J. OPERATOR THEORY
68:1(2012), 179–203

UNITARY EQUIVALENCE OF A MATRIX TO ITS TRANSPOSE

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Communicated by Nikolai K. Nikolski

ABSTRACT. Motivated by a problem of Halmos, we obtain a canonical decomposition for complex matrices which are unitarily equivalent to their transpose (UET). Surprisingly, the naïve assertion that a matrix is UET if and only if it is unitarily equivalent to a complex symmetric matrix holds for matrices 7×7 and smaller, but fails for matrices 8×8 and larger.

KEYWORDS: *Complex symmetric matrix, complex symmetric operator, unitary orbit, unitary equivalence, linear preserver, skew-Hamiltonian, numerical range, transpose, UECSM.*

MSC (2000): 15A57, 47A30.

1. INTRODUCTION

In his problem book ([16], Proposition 159) Halmos asks whether every square complex matrix is unitarily equivalent to its transpose (UET). For example, every finite Toeplitz matrix is unitarily equivalent to its transpose via the permutation matrix which reverses the order of the standard basis. Upon appealing to the Jordan canonical form, it follows that every square complex matrix T is *similar* to its transpose T^{t}. Thus similarity invariants are insufficient to handle Halmos' problem.

In his discussion, Halmos introduces the single counterexample

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1}$$

and proves that it is not UET. A more recent example, due to George and Ikrahmov [14], is

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 0 & 2 & 5 \end{pmatrix}. \tag{1.2}$$

While settling Halmos' original question in the negative, (1.1) and (1.2) are only isolated examples. Our present aim is to obtain a complete characterization and canonical decomposition of those matrices which are UET.

From some perspectives, every square matrix is close to being UET. Indeed, it is known that for each $T \in M_n(\mathbb{C})$ there exist unitary matrices U and V such that $T = UT^{\mathrm{t}}V$ [15]. In fact, Taussky and Zassenhaus proved that for any field $\mathbb{K}$ each $T \in M_n(\mathbb{K})$ is similar to its transpose [29] (for congruence see [7]).

In addition to Halmos' question, this article is partially motivated by the recent explosion in work on linear preservers. In particular, linear maps on $M_n(\mathbb{C})$ of the form $\phi(T) = UT^{\mathrm{t}}U^*$ (where U is unitary) feature prominently in the literature [3], [5], [13], [18], [19], [20], [25]. Our work completely classifies the fixed points of such maps.

Halmos' example (1.1) recently appeared in another context which also motivated the authors to consider his problem. It is well-known that every square complex matrix is *similar* to a complex symmetric matrix (i.e., $T = T^{\mathrm{t}}$) ([17], Theorem 4.4.9). Indeed, various complex symmetric canonical forms have been proposed throughout the years [6], [8], [17], [26], [33]. It turns out that Halmos' matrix (1.1) was among the first several matrices demonstrated to be *not* unitarily equivalent to a complex symmetric matrix (UECSM) ([12], Example 1; see also Example 6 of [11] and Example 1 of [31]).

In [1] it was remarked in passing that

$$T \text{ is UECSM} \quad \Rightarrow \quad T \text{ is UET}, \tag{1.3}$$

raising the question of whether the converse is also true. It turns out that Vermeer had already studied this problem over $\mathbb{R}$ a few years earlier and provided an 8×8 counterexample ([31], Example 3). In fact, we prove that there are no smaller counterexamples: the implication (1.3) can be reversed for matrices 7×7 and smaller, but not for matrices 8×8 or larger.

To state our main results, we require a few definitions.

DEFINITION 1.1. A $2d \times 2d$ block matrix of the form

$$T = \begin{pmatrix} A & B \\ D & A^{\mathrm{t}} \end{pmatrix} \tag{1.4}$$

where $B^{\mathrm{t}} = -B$ and $D^{\mathrm{t}} = -D$ is called *skew-Hamiltonian* (SHM). A matrix $T \in M_{2d}(\mathbb{C})$ that is unitarily equivalent to a skew-Hamiltonian matrix is called UESHM.

Needless to say, the matrices A, B, and D in (1.4) are necessarily $d \times d$. A short computation reveals that if T is skew-Hamiltonian, then

$$T = -\Omega T^{\mathrm{t}} \Omega = \Omega T^{\mathrm{t}} \Omega^*, \tag{1.5}$$

where

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1.6}$$

In particular, it follows immediately from (1.5) that every SHM is UET.

We are now ready to state our main result:

THEOREM 1.2. *A matrix* $T \in M_n(\mathbb{C})$ *is* UET *if and only if it is unitarily equivalent to a direct sum of (some of the summands may be absent):*

(i) *Irreducible complex symmetric matrices (*CSMs*).*

(ii) *Irreducible skew-Hamiltonian matrices (*SHMs*). Such matrices are necessarily* 8×8 *or larger.*

(iii) $2d \times 2d$ *blocks of the form*

$$\begin{pmatrix} A & 0 \\ 0 & A^{\mathrm{t}} \end{pmatrix} \tag{1.7}$$

where $A \in M_d(\mathbb{C})$ *is irreducible and neither* UECSM *nor* UESHM. *Such matrices are necessarily* 6×6 *or larger.*
Moreover, the unitary orbits of the three classes described above are pairwise disjoint.

We use the term *irreducible* in the operator-theoretic sense. Namely, a matrix $T \in M_n(\mathbb{C})$ is called irreducible if T is not unitarily equivalent to a direct sum $A \oplus B$ where $A \in M_d(\mathbb{C})$ and $B \in M_{n-d}(\mathbb{C})$ for some $1 < d < n$. Equivalently, T is irreducible if and only if the only normal matrices commuting with T are multiples of the identity. In the following, we shall denote unitary equivalence by $\cong$.

2. SOME COROLLARIES OF THEOREM 1.2

The proof of Theorem 1.2 requires a number of preliminary results and is consequently deferred until Section 8. We focus here on a few immediate corollaries.

COROLLARY 2.1. *If* $T \in M_n(\mathbb{C})$ *is irreducible and* UET, *then* T *is either* UECSM *or* UESHM.

COROLLARY 2.2. *If* $n \leqslant 5$ *and* $T \in M_n(\mathbb{C})$ *is* UET, *then* T *is unitarily equivalent to a direct sum of irreducible complex symmetric matrices.*

Our next corollary implies that the converse of the implication (1.3) holds for matrices 7×7 and smaller. On the other hand, it is possible to show (see Section 7) that the converse fails for matrices 8×8 and larger.

COROLLARY 2.3. *If* $n \leqslant 7$ *and* $T \in M_n(\mathbb{C})$ *is* UET, *then* T *is* UECSM.

Proof. By Theorem 1.2, T is unitarily equivalent to a direct sum of blocks of type I or III. It turns out (see Lemma 5.1) that any matrix of the form (1.7) is UECSM (although clearly not irreducible and hence not of type I) whence T is itself UECSM. ∎

We close this section with a few remarks about 2×2 and 3×3 matrices. For 2×2 matrices it has long been known [22] that $A \cong B$ if and only if $\Phi(A) = \Phi(B)$ where $\Phi : M_2(\mathbb{C}) \to \mathbb{C}^3$ is the function

$$\Phi(X) = (\operatorname{tr} X, \operatorname{tr} X^2, \operatorname{tr} X^* X).$$

Since $\Phi(X) = \Phi(X^{\mathrm{t}})$ for all $X \in M_2(\mathbb{C})$ we immediately obtain the following useful lemma from Corollary 2.3.

LEMMA 2.4. *Every* 2×2 *matrix is* UECSM.

For other proofs of the preceding lemma see Corollary 3 of [1], Corollary 3.3 of [4], [10], Example 6 of [11], Corollary 1 of [12], p. 477 of [21], or Corollary 3 of [30].

Based on Specht's Criterion [28], Pearcy obtained a list of nine words in X and X^* so that $A, B \in M_3(\mathbb{C})$ are unitarily equivalent if the traces of these words are equal for $X = A$ and $X = B$ [23]. Later, Sibirskiĭ [27] showed that two of these words are unnecessary and, moreover, that $A \cong B$ if and only if $\Phi(A) = \Phi(B)$ where $\Phi : M_3(\mathbb{C}) \to \mathbb{C}^7$ is the function defined by

$$\Phi(X) = (\operatorname{tr} X,\ \operatorname{tr} X^2,\ \operatorname{tr} X^3,\ \operatorname{tr} X^* X,\ \operatorname{tr} X^* X^2,\ \operatorname{tr} X^{*2} X^2,\ \operatorname{tr} X^* X^2 X^{*2} X). \tag{2.1}$$

The preceding can be used to develop a simple test for checking whether a 3×3 matrix is UECSM. Some general approaches to this problem in higher dimensions can be found in [1], [10], [30] (see also Theorem 3 of [31]).

