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PRODUCT BETWEEN ULTRAFILTERS AND APPLICATIONS TO CONNES' EMBEDDING PROBLEM

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ABSTRACT. In this paper we want to apply the notion of product between ultrafilters to answer several questions which arise around Connes' embedding problem. We shall prove that an ultraproduct of hyperlinear groups is still hyperlinear and consequently the von Neumann algebra of the free group with uncountable many generators is embeddable into R^ω. This follows also from a general construction that allows, starting from an hyperlinear group, to find a family of hyperlinear groups. We shall prove also that the cross product of an hyperlinear group via a profinite action is embeddable into R^ω.

KEYWORDS: *Hyperlinear groups, Connes' embedding problem, product of ultrafilters.*

MSC (2000): Primary 47; Secondary 20.

1. PRELIMINARIES

We start by introducing the notion of product between ultrafilters. It is already known in Model Theory (see, for example, [2]), but it seems nobody applied it to Operator Algebras.

DEFINITION 1.1. Let $\mathcal{U}, \mathcal{V}$ be two ultrafilters respectively on I and J. The tensor product $\mathcal{U} \otimes \mathcal{V}$ is the ultrafilter on $I \times J$ defined by setting

$$X \in \mathcal{U} \otimes \mathcal{V} \Leftrightarrow \{i \in I : \{j \in J : (i,j) \in X\} \in \mathcal{V}\} \in \mathcal{U}.$$

Observe that this is indeed a maximal filter, i.e. an ultrafilter.

REMARK 1.2. This definition is equivalent to the following one:

$$X \in \mathcal{U} \otimes \mathcal{V} \Leftrightarrow \exists A \in \mathcal{U} \text{ such that } \forall i \in A, \pi_J(X \cap \pi_I^{-1}(i)) \in \mathcal{V}$$

where π_I, π_J are the projections of $I \times J$ on the first and second component.

We prefer this second definition since it is easier to apply to prove the following

THEOREM 1.3. *Let* $\{x_i^j\}_{(i,j)\in I\times J} \subseteq \mathbb{R}$ *be bounded. Then*

$$\lim_{i\to\mathcal{U}}\lim_{j\to\mathcal{V}} x_i^j = \lim_{(i,j)\to\mathcal{U}\otimes\mathcal{V}} x_i^j.$$

Proof. Let $x = \lim_{i\to\mathcal{U}}\lim_{j\to\mathcal{V}} x_i^j$. Fixed $\varepsilon > 0$, we notice from the definitions that

$$A = \left\{ i \in I : \left| \lim_{j\to\mathcal{V}} x_i^j - x \right| < \frac{\varepsilon}{2} \right\} \in \mathcal{U} \quad \text{and} \quad A_i = \left\{ j \in J : \left| x_i^j - \lim_{j\to\mathcal{V}} x_i^j \right| < \frac{\varepsilon}{2} \right\} \in \mathcal{V}.$$

Combining these two (by triangle inequality) we get

$$X = \{(i,j) \in I \times J : i \in A, j \in A_i\} \subseteq \{(i,j) \in I \times J : |x_i^j - x| < \varepsilon\}.$$

Since $X \in \mathcal{U}\otimes\mathcal{V}$ and ε was arbitrary, it follows the thesis. ∎

NOTATION 1.4. By ω, ω' we shall denote free ultrafilters on $\mathbb{N}$. R stands for the hyperfinite type II_1 factor. We shall use the classical notation R^ω for the ultrapower of R with regard to ω and denote by τ its trace. By $L(G)$ we denote the von Neumann group algebra of G.

2. MAIN SECTION

The main result is actually an easy consequence of Theorem 1.3, but it gives a tool to pass by the limit on representations. We shall give some applications of this procedure.

