection between Θ_k^2 and positive semidefinite tracial moment matrices of order k. If every tracial moment matrix $M_k(y) \succeq 0$ of order k has a tracial representation then every trace-positive polynomial of degree at most $2k$ lies in Θ_k^2. In fact:

THEOREM 4.4. *Let $k \in \mathbb{N}$. All truncated tracial sequences $(y_w)_{\leqslant 2k}$ with* positive definite *tracial moment matrix $M_k(y)$ have a tracial moment representation* (3.3) *if and only if all trace-positive polynomials of degree $\leqslant 2k$ are elements of Θ_k^2.*

For the proof we need some preliminary work.

LEMMA 4.5. *Θ_k^2 is a closed convex cone in $\mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k}$.*

Proof. Endow $\mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k}$ with a norm $\|\text{␣}\|$ and the quotient space $\mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k}/\underset{\sim}{\mathrm{cyc}}$ with the quotient norm

$$\|\pi(f)\| := \inf\{\|f + h\| : h \overset{\mathrm{cyc}}{\sim} 0\}, \quad f \in \mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k}. \tag{4.2}$$

Here $\pi : \mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k} \to \mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k}/\underset{\sim}{\mathrm{cyc}}$ denotes the quotient map. (Note: due to the finite-dimensionality of $\mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k}$, the infimum on the right-hand side of (4.2) is attained.)

Since $\Theta_k^2 = \pi^{-1}(\pi(\Theta_k^2))$, it suffices to show that $\pi(\Theta_k^2)$ is closed. Let $d_k = \dim \mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k}$. Since by Carathéodory's theorem ([2], p. 10) each element $f \in \mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k}$ can be written as a convex combination of $d_k + 1$ elements of $\mathbb{R}\langle \underline{X}\rangle_{\leqslant 2k}$, the image of

$$\begin{aligned} \varphi : (\mathbb{R}\langle \underline{X}\rangle_{\leqslant k})^{d_k+1} &\to \mathbb{R}\langle \underline{X}\rangle_{2k}/\underset{\sim}{\mathrm{cyc}} \\ (g_i)_{i=0,\dots,d_k} &\mapsto \pi\Big(\sum_{i=0}^{d_k} g_i^* g_i\Big) \end{aligned}$$

equals $\pi(\Sigma_k^2) = \pi(\Theta_k^2)$. In $(\mathbb{R}\langle \underline{X}\rangle_{\leqslant k})^{d_k+1}$ we define $\mathcal{S} := \{g = (g_i) : \|g\| = 1\}$. Note that $\mathcal{S}$ is compact, thus $V := \varphi(\mathcal{S}) \subseteq \pi(\Theta_k^2)$ is compact as well. Since

$0 \notin \mathcal{S}$, and a sum of hermitian squares cannot be cyclically equivalent to 0 by Lemma 3.2(b) in [16], we see that $0 \notin V$.

Let $(f_\ell)_\ell$ be a sequence in $\pi(\Theta_k^2)$ which converges to $\pi(f)$ for some $f \in \mathbb{R}\langle \underline{X} \rangle_{\leqslant 2k}$. Write $f_\ell = \lambda_\ell v_\ell$ for $\lambda_\ell \in \mathbb{R}_{\geqslant 0}$ and $v_\ell \in V$. Since V is compact there exists a subsequence $(v_{\ell_j})_j$ of v_ℓ converging to $v \in V$. Then

$$\lambda_{\ell_j} = \frac{\|f_{\ell_j}\|}{\|v_{\ell_j}\|} \xrightarrow{j\to\infty} \frac{\|f\|}{\|v\|}.$$

Thus $f_\ell \to f = (\|f\|/\|v\|)v \in \pi(\Theta_k^2)$. ∎

DEFINITION 4.6. To a truncated tracial sequence $(y_w)_{\leqslant k}$ we associate the (*tracial*) *Riesz functional* $L_y : \mathbb{R}\langle \underline{X} \rangle_{\leqslant k} \to \mathbb{R}$ defined by

$$L_y(p) := \sum_w p_w y_w \quad \text{for } p = \sum_w p_w w \in \mathbb{R}\langle \underline{X} \rangle_{\leqslant k}.$$