PROPOSITION 2.5. *If* $X \in M_3(\mathbb{C})$, *then* X *is* UECSM *if and only if*

$$\operatorname{tr} X^* X^2 X^{*2} X = \operatorname{tr} X X^{*2} X^2 X^*. \tag{2.2}$$

Proof. Since a 3×3 matrix is UECSM if and only if it is UET (by Theorem 1.2), it suffices to prove that (2.2) is equivalent to asserting that $X \cong X^{\mathrm{t}}$. This in turn is equivalent to proving that $\Phi(X) = \Phi(X^{\mathrm{t}})$ for the function $\Phi : M_3(\mathbb{C}) \to \mathbb{C}^7$ defined by (2.1). Let $\phi_1(X), \phi_2(X), \ldots, \phi_7(X)$ denote the entries of (2.1) and note that $\phi_i(X) = \phi_i(X^{\mathrm{t}})$ for $i = 1, 2, 3$. Moreover, a short computation shows that

$$\begin{aligned}
\phi_4(X) &= \operatorname{tr} X^* X = \operatorname{tr} X X^* = \operatorname{tr} X^{*\mathrm{t}} X^{\mathrm{t}} = \operatorname{tr} X^{\mathrm{t}*} X^{\mathrm{t}} = \phi_4(X^{\mathrm{t}}), \\
\phi_5(X) &= \operatorname{tr} X^* X^2 = \operatorname{tr} X^2 X^* = \operatorname{tr} X^{*\mathrm{t}} X^{2^{\mathrm{t}}} = \operatorname{tr} X^{\mathrm{t}*} X^{\mathrm{t}^2} = \phi_5(X^{\mathrm{t}}), \\
\phi_6(X) &= \operatorname{tr} X^{*2} X^2 = \operatorname{tr}(X^2)^{\mathrm{t}} (X^*)^{2^{\mathrm{t}}} = \operatorname{tr}(X^{\mathrm{t}})^2 (X^{\mathrm{t}})^{*2} = \operatorname{tr}\, (X^{\mathrm{t}})^{*2} (X^{\mathrm{t}})^2 = \phi_6(X^{\mathrm{t}}).
\end{aligned}$$

Thus X is UECSM if and only if $\phi_7(X) = \phi_7(X^{\mathrm{t}})$. Since

$$\phi_7(X^{\mathrm{t}}) = \operatorname{tr} X^{\mathrm{t}*} X^{\mathrm{t}^2} (X^{\mathrm{t}})^{*2} X^{\mathrm{t}} = X X^{*2} X^2 X^*$$

the desired result follows. ∎

Pearcy also proved that one need only check traces of words of length $\leqslant 2n^2$ to test two $n \times n$ matrices for unitary equivalence ([24], Theorem 2). As Corollary 2.3 and Proposition 2.5 suggest, a similar algorithm could be developed for $n \leqslant 7$. See [9] for a discussion of the 4×4 case.

3. BUILDING BLOCKS OF TYPE I: UECSMs

In this section we gather together some information about UECSMs that will be necessary in what follows. We first require the notion of a conjugation.

DEFINITION 3.1. A function $C : \mathbb{C}^n \to \mathbb{C}^n$ is called a *conjugation* if it has the following three properties:
(i) C is conjugate-linear;
(ii) C is isometric: $\langle x, y\rangle = \langle Cy, Cx\rangle$ for all $x, y \in \mathbb{C}^n$;
(iii) C is involutive: $C^2 = I$.

In light of the polarization identity, condition (ii) in the preceding definition is equivalent to asserting that $\|Cx\| = \|x\|$ for all $x \in \mathbb{C}^n$. Let us now observe that T is a complex symmetric matrix (i.e., $T = T^{\mathrm{t}}$) if and only if $T = JT^*J$, where J denotes the *canonical conjugation*

$$J(z_1, z_2, \ldots, z_n) = (\overline{z}_1, \overline{z}_2, \ldots, \overline{z}_n) \tag{3.1}$$

on $\mathbb{C}^n$. Moreover, we also have

$$\overline{T} = JTJ, \tag{3.2}$$

$$T^{\mathrm{t}} = JT^*J, \tag{3.3}$$

where $\overline{T}$ is the entry-by-entry complex conjugate of T (these relations have long been used in the study of Hankel operators [2]).

LEMMA 3.2. *C is a conjugation on $\mathbb{C}^n$ if and only if $C = UJ$ where U is a complex symmetric (i.e., $U = U^{\mathrm{t}}$) unitary matrix.*

Proof. If C is a conjugation on $\mathbb{C}^n$, then $U = CJ$ is an isometric linear map and hence unitary. It follows from (3.2) that $U\overline{U} = UJUJ = C^2 = I$ whence $\overline{U} = U^*$ so that $U = U^{\mathrm{t}}$ as claimed. Conversely, if U is a complex symmetric unitary matrix then $C = UJ$ is conjugate-linear, isometric, and satisfies $C^2 = I$ by a similar computation. ∎

The relevance of conjugations to our work lies in the following lemma (the equivalence of (i) and (ii) was first noted in Theorem 3 of [31]).

LEMMA 3.3. *For $T \in M_n(\mathbb{C})$, the following are equivalent:*
(i) *T is* UECSM;
(ii) *$T = UT^{\mathrm{t}}U^*$ for some complex symmetric unitary matrix U;*
(iii) *$T = CT^*C$ for some conjugation C on $\mathbb{C}^n$.*

In particular, if T is UECSM, *then T is* UET.

Proof. (i) $\Rightarrow$ (ii) If Q is unitary and $Q^*TQ = S$ is complex symmetric, then $Q^*TQ = (Q^*TQ)^{\mathrm{t}} = Q^{\mathrm{t}}T^{\mathrm{t}}\overline{Q}$ whence $T = UT^{\mathrm{t}}U^*$ where $U = QQ^{\mathrm{t}}$.

(ii) $\Rightarrow$ (iii) If $T = UT^{\mathrm{t}}U^*$ where $U = U^{\mathrm{t}}$ is unitary, then $C = UJ$ is a conjugation by Lemma 3.2. Since $U = CJ$ and $U^* = JC$, it follows from (3.3) that $T = UT^{\mathrm{t}}U^* = CJT^{\mathrm{t}}JC = CT^*C$.

(iii) $\Rightarrow$ (i) Suppose that $T = CT^*C$ for some conjugation C on $\mathbb{C}^n$. By Lemma 1 of [11] there exists an orthonormal basis $e_1, e_2, \ldots, e_n$ such that $Ce_i = e_i$ for $i = 1, 2, \ldots, n$. Let $Q = (e_1|e_2|\cdots|e_n)$ be the unitary matrix whose columns are these basis vectors. The matrix $S = Q^*TQ$ is complex symmetric since the ijth entry $[S]_{ij}$ of S satisfies

$$[S]_{ij} = \langle Te_j, e_i \rangle = \langle CT^*Ce_j, e_i \rangle = \langle e_i, T^*e_j \rangle = \langle Te_i, e_j \rangle = [S]_{ji}. \quad \blacksquare$$

In order to obtain the decomposition guaranteed by Theorem 1.2, we must be able to break apart matrices which are UECSM into direct sums of simpler matrices. Unfortunately, the class UECSM is not closed under restrictions to direct summands (see Example 5.2 in Section 5), making our task more difficult. We begin by considering a special case where such a reduction is possible.

LEMMA 3.4. *Suppose that $T \in M_n(\mathbb{C})$ is* UECSM *and let C be a conjugation for which $T = CT^*C$. If $\mathcal{M}$ is a proper, nontrivial subspace of $\mathbb{C}^n$ that reduces T and satisfies $\mathcal{M} \cap C\mathcal{M} \neq \{0\}$, then $T \cong T_1 \oplus T_2$ where T_1 and T_2 are* UECSM.