PROPOSITION 2.1. *Let* ω, ω' *be two ultrafilters on* $\mathbb{N}$. *Then*

$$(R^\omega)^{\omega'} \cong R^{\omega\otimes\omega'}.$$

Proof. Those von Neumann algebras have the same algebraic structure. So we only have to prove that they have the same trace. It is just a consequence of Theorem 1.3. ∎

We want to apply this result to hyperlinear groups. In order to fully benefit from it we will introduce the notion of hyperlinear pair.

DEFINITION 2.2. By a *central pair* we mean (G, φ), where G is a group and $\varphi : G \to \mathbb{C}$ is a positive defined function, central (i.e. constant on conjugacy classes) and $\varphi(e) = 1$. Let $\mathrm{Cen}(G)$ be the set of those functions on G.

REMARK 2.3. An important element of $\mathrm{Cen}(G)$ is the function δ_e, defined by setting $\delta_e(g) = 0, \forall g \neq e$.

REMARK 2.4. If (G, φ) is a central pair then we have a canonical bi-invariant and bounded metric induced on G by:

$$d(g,h)^2 = 2 - \varphi(g^{-1}h) - \varphi(h^{-1}g) \quad \forall g, h \in G.$$

We recall that one can define the notion of ultraproduct of groups with bi-invariant metric (see [8]). We can use this definition for our particular case of central pairs.

DEFINITION 2.5. Let $(G_n, \varphi_n)_{n\in\mathbb{N}}$ be a sequence of central pairs and ω an ultrafilter. By the *ultraproduct* of the family we mean the central pair:

$$(G, \varphi) = \Big(\prod_n G_n/N, \lim_\omega \varphi_n\Big),$$

where $\prod_n G_n$ is just the cartesian product and $N = \Big\{(g_n) \in \prod_n G_n : \lim_\omega \varphi_n(g_n) \to 1\Big\}$.

We shall denote by $\prod_\omega (G_n, \varphi_n)$ the ultraproduct of central pairs.

NOTE 2.6. It is easy to recognize that our definition of N coincides with the classical one: $N = \Big\{(g_n)_n \in \prod G_n : \lim_\omega d_n(g_n, e_n) \to 0\Big\}$.

DEFINITION 2.7. A central pair (G, φ) is called *hyperlinear* if there exists an homomorphism $\theta_\varphi : G \to U(R^\omega)$ such that

$$\tau(\theta_\varphi(g)) = \varphi(g) \quad \forall g \in G.$$

Let $\mathrm{Hyp}(G) = \{\phi \in \mathrm{Cen}(G) : (G, \phi) \text{ is a hyperlinear pair}\}$.

REMARK 2.8. We recall the original definition by Rădulescu: a countable i.c.c. group G is called hyperlinear if there exists a monomorphism $G \to U(R^\omega)$. It happens if and only if $\delta_e \in \mathrm{Hyp}(G)$ (see Proposition 2.5 of [9]).

REMARK 2.9. If (G, φ) is a hyperlinear pair, then (G, φ) is also a central pair and the induced distance is just the distance in norm 2 in R^ω.

We can now use Proposition 2.1 in order to get the following

PROPOSITION 2.10. *An ultraproduct of hyperlinear pairs is a hyperlinear pair.*

Proof. Take a sequence (G_n, φ_n) of hyperlinear pairs and just embed each pair in an R^ω. The ultraproduct of the family with respect to ω' will sit inside $(R^\omega)^{\omega'} \cong R^{\omega\otimes\omega'}$.

In case we cannot find a "good" ω for all hyperlinear pairs, we just need to adapt our notion of product between two ultrafilters to a notion of ultraproduct of ultrafilters. We shall not do this, as it is just a technical trick and assuming continuum hypothesis these R^ω are isomorphic between themselves anyway. ∎

In order to give some information on the structure of $\mathrm{Hyp}(G)$, we recall that a *monoid* is a set with a binary associative operation admitting a neutral element. If $(X, \cdot)$ is a monoid, an element $x \in X$ is called an *annihilator* if $x \cdot y = y \cdot x = x, \forall y \in X$. The set of annihilators of X is denoted by $0(X)$. Clearly $\mathrm{Cen}(G)$ is a monoid with respect to the pointwise product and $\delta_e \in 0(\mathrm{Cen}(G))$.