We say that L_y is *strictly positive* ($L_y > 0$), if

$$L_y(p) > 0 \text{ for all trace-positive } p \in \mathbb{R}\langle \underline{X} \rangle_{\leqslant k},\ p \overset{\mathrm{cyc}}{\not\sim} 0.$$

If $L_y(p) \geqslant 0$ for all trace-positive $p \in \mathbb{R}\langle \underline{X} \rangle_{\leqslant k}$, then L_y is *positive* ($L_y \geqslant 0$).

Equivalently, a tracial Riesz functional L_y is positive (respectively, strictly positive) if and only if the map $\overline{L}_y$ it induces on $\mathbb{R}\langle \underline{X} \rangle_{\leqslant k}/\underset{\sim}{\mathrm{cyc}}$ is nonnegative (respectively, positive) on the nonzero images of trace-positive polynomials in $\mathbb{R}\langle \underline{X} \rangle_{\leqslant k}/\underset{\sim}{\mathrm{cyc}}$.

We shall prove that strictly positive Riesz functionals lie in the interior of the cone of positive Riesz functionals, and that truncated tracial sequences y with *strictly* positive L_y are truncated tracial moment sequences (Theorem 4.8 below). These results are motivated by and resemble the results of Fialkow and Nie in Section 2 in [11] in the commutative context.

LEMMA 4.7. *If $L_y > 0$ then there exists an $\varepsilon > 0$ such that $L_{\widetilde{y}} > 0$ for all $\widetilde{y}$ with $\|y - \widetilde{y}\|_1 < \varepsilon$.*

Proof. Let $\mathcal{C}_k$ be the convex cone of trace-positive polynomials of degree at most $2k$. We equip $\mathbb{R}\langle \underline{X} \rangle_{\leqslant 2k}/\underset{\sim}{\mathrm{cyc}}$ with a quotient norm as in (4.2). Then

$$\mathcal{S} := \{\pi(p) \in \mathbb{R}\langle \underline{X} \rangle_{\leqslant 2k}/\underset{\sim}{\mathrm{cyc}} : p \in \mathcal{C}_k,\ \|\pi(p)\| = 1\}$$

is compact. By a scaling argument, it suffices to show that $\overline{L}_{\widetilde{y}} > 0$ on $\mathcal{S}$ for $\widetilde{y}$ close to y. The map $y \mapsto \overline{L}_y$ is linear between finite-dimensional vector spaces. Thus

$$|\overline{L}_{y'}(\pi(p)) - \overline{L}_{y''}(\pi(p))| \leqslant C\|y' - y''\|_1$$

for all $\pi(p) \in \mathcal{S}$, truncated tracial moment sequences y', y'', and some $C \in \mathbb{R}_{>0}$.

Since $\overline{L}_y$ is continuous and strictly positive on $\mathcal{S}$, there exists an $\varepsilon > 0$ such that $\overline{L}_y(\pi(p)) \geqslant 2\varepsilon$ for all $\pi(p) \in \mathcal{S}$. Let $\widetilde{y}$ satisfy $\|y - \widetilde{y}\|_1 < \varepsilon/C$. Then

$$\overline{L}_{\widetilde{y}}(\pi(p)) \geqslant \overline{L}_y(\pi(p)) - C\|y - \widetilde{y}\|_1 \geqslant \varepsilon > 0. \qquad \blacksquare$$

THEOREM 4.8. *Let $y = (y_w)_{\leqslant k}$ be a truncated tracial sequence of order k. If $L_y > 0$, then y is a truncated tracial moment sequence.*

Proof. We show first that $y \in \overline{T}$, where $\overline{T}$ is the closure of

$$T = \Big\{ (y_w)_{\leqslant k} : \exists \underline{A}^{(i)} \ \exists \lambda_i \in \mathbb{R}_{\geqslant 0} : \ y_w = \sum \lambda_i \operatorname{Tr}(w(\underline{A}^{(i)})) \Big\}.$$