Proof. Let us initially regard T as a linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$ and work in a coordinate-free manner until the conclusion of the proof. Since $\mathcal{M}$ reduces $T = CT^*C$, a short computation reveals that

$$T(C\mathcal{M}) = (TC)\mathcal{M} = (CT^*)\mathcal{M} = C(T^*\mathcal{M}) \subseteq C\mathcal{M} \tag{3.4}$$

whence $C\mathcal{M}$ is T-invariant. Upon replacing T with T^* in (3.4) we find that $C\mathcal{M}$ is also T^*-invariant whence $C\mathcal{M}$ reduces T. It follows that the subspaces

$$\mathcal{H}_1 = \mathcal{M} \cap C\mathcal{M}, \quad \mathcal{H}_2 = \mathcal{H}_1^{\perp},$$

both reduce T. Observe that $\mathcal{H}_1, \mathcal{H}_2$ are both proper and nontrivial by assumption. Moreover, note that $\mathcal{H}_1$ is C-invariant by definition and, since $C^2 = I$, it follows that $C\mathcal{H}_1 = \mathcal{H}_1$. In light of the fact that C is isometric it also follows that $C\mathcal{H}_2 = \mathcal{H}_2$. With respect to the orthogonal decomposition $\mathbb{C}^n = \mathcal{H}_1 \oplus \mathcal{H}_2$, we have the block-operator decompositions $T = T_1 \oplus T_2$ and $C = C_1 \oplus C_2$ where C_1, C_2 are conjugations on $\mathcal{H}_1, \mathcal{H}_2$, respectively. Expanding the identity $T = CT^*C$ in terms of block operators reveals that $T_1 = C_1T_1^*C_1$ and $T_2 = C_2T_2^*C_2$. Upon identifying T_1, T_2 as matrices computed with respect to some orthonormal bases of $\mathcal{H}_1, \mathcal{H}_2$, respectively, we conclude that T_1 and T_2 are UECSM by Lemma 3.3. $\blacksquare$

If a matrix which is UECSM is reducible and the preceding lemma does not apply, then we have the following decomposition.

LEMMA 3.5. *Suppose that $T = A \oplus B \in M_n(\mathbb{C})$ is* UECSM, *$A \in M_d(\mathbb{C})$, $B \in M_{n-d}(\mathbb{C})$, and let C be a conjugation on $\mathbb{C}^n$ for which $T = CT^*C$. If $\mathcal{M} \cap C\mathcal{M} = \{0\}$ for every proper reducing subspace $\mathcal{M}$ of T, then $n = 2d$ and $T \cong A \oplus A^{\mathrm{t}}$ where $A \in M_d(\mathbb{C})$ is irreducible.*

Proof. Since $T = A \oplus B$, the subspaces $\mathcal{M} = \mathbb{C}^d \oplus \{0\}$ and $\mathcal{M}^\perp = \{0\} \oplus \mathbb{C}^{n-d}$ reduce T. By hypothesis and the fact that $C(\mathcal{M}^\perp) = (C\mathcal{M})^\perp$ it follows that

$$\mathcal{M} \cap C\mathcal{M} = \{0\}, \tag{3.5}$$

$$\mathcal{M}^\perp \cap C(\mathcal{M}^\perp) = \{0\}. \tag{3.6}$$

Since $\dim C\mathcal{M} = \dim \mathcal{M} = d$, it follows from (3.5) that $2d \leqslant n$. Similarly, since $\dim C(\mathcal{M}^\perp) = \dim \mathcal{M}^\perp = n - d$, it follows from (3.6) that $2(n-d) \leqslant n$. Putting these two inequalities together reveals that $n = 2d$. In fact, a similar argument shows that *every* proper, nontrivial reducing subspace of T is of dimension d. Moreover, it also follows that A and B are both irreducible, since otherwise T would have a nontrivial reducing subspace of dimension $< d$.

Letting

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{3.7}$$

denote the orthogonal projection onto $\mathcal{M}$, we note that $R = CPC$ is the orthogonal projection onto $C\mathcal{M}$. Moreover, since $T = CT^*C$ it follows that $C\mathcal{M}$ reduces T. Writing

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^* & R_{22} \end{pmatrix}, \tag{3.8}$$

where $R_{11}^* = R_{11}$ and $R_{22}^* = R_{22}$, and then expanding out the equation $RT = TR$ block-by-block we find that $AR_{11} = R_{11}A$, $BR_{22} = R_{22}B$, and

$$AR_{12} = R_{12}B. \tag{3.9}$$

Since A and B are both irreducible, it follows that $R_{11} = \alpha I$ and $R_{22} = \beta I$ where $0 \leqslant \alpha, \beta \leqslant 1$ (since R is an orthogonal projection). Since $R^2 = R$ we also have

$$\begin{pmatrix} \alpha^2 I + R_{12}R_{12}^* & (\alpha+\beta)R_{12} \\ (\alpha+\beta)R_{12}^* & \beta^2 I + R_{12}^*R_{12} \end{pmatrix} = \begin{pmatrix} \alpha I & R_{12} \\ R_{12}^* & \beta I \end{pmatrix}, \tag{3.10}$$

which presents three distinct possibilities.

Case 1. Suppose that $\alpha = 1$. Looking at the $(1,1)$ entry in (3.10), we find that $R_{12}R_{12}^* = 0$ whence $R_{12} = 0$. From the $(2,2)$ entry in (3.10) we find that $\beta^2 = \beta$ whence either $\beta = 0$ or $\beta = 1$. Both cases are easy to dispatch.

(i) If $\beta = 0$, then $R = P$, the orthogonal projection (3.7) onto $\mathcal{M} = \mathbb{C}^d \oplus \{0\}$. Since R is the orthogonal projection onto $C\mathcal{M}$ we have $\mathcal{M} = C\mathcal{M}$, which contradicts the hypothesis that $\mathcal{M} \cap C\mathcal{M} = \{0\}$.

(ii) If $\beta = 1$, then $R = I$ which contradicts the fact that $\dim C\mathcal{M} = d$.

Case 2. Suppose that $\alpha = 0$. As before, we find that $R_{12} = 0$ and $\beta^2 = \beta$. Since $\beta = 0$ leads to the contradiction $R = 0$, it follows that $\beta = 1$ and

$$R = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

the orthogonal projection onto $\mathcal{M}^\perp = \{0\} \oplus \mathbb{C}^d$ (i.e., $C\mathcal{M} = \mathcal{M}^\perp$). With respect to the orthogonal decomposition $\mathbb{C}^{2n} = \mathbb{C}^d \oplus \mathbb{C}^d$ we have

$$C = \begin{pmatrix} 0 & UJ \\ U^{\mathrm{t}}J & 0 \end{pmatrix}$$

where U is a unitary matrix and J is the canonical conjugation on $\mathbb{C}^d$ (by Lemma 3.2). Expanding the equality $T = CT^*C$ block-by-block reveals that $A = UJB^*U^{\mathrm{t}}J = UB^{\mathrm{t}}U^*$, from which we conclude that $A \cong B^{\mathrm{t}}$. Thus $T \cong A \oplus A^{\mathrm{t}}$, as claimed.

Case 3. Suppose that $0 < \alpha < 1$. In this case an examination of the $(1,1)$ entry in (3.10) reveals that $R_{12} \neq 0$. Looking next at the $(1,2)$ entry in (3.10) we find that $\beta = 1 - \alpha$ from which

$$R_{12}^* R_{12} = R_{12} R_{12}^* = \alpha(1-\alpha)I$$

follows upon consideration of the $(1,1)$ and $(2,2)$ entries of (3.10). In other words, $R_{12} = \sqrt{\alpha(1-\alpha)}U^*$ for some $d \times d$ unitary matrix U so that

$$R = \begin{pmatrix} \alpha I & \sqrt{\alpha(1-\alpha)}U^* \\ \sqrt{\alpha(1-\alpha)}U & 1-\alpha \end{pmatrix}.$$

By (3.9) it also follows that $UA = BU$ whence $A \cong B$.