PROPOSITION 2.11. $\mathrm{Hyp}(G)$ *is a submonoid of* $\mathrm{Cen}(G)$*. It is closed under ultralimits and convex combinations. Moreover, for G countable,* $0(\mathrm{Hyp}(G)) \neq \emptyset$ *if and only if G is hyperlinear in the classical sense of Rădulescu.*

Proof. The constant function 1 forms with G a hyperlinear pair via the trivial representation. $\mathrm{Hyp}(G)$ is closed under pointwise multiplication because $R^\omega \otimes R^\omega \subset R^\omega$, $\tau(x \otimes y) = \tau(x)\tau(y)$ and so $\theta_{\varphi\cdot\psi} = \theta_\varphi \otimes \theta_\psi$ will do the work.

For the second part note that $\left(G, \lim_\omega \varphi_n\right) \subset \prod_\omega (G, \varphi_n)$ and use our last proposition. For convex combination define an homomorphism of G in $R^\omega \oplus R^\omega$ with the same convex combination of traces.

At last, if G is hyperlinear, then $\delta_e \in \mathrm{Hyp}(G)$ and it is an annihilator. Conversely, let $\psi \in 0(\mathrm{Hyp}(G))$, then $\psi(g)\varphi(g) = \psi(g)$, for all $\varphi \in \mathrm{Hyp}(G)$ and for all $g \in G$. We want to prove that $\psi = \delta_e$. Suppose that there exists $g \in G, g \neq e$ such that $\psi(g) \neq 0$, then $\varphi(g) = 1$ for all $\varphi \in \mathrm{Hyp}(G)$. This is impossible by virtue of Proposition 2.13. ∎

COROLLARY 2.12. An i.c.c. group G embeds in $U(R^\omega)$ if and only if $L(G)$ embeds into R^ω.

Proof. If $L(G) \subseteq R^\omega$ then clearly $G \subset U(R^\omega)$. Conversely, let $\theta : G \to U(R^\omega)$ an embedding. Let τ be the normalized trace on R^ω. Then $|\varphi(g)| = |\tau(\theta(g))| < 1$ for any $g \neq e$ (since G is i.c.c.) and $\varphi \in \mathrm{Hyp}(G)$. Because of Proposition 2.11 we have that $\varphi^n \in \mathrm{Hyp}(G)$ and $\lim_{n\to\omega} \varphi^n \in \mathrm{Hyp}(G)$.

Now $|\varphi(g)| < 1$ for $g \neq e$ so $\lim_n \varphi(g)^n = 0$. This means that $\lim_n \varphi^n = \delta_e$, so $\delta_e \in \mathrm{Hyp}(G)$. This is equivalent to $L(G)$ embeds in R^ω. ∎

This is a simplification of the initial proof given by Rădulescu in [9] and also note that our proof does not need the countability of G.

PROPOSITION 2.13. *A countable group G is hyperlinear if and only if for any* $g \in G \setminus \{e\}$ *there is a hyperlinear pair* (G, φ_g) *such that* $|\varphi_g(g)| < 1$.

Proof. The *only if* part is trivial. Conversely, we need to show that $\delta_e \in \mathrm{Hyp}(G)$. Take $G = \bigcup_n F_n$, with F_n an increasing sequence of finite subsets of G. Define $\varphi_{F_n} = \prod_{g\in F_n} \varphi_g$. According to Proposition 2.11 $\varphi_{F_n} \in \mathrm{Hyp}(G)$ and by the same proposition so is $\varphi = \lim_{n\to\omega} \varphi_{F_n}$.