Assume $L_y > 0$ but $y \notin \overline{T}$. Since $\overline{T}$ is a closed convex cone in $\mathbb{R}^\eta$ (for some $\eta \in \mathbb{N}$), by the Minkowski separation theorem there exists a vector $\vec{x}p \in \mathbb{R}^\eta$ such that $\vec{x}p^* y < 0$ and $\vec{x}p^* w \geqslant 0$ for all $w \in \overline{T}$. The noncommutative polynomial corresponding to $\vec{x}p$ is trace positive since $\vec{x}p^* z \geqslant 0$ for all $z \in \overline{T}$. Thus $0 < L_y(p) = \vec{x}p^* y < 0$, a contradiction.

By Lemma 4.7, $y \in \operatorname{int}(\overline{T})$. Thus $y \in \operatorname{int}(\overline{T}) \subseteq T$, see e.g. Theorem 25.20 in [3]. $\blacksquare$

We remark that assuming only non-strict positivity of L_y in Theorem 4.8 would not suffice for the existence of a tracial moment representation (3.3) for y. This is a consequence of Example 3.5.

Proof of Theorem 4.4. To show sufficiency, assume $f = \sum\limits_w f_w w \in \mathbb{R}\langle \underline{X} \rangle_{\leqslant 2k}$ is trace-positive but $f \notin \Theta_k^2$. By Lemma 4.5, Θ_k^2 is a closed convex cone in $\mathbb{R}\langle \underline{X} \rangle_{\leqslant 2k}$, thus by the Minkowski separation theorem we find a hyperplane which separates f and Θ_k^2. That is, there is a linear form $L : \mathbb{R}\langle \underline{X} \rangle_{\leqslant 2k} \to \mathbb{R}$ such that $L(f) < 0$ and $L(p) \geqslant 0$ for $p \in \Theta_k^2$. In particular, $L(f) = 0$ for all $f \overset{\text{cyc}}{\sim} 0$, i.e., without loss of generality, L is tracial. Since there are tracial states strictly positive on $\Sigma_k^2 \setminus \{0\}$, we may assume $L(p) > 0$ for all $p \in \Theta_k^2$, $p \overset{\text{cyc}}{\nsim} 0$. Hence the bilinear form given by

$$(p, q) \mapsto L(pq)$$

can be written as $L(pq) = \vec{x}q^* M \vec{x}p$ for some truncated tracial moment matrix $M \succ 0$. By assumption, the corresponding truncated tracial sequence y has a tracial moment representation

$$y_w = \sum \lambda_i \operatorname{Tr}(w(\underline{A}^{(i)}))$$

for some tuples $A^{(i)}$ of symmetric matrices $A_j^{(i)}$ and $\lambda_i \in \mathbb{R}_{\geqslant 0}$ which implies the contradiction

$$0 > L(f) = \sum \lambda_i \operatorname{Tr}(f(\underline{A}^{(i)})) \geqslant 0.$$

Conversely, if the second statement holds true, then $L_y > 0$ if and only if $M(y) \succ 0$. Thus a positive definite moment matrix $M(y)$ defines a strictly positive functional L_y which by Theorem 4.8 has a tracial representation. $\blacksquare$

As mentioned above, the Motzkin polynomial M_{nc} is trace-positive but we also have $M_{\mathrm{nc}} \notin \Theta^2$. Thus by Theorem 4.4 there exists at least one truncated tracial moment matrix which is positive definite but has no tracial representation.