Now recall that $R = CPC$ is the orthogonal projection onto $C\mathcal{M}$ and that $C = SJ$ for some $n \times n$ complex symmetric unitary matrix

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{t}} & S_{22} \end{pmatrix}.$$

Writing $CP = RC$ as $SJP = RSJ$ (where J denotes the canonical conjugation on $\mathbb{C}^n$) we note that $JP = PJ$ by (3.7) and conclude that $SP = RS$. In other words,

$$\begin{pmatrix} S_{11} & 0 \\ S_{12}^{\mathrm{t}} & 0 \end{pmatrix} = \begin{pmatrix} \alpha S_{11} + \sqrt{\alpha(1-\alpha)}U^* S_{12}^{\mathrm{t}} & \alpha S_{12} + \sqrt{\alpha(1-\alpha)}U^* S_{22} \\ \sqrt{\alpha(1-\alpha)}US_{11} + (1-\alpha)S_{12}^{\mathrm{t}} & \sqrt{\alpha(1-\alpha)}US_{12} + (1-\alpha)S_{22} \end{pmatrix}. \tag{3.11}$$

Examining the $(1,1)$ entry of (3.11) and using the fact that $S_{11}^{\mathrm{t}} = S_{11}$ we find that

$$S_{12} = \sqrt{\frac{1-\alpha}{\alpha}}\, S_{11} U^{\mathrm{t}}. \tag{3.12}$$

The $(1,2)$ entry of (3.11) now tells us that

$$S_{12} = -\sqrt{\frac{1-\alpha}{\alpha}}\, U^* S_{22}. \tag{3.13}$$

From (3.12) and (3.13) we have $S_{22} = -US_{11}U^{\mathrm{t}}$ and hence

$$S = \begin{pmatrix} S_{11} & \sqrt{\frac{1-\alpha}{\alpha}}S_{11}U^{\mathrm{t}} \\ \sqrt{\frac{1-\alpha}{\alpha}}US_{11} & -US_{11}U^{\mathrm{t}} \end{pmatrix}.$$

Recalling that $S = S^{\mathrm{t}}$ is unitary, we have $I = S^*S = SS^*$, which can be expanded block-by-block to reveal that $S_{11}S_{11}^* = S_{11}^*S_{11} = \alpha I$. In other words, $S_{11} = \sqrt{\alpha}\,V$ for some $d \times d$ complex symmetric unitary matrix V. Thus S assumes the form

$$S = \begin{pmatrix} \sqrt{\alpha}\,V & \sqrt{1-\alpha}\,VU^{\mathrm{t}} \\ \sqrt{1-\alpha}\,UV & -\sqrt{\alpha}\,UVU^{\mathrm{t}} \end{pmatrix}.$$

Since $TC = CT^*$ and $C = SJ$ we have $TS = ST^{\mathrm{t}}$, which yields

$$\begin{pmatrix} \sqrt{\alpha}\,AV & \sqrt{1-\alpha}\,AVU^{\mathrm{t}} \\ \sqrt{1-\alpha}\,BUV & -\sqrt{\alpha}BUVU^{\mathrm{t}} \end{pmatrix} = \begin{pmatrix} \sqrt{\alpha}\,VA^{\mathrm{t}} & \sqrt{1-\alpha}\,VU^{\mathrm{t}}B^{\mathrm{t}} \\ \sqrt{1-\alpha}\,UVA^{\mathrm{t}} & -\sqrt{\alpha}\,UVU^{\mathrm{t}}B^{\mathrm{t}} \end{pmatrix}.$$

From the preceding we find $A \cong A^{\mathrm{t}}$, $B \cong B^{\mathrm{t}}$ (i.e., A and B are UET), and $A \cong B^{\mathrm{t}}$. In particular, $T \cong A \oplus A^{\mathrm{t}}$, as claimed. ∎

Putting the preceding lemmas together we obtain the following proposition.

PROPOSITION 3.6. *If $T \in M_n(\mathbb{C})$ is* UECSM, *then T is unitarily equivalent to a direct sum of matrices, each of which is either*

(i) *an irreducible complex symmetric matrix;*

(ii) *a block matrix of the form $A \oplus A^{\mathrm{t}}$ where A is irreducible.*

Proof. We proceed by induction on n. The case $n = 1$ is trivial. For our induction hypothesis, suppose that the theorem holds for $M_k(\mathbb{C})$ with $k = 1, 2, \ldots, n-1$. Now suppose that $T \in M_n(\mathbb{C})$ is UECSM. If T is irreducible, then it is already of the form (i) and there is nothing to prove. Let us therefore assume that T is reducible. There are now two possibilities:

Case 1. If T has a proper, nontrivial reducing subspace $\mathcal{M}$ such that $\mathcal{M} \cap C\mathcal{M} \neq \{0\}$, then Lemma 3.4 asserts that $T \cong A \oplus B$ where A and B are UECSM. The desired conclusion now follows by the induction hypothesis.

Case 2. If every proper, nontrivial reducing subspace $\mathcal{M}$ of T satisfies $\mathcal{M} \cap C\mathcal{M} = \{0\}$, then T satisfies the hypotheses of Lemma 3.5. Therefore $n = 2d$ and $T \cong A \oplus A^{\mathrm{t}}$ for some irreducible $A \in M_d(\mathbb{C})$. ∎

4. BUILDING BLOCK II: UESHMs

As we noted in the introduction, there exist matrices which are UET but *not* UECSM. In order to characterize those matrices which are UET, we must introduce a new family of matrices along with the following definition.

DEFINITION 4.1. A function $K : \mathbb{C}^n \to \mathbb{C}^n$ is called an *anticonjugation* if it satisfies the following three properties:

(i) K is conjugate-linear;

(ii) K is isometric: $\langle x, y\rangle = \langle Ky, Kx\rangle$ for all $x, y \in \mathbb{C}^n$;

(iii) K is skew-involutive: $K^2 = -I$.

Henceforth, the capital letter K will be reserved exclusively to denote anticonjugations. The proof of the following lemma is virtually identical to that of Lemma 3.2 and is therefore omitted.

LEMMA 4.2. *K is an anticonjugation on $\mathbb{C}^n$ if and only if $K = SJ$ where S is a skew-symmetric (i.e., $S = -S^{\mathrm{t}}$) unitary matrix.*

Unlike conjugations, which can be defined on spaces of arbitrary dimension, anticonjugations can act only on spaces of *even* dimension.

LEMMA 4.3. *If n is odd, then there does not exist an anticonjugation K on $\mathbb{C}^n$.*

Proof. By Lemma 4.2 it suffices to prove that there does not exist an $n \times n$ skew-symmetric unitary matrix S. If S is such a matrix, then $\det S = \det(S^{\mathrm{t}}) = \det(-S) = (-1)^n \det S = -\det S$ since n is odd. This implies that $\det S = 0$, a contradiction. ∎

Observe that skew-symmetric unitaries and their corresponding anticonjugations exist when $n = 2d$ is even. For example, let S in Lemma 4.2 be given by

$$S = \bigoplus_{i=1}^{d} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

LEMMA 4.4. *If K is an anticonjugation on $\mathbb{C}^{2d}$, then there is an orthonormal basis $e_1, e_2, \ldots, e_{2d}$ of $\mathbb{C}^{2d}$ such that*

$$Ke_i = \begin{cases} e_{i+d} & \text{if } 1 \leqslant i \leqslant d, \\ -e_{i-d} & \text{if } d+1 \leqslant i \leqslant 2d. \end{cases} \tag{4.1}$$

Proof. The desired basis can be constructed inductively. First note that we have $\langle x, Kx\rangle = \langle K^2x, Kx\rangle = -\langle x, Kx\rangle$ whence

$$\langle x, Kx\rangle = 0 \tag{4.2}$$

for every $x \in \mathbb{C}^{2d}$. Now let e_1 be any unit vector, set $e_{d+1} = Ke_1$, and note that $Ke_{d+1} = -e_1$ since $K^2 = -I$. In light of (4.2) the set $\{e_1, e_{d+1}\}$ is orthonormal. Next select a unit vector e_2 in $\{e_1, e_{d+1}\}^\perp$ and let $e_{d+2} = Ke_2$. A few additional computations reveal that the set $\{e_1, e_2, e_{d+1}, e_{d+2}\}$ is orthonormal:

$$\langle e_{d+2}, e_1\rangle = \langle Ke_1, Ke_{d+2}\rangle = -\langle e_{d+1}, e_2\rangle = 0,$$
$$\langle e_{d+1}, e_{d+2}\rangle = \langle Ke_{d+2}, Ke_{d+1}\rangle = \langle e_2, e_1\rangle = 0.$$

Continuing in this fashion we obtain the desired orthonormal basis. ∎

LEMMA 4.5. *For $T \in M_n(\mathbb{C})$ the following are equivalent:*

(i) *T is* UESHM*;*

(ii) *$T = UT^{\mathrm{t}}U^*$ for some skew-symmetric unitary matrix U;*

(iii) *$T = -KT^*K$ for some anticonjugation K on C^n.*

In particular, if T is UESHM*, then T is* UET*. Moreover, for any of* (i)*,* (ii)*, or* (iii) *to hold n must be even.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $Q^*TQ = S$ where Q is unitary and S is skew-Hamiltonian. By (1.5) we have $S = \Omega S^{\mathrm{t}} \Omega^{\mathrm{t}}$ where Ω denotes the matrix (1.6). It follows that

$$Q^*TQ = S = \Omega S^{\mathrm{t}}\Omega^{\mathrm{t}} = (\Omega S \Omega^{\mathrm{t}})^{\mathrm{t}} = (\Omega Q^* T Q \Omega^{\mathrm{t}})^{\mathrm{t}} = \Omega Q^{\mathrm{t}} T^{\mathrm{t}} \overline{Q} \Omega^{\mathrm{t}}$$

whence $T = UT^{\mathrm{t}}U^*$ where $U = Q\Omega Q^{\mathrm{t}}$ is a skew-symmetric unitary matrix.