Now because of the hypothesis $|\varphi_g(g)| < 1$ and because of F_n is an increasing sequence we deduce $|\varphi(g)| < 1$. As in the above corollary we now have $\delta_e = \lim_n \varphi^n$, so $\delta_e \in \mathrm{Hyp}(G)$. ∎

We end this section by presenting a motivation for our definition of $\mathrm{Hyp}(G)$.

PROPOSITION 2.14. *If* $\mathrm{Cen}(F_\infty) = \mathrm{Hyp}(F_\infty)$ *then every countable group is hyperlinear.*

Proof. Let G be a countable group. Let H be a normal subgroup of F_∞ such that $G \cong F_\infty/H$. Let $\varphi_H : F_\infty \to \mathbb{C}$ be the characteristic function of H. We shall prove that $\varphi_H \in \mathrm{Cen}(F_\infty)$. It is easy to see that $\delta_e \in \mathrm{Hyp}(G)$ if and only if $\varphi_H \in \mathrm{Hyp}(F_\infty)$. This will finish the proof.

Now H is normal in F_∞. So for any $g, h \in F_\infty$, $h \in H$ if and only if $ghg^{-1} \in H$. This proves that φ_H is central. To prove that it is also positive defined take $g_1, \ldots, g_n \in F_\infty$. Consider the matrix $\{\varphi_H(g_i^{-1}g_j)\}_{i,j}$ and notice that is the matrix of an equivalence relation on a set with n elements (because H is a subgroup). By permuting elements $(g_i)_i$ we can assume that is a block matrix. This means that $\sum\limits_{i,j=1}^{n} \overline{\lambda}_i\lambda_j\varphi(g_i^{-1}g_j)$ is nonnegative. So φ_H is positive defined. ∎

NOTE 2.15. Our sets $\mathrm{Cen}(G)$ and $\mathrm{Hyp}(G)$ can be generalized to a type II_1 factor instead of just group algebras. Let M be such a factor and consider $B = \{x_n\}_{n\in\mathbb{N}} \subset M$ a basis in $L^2(M, \mathrm{tr})$. Suppose that $x_0 = \mathrm{id}$. We shall consider now $\varphi : B \to \mathbb{C}$ such that $\varphi(x_0) = 1$ and the linear extension of φ to M is positive and tracial (may not be faithful). The problem is that such a linear extension may not be well defined. We formalize this as follows: $\varphi \in \mathrm{Cen}(M)$ if and only if whenever $\varphi(x^*x)$ is well defined then so is $\varphi(xx^*)$ and $\varphi(x^*x) = \varphi(xx^*) \geqslant 0$.

For $\varphi \in \mathrm{Cen}(M)$ we can define M_φ by the GNS-construction. We define $\varphi \in \mathrm{Hyp}(M)$ if and only if this M_φ is embedable in R^ω. As we saw, for $M = L(G)$ and φ_H for H a normal subgroup of G then $L(G)_{\varphi_H} = L(G/H)$.

As another example we may take the crossed product $M = L^\infty(X) \rtimes G$ of a non-free measure preserving action. Take $\{f_i : i \in \mathbb{N}\}$ a basis for $L^\infty(X)$ and $B = \{f_iu_g : i \in \mathbb{N},\ g \in G\}$. Define $\varphi(f_iu_g) = \int\limits_{X_g} f_i$ where $X_g = \{x \in X : gx = x\}$. Then $M_\varphi = M(E_G)$, the Feldmann–Moore construction for the equivalence relation induced by G on X.

3. OTHER APPLICATIONS

3.1. CONSTRUCTION OF UNCOUNTABLE HYPERLINEAR GROUPS.

Now we want to present a construction that, starting from an hyperlinear group G, allows to construct a family of countable and uncountable hyperlinear groups. An easy application of this construction is that the von Neumann algebra of the free group with uncountable many generators $\mathbb{F}_{\aleph_c}$ is embeddable into R^ω. The Hilbert–Schmidt distance between two distinct universal unitaries of $\mathbb{F}_{\aleph_c}$ will be equal to $\sqrt{2}$, giving another proof of the non-separability of R^ω.