EXAMPLE 4.9. Taking the index set

$$(1, X, Y, X^2, XY, YX, Y^2, X^2Y, XY^2, YX^2, Y^2X, X^3, Y^3, XYX, YXY),$$

the matrix

$$M_3(y) := \begin{pmatrix}
1 & 0 & 0 & \frac{7}{4} & 0 & 0 & \frac{7}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{7}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{19}{16} & 0 & \frac{19}{16} & \frac{21}{4} & 0 & 0 & 0 \\
0 & 0 & \frac{7}{4} & 0 & 0 & 0 & 0 & \frac{19}{16} & 0 & \frac{19}{16} & 0 & 0 & \frac{21}{4} & 0 & 0 \\
\frac{7}{4} & 0 & 0 & \frac{21}{4} & 0 & 0 & \frac{19}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{19}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{19}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{7}{4} & 0 & 0 & \frac{19}{16} & 0 & 0 & \frac{21}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{19}{16} & 0 & 0 & 0 & 0 & \frac{9}{8} & 0 & \frac{5}{6} & 0 & 0 & \frac{9}{8} & 0 & 0 \\
0 & \frac{19}{16} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{8} & 0 & \frac{5}{6} & \frac{9}{8} & 0 & 0 & 0 \\
0 & 0 & \frac{19}{16} & 0 & 0 & 0 & 0 & \frac{5}{6} & 0 & \frac{9}{8} & 0 & 0 & \frac{9}{8} & 0 & 0 \\
0 & \frac{19}{16} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{6} & 0 & \frac{9}{8} & \frac{9}{8} & 0 & 0 & 0 \\
0 & \frac{21}{4} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{8} & 0 & \frac{9}{8} & 51 & 0 & 0 & 0 \\
0 & 0 & \frac{21}{4} & 0 & 0 & 0 & 0 & \frac{9}{8} & 0 & \frac{9}{8} & 0 & 0 & 51 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{6}
\end{pmatrix}$$

is a tracial moment matrix of degree 3 in 2 variables and is positive definite. But

$$L_y(M_{\mathrm{nc}}) = M_{\mathrm{nc}}(y) = -\frac{5}{16} < 0.$$

Thus y is not a truncated tracial moment sequence, since otherwise $L_y(p) \geqslant 0$ for all trace-positive polynomials $p \in \mathbb{R}\langle X, Y\rangle_{\leqslant 6}$.

On the other hand, the (free) noncommutative moment problem is always solvable for positive definite moment matrices; cf. Theorem 2.1 in [24]. In our example this means there are symmetric matrices $A, B \in \mathbb{R}^{15\times 15}$ and a vector $v \in \mathbb{R}^{15}$ such that, for all $w \in \langle X, Y\rangle_{\leqslant 3}$,

$$y_w = \langle w(A, B)v, v\rangle.$$

REMARK 4.10. A trace-positive polynomial $f \in \mathbb{R}\langle \underline{X}\rangle$ of degree $2k$ lies in Θ_k^2 if and only if $L_y(f) \geqslant 0$ for all truncated tracial sequences $y = (y_w)_{\leqslant 2k}$ with $M_k(y) \succeq 0$. This condition is obviously satisfied if all truncated tracial sequences $y = (y_w)_{\leqslant 2k}$ with $M_k(y) \succeq 0$ have a tracial representation.

We can prove that trace-positive binary quartics, i.e., polynomials of degree 4 in $\mathbb{R}\langle X, Y\rangle$, lie in Θ_2^2 [7]. This implies by Theorem 4.4 that truncated tracial sequences of degree 4 with a positive definite tracial moment matrix have a tracial moment representation. Example 3.5 shows that a polynomial f can satisfy

$L_y(f) \geqslant 0$ for all truncated tracial sequences $y = (y_w)_{\leqslant 2k}$ although there are truncated tracial sequences with $M_k(y) \succeq 0$ that do not have a tracial representation.