(ii) $\Rightarrow$ (iii) Suppose that $T = UT^{\mathrm{t}}U^*$ where $U = -U^{\mathrm{t}}$ is a unitary matrix. We claim that $K = UJ$ is an anticonjugation. Indeed, K is conjugate-linear, isometric, and satisfies

$$K^2 = (UJ)(UJ) = U(JUJ) = U\overline{U} = U(-U^*) = -I.$$

Putting this all together we find that

$$T = UT^{\mathrm{t}}U^* = U(JT^*J)(-\overline{U}) = -(UJ)T^*(UJ) = -KT^*K,$$

as desired.

(iii) $\Rightarrow$ (i) By Lemma 4.3 we know that n is even, say $n = 2d$. Let $e_1, e_2, \ldots, e_{2d}$ be the orthonormal basis provided by Lemma 4.4 and let $Q = (e_1|e_2|\cdots|e_{2d})$ be the $2d \times 2d$ unitary matrix whose columns are the basis vectors $e_1, e_2, \ldots, e_{2d}$. The ijth entry $[S]_{ij}$ of the matrix $S = Q^*TQ$ is given by the formula

$$\begin{aligned} [S]_{ij} &= \langle Te_j, e_i\rangle = \langle -KT^*Ke_j, e_i\rangle = \langle Ke_i, T^*Ke_j\rangle = \langle TKe_i, Ke_j\rangle \\ &= \left\{ \begin{array}{rcll} \langle Te_{i+d}, e_{j+d}\rangle &=& [S]_{j+d,i+d} & \text{if } 1 \leqslant i,j \leqslant d, \\ -\langle Te_{i+d}, e_{j-d}\rangle &=& -[S]_{j-d,i+d} & \text{if } 1 \leqslant i \leqslant d < j \leqslant 2d, \\ -\langle Te_{i-d}, e_{j+d}\rangle &=& -[S]_{j+d,i-d} & \text{if } 1 \leqslant j \leqslant d < i \leqslant 2d, \\ \langle Te_{i-d}, e_{j-d}\rangle &=& [S]_{j-d,i-d} & \text{if } d \leqslant i,j \leqslant 2d. \end{array} \right. \end{aligned}$$

In other words, S is of the form (1.4) whence T is UESHM. ∎

Putting the preceding material together, we obtain the following:

PROPOSITION 4.6. *If $T \in M_n(\mathbb{C})$ is* UESHM*, then T is unitarily equivalent to a direct sum of matrices, each of which is either*

(i) *an irreducible skew-Hamiltonian matrix;*

(ii) *a block matrix of the form $A \oplus A^{\mathrm{t}}$ where A is irreducible.*

Proof. The proofs of Lemmas 3.4 and 3.5 go through mutatis mutandis with K in place of C and $T = -KT^*K$ in place of $T = CT^*C$. One then mimics the proof of Proposition 3.6. ∎

Before proceeding, let us remark that it is possible to show that every skew-Hamiltonian matrix is *similar* to a block matrix of the form $A \oplus A^{\mathrm{t}}$ [32].

5. BUILDING BLOCK III: MATRICES OF THE FORM $A \oplus A^{\mathrm{t}}$

One of the difficulties in establishing Theorem 1.2 is the fact that there is a nontrivial overlap between the classes UECSM and UESHM. As Propositions 3.6 and 4.6 suggest, matrices of the form $A \oplus A^{\mathrm{t}}$ are both UECSM and UESHM.

LEMMA 5.1. *If $A \in M_d(\mathbb{C})$, then*

$$T = \begin{pmatrix} A & 0 \\ 0 & A^{\mathrm{t}} \end{pmatrix} \tag{5.1}$$

is both UECSM *and* UESHM. *In particular, any matrix of the form* (5.1) *is* UET.

Proof. Letting J denote the canonical conjugation (3.1) on $\mathbb{C}^n$, simply observe that $T = CT^*C$ and $T = -KT^*K$ where

$$C = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}.$$

Now appeal to Lemma 3.3 and Lemma 4.5. ∎

EXAMPLE 5.2. The matrix

$$T = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right)$$

is of the form (5.1) and hence is UET by Lemma 5.1. However, T has Halmos' matrix (1.1) as a direct summand and that specific matrix is known *not* to be UET ([16], Proposition 159). This example indicates that we cannot take a block diagonal matrix which is UET and conclude that the direct summands are also UET.

Lemma 5.1 asserts that matrices of the form $A \oplus A^{\mathrm{t}}$ are both UECSM and UESHM. However, by the nature of their construction such matrices are reducible. On the other hand, for *irreducible* matrices we have the following lemma.

LEMMA 5.3. *An irreducible matrix cannot be both* UESHM *and* UECSM.

Proof. Suppose toward a contradiction that T is an irreducible matrix which is both UECSM and UESHM. Since T^* also shares these same properties, by Lemma 3.3 and Lemma 4.5 there is a conjugation C and an anticonjugation K such that

$$CTC = T^* = -KTK. \tag{5.2}$$

Since $C^2 = I$ and $K^2 = -I$ we conclude from (5.2) that $(CK)T = T(CK)$ whence T commutes with the unitary operator $U = CK$. Since T is irreducible we conclude that $CK = \alpha I$ for some unimodular constant α. Multiplying both sides of the preceding by C we obtain $K = \overline{\alpha}C$ from which we obtain the contradiction

$$-I = K^2 = (\overline{\alpha}C)(\overline{\alpha}C) = |\alpha|^2 C^2 = I. \quad \blacksquare$$

We should pause to remark that Lemma 5.3 does not give a contradiction in the 2×2 case. Although every 2×2 matrix is UECSM by Lemma 2.4, every 2×2 matrix which is UESHM must actually be scalar (and hence reducible) since it is of the form $A \oplus A^{\mathrm{t}}$ for some 1×1 matrix A.

The following proposition provides a partial converse to the implication (1.3):

PROPOSITION 5.4. *If $T \in M_n(\mathbb{C})$ is irreducible and* UET, *then precisely one of the following is true:*

(i) *T is* UECSM*;*

(ii) *T is* UESHM.

If in addition n is odd, then T must be UECSM.

Proof. Suppose that $T = UT^{\mathrm{t}}U^*$ for some unitary matrix U. Taking the transpose of the preceding we obtain $T^{\mathrm{t}} = \overline{U}TU^{\mathrm{t}}$ whence

$$(U\overline{U})T = T(U\overline{U}).$$

Since T is irreducible and $U\overline{U}$ is unitary, it follows that $U\overline{U} = \alpha I$ for some $|\alpha| = 1$. From this we conclude that $U = \alpha U^{\mathrm{t}}$ whence $U^{\mathrm{t}} = \alpha U$ follows upon transposition.

Putting this all together, we conclude that $U = \alpha^2 U$ and consequently $\alpha^2 = 1$. If $\alpha = 1$, then $U = U^{\mathrm{t}}$ whence T is UECSM by Lemma 3.3.

On the other hand, if $\alpha = -1$, then $U = -U^{\mathrm{t}}$ whence T is UESHM by Lemma 4.5. The final statement also follows from Lemma 4.5. $\blacksquare$

Putting the preceding material together we obtain the following.

PROPOSITION 5.5. *If $T = A \oplus A^{\mathrm{t}}$, then T is unitarily equivalent to a direct sum of matrices, each of which is either*

(i) *an irreducible complex symmetric matrix;*

(ii) *an irreducible skew-Hamiltonian matrix;*

(iii) *a block matrix of the form $A_i \oplus A_i^{\mathrm{t}}$ where A_i is irreducible and neither* UECSM *nor* UESHM.

Proof. Suppose that A is irreducible. If A is not UET, then we can conclude that A is neither UECSM (by Lemma 3.3) nor UESHM (by Lemma 4.5). Thus T is already of the form (iii). On the other hand, if A is UET, then A is either UECSM or UESHM by Proposition 5.4. In either case, T is of the desired form.

$$\begin{tikzcd}
\boxed{T \text{ is UECSM}} \arrow[r, Rightarrow, "\text{Lem. 3.3}"] \arrow[d, Leftrightarrow, "\text{Lem. 3.3}"] & \boxed{T \text{ is UET}} & \boxed{T \text{ is UESHM}} \arrow[l, Rightarrow, "\text{Lem. 4.5}"'] \arrow[d, Leftrightarrow, "\text{Lem. 4.5}"] \\
\boxed{\begin{array}{c} T = CT^*C \\ \text{for some con-} \\ \text{jugation } C \end{array}} & \boxed{T \cong A \oplus A^{\mathrm{t}}} \arrow[l, Rightarrow, "\text{Lem. 5.1}"'] \arrow[r, Rightarrow, "\text{Lem. 5.1}"] \arrow[u, Rightarrow, "\text{Lemma 5.1}"] & \boxed{\begin{array}{c} T = -KT^*K \\ \text{for some anti-} \\ \text{conjugation } K \end{array}}
\end{tikzcd}$$

FIGURE 1. Relationship between the classes UET, UECSM, and UESHM. In general, the one-way implications cannot be reversed.