DEFINITION 3.1. Let G be a countable group with generators $g_1, g_2, \ldots$. Let $\Im$ be a family of infinite subsets of $\mathbb{N}$ such that $F_1, F_2 \in \Im$ implies $F_1 \cap F_2$ is finite. Now let $F = \{f_1, f_2, \ldots\} \in \Im$, define the sequence $(g_n^F)_n = g_{f_n}$. Let g^F be the sequence g_n^F modulo ω. We can multiply g^{F_1}, g^{F_2} component-wise, by using the relations on G. The group generated by the elements g^F is denoted by $G(\omega,\Im)$.

Notice that $G(\omega,\Im)$ does not depend only on ω and $\Im$, but also on the set of generators chosen.

REMARK 3.2. The generators g^F of $G(\omega,\Im)$ are different elements in $G(\omega,\Im)$. This is because $g_n^{F_1} = g_n^{F_2}$ holds only for a finite number of indexes, by the definition of $\Im$. Since a free ultrafilter does not contain finite sets, g^{F_1} and g^{F_2} must be different.

REMARK 3.3. $G(\omega,\Im)$ can be countable (if the family $\Im$ is countable), but also uncountable. Indeed one can use the Zorn's lemma to prove the existence of an uncountable family $\Im$ which verifies the property $F_1, F_2 \in \Im$ implies $F_1 \cap F_2$ is finite. An elegant example privately suggested by Ozawa is the following: take $t \in [\frac{1}{10}, 1)$, for example $t = 0{,}132483\ldots$, define

$$I_t = \{1, 13, 132, 1324, 13248, 132483, \ldots\}$$

i.e. I_t is the set of the approximation of t. Then $\{I_t\}_{t\in[\frac{1}{10},1)}$ is an uncountable family of subsets of $\mathbb{N}$ such that $I_t \cap I_s$ is finite for all $t \neq s$.

PROPOSITION 3.4. *If G is hyperlinear, then also $G(\omega,\Im)$ is hyperlinear.*

Proof. We want to prove that $G(\omega,\Im) \subset \prod_\omega(G,\delta)$ and that the latter is a hyperlinear pair because of Proposition 2.10. Moreover we shall prove that if in an ultraproduct of central pairs just δ_e appears, then the central positive defined function of the ultraproduct will also be δ_e. This two affirmations will show that $\delta_e \in \mathrm{Hyp}(G(\omega,\Im))$, i.e. $G(\omega,\Im)$ is hyperlinear.

Recall that $\prod_\omega(G_n,\varphi_n) = \left(\prod_n G_n/N, \lim_\omega \varphi_n\right)$, where $\prod_n G_n$ is just the cartesian product and $N = \left\{(g_n) \in \prod_n G_n : \lim_\omega \varphi_n(g_n) \to 1\right\}$. So let G_n a copy of G and $\varphi_n = \delta_e$ for each n. Then $\lim_\omega \varphi_n \in \{0,1\}$. If this limit is 1 for some element, then that element is in N i.e. it is the identity in the ultraproduct. So indeed $\lim_\omega \delta_e = \delta_e$ proving our second affirmation.

Now from the construction of $G(\omega,\Im)$ we see that $G(\omega,\Im) \subset \prod_n G_n$. If an element $g = (g_n)_n$ of $G(\omega,\Im)$ is in N then $\lim_\omega \delta_e(g_n) = 1$ meaning that $g_n = e$ in G for any n in a set in ω. From the definition of $G(\omega,\Im)$ this means that $g = e$. We proved that $G(\omega,\Im) \subset \prod_\omega(G,\delta)$. ∎

It is well known that $\mathbb{F}_\infty$, free group with countable many generators is hyperlinear. We shall denote with $\mathbb{F}_{\aleph_c}$ the free group with $\aleph_c$ many generators (set of continuum power).