5. APPENDIX. PROOFS OF THE CLAIMS MADE IN EXAMPLES 3.4 AND 3.5

EXAMPLE 3.4 REVISITED. We take the index set $J = (1, X, X^2, X^3, X^4)$ and $y = (1, 1-\sqrt{2}, 1, 1-\sqrt{2}, 1)$. Then there is no symmetric matrix $A \in \mathbb{R}^{t\times t}$ for any $t \in \mathbb{N}$ such that

$$y_w = \operatorname{Tr}(w(A)) \quad \text{for all } w \in J. \tag{5.1}$$

Without loss of generality we can choose A to be diagonal with diagonal elements $a_1, \dots, a_t$. Then $y_w = \operatorname{Tr}(w(A))$ if and only if the following equations hold:

$$\sum_{i=1}^{t} a_i = \sum_{i=1}^{t} a_i^3 = (1-\sqrt{2})t, \tag{5.2}$$

$$\sum_{i=1}^{t} a_i^2 = \sum_{i=1}^{t} a_i^4 = t. \tag{5.3}$$

In the general means inequality

$$\frac{\sum\limits_{i=1}^{t} x_i}{t} \geqslant \sqrt{\frac{\sum\limits_{i=1}^{t} x_i^2}{t}}$$

for the arithmetic and the quadratic mean of $x = (x_1, \dots, x_t) \in \mathbb{R}^t_{\geqslant 0}$, equality holds if and only if all the x_i are the same. Hence (5.3) rewritten as

$$\frac{\sum a_i^2}{t} = 1 = \sqrt{\frac{\sum a_i^4}{t}},$$

gives $a_1^2 = \cdots = a_t^2 = 1$. Therefore,

$$\sum_{i=1}^{t} a_i = \sum_{i=1}^{t} a_i^3 \in \mathbb{Z}.$$

Since $(1-\sqrt{2})t \notin \mathbb{Z}$, this contradicts (5.3) and there is no representation (5.1) of y.

EXAMPLE 3.5 REVISITED. The truncated tracial moment matrix

$$M_2(y) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 4 \end{pmatrix}$$

is positive semidefinite but with respect to the index set

$$(1, X, Y, X^2, XY, YX, Y^2),$$

y has no tracial moment representation (3.3).

Assume $y_w = \sum_{i=1}^{N} \lambda_i \operatorname{Tr}(w(A_1^{(i)}, A_2^{(i)}))$ for some symmetric matrices $A_j^{(i)}$ and $\lambda_i \in \mathbb{R}_{\geqslant 0}$ with $\sum_i \lambda_i = 1$. Setting

$$T^{(i)} := (\operatorname{Tr}(u^* v(A_1^{(i)}, A_2^{(i)})))_{u,v}$$

we have $M_2(y) = \sum_{i=1}^{N} \lambda_i T^{(i)}$. Each $T^{(i)}$ is positive semidefinite, thus in particular $T_{22}^{(i)} = T_{33}^{(i)} = T_{23}^{(i)} =: t_i$ holds for all $i = 1, \ldots, N$. Let d_i be the size of the symmetric matrices $A_j^{(i)}$, $j = 1, 2$. From

$$\frac{1}{d_i^2}\langle A_1^{(i)}, A_1^{(i)}\rangle\langle A_2^{(i)}, A_2^{(i)}\rangle = \operatorname{Tr}({A_1^{(i)}}^2)\operatorname{Tr}({A_2^{(i)}}^2) = t_i^2 = (\operatorname{Tr}(A_1^{(i)} A_2^{(i)}))^2 = \frac{1}{d_i^2}\langle A_1^{(i)}, A_2^{(i)}\rangle^2$$

we obtain by the Cauchy–Schwarz inequality that $A_1^{(i)} = \alpha_i A_2^{(i)}$ for some $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, N$. But then we derive the contradiction

$$\begin{aligned} 2 = M_2(y)_{55} = \sum_{i=1}^{N} \lambda_i T_{55}^{(i)} &= \sum \lambda_i \operatorname{Tr}({A_1^{(i)}}^2 {A_2^{(i)}}^2) = \sum \lambda_i \alpha_i^2 \operatorname{Tr}({A_2^{(i)}}^4) \\ &= \sum \lambda_i \operatorname{Tr}(A_1^{(i)} A_2^{(i)} A_1^{(i)} A_2^{(i)}) = M_2(y)_{45} = 1. \end{aligned}$$

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