If A is reducible, then write $A \cong A_1 \oplus A_2 \oplus \cdots \oplus A_r$ where $A_1, A_2, \ldots, A_r$ are irreducible and note that

$$\begin{aligned} T = A \oplus A^{\mathrm{t}} &\cong (A_1 \oplus A_2 \oplus \cdots \oplus A_r) \oplus (A_1^{\mathrm{t}} \oplus A_2^{\mathrm{t}} \oplus \cdots \oplus A_r^{\mathrm{t}}) \\ &\cong (A_1 \oplus A_1^{\mathrm{t}}) \oplus (A_2 \oplus A_2^{\mathrm{t}}) \oplus \cdots \oplus (A_r \oplus A_r^{\mathrm{t}}). \end{aligned}$$

Now apply the first portion of the proof to each of the matrices $T_i = A_i \oplus A_i^{\mathrm{t}}$. ∎

6. SHMs IN DIMENSIONS 2, 4, AND 6 ARE REDUCIBLE

It turns out that skew-Hamiltonian matrices in dimensions 2, 4, and 6 are reducible whereas irreducible SHMs exist in dimensions $8, 10, 12, \ldots$ (see Section 7). This issue is highly nontrivial, for it is known that every skew-Hamiltonian matrix is *similar* to a block matrix of the form $A \oplus A^{\mathrm{t}}$ [32].

Since every 2×2 SHM is obviously scalar, we consider in this section the 4×4 and 6×6 cases.

LEMMA 6.1. *If $T \in M_4(\mathbb{C})$ is* SHM, *then T is reducible.*

Proof. Suppose that $T \in M_4(\mathbb{C})$ is of the form (1.4). By interchanging the second and third rows, and then the second and third columns of T, we may further assume that T is of the form

$$T = \begin{pmatrix} \lambda_1 I & X \\ \widetilde{X} & \lambda_2 I \end{pmatrix}, \tag{6.1}$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$, X is a 2×2 matrix, and $\widetilde{X}$ denotes the *adjugate* of X. In other words, we have

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \widetilde{X} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

where $X\widetilde{X} = \widetilde{X}X = (\det X)I$. Note that the adjugate operation satisfies $\widetilde{XY} = \widetilde{Y}\widetilde{X}$ and $\widetilde{X} = (\det X)X^{-1}$ if X is invertible.

Now write $X = UDW^*$ where U, W are unitary matrices satisfying $\det U = \det W = 1$ and D is a diagonal matrix having *complex* entries (this factorization can easily be obtained from the singular value decomposition of X). Plugging this into (6.1) we find that

$$
\begin{aligned}
T &= \begin{pmatrix} \lambda_1 I & UDW^* \\ \widetilde{UDW^*} & \lambda_2 I \end{pmatrix} = \begin{pmatrix} \lambda_1 I & UDW^* \\ W\widetilde{D}U^* & \lambda_2 I \end{pmatrix} \\
&= \begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} \lambda_1 I & D \\ \widetilde{D} & \lambda_2 I \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & W^* \end{pmatrix}.
\end{aligned}
$$

Writing

$$D = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix},$$

where $\xi_1, \xi_2 \in \mathbb{C}$, we find that

$$T \cong \begin{pmatrix} \lambda_1 I & D \\ \widetilde{D} & \lambda_2 I \end{pmatrix} = \left(\begin{array}{cc|cc} \lambda_1 & 0 & \xi_1 & 0 \\ 0 & \lambda_1 & 0 & \xi_2 \\ \hline \xi_2 & 0 & \lambda_2 & 0 \\ 0 & \xi_1 & 0 & \lambda_2 \end{array}\right) \cong \left(\begin{array}{cc|cc} \lambda_1 & \xi_1 & 0 & 0 \\ \xi_2 & \lambda_2 & 0 & 0 \\ \hline 0 & 0 & \lambda_1 & \xi_2 \\ 0 & 0 & \xi_1 & \lambda_2 \end{array}\right). \quad \blacksquare$$

While it is true that every 6×6 SHM is reducible, the proof is not nearly as simple as that of Lemma 6.1. Unfortunately, we were unable to come up with a proof that did not involve a significant amount of symbolic computation. Nevertheless, the techniques involved are relatively simple and the motivated reader should have no trouble verifying the calculations described below.

LEMMA 6.2. *If $T \in M_6(\mathbb{C})$ is* SHM, *then T is reducible.*

Proof. Let T be a 6×6 matrix of the form (1.4). We intend to show that T commutes with a non-scalar selfadjoint matrix Q. However, attempting to consider $QT = TQ$ as a system of $36 \times 2 = 72$ real equations in $15 \times 2 + 6 = 36$ real variables is not computationally feasible since the resulting 72×36 system is symbolic, rather than numeric. Several simplifications are needed before our problem becomes tractable.

First let us note that if $U \in M_3(\mathbb{C})$ is unitary then

$$\begin{pmatrix} U & 0 \\ 0 & \overline{U} \end{pmatrix} \begin{pmatrix} A & B \\ D & A^{\mathrm{t}} \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & U^{\mathrm{t}} \end{pmatrix} = \begin{pmatrix} UAU^* & UBU^{\mathrm{t}} \\ \overline{U}DU^* & (UAU^*)^{\mathrm{t}} \end{pmatrix} \tag{6.2}$$

where UBU^{t} and $\overline{U}DU^*$ are skew-symmetric. Without loss of generality, we may therefore assume that A is upper-triangular. Furthermore, by scaling and subtracting a multiple of the identity from T we can also assume that at most one of the diagonal entries of A is non-real while the others are either 0 or 1.

Appealing to (6.2) again where U is now a suitable diagonal unitary matrix, we can further arrange things so that that the skew-symmetric matrix UBU^{t} has only real entries. The nonzero off-diagonal entries of the upper-triangular matrix UAU^* may change depending on U, but the diagonal entries are unaffected. Thus we may further assume that B has only real entries.

Now recall from (1.5) that T is SHM if and only if $T = -\Omega T^{\mathrm{t}}\Omega$ where Ω denotes the matrix (1.6). If $Q = Q^*$ and $QT = TQ$, then clearly $Q^{\mathrm{t}}T^{\mathrm{t}} = T^{\mathrm{t}}Q^{\mathrm{t}}$ whence $\Omega Q^{\mathrm{t}}\Omega$ also commutes with T. Let us therefore consider selfadjoint matrices Q which satisfy $Q = \Omega Q^{\mathrm{t}}\Omega$. In other words, we restrict our attention to matrices of the form

$$Q = \begin{pmatrix} X & Y \\ Y^* & -X^{\mathrm{t}} \end{pmatrix} \tag{6.3}$$

where $X = X^*$ and $Y = Y^{\mathrm{t}}$. There are now a total of $9 + 12 = 21$ real unknowns arising from the components of X and Y.

Expanding out the system $QT = TQ$ we obtain

$$XA - AX + YD - BY^* = 0, \tag{6.4}$$

$$XB + BX^{\mathrm{t}} + YA^{\mathrm{t}} - AY = 0, \tag{6.5}$$

$$DX + X^{\mathrm{t}}D + A^{\mathrm{t}}Y^* - Y^*A = 0, \tag{6.6}$$

$$A^{\mathrm{t}}X^{\mathrm{t}} - X^{\mathrm{t}}A^{\mathrm{t}} - DY + Y^*B = 0, \tag{6.7}$$

which yields a system of 72 real equations in 21 unknowns. However it is clear that (6.4) and (6.7) are transposes of each other and hence (6.7) can be ignored. Moreover, (6.5) and (6.6) are skew-symmetric and hence we obtain a system of $18 + 6 + 6 = 30$ real equations in 21 unknowns.