COROLLARY 3.5. $\mathbb{F}_{\aleph_c}$ is hyperlinear. In particular R^ω is not separable.

Proof. If $\mathrm{Card}(\Im) = \aleph_c$ then $\mathbb{F}_\infty(\omega, \Im) = \mathbb{F}_{\aleph_c}$, and we can apply the previous proposition.

Representing $L(\mathbb{F}_{\aleph_c})$ on R^ω, the Hilbert–Schmidt distance between two elements of $\mathbb{F}_{\aleph_c}$ will be $\sqrt{2}$. Separability in the weak or in the strong topology is the same and the last one coincide with the Hilbert–Schmidt topology on the bounded sets (see [6]). ∎

NOTE 3.6. We want to underline the importance of inseparability of R^ω in connection to Connes' embedding conjecture: every separable type II_1 factor can be embedded into R^ω (see [1]). Ge and Hadwin proved in [4] that the Continuum Hypothesis implies that $R^\omega \cong R^{\omega'}$ for any pair of free ultrafilters on $\mathbb{N}$. So, Connes' embedding conjecture and Continuum Hypothesis imply the existence of a universal type II_1 factor, universal meaning that it contains a copy of each type II_1 factor. Ozawa proved in [7] that such a universal factor cannot be separable. So, if R^ω were separable, Connes' embedding conjecture would be false or undecidable.

PROBLEM 3.7. What kind of groups have the shape $G(\omega, \Im)$? Is it true that if $\{R_a\}_{a\in A}$ is the set of distinct relations on G and $B \subseteq A$, then there exist ω and $\Im$ such that the set of relations of $G(\omega, \Im)$ is $\{R_a\}_{a\in B}$?

3.2. CROSS PRODUCT VIA PROFINITE ACTIONS.

We want to apply Proposition 2.1 also to some other type II_1 factors. For this we ask ourselves when the crossed product $L^\infty(X) \rtimes_\alpha G$ for a free action α embeds in R^ω. Of course when this happens G has to be hyperlinear. We shall prove the converse in the easy case in which α is profinite.

DEFINITION 3.8. Let α be an action of a group G on a von Neumann algebra P. Then α is called *profinite* if there is an increasing sequence of finite dimensional G-invariant subalgebras $A_1 \subset A_2 \subset \cdots$ such that $P = \left(\bigcup_n A_n\right)''$.

PROPOSITION 3.9. *Let G be a hyperlinear group and α be a profinite action of G on X. Then $L^\infty(X) \rtimes_\alpha G$ is embeddable into R^ω.*

Proof. The crossed product is generated on $L^2(X) \otimes l^2G$ by the operators $\alpha(g) \otimes \lambda(g)$ for $g \in G$ and $m_f \otimes 1$ for $f \in L^\infty(X)$ (here λ is the regular representation of G on l^2G and m_f is the multiplication operator).

Let $L^\infty(X) = \left(\bigcup_n A_n\right)''$ with A_n G-invariant and finite dimensional. We can then form $A_n \rtimes_\alpha G$ and $L^\infty(X) \rtimes_\alpha G = \left(\bigcup_n A_n \rtimes_\alpha G\right)''$. Looking at the above definition of crossed product we can deduce that $A_n \rtimes_\alpha G \subset M_{k_n} \otimes L(G)$. Here entered the fact that A_n is finite dimensional. Now, because G is hyperlinear

$M_{k_n} \otimes L(G) \subset R \otimes R^\omega \subset R^\omega$. We can then embed $\bigcup_n A_n \rtimes_\alpha G$ in $(R^\omega)^{\omega'}$ so that $L^\infty(X) \rtimes_\alpha G \subset R^{\omega\otimes\omega'}$. ∎

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