With the preceding reductions in hand, it becomes possible to compute the rank of the system symbolically via `Mathematica`. In particular, the `MatrixRank` command computes the rank of a symbolic matrix under the assumption that the distinct symbols appearing as coefficients in the system are linearly independent. By considering separately the cases where A has either one, two, or three distinct eigenvalues one can conclude that the rank of our system is always $\leqslant 20$ whence a nontrivial solution to our problem exists. By (6.3) it is clear that the resulting Q cannot be a multiple of the identity whence T is reducible. ∎

7. IRREDUCIBLE SHMs EXIST IN DIMENSIONS $n = 8, 10, 12, \ldots$

We now turn our attention to the task of proving that irreducible SHMs exist in dimensions $n = 8, 10, 12, \ldots$. As we have mentioned before, this is nontrivial due to the fact that every skew-Hamiltonian matrix *similar* to a block matrix of the form $A \oplus A^{\mathrm{t}}$ [32].

PROPOSITION 7.1. *For each $d \geqslant 4$ the $2d \times 2d$ matrix*

$$T = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & d \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & 1 & \ddots & 1 & 1 \\ -1 & -1 & 0 & \ddots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ -1 & -1 & -1 & \ddots & 0 & 1 \\ -1 & -1 & -1 & \cdots & -1 & 0 \end{pmatrix}, \tag{7.1}$$

is skew-Hamiltonian and irreducible.

Proof. Since $A = A^{\mathrm{t}}$ and $B^{\mathrm{t}} = -B$, it is clear that T is SHM. It therefore suffices to prove that T is irreducible. To this end, suppose that Q is a selfadjoint matrix which satisfies $QT = TQ$. Writing

$$Q = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}, \tag{7.2}$$

where $X = X^*$ and $Z = Z^*$, we find that

$$XA = AX + BY^*, \tag{7.3}$$

$$XB + YA = AY + BZ, \tag{7.4}$$

$$Y^*A = AY^*, \tag{7.5}$$

$$Y^*B + ZA = AZ. \tag{7.6}$$

Examining (7.5) entry-by-entry reveals that Y^* is diagonal. In particular, $YA = AY$ and hence

$$XB = BZ \tag{7.7}$$

follows from (7.4). By (7.3) we have

$$\underbrace{XA - AX}_{\text{skew-selfadjoint}} = BY^*. \tag{7.8}$$

Since Y^* is diagonal, a short computation shows that BY^* is skew-selfadjoint if and only if $Y^* = \alpha I$ for some $\alpha \in \mathbb{R}$ (this requires $d \geqslant 3$). We may therefore rewrite (7.8) as

$$XA - AX = \alpha B. \tag{7.9}$$

Equation (7.6) now assumes the similar form

$$ZA - AZ = -\alpha B. \tag{7.10}$$

Adding (7.9) and (7.10) together we find that

$$(X + Z)A = A(X + Z)$$

whence, since A is diagonal and has distinct eigenvalues, the matrix $X + Z = D$ is also diagonal. Plugging this into (7.7) yields

$$\underbrace{XB + BX}_{\text{skew-selfadjoint}} = BD.$$

The same reasoning employed in analyzing (7.8) now reveals that $D = 2\delta I$ for some $\delta \in \mathbb{R}$. Since $X + Z = 2\delta I$ we conclude that

$$(X - \delta I) = -(Z - \delta I). \tag{7.11}$$

At this point we observe that $Q + \delta I$ also commutes with T, and hence upon making the substitutions $X \mapsto X + \delta I$ and $Z \mapsto Z + \delta I$ in (7.2) we may assume that $X = -Z$. Plugging this into (7.7) we see that

$$XB = -BX. \tag{7.12}$$

From equations (7.9) and (7.12) we shall derive a number of constraints upon the entries of X which can be shown to be mutually incompatible unless $X = Y = 0$.

Examining (7.9) entry-by-entry, we find that the ijth entry x_{ij} of X is given by

$$x_{ij} = \frac{\alpha}{|j - i|}$$

for $i \neq j$. Thus

$$X = \begin{pmatrix} x_{11} & \alpha & \frac{\alpha}{2} & \frac{\alpha}{3} & \cdots & \frac{\alpha}{d-1} \\ \alpha & x_{22} & \alpha & \frac{\alpha}{2} & \cdots & \frac{\alpha}{d-2} \\ \frac{\alpha}{2} & \alpha & x_{33} & \alpha & \cdots & \frac{\alpha}{d-3} \\ \frac{\alpha}{3} & \frac{\alpha}{2} & \alpha & x_{44} & \cdots & \frac{\alpha}{d-4} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \frac{\alpha}{d-1} & \frac{\alpha}{d-2} & \frac{\alpha}{d-3} & \frac{\alpha}{d-4} & \cdots & x_{dd} \end{pmatrix} \tag{7.13}$$

where the diagonal entries $x_{11}, x_{22}, \ldots, x_{dd}$ are to be determined.

On the other hand, from (7.12) it follows that

$$(UX)B = B(UX) \tag{7.14}$$

since $UBU = -B$ where

$$U = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

From (7.14) we wish to conclude that $UX = p(B)$ for some polynomial $p(z)$. This will follow if we can show that B has distinct eigenvalues.

A short computation first reveals

$$2(B+I)^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

whence

$$I - 2(B+I)^{-1} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}}_{L}.$$

Cofactor expansion along the first column shows that L has the characteristic polynomial $z^d + 1$, whose roots are precisely the dth roots of -1. From this we conclude that the skew-symmetric matrix B has d distinct eigenvalues (namely the numbers $(1+\zeta_i)/(1-\zeta_i)$ where $\zeta_1, \zeta_2, \ldots, \zeta_d$ are the dth roots of -1).

Since 1 is not an eigenvalue of L it follows that

$$B = 2(I-L)^{-1} - I = q(L)$$

is a polynomial in L (the fact that $(I-L)^{-1}$ is a polynomial in L follows from the Cayley-Hamilton theorem). Thus $UX = p(B) = p(q(L))$ is also a polynomial in L. Now observe that $L^d = -I$ and that each of the matrices $I, L, L^2, \ldots, L^{d-1}$ is a Toeplitz matrix. Therefore UX is a Toeplitz matrix whence X is a Hankel matrix.

Recalling that $d \geqslant 4$ and looking back to the explicit formula (7.13) for X, we see that X cannot be a Hankel matrix unless $\alpha = 0$ and $x_{11} = x_{22} = \cdots = x_{dd} = 0$. In other words, it must be the case that $X = Y = 0$ whence $Q = 0$ (note that we considered $Q + \delta I$ in place of Q after the step (7.11)). Thus T is irreducible. ∎

Observe that the argument above fails if $d \leqslant 3$ since setting $x_{22} = \alpha/2$ in (7.13) yields a Hankel matrix. In this case, one can verify directly that T commutes with

$$\left(\begin{array}{ccc|ccc} -1 & 1 & \frac{1}{2} & 1 & 0 & 0 \\ 1 & \frac{1}{2} & 1 & 0 & 1 & 0 \\ \frac{1}{2} & 1 & -1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & -1 & 1 \end{array}\right).$$

As anticipated in the proof of Lemma 6.2, the preceding matrix is of the form (6.3).

8. PROOF OF THEOREM 1.2

This entire section is devoted to the proof of Theorem 1.2. For the sake of readability, it is divided into several subsections.

8.1. THE SIMPLE DIRECTION. One implication of the proof is now trivial. If T is unitarily equivalent to a direct sum of CSMs, SHMs, and matrices of the form $A \oplus A^{\mathrm{t}}$, then T is UET by Lemmas 3.3, 4.5, and 5.1.

8.2. INITIAL SETUP. We now focus on the more difficult implication of Theorem 1.2. If $T \in M_n(\mathbb{C})$ is UET, then there exists a unitary matrix U such that

$$T = UT^{\mathrm{t}}U^*. \tag{8.1}$$

Taking the transpose of the preceding and solving for T^{t} we obtain $T^{\mathrm{t}} = \overline{U}TU^{\mathrm{t}}$ whence

$$(U\overline{U})T = T(U\overline{U}). \tag{8.2}$$

In light of (8.2) we are therefore led to consider the unitary matrix

$$V = U\overline{U}.$$

The following lemma lists several restrictions on the eigenvalues of V which will be useful in what follows. We denote by $\sigma(A)$ the set of eigenvalues of a matrix $A \in M_n(\mathbb{C})$.

LEMMA 8.1. *If U is a unitary matrix and $V = U\overline{U}$, then* $\det V = 1$ *and* $\sigma(V) = \overline{\sigma(V)}$. *In particular, the eigenvalues of V are restricted to:*

(i) 1;

(ii) -1, *with even multiplicity;*

(iii) *complex conjugate pairs $\lambda, \overline{\lambda}$ where $\lambda \neq \pm 1$ and both λ and $\overline{\lambda}$ have the same multiplicity.*

Proof. Since U is unitary it follows that

$$\det V = \det U\overline{U} = (\det U)(\det \overline{U}) = (\det U)\overline{(\det U)} = |\det U|^2 = 1.$$

Since U is invertible, $U\overline{U}$ and $\overline{U}U$ have the same characteristic polynomial whence

$$\dim\ker(U\overline{U} - zI) = \dim\ker(\overline{U}U - \overline{z}I) = \dim\ker(U\overline{U} - \overline{z}I)$$

holds for all $z \in \mathbb{C}$. Among other things, this establishes (iii) and shows that $\sigma(V) = \overline{\sigma(V)}$. Conditions (i) and (ii) now follow from (iii) and the fact that $\det V = 1$. ∎

Now observe that

$$U\overline{V} = VU \tag{8.3}$$

holds since both sides of (8.3) equal $U\overline{U}U$. By the Spectral Theorem, there exists a unitary matrix W so that

$$V = W^* D W \tag{8.4}$$

where (by Lemma 8.1) D is a block diagonal matrix of the form

$$D = \left(\begin{array}{c|c|cc|c|cc} I & & & & & & \\ \hline & -I & & & & & \\ \hline & & \lambda_1 I & & & & \\ & & & \overline{\lambda_1} I & & & \\ \hline & & & & \ddots & & \\ \hline & & & & & \lambda_r I & \\ & & & & & & \overline{\lambda_r} I \end{array}\right) \tag{8.5}$$

for some distinct unimodular constants $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that

(i) $\lambda_i \neq \pm 1$ for $i = 1, 2, \ldots, r$;

(ii) $\lambda_i \neq \overline{\lambda_j}$ for $1 \leqslant i, j \leqslant r$.

Let us make several remarks about the matrix (8.5). First, it is possible that some of the blocks may be absent depending upon V. Second, the blocks corresponding to conjugate eigenvalues λ_i and $\overline{\lambda_i}$ must be of the same size by Lemma 8.1.

8.3. THE MATRIX Q. Substituting (8.4) into (8.3) we find that

$$Q\overline{D} = DQ \tag{8.6}$$

where

$$Q = WUW^{\mathrm{t}} \tag{8.7}$$

is unitary. In light of (8.6) and the structure of D given in (8.5), a short matrix computation reveals that

$$Q = \left(\begin{array}{c|c|cc|c|cc} Q_+ & & & & & & \\ \hline & Q_- & & & & & \\ \hline & & 0 & Y_1 & & & \\ & & X_1 & 0 & & & \\ \hline & & & & \ddots & & \\ \hline & & & & & 0 & Y_r \\ & & & & & X_r & 0 \end{array}\right) \tag{8.8}$$

where $Q_+, Q_-, X_1, Y_1, \ldots, X_r, Y_r$ are unitary matrices. Next observe that

$$Q\overline{Q} = (WUW^{\mathrm{t}})(\overline{WU}W^*) = WU\overline{U}W^* = WVW^* = D$$

by (8.7) and (8.4). Using the fact that Q is unitary we conclude from the preceding that

$$Q = DQ^{\mathrm{t}}. \tag{8.9}$$

This gives us further insight into the structure of Q. Using the block matrix decompositions (8.5) and (8.8) and examining (8.9) block-by-block we conclude that

(i) $Q_+ = Q_+^{\mathrm{t}}$ (i.e., Q_+ is complex symmetric and unitary);

(ii) $Q_- = -Q_-^{\mathrm{t}}$ (i.e., Q_- is skew-symmetric and unitary);

(iii) $Y_i = \lambda_i X_i^{\mathrm{t}}$ where X_i is unitary and $\lambda_i \neq \pm 1$ for $i = 1, 2, \ldots, r$.

Thus we may write

$$Q = \left(\begin{array}{c|c|cc|c|cc} Q_+ & & & & & & \\ \hline & Q_- & & & & & \\ \hline & & 0 & \lambda_1 X_1^{\mathrm{t}} & & & \\ & & X_1 & 0 & & & \\ \hline & & & & \ddots & & \\ \hline & & & & & 0 & \lambda_r X_r^{\mathrm{t}} \\ & & & & & X_r & 0 \end{array}\right). \tag{8.10}$$

It is important to note that some of the blocks in (8.10) may be absent, depending upon the decomposition (8.5) of D.

8.4. SIMPLIFYING THE PROBLEM. Now that we have a concrete description of Q in hand, let us return to our original equation (8.1). In light of (8.7) we see that

$$U = W^* Q \overline{W}$$

whence by (8.1) it follows that

$$T = (W^* Q \overline{W}) T^{\mathrm{t}} (W^{\mathrm{t}} Q^* W) = W^* Q (W T W^*)^{\mathrm{t}} Q^* W.$$

Rearranging this reveals that

$$(W T W^*) Q = Q (W T W^*)^{\mathrm{t}}.$$

Since $T \cong WTW^*$, it follows that in order to characterize, up to unitary equivalence, those matrices T which are UET we need only consider those T which satisfy

$$TQ = QT^{\mathrm{t}} \tag{8.11}$$

where Q is a unitary matrix of the special form (8.10) satisfying conditions (i), (ii), and (iii) above.

8.5. DESCRIBING T. Taking the transpose of (8.11), solving for T^{t}, and substituting the result back into (8.11) reveals that $(Q\overline{Q})T = T(Q\overline{Q})$. Thus T is actually block-diagonal, the sizes of the corresponding blocks being determined by the decomposition (8.10) of Q itself. Returning to (8.11), we find that T has the form

$$T = \left(\begin{array}{c|c|cc|c|cc}
T_+ & & & & & & \\ \hline
& T_- & & & & & \\ \hline
& & A_1 & 0 & & & \\
& & 0 & X_1 A_1^{\mathrm{t}} X_1^* & & & \\ \hline
& & & & \ddots & & \\ \hline
& & & & & A_r & 0 \\
& & & & & 0 & X_r A_r^{\mathrm{t}} X_r^*
\end{array}\right) \tag{8.12}$$

where

(i) $T_+ = Q_+ T_+^{\mathrm{t}} Q_+^*$. Since Q_+ is a symmetric unitary matrix, Lemma 3.3 implies that T_+ is UECSM.

(ii) $T_- = Q_- T_-^{\mathrm{t}} Q_-^*$. Since Q_- is a skew-symmetric unitary matrix, Lemma 4.5 implies that T_- is UESHM.

(iii) $A_1, A_2, \ldots, A_r$ are arbitrary.

Obtaining (i) and (ii) from (8.11) is straightforward. Let us say a few words about (iii). Examining (8.11) we obtain r equations of the form

$$\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \begin{pmatrix} 0 & \lambda_i X_i^{\mathrm{t}} \\ X_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda_i X_i^{\mathrm{t}} \\ X_i & 0 \end{pmatrix} \begin{pmatrix} A_i^{\mathrm{t}} & C_i^{\mathrm{t}} \\ B_i^{\mathrm{t}} & D_i^{\mathrm{t}} \end{pmatrix}.$$

This leads to $B_i X_i = \lambda_i X_i^{\mathrm{t}} B_i^{\mathrm{t}} = \lambda_i (B_i X_i)^{\mathrm{t}}$ whence $B_i X_i = \lambda_i^2 B_i X_i$. Since $\lambda_i \neq \pm 1$ and X_i is invertible we conclude that $B_i = 0$. Similarly we find that $C_i = 0$ and $D_i = X_i A_i^{\mathrm{t}} X_i^*$.

It is evident at this point that T is unitarily equivalent to the direct sum of a CSM, an SHM, and blocks of the form $A \oplus A^{\mathrm{t}}$. As usual, some of the blocks in (8.12) may be absent. The exact form of (8.12) depends upon the decomposition (8.10).

By Proposition 3.6, the block T_+ is unitarily equivalent to a direct sum of matrices of type I or III. Similarly, Proposition 4.6 asserts that T_- is unitarily equivalent to a direct sum of matrices of type II or III. Finally, Proposition 5.5 permits us to decompose the resulting type III blocks and the 2×2 block matrices from (8.12) into a direct sum of matrices of type I, II, or III.

The restrictions on the dimensions of type II and type III blocks follow immediately from the results of Sections 6 and 7 and Lemma 2.4. This establishes the existence of the desired decomposition.

For the final statement of Theorem 1.2, first note that a matrix of type III is reducible and hence cannot belong to the unitary orbit of a type I or type II matrix. That the unitary orbits of a type I matrix and a type II matrix are disjoint follows from Lemma 5.3. ∎

Acknowledgements. The first author was partially supported by NSF Grants DMS-0638789 and DMS-1001614. The second author was partially supported by a NSF Graduate Research Fellowship.

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Received February 8, 2